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A ROBIN-ROBIN NON-OVERLAPPING DOMAIN DECOMPOSITION METHOD FOR AN ELLIPTIC BOUNDARY CONTROL PROBLEM

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Abstract. A Robin-Robin non-overlapping domain decomposition method for an optimal boundary control problem associated with an elliptic boundary value problem is presented. The existence of the whole domain and subdomain optimal solutions is proven. The convergence of the subdomain optimal solutions to the whole domain optimal solution is shown. The optimality system is derived and a gradient-type method is defined for finding the optimal solution. A theoretic convergence result for the gradient method is established. The finite element version of the Robin-Robin non-overlapping domain decomposition method is analyzed and some numerical results by the method on both serial and parallel computers (using MPI) are presented.

Key Words. Nonoverlapping domain decomposition method, elliptic boundary control problem, elliptic boundary value problem, MPI.

1. Introduction

Domain decomposition methods have the subject of extensive study in the last few decades; see, e.g., www.ddm.org. An important class of non-overlapping domain decomposition method is the Robin-Robin type methods based on successive exchanges of interface Robin data [10, 11, 7]. In this paper we design and analyze a Robin-Robin non-overlapping domain decomposition method for solving an optimal boundary control problem constrained by the second order elliptic partial differential equation (PDE). In addition, we develop both serial and parallel (MPI) codes to give some numerical results.

The content of this paper is as follows. In Section 2.1, we introduce the whole-domain and subdomain optimal boundary control problems. In Section 2.2, we prove the existence of the whole domain optimal solution and the subdomain optimal solution. In Section 2.3, we show that the subdomain optimal solution converges weakly to the whole domain optimal solution. In Section 2.4, we use the method of Lagrange multiplier to derive the optimality system of equations. In Section 2.5, we define a gradient method for our optimal boundary control problem on the subdomain and prove the theoretic convergence of the method. In Section 3, we analyzed the finite element version of the Robin-Robin non-overlapping domain decomposition method in the same way as we did the continuous version. Finally, in Section 4, we use both serial and parallel computers to present numerical results. In the parallel computing, we use MPI, the Message Passing Interface, for the communication between computer processors.

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2. Optimal boundary control problems

In this chapter, we solve an optimal control problem constrained by the general second order elliptic PDE under the Neumann boundary condition using the Robin type non-overlapping domain decomposition method (DDM).

2.1. The model problem. We consider the general second order elliptic PDE under the Neumann boundary condition:

$$-\text{div}\,[A(\mathbf{x})\nabla u] + \mathbf{b}(\mathbf{x})\cdot\nabla u + c(\mathbf{x})u = f \quad \text{in } \Omega, \qquad [A(\mathbf{x})\nabla u]\cdot\mathbf{n} = p \quad \text{on } \Gamma, \tag{2.1}$$

where Ω is an open, bounded subset of $\mathbb{R}^2$ with boundary Γ, $u : \bar{\Omega} \to \mathbb{R}$ is the unknown, A is a symmetric-matrix-valued $L^\infty(\Omega)$ function that is uniformly positive definite, $\mathbf{b}$ is a vector-valued $L^\infty(\Omega)$ function, c is a real-valued $L^\infty(\Omega)$ function, $f \in L^2(\Omega)$, $\mathbf{n}$ is the outward normal to Ω, and $p \in L^2(\Gamma)$ is a flexible boundary input data called a boundary control.

Here, we optimize the following cost functional subject to (2.1):

$$\mathcal{J}_\beta(u,p) = \frac{1}{2}\int_\Omega |u-U|^2 d\Omega + \frac{\beta}{2}\int_\Gamma p^2 d\Gamma, \tag{2.2}$$

where U is a given target solution and β is a positive constant.

In this paper, in order to minimize (2.2) using DDM, we partition the whole domain Ω into two subdomains Ω_1 and Ω_2. Then we denote a new boundary by Γ_0, separate the original boundary into $\Gamma_1 = \partial\Omega_1\backslash\Gamma_0$ and $\Gamma_2 = \partial\Omega_2\backslash\Gamma_0$. (see Figure 2.1)

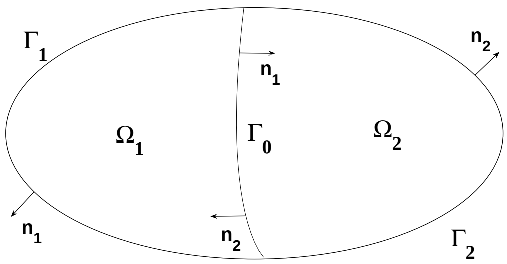

FIGURE 2.1. Two subdomains

In Figure 2.1, $\mathbf{n}_i$ is the unit outward normal to $\partial\Omega_i$.

Before we solve this problem, we introduce notation:

$$[u,v]_\mathcal{D} = \int_\mathcal{D} uv\; d\mathcal{D} \quad \forall u,v \in L^2(\mathcal{D}) \quad \text{and}$$

$$a[u,v] = \int_\Omega [A(\mathbf{x})\nabla u\cdot\nabla v + (\mathbf{b}(\mathbf{x})\cdot\nabla u)v + c(\mathbf{x})uv]\; d\Omega \quad \forall u,v \in H^1(\Omega),$$

where $H^1(\Omega)$ is the standard Sobolev space (see [1]).

Under the notation, we have the weak formulation of (2.1): seek $u \in H^1(\Omega)$ such that

$$a[u,v] = [f,v]_\Omega + [p,v]_\Gamma \quad \forall v \in H^1(\Omega). \tag{2.3}$$

We assume here that, throughout the paper, our bilinear forms are coersive; e.g., in (2.3), there is a constant $C > 0$ such that

$$a[u,u] \geq C\|u\|^2_{1,\Omega} \quad \forall u \in H^1(\Omega) \tag{2.4}$$

to ensure the existence of the solution of our PDE.

In order to solve the problem using DDM, we define subdomain problems with the Robin boundary condition on the interface Γ_0: for a $\lambda > 0$,

$$
\begin{aligned}
-\mathrm{div}\,[A(\mathbf{x})\nabla u_i] + \mathbf{b}(\mathbf{x})\cdot\nabla u_i + c(\mathbf{x})u_i &= f_i \quad \text{in } \Omega_i,\\
[A(\mathbf{x})\nabla u_i]\cdot\mathbf{n}_i &= p_i \quad \text{on } \Gamma_i, \quad \text{and}\\
u_i + \lambda[A(\mathbf{x})\nabla u_i]\cdot\mathbf{n}_i &= g_i \quad \text{on } \Gamma_0,
\end{aligned}
\tag{2.5}
$$

where $g_i \in L^2(\Gamma_0)$ is another control. Then we have the weak formulation of (2.5): seek a $u_i \in H^1(\Omega_i)$ such that

$$
a_i[u_i, v_i] + \lambda^{-1}[u_i, v_i]_{\Gamma_0} = [f_i, v_i]_{\Omega_i} + [p_i, v_i]_{\Gamma_i} + \lambda^{-1}[g_i, v_i]_{\Gamma_0} \quad \forall v_i \in H^1(\Omega_i). \tag{2.6}
$$

In two subdomains problem, the cost functional $\mathcal{J}_\beta$ becomes

$$
\mathcal{K}_\beta(u_1, u_2, p_1, p_2) = \frac{1}{2}\sum_{i=1}^{2}\int_{\Omega_i} |u_i - U|^2 d\Omega + \frac{\beta}{2}\sum_{i=1}^{2}\int_{\Gamma_i} p_i^2 d\Gamma. \tag{2.7}
$$

Note that we need transmission conditions to have the identical solution from the whole domain and subdomain problems. In our case, the conditions are $u_1 = u_2$ and $[A(\mathbf{x})\nabla u_1]\cdot\mathbf{n}_1 = -[A(\mathbf{x})\nabla u_2]\cdot\mathbf{n}_2$ on Γ_0. To satisfy this, we should make $u_1 - u_2 = 0$ and choose g_1 and g_2 such that $g_1 + g_2 - u_1 - u_2 = 0$ on Γ_0. For this, we consider a new functional as follows:

$$
\mathcal{G}(u_1, u_2, g_1, g_2) = \int_{\Gamma_0} |u_1 - u_2|^2 + |g_1 + g_2 - u_1 - u_2|^2 d\Gamma. \tag{2.8}
$$

We, thus, combine two functionals (2.7) and (2.8) with a positive constant σ:

$$
\mathcal{E}_{\beta\sigma}(u_1, u_2, p_1, p_2, g_1, g_2) = \mathcal{K}_\beta(u_1, u_2, p_1, p_2) + \frac{1}{2\sigma}\mathcal{G}(u_1, u_2, g_1, g_2). \tag{2.9}
$$

Then we set our functional to be optimized using boundary controls p_i and g_i with additional δ term to ensure the convergence in Section 2.3:

$$
\mathcal{E}_{\beta\delta\sigma}(u_1, u_2, p_1, p_2, g_1, g_2) = \mathcal{E}_{\beta\sigma}(u_1, u_2, p_1, p_2, g_1, g_2) + \frac{\delta}{2}\sum_{i=1}^{2}\int_{\Gamma_0} g_i^2 d\Gamma \tag{2.10}
$$

where δ is a positive constant.

2.2. The existence of an optimal solution. In this section, we prove the existence of an optimal solution on the whole domain and then on the subdomains. Let

$$
\mathcal{U}_{ad} = \{(u, p) \in H^1(\Omega)\times L^2(\Gamma) \mid (2.3) \text{ satisfied and } \mathcal{J}_\beta(u, p) < \infty\} \tag{2.11}
$$

be the admissibility set and $(\hat{u}, \hat{p}) \in \mathcal{U}_{ad}$ be an *optimal solution* of $\mathcal{J}_\beta$ if there exists $\epsilon > 0$ such that $\mathcal{J}_\beta(\hat{u}, \hat{p}) \le \mathcal{J}_\beta(u, p)$ for all $(u, p) \in \mathcal{U}_{ad}$ satisfying $\|u - \hat{u}\|_{1,\Omega} + \|p - \hat{p}\|_{0,\Gamma} \le \epsilon$. Then we have the following theorem for the optimal solution on the whole domain.

Theorem 2.1. *There is a unique optimal solution* $(\hat{u}, \hat{p}) \in \mathcal{U}_{ad}$.

PROOF: Note that there is a minimizing sequence $\{(u^{(n)}, p^{(n)})\}$ in $\mathcal{U}_{ad}$ such that

$$
\lim_{n\to\infty} \mathcal{J}_\beta(u^{(n)}, p^{(n)}) = \inf_{(u,p)\in\mathcal{U}_{ad}} \mathcal{J}_\beta(u, p). \tag{2.12}
$$

From (2.11), we have a sequence $\{p^{(n)}\}$ that is uniformly bounded in $L^2(\Gamma)$. Note also that there is a constant $C > 0$ such that

$$
\|u^{(n)}\|_{1,\Omega} \le C(\|f\|_{0,\Omega} + \|p^{(n)}\|_{0,\Gamma}). \tag{2.13}
$$

Thus, $\{u^{(n)}\}$ is uniformly bounded in $H^1(\Omega)$. Consequently, there is a subsequence $\{(u^{(n_i)}, p^{(n_i)})\}$ such that

$$u^{(n_i)} \rightharpoonup \hat{u} \text{ weakly} \quad \text{in} \quad H^1(\Omega) \quad \text{and} \quad p^{(n_i)} \rightharpoonup \hat{p} \text{ weakly} \quad \text{in} \quad L^2(\Gamma) \tag{2.14}$$

for some $(\hat{u}, \hat{p}) \in H^1(\Omega) \times L^2(\Gamma)$. Thus, we have

$$a[\hat{u}, v] = \lim_{n_i \to \infty} a[u^{(n_i)}, v] = [f, v]_\Omega + \lim_{n_i \to \infty} [p^{(n_i)}, v]_\Gamma = [f, v]_\Omega + [\hat{p}, v]_\Gamma \tag{2.15}$$

Hence, $(\hat{u}, \hat{p}) \in \mathcal{U}_{ad}$.

Since $\mathcal{J}_\beta$ is weakly lower semicontinuous (see [3]), (2.14) implies that

$$\mathcal{J}_\beta(\hat{u}, \hat{p}) \le \lim_{n_i \to \infty} \inf \mathcal{J}_\beta(u^{(n_i)}, p^{(n_i)}) = \inf_{(u,p) \in \mathcal{U}_{ad}} \mathcal{J}_\beta(u, p) \tag{2.16}$$

Therefore, $(\hat{u}, \hat{p})$ is an optimal solution.

On the other hand, uniqueness follows from the strict convexity of the functional, the convexity of $\mathcal{U}_{ad}$, and the linearity of the constraints. □

We now define the admissibility set for the subdomain optimal control problem

$$\begin{aligned} \mathcal{W}_{ad} = &\{(u_1, u_2, p_1, p_2, g_1, g_2) \in H^1(\Omega_1) \times H^1(\Omega_2) \times L^2(\Gamma_1) \times L^2(\Gamma_2) \times [L^2(\Gamma_0)]^2 \\ &\text{such that (2.6) satisfied and } \mathcal{E}_{\beta\delta\sigma}(u_1, u_2, p_1, p_2, g_1, g_2) < \infty\} \end{aligned} \tag{2.17}$$

and another bilinear form $b_i[u_i, v_i] := a_i[u_i, v_i] + \lambda^{-1}[u_i, v_i]_{\Gamma_0}$. Note that there is a unique solution of (2.6) and that there is a positive constants C_i such that

$$\|u_i\|_{1,\Omega_i} \le C_i(\|f_i\|_{0,\Omega_i} + \|p_i\|_{0,\Gamma_i} + \|g_i\|_{0,\Gamma_0}). \tag{2.18}$$

Then the existence of the solution for the optimization problem on the subdmains follows.

Theorem 2.2. *There is a unique optimal solution in* $\mathcal{W}_{ad}$.

2.3. Convergence of the subdomain optimal solution to whole domain optimal solution. In this section, we show that solutions of $\mathcal{E}_{\beta\delta\sigma}$ converges weakly to the optimal solution of $\mathcal{J}_\beta$ as $\delta, \sigma \to 0$. Here we introduce notation $(u, v)^{\beta\delta\sigma}$ which means $(u^{\beta\delta\sigma}, v^{\beta\delta\sigma})$ and $(u \pm v)^{\beta\delta\sigma}$ which means $(u^{\beta\delta\sigma} \pm v^{\beta\delta\sigma})$ for simplicity.

Theorem 2.3. *For each* $\delta, \sigma > 0$, *let* $(u_1, u_2, p_1, p_2, g_1, g_2)^{\beta\delta\sigma} \in \mathcal{W}_{ad}$ *denote an optimal solution of* $\mathcal{E}_{\beta\delta\sigma}$. *Let* $(u, p)^\beta$ *be the optimal solution of* $\mathcal{J}_\beta$, $u^\beta|_{\Omega_i} = u_i^\beta$, $p^\beta|_{\Gamma_i} = p_i^\beta$, *and* $g_i^\beta = u_i^\beta + \lambda[A(\mathbf{x})\nabla u_i^\beta] \cdot \mathbf{n}_i$ *for* $i = 1, 2$. *Then* $(u_1, u_2, p_1, p_2)^{\beta\delta\sigma}$ *converges to* $(u_1, u_2, p_1, p_2)^\beta$ *weakly as* $\delta, \sigma \to 0$.

Proof: Suppose that $\{(u_1, u_2, p_1, p_2, g_1, g_2)^{\beta\delta\sigma}\}$ is a sequence of optimal solutions and that $\delta, \sigma \to 0$. Then we have

$$\mathcal{E}_{\beta\delta\sigma}(u_1, u_2, p_1, p_2, g_1, g_2)^{\beta\delta\sigma} \le \mathcal{E}_{\beta\delta\sigma}(u_1, u_2, p_1, p_2, g_1, g_2)^\beta \text{ for each } \delta, \sigma > 0.$$

I.e., we have for any $\delta, \sigma > 0$,

$$\mathcal{K}_\beta(u_1, u_2, p_1, p_2)^{\beta\delta\sigma} + \frac{\delta}{2} \sum_{i=1}^{2} \int_{\Gamma_0} (g_i^{\beta\delta\sigma})^2 d\Gamma \le \mathcal{E}_{\beta\delta\sigma}(u_1, u_2, p_1, p_2, g_1, g_2)^\beta \quad \text{and}$$

$$\frac{1}{2\sigma} \int_{\Gamma_0} |(u_1 - u_2)^{\beta\delta\sigma}|^2 + |(g_1 + g_2 - u_1 - u_2)^{\beta\delta\sigma}|^2 d\Gamma \le \mathcal{E}_{\beta\delta\sigma}(u_1, u_2, p_1, p_2, g_1, g_2)^\beta.$$

Note that

$$\mathcal{E}_{\beta\delta\sigma}(u_1,u_2,p_1,p_2,g_1,g_2)^\beta = \mathcal{K}_\beta(u_1,u_2,p_1,p_2)^\beta + \frac{\delta}{2}\sum_{i=1}^{2}\int_{\Gamma_0}(g_i^\beta)^2 d\Gamma$$
$$= \mathcal{J}_\beta(u,p)^\beta + \frac{\delta}{2}\sum_{i=1}^{2}\int_{\Gamma_0}(g_i^\beta)^2 d\Gamma.$$

Thus, we have for any $\delta,\sigma>0$,

$$\mathcal{K}_\beta(u_1,u_2,p_1,p_2)^{\beta\delta\sigma} + \frac{\delta}{2}\sum_{i=1}^{2}\int_{\Gamma_0}(g_i^{\beta\delta\sigma})^2 d\Gamma \leq \mathcal{J}_\beta(u,p)^\beta + \frac{\delta}{2}\sum_{i=1}^{2}\int_{\Gamma_0}(g_i^\beta)^2 d\Gamma \quad \text{and} \tag{2.19}$$

$$\frac{1}{2\sigma}\int_{\Gamma_0}|(u_1-u_2)^{\beta\delta\sigma}|^2 + |(g_1+g_2-u_1-u_2)^{\beta\delta\sigma}|^2 d\Gamma \leq \mathcal{J}_\beta(u,p)^\beta + \frac{\delta}{2}\sum_{i=1}^{2}\int_{\Gamma_0}(g_i^\beta)^2 d\Gamma. \tag{2.20}$$

From (2.17), there are sequences $\{p_i^{\beta\delta\sigma}\}$ and $\{g_i^{\beta\delta\sigma}\}$ that are uniformly bounded in $L^2(\Gamma_i)$ and $L^2(\Gamma_0)$ respectively. By (2.18), $\{u_i^{\beta\delta\sigma}\}$ is also uniformly bounded in $H^1(\Omega_i)$. Hence, there exist subsequences of $\{u_i^{\beta\delta\sigma}\},\{p_i^{\beta\delta\sigma}\}$, and $\{g_i^{\beta\delta\sigma}\}$ that converge weakly to some $(u_1,u_2,p_1,p_2,g_1,g_2)^* \in H^1(\Omega_1)\times H^1(\Omega_2)\times L^2(\Gamma_1)\times L^2(\Gamma_2)\times[L^2(\Gamma_0)]^2$.

Then by passing to the limit, we have

$$a_i[u_i^*,v_i] + \lambda^{-1}[u_i^*,v_i]_{\Gamma_0} = [f_i,v_i]_{\Omega_i} + [p_i^*,v_i]_{\Gamma_i} + \lambda^{-1}[g_i^*,v_i]_{\Gamma_0} \quad \forall v_i\in H^1(\Omega_i).$$

In (2.20), $\int_{\Gamma_0}|(u_1-u_2)^*|^2 + |(g_1+g_2-u_1-u_2)^*|^2 d\Gamma \to 0$ as $\sigma\to 0$; i.e., the transmission conditions hold. Hence, $\mathcal{K}_\beta(u_1,u_2,p_1,p_2)^* = \mathcal{J}_\beta(u,p)^*$.

In (2.19), $\mathcal{J}_\beta(u,p)^* \leq \mathcal{J}_\beta(u,p)^\beta$ as $\delta\to 0$. Then by uniqueness of the optimal solution, we have $(u,p)^* = (u,p)^\beta$.

Let $\{(\bar{u_1},\bar{u_2},\bar{p_1},\bar{p_2},\bar{g_1},\bar{g_2})^{\beta\delta\sigma}\}$ be a subsequence of $\{(u_1,u_2,p_1,p_2,g_1,g_2)^{\beta\delta\sigma}\}$. And we apply the above argument in the theorem to this subsequence. Then we have that $(u_1,u_2,p_1,p_2)^{\beta\delta\sigma}$ converges weakly to $(u_1,u_2,p_1,p_2)^\beta$ in $H^1(\Omega_1)\times H^1(\Omega_2)\times L^2(\Gamma_1)\times L^2(\Gamma_2)$. □

Corollary 2.4. *Under the same hypothesis in above theorem, for a given U, we assume that $\mathcal{J}_\beta(u^\beta,p^\beta)=0$. Then we have the weak convergence of the subdomain optimal solution to whole domain optimal solution as $\delta\to 0$ and $\sigma\to\infty$.*

2.4. The method of Lagrange multiplier. In this section, we use the method of Lagrange multiplier to derive the optimality system of equations for our optimal boundary control problem. Throughout this section, we use $\mathcal{E}$ instead of $\mathcal{E}_{\beta\delta\sigma}(u_1,u_2,p_1,p_2,g_1,g_2)$ for simplicity if we need. We define the Lagrangian

$$\mathcal{L}(u_1,u_2,p_1,p_2,g_1,g_2,\xi_1,\xi_2) = \mathcal{E}_{\beta\delta\sigma}(u_1,u_2,p_1,p_2,g_1,g_2) - \sum_{i=1}^{2}\left(a_i[u_i,\xi_i] + \lambda^{-1}[u_i,\xi_i]_{\Gamma_0} - [f_i,\xi_i]_{\Omega_i} - [p_i,\xi_i]_{\Gamma_i} - \lambda^{-1}[g_i,\xi_i]_{\Gamma_0}\right),$$

where

$$\mathcal{E}_{\beta\delta\sigma}(u_1,u_2,p_1,p_2,g_1,g_2) = \frac{1}{2}\sum_{i=1}^{2}\int_{\Omega_i}|u_i-U|^2\,d\Omega + \frac{\beta}{2}\sum_{i=1}^{2}\int_{\Gamma_i}p_i^2\,d\Gamma$$
$$+\frac{\delta}{2}\sum_{i=1}^{2}\int_{\Gamma_0}g_i^2\,d\Gamma + \frac{1}{2\sigma}\int_{\Gamma_0}|u_1-u_2|^2+|g_1+g_2-u_1-u_2|^2\,d\Gamma.$$

Note that by taking the first variations of $\mathcal{L}$ with respect to ξ_i and then setting the result equation to zero, we obtain the weak formulation (2.6).

Similarly, with respect to u_i, we have

$$[u_i-U,v_i]_{\Omega_i}+\frac{1}{\sigma}[2u_i-g_1-g_2,v_i]_{\Gamma_0} = a_i[\xi_i,v_i]+\lambda^{-1}[\xi_i,v_i]_{\Gamma_0} \quad \forall v_i\in H^1(\Omega_i), \tag{2.21}$$

with respect to p_i,

$$\beta[p_i,w_i]_{\Gamma_i} = -[\xi_i,w_i]_{\Gamma_i} \quad \forall w_i\in L^2(\Gamma_i). \tag{2.22}$$

and with respect to g_i,

$$\delta[g_i,z_i]_{\Gamma_0}+\frac{1}{\sigma}[g_1+g_2-u_1-u_2,z_i]_{\Gamma_0} = -\lambda^{-1}[\xi_i,z_i]_{\Gamma_0} \quad \forall z_i\in L^2(\Gamma_0). \tag{2.23}$$

Here equations (2.6) and (2.21)-(2.23) are called the optimality system of equations. We now define the first derivatives of $\mathcal{E}$ with respect to p_i and g_i through their actions:

$$\left\langle\frac{\partial\mathcal{E}}{\partial p_i},\tilde{p_i}\right\rangle = [u_i-U,\tilde{u_i}]_{\Omega_i}+\beta[p_i,\tilde{p_i}]_{\Gamma_i}+\frac{1}{\sigma}[2u_i-g_1-g_2,\tilde{u_i}]_{\Gamma_0}, \tag{2.24}$$

where $\tilde{u_i}\in H^1(\Omega_i)$ is the solution of

$$a_i[\tilde{u_i},v]+\lambda^{-1}[\tilde{u_i},v]_{\Gamma_0} = [\tilde{p_i},v]_{\Gamma_i} \quad \forall v\in H^1(\Omega_i) \tag{2.25}$$

and

$$\left\langle\frac{\partial\mathcal{E}}{\partial g_i},\tilde{g_i}\right\rangle = [u_i-U,\hat{u}_i]_{\Omega_i}+\delta[g_i,\tilde{g_i}]_{\Gamma_0}$$
$$+\frac{1}{\sigma}[2u_i-g_1-g_2,\hat{u}_i]_{\Gamma_0}+\frac{1}{\sigma}[g_1+g_2-u_1-u_2,\tilde{g_i}]_{\Gamma_0}, \tag{2.26}$$

where $\hat{u}_i\in H^1(\Omega_i)$ is the solution of

$$a_i[\hat{u}_i,v]+\lambda^{-1}[\hat{u}_i,v]_{\Gamma_0} = \lambda^{-1}[\tilde{g_i},v]_{\Gamma_i} \quad \forall v\in H^1(\Omega_0). \tag{2.27}$$

Now let ξ_i be solution of (2.21) and set $v_i=\tilde{u_i}$ in (2.21), where $\widetilde{u_i}$ is the solution of (2.25) and $v=\xi_i$ in (2.25). Then we have

$$[u_i-U,\tilde{u_i}]_{\Omega_i}+\frac{1}{\sigma}[2u_i-g_1-g_2,\tilde{u_i}]_{\Gamma_0} = a_i[\xi_i,\tilde{u_i}]+\lambda^{-1}[\xi_i,\tilde{u_i}]_{\Gamma_0} \tag{2.28}$$

and

$$a_i[\tilde{u_i},\xi_i]+\lambda^{-1}[\tilde{u_i},\xi_i]_{\Gamma_0} = [\tilde{p_i},\xi_i]_{\Gamma_i}. \tag{2.29}$$

Combining (2.28) and (2.29) yields

$$[\tilde{p_i},\xi_i]_{\Gamma_i} = [u_i-U,\tilde{u_i}]_{\Omega_i}+\frac{1}{\sigma}[2u_i-g_1-g_2,\tilde{u_i}]_{\Gamma_0}. \tag{2.30}$$

Substituting (2.30) in (2.24) yields

$$\left\langle\frac{\partial\mathcal{E}}{\partial p_i},\tilde{p_i}\right\rangle = [\xi_i,\tilde{p_i}]_{\Gamma_i}+\beta[p_i,\tilde{p_i}]_{\Gamma_i}. \tag{2.31}$$

Similarly we have

$$\left\langle \frac{\partial \mathcal{E}}{\partial g_i}, \tilde{g}_i \right\rangle = \delta[g_i, \tilde{g}_i]_{\Gamma_0} + \lambda^{-1}[\xi_i, \tilde{g}_i]_{\Gamma_0} + \frac{1}{\sigma}[g_1 + g_2 - u_1 - u_2, \tilde{g}_i]_{\Gamma_0}. \tag{2.32}$$

Consequently, from (2.31) and (2.32), we have the following formulas

$$\frac{\partial \mathcal{E}}{\partial p_i} = \xi_i + \beta p_i \tag{2.33}$$

and

$$\frac{\partial \mathcal{E}}{\partial g_i} = \delta g_i + \lambda^{-1}\xi_i + \frac{1}{\sigma}(g_1 + g_2 - u_1 - u_2) \tag{2.34}$$

for an iterative approximation method in the next section.

2.5. A gradient method. In this section, we define an iterative method using the gradient from Section 2.4 to solve our boundary control problem and show the convergence of the method.

The kth step iteration formula for the gradient method is given by

$$\begin{aligned}(p_1^{(k+1)}, p_2^{(k+1)}, g_1^{(k+1)}, g_2^{(k+1)}) = (&p_1^{(k)}, p_2^{(k)}, g_1^{(k)}, g_2^{(k)}) \\ &- \alpha(\beta p_1^{(k)} + \xi_1^{(k)}, \beta p_2^{(k)} + \xi_2^{(k)}, \\ &\delta g_1^{(k)} + \frac{1}{\sigma}(g_1^{(k)} + g_2^{(k)} - u_1^{(k)} - u_2^{(k)}) + \lambda^{-1}\xi_1^{(k)}, \\ &\delta g_2^{(k)} + \frac{1}{\sigma}(g_1^{(k)} + g_2^{(k)} - u_1^{(k)} - u_2^{(k)}) + \lambda^{-1}\xi_2^{(k)}),\end{aligned} \tag{2.35}$$

where α is a positive constant, $u_i^{(k)}$ is from

$$\begin{aligned}&a_i[u_i^{(k)}, v_i] + \lambda^{-1}[u_i^{(k)}, v_i]_{\Gamma_0} \\ &\quad = [f_i, v_i]_{\Omega_i} + [p_i^{(k)}, v_i]_{\Gamma_i} + \lambda^{-1}[g_i^{(k)}, v_i]_{\Gamma_0} \quad \forall v_i \in H^1(\Omega_i)\end{aligned} \tag{2.36}$$

and $\xi_i^{(k)}$ is from

$$\begin{aligned}&a_i[\xi_i^{(k)}, v_i] + \lambda^{-1}[\xi_i^{(k)}, v_i]_{\Gamma_0} \\ &\quad = [u_i^{(k)} - U, v_i]_{\Omega_i} + \frac{1}{\sigma}[2u_i^{(k)} - g_1^{(k)} - g_2^{(k)}, v_i]_{\Gamma_0} \quad \forall v_i \in H^1(\Omega_i).\end{aligned} \tag{2.37}$$

We state the following theorem to show the convergence of the method.

Theorem 2.5. *Let X be a Hilbert space equipped with the inner product $(\cdot,\cdot)_X$ and norm $\|\cdot\|_X$. Suppose that $\mathcal{M}$ is a functional on X such that*
1. $\mathcal{M}$ has a local minimum at $\hat{x}$ and is twice differentiable in an open ball B centered at $\hat{x}$;
2. $|\langle \mathcal{M}''(u), (x, y)\rangle| \le M\|x\|_X\|y\|_X \quad \forall u \in B,\ x \in X,\ y \in X$;
3. $|\langle \mathcal{M}''(u), (x, x)\rangle| \ge m\|x\|_X^2 \quad \forall u \in B,\ x \in X$,
where M and m are positive constants. Let R denote Riesz map. Choose $x^{(0)}$ sufficiently close to $\hat{x}$ and choose a sequence ρ_k such that $0 < \rho_* \le \rho_k \le \rho^* < 2m/M^2$. Then the sequence $x^{(k+1)}$ defined by

$$x^{(k+1)} = x^{(k)} - \rho_k R\mathcal{M}'(x^{(k)}) \qquad \text{for} \quad k = 0, 1, 2, \cdots,$$

converges to $\hat{x}$.

PROOF: see [2].

We calculate the second derivatives of $\mathcal{E}$ to determine positive constants M and m in Theorem 2.5.

$$\left\langle \frac{\partial^2\mathcal{E}}{\partial p_i^2}, (\tilde{p_i}, p_i) \right\rangle = [u_i, \tilde{u_i}]_{\Omega_i} + \beta[p_i, \tilde{p_i}]_{\Gamma_i} + \frac{1}{\sigma}[2u_i, \tilde{u_i}]_{\Gamma_0},$$
$$\left\langle \frac{\partial^2\mathcal{E}}{\partial p_1 \partial p_2}, (\tilde{p_1}, p_2) \right\rangle = 0,$$
$$\left\langle \frac{\partial^2\mathcal{E}}{\partial p_2 \partial p_1}, (\tilde{p_2}, p_1) \right\rangle = 0,$$
$$\left\langle \frac{\partial^2\mathcal{E}}{\partial g_i^2}, (\tilde{g_i}, g_i) \right\rangle = [\check{u}_i, \hat{u}_i]_{\Omega_i} + \delta[g_i, \tilde{g_i}]_{\Gamma_0} + \frac{1}{\sigma}[2\check{u}_i - g_i, \hat{u}_i]_{\Gamma_0} + \frac{1}{\sigma}[-\check{u}_i + g_i, \tilde{g_i}]_{\Gamma_0},$$
$$\left\langle \frac{\partial^2\mathcal{E}}{\partial g_1 \partial g_2}, (\tilde{g_1}, g_2) \right\rangle = \frac{1}{\sigma}[-g_2, \hat{u}_1]_{\Gamma_0} + \frac{1}{\sigma}[-\check{u}_2 + g_2, \tilde{g_1}]_{\Gamma_0},$$
$$\left\langle \frac{\partial^2\mathcal{E}}{\partial g_2 \partial g_1}, (\tilde{g_2}, g_1) \right\rangle = \frac{1}{\sigma}[-g_1, \hat{u}_2]_{\Gamma_0} + \frac{1}{\sigma}[-\check{u}_1 + g_1, \tilde{g_2}]_{\Gamma_0},$$
$$\left\langle \frac{\partial^2\mathcal{E}}{\partial p_i \partial g_i}, (\tilde{p_i}, g_i) \right\rangle = [\check{u}_i, \tilde{u_i}]_{\Omega_i} + \frac{1}{\sigma}[2\check{u}_i - g_i, \tilde{u_i}]_{\Gamma_0},$$
$$\left\langle \frac{\partial^2\mathcal{E}}{\partial g_i \partial p_i}, (\tilde{g_i}, p_i) \right\rangle = [u_i, \hat{u}_i]_{\Omega_i} + \frac{1}{\sigma}[2u_i, \hat{u}_i]_{\Gamma_0} + \frac{1}{\sigma}[-u_i, \tilde{g_i}]_{\Gamma_0},$$
$$\left\langle \frac{\partial^2\mathcal{E}}{\partial p_1 \partial g_2}, (\tilde{p_1}, g_2) \right\rangle = \frac{1}{\sigma}[-g_2, \tilde{u_1}]_{\Gamma_0},$$
$$\left\langle \frac{\partial^2\mathcal{E}}{\partial p_2 \partial g_1}, (\tilde{p_2}, g_1) \right\rangle = \frac{1}{\sigma}[-g_1, \tilde{u_2}]_{\Gamma_0},$$
$$\left\langle \frac{\partial^2\mathcal{E}}{\partial g_1 \partial p_2}, (\tilde{g_1}, p_2) \right\rangle = \frac{1}{\sigma}[-u_2, \tilde{g_1}]_{\Gamma_0}, \quad \text{and}$$
$$\left\langle \frac{\partial^2\mathcal{E}}{\partial g_2 \partial p_1}, (\tilde{g_2}, p_1) \right\rangle = \frac{1}{\sigma}[-u_1, \tilde{g_2}]_{\Gamma_0},$$

where $\tilde{u_i}, u_i, \hat{u}_i, \check{u}_i \in H^1(\Omega_i)$ are the solutions of

$$\begin{aligned} b_i[\tilde{u_i}, v] &= [\tilde{p_i}, v]_{\Gamma_i} \quad \forall v \in H^1(\Omega_i), \\ b_i[u_i, v] &= [p_i, v]_{\Gamma_i} \quad \forall v \in H^1(\Omega_i), \\ b_i[\hat{u}_i, v] &= \lambda^{-1}[\tilde{g_i}, v]_{\Gamma_0} \quad \forall v \in H^1(\Omega_i), \text{ and} \\ b_i[\check{u}_i, v] &= \lambda^{-1}[g_i, v]_{\Gamma_0} \quad \forall v \in H^1(\Omega_i). \end{aligned}$$

Note that there exist constants C's such that

$$\begin{aligned} \|\tilde{u_i}\|_{1,\Omega_i} &\le C\|\tilde{p_i}\|_{0,\Gamma_i}, \\ \|u_i\|_{1,\Omega_i} &\le C\|p_i\|_{0,\Gamma_i}, \\ \|\hat{u}_i\|_{1,\Omega_i} &\le C\|\tilde{g_i}\|_{0,\Gamma_0}, \text{ and} \\ \|\check{u}_i\|_{1,\Omega_i} &\le C\|g_i\|_{0,\Gamma_0}, \end{aligned}$$

respectively.

Also note that in our calculations, $\nabla^2\mathcal{E}$ is 4 by 4 matrix, $x = (p_1, p_2, g_1, g_2)^T$, and $y = (\tilde{p_1}, \tilde{p_2}, \tilde{g_1}, \tilde{g_2})^T$. Thus, $\langle \nabla^2\mathcal{E}, (x, y)\rangle$ is defined by $\langle \nabla^2\mathcal{E}x, y\rangle$ or $\langle x, \nabla^2\mathcal{E}y\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the scalar product. Hence, we have

$$\begin{aligned}
< \nabla^2\mathcal{E},(x,y) >=&\sum_{i=1}^{2}([u_i,\tilde{u_i}]_{\Omega_i}+\beta[p_i,\tilde{p_i}]_{\Gamma_i}+\frac{1}{\sigma}[2u_i,\tilde{u_i}]_{\Gamma_0}+[\check{u}_i,\tilde{u_i}]_{\Omega_i}\\
&+\frac{1}{\sigma}[2\check{u}_i-g_i,\tilde{u_i}]_{\Gamma_0}+[u_i,\hat{u_i}]_{\Omega_i}+\frac{1}{\sigma}[2u_i,\hat{u_i}]_{\Gamma_0}+\frac{1}{\sigma}[-u_i,\tilde{g_i}]_{\Gamma_0}\\
&+[\check{u}_i,\hat{u_i}]_{\Omega_i}+\delta[g_i,\tilde{g_i}]_{\Gamma_0}+\frac{1}{\sigma}[2\check{u}_i-g_i,\hat{u_i}]_{\Gamma_0}+\frac{1}{\sigma}[-\check{u}_i\\
&+g_i,\tilde{g_i}]_{\Gamma_0})+\frac{1}{\sigma}([-g_2,\tilde{u_1}]_{\Gamma_0}+[-g_1,\tilde{u_2}]_{\Gamma_0}+[-u_2,\tilde{g_1}]_{\Gamma_0}\\
&+[-u_1,\tilde{g_2}]_{\Gamma_0}+[-g_2,\hat{u_1}]_{\Gamma_0}+[-\check{u}_2+g_2,\tilde{g_1}]_{\Gamma_0}+[-g_1,\hat{u_2}]_{\Gamma_0}\\
&+[-\check{u}_1+g_1,\tilde{g_2}]_{\Gamma_0}),\ \text{and, hence,}
\end{aligned}$$

$$\begin{aligned}
| < \nabla^2\mathcal{E},(x,y) > | \le&\sum_{i=1}^{2}\{C^2(\|p_i\|_{0,\Gamma_i}\|\tilde{p_i}\|_{0,\Gamma_i}+\|p_i\|_{0,\Gamma_i}\|\tilde{g_i}\|_{0,\Gamma_0}+\|g_i\|_{0,\Gamma_0}\|\tilde{g_i}\|_{0,\Gamma_0}\\
&+\|g_i\|_{0,\Gamma_0}\|\tilde{p_i}\|_{0,\Gamma_i})+\frac{2C^2}{\sigma}(\|p_i\|_{0,\Gamma_i}\|\tilde{p_i}\|_{0,\Gamma_i}+\|p_i\|_{0,\Gamma_i}\|\tilde{g_i}\|_{0,\Gamma_0}\\
&+\|g_i\|_{0,\Gamma_0}\|\tilde{g_i}\|_{0,\Gamma_0}+\|g_i\|_{0,\Gamma_0}\|\tilde{p_i}\|_{0,\Gamma_i})+\frac{C}{\sigma}(\|p_i\|_{0,\Gamma_i}\|\tilde{g_i}\|_{0,\Gamma_0}\\
&+\|g_i\|_{0,\Gamma_0}\|\tilde{p_i}\|_{0,\Gamma_i}+2\|g_i\|_{0,\Gamma_0}\|\tilde{g_i}\|_{0,\Gamma_0})+\beta\|p_i\|_{0,\Gamma_i}\|\tilde{p_i}\|_{0,\Gamma_i}\\
&+\delta\|g_i\|_{0,\Gamma_0}\|\tilde{g_i}\|_{0,\Gamma_0}\}+\frac{1}{\sigma}(\|g_1\|_{0,\Gamma_0}\|\tilde{g_1}\|_{0,\Gamma_0}+\|g_1\|_{0,\Gamma_0}\|\tilde{g_2}\|_{0,\Gamma_0}\\
&+\|g_2\|_{0,\Gamma_0}\|\tilde{g_1}\|_{0,\Gamma_0}+\|g_2\|_{0,\Gamma_0}\|\tilde{g_2}\|_{0,\Gamma_0})+\frac{C}{\sigma}(\|p_1\|_{0,\Gamma_1}\|\tilde{g_2}\|_{0,\Gamma_0}\\
&+\|p_2\|_{0,\Gamma_2}\|\tilde{g_1}\|_{0,\Gamma_0}+2\|g_1\|_{0,\Gamma_0}\|\tilde{g_2}\|_{0,\Gamma_0}+2\|g_2\|_{0,\Gamma_0}\|\tilde{g_1}\|_{0,\Gamma_0}\\
&+\|g_1\|_{0,\Gamma_0}\|\tilde{p_2}\|_{0,\Gamma_2}+\|g_2\|_{0,\Gamma_0}\|\tilde{p_1}\|_{0,\Gamma_1})\\
\le&\left(C^2+\frac{2C^2}{\sigma}+\frac{2C}{\sigma}+\beta+\delta+\frac{1}{\sigma}+\frac{2C}{\sigma}\right)\|x\|\|y\|\le M\|x\|\|y\|,
\end{aligned}$$

where

$$\begin{aligned}
M&=7\max\{C^2,\ 2C^2/\sigma,\ 2C/\sigma,\ \beta,\ \delta,\ 1/\sigma\},\\
\|x\|&=\sqrt{\|p_1\|^2_{0,\Gamma_1}+\|p_2\|^2_{0,\Gamma_2}+\|g_1\|^2_{0,\Gamma_0}+\|g_2\|^2_{0,\Gamma_0}},\quad\text{and}\\
\|y\|&=\sqrt{\|\tilde{p_1}\|^2_{0,\Gamma_1}+\|\tilde{p_2}\|^2_{0,\Gamma_2}+\|\tilde{g_1}\|^2_{0,\Gamma_0}+\|\tilde{g_2}\|^2_{0,\Gamma_0}}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
< \nabla^2\mathcal{E},(x,x) >=&\sum_{i=1}^{2}([u_i,u_i]_{\Omega_i}+\beta[p_i,p_i]_{\Gamma_i}+\frac{1}{\sigma}[2u_i,u_i]_{\Gamma_0}+[\check{u}_i,u_i]_{\Omega_i}\\
&+\frac{1}{\sigma}[2\check{u}_i-g_i,u_i]_{\Gamma_0}+[u_i,\check{u}_i]_{\Omega_i}+\frac{1}{\sigma}[2u_i,\check{u}_i]_{\Gamma_0}+\frac{1}{\sigma}[-u_i,g_i]_{\Gamma_0}\\
&+[\check{u}_i,\check{u}_i]_{\Omega_i}+\delta[g_i,g_i]_{\Gamma_0}+\frac{1}{\sigma}[2\check{u}_i-g_i,\check{u}_i]_{\Gamma_0}+\frac{1}{\sigma}[-\check{u}_i+g_i,g_i]_{\Gamma_0})\\
&+\frac{1}{\sigma}([-g_2,u_1]_{\Gamma_0}+[-g_1,u_2]_{\Gamma_0}+[-u_2,g_1]_{\Gamma_0}+[-u_1,g_2]_{\Gamma_0}\\
&+[-g_2,\check{u}_1]_{\Gamma_0}+[-\check{u}_2+g_2,g_1]_{\Gamma_0}+[-g_1,\check{u}_2]_{\Gamma_0}+[-\check{u}_1+g_1,g_2]_{\Gamma_0}).
\end{aligned}$$

Then we have

$$\begin{aligned}
| < \nabla^2 \mathcal{E}, (x,x) > | \geq & | \sum_{i=1}^{2} ([u_i + \check{u}_i, u_i + \check{u}_i]_{\Omega_i} + \frac{1}{\sigma}[u_i + \check{u}_i, u_i + \check{u}_i]_{\Gamma_0} \\
& + \frac{1}{\sigma}[u_i + \check{u}_i - g_1 - g_2, u_i + \check{u}_i - g_1 - g_2]_{\Gamma_0} + \beta[p_i, p_i]_{\Gamma_i} \\
& + \delta[g_i, g_i]_{\Gamma_0}) - \frac{2}{\sigma}([g_1, g_1]_{\Gamma_0} + [g_2, g_2]_{\Gamma_0})| \\
\geq & | \sum_{i=1}^{2} [u_i + \check{u}_i, u_i + \check{u}_i]_{\Omega_i} + \frac{1}{\sigma}[u_i + \check{u}_i, u_i + \check{u}_i]_{\Gamma_0} \\
& + \frac{1}{\sigma}[u_i + \check{u}_i - g_1 - g_2, u_i + \check{u}_i - g_1 - g_2]_{\Gamma_0} + \beta[p_i, p_i]_{\Gamma_i} \\
& + (\delta - \frac{2}{\sigma})[g_i, g_i]_{\Gamma_0}| \\
\geq & m\|x\|^2,
\end{aligned}$$

where $m = \min\{\beta, \delta - 2/\sigma\}$. Hence the sufficient conditions of Theorem 2.5 hold for our method.

3. The finite element version

The finite element version of what we have done can be analyzed in the same way as the continuous version. So we will talk about the finite element version here without proofs. Also we are going to talk about the finite element approximation of our optimization problem under some assumptions.

We assume that Ω is a two dimensional polygon or a three dimensional polyhedron. Consider regular triangulations $\mathcal{T}^h(\Omega)$ of Ω such that no element of the triangulations crosses the interface Γ_0. Let $X^h \subset H^1(\Omega)$ be a family of finite dimensional spaces of functions and we set $X_i^h = X^h|_{\Omega_i}$ for $i = 1, 2$, $P^h = P_1^h \times P_2^h \equiv X_1^h|_{\Gamma_1} \times X_2^h|_{\Gamma_2}$, and $G^h = G_1^h \times G_2^h \equiv X_1^h|_{\Gamma_0} \times X_2^h|_{\Gamma_0}$. We assume that X^h and X_i^h satisfy standard approximation properties; see [**?**].

3.1. Minimization of the discrete problems. The finite element version of (2.1) is

(3.1)

$$-\text{div}\,[A(\mathbf{x})\nabla u^h] + \mathbf{b}(\mathbf{x}) \cdot \nabla u^h + c(\mathbf{x})u^h = f \quad \text{in } \Omega, \qquad [A(\mathbf{x})\nabla u^h] \cdot \mathbf{n} = p^h \quad \text{on } \Gamma,$$

where $u^h \in X^h$ and $p^h \in P^h$ and also the finite version of (2.2) is described as follows: for a given U,

$$\mathcal{J}_\beta(u^h, p^h) = \frac{1}{2}\int_\Omega |u^h - U|^2 d\Omega + \frac{\beta}{2}\int_\Gamma (p^h)^2 d\Gamma, \tag{3.2}$$

where β is a positive constant.

Let $f \in L^2(\Omega)$. For given $p^h \in P^h$ and $g^h \in G^h$, the discrete Robin type boundary value problems are defined over subdomains (for a fixed positive λ):

$$\begin{aligned}
-\text{div}\,[A(\mathbf{x})\nabla u_i^h] + \mathbf{b}(\mathbf{x}) \cdot \nabla u_i^h + c(\mathbf{x})u_i^h &= f_i \quad \text{in } \Omega_i, \\
[A(\mathbf{x})\nabla u_i^h] \cdot \mathbf{n}_i &= p_i^h \quad \text{on } \Gamma_i, \quad \text{and} \\
u_i^h + \lambda[A(\mathbf{x})\nabla u_i^h] \cdot \mathbf{n}_i &= g_i^h \quad \text{on } \Gamma_0
\end{aligned} \tag{3.3}$$

for $i = 1, 2$ are well posed in the sense of the following weak formulation: seek a $u_i^h \in X_i^h$ for $i = 1, 2$ such that

$$a_i[u_i^h, v_i^h] + \lambda^{-1}[u_i^h, v_i^h]_{\Gamma_0} = [f_i, v_i^h]_{\Omega_i} + [p_i^h, v_i^h]_{\Gamma_i} + \lambda^{-1}[g_i^h, v_i^h]_{\Gamma_0} \quad \forall v_i^h \in X_i^h. \tag{3.4}$$

Now we have the finite element version of (2.7) and (2.10).

$$\mathcal{K}_\beta(u_1^h, u_2^h, p_1^h, p_2^h) = \frac{1}{2}\sum_{i=1}^{2}\int_{\Omega_i} |u_i^h - U|^2 d\Omega + \frac{\beta}{2}\sum_{i=1}^{2}\int_{\Gamma_i} (p_i^h)^2 d\Gamma. \tag{3.5}$$

$$\begin{aligned}\mathcal{E}_{\beta\delta\sigma}(u_1^h, u_2^h, p_1^h, p_2^h, g_1^h, g_2^h) &= \mathcal{K}_\beta(u_1^h, u_2^h, p_1^h, p_2^h) + \frac{\delta}{2}\sum_{i=1}^{2}\int_{\Gamma_0} (g_i^h)^2 d\Gamma \\ &+ \frac{1}{2\sigma}\int_{\Gamma_0} |u_1^h - u_2^h|^2 + |g_1^h + g_2^h - u_1^h - u_2^h|^2 d\Gamma,\end{aligned} \tag{3.6}$$

3.2. The existence of the discrete optimal solution. We know that a weak formulation of (3.1) is as follows: seek $u^h \in X^h$ such that

$$a[u^h, v^h] = [f, v^h]_\Omega + [p^h, v^h]_\Gamma \quad \forall v^h \in X^h. \tag{3.7}$$

Define $\mathcal{U}_{ad}^h$ and $\mathcal{W}_{ad}^h$ as follows:

$$\mathcal{U}_{ad}^h = \{(u^h, p^h) \in X^h \times P^h \text{ such that (3.7) satisfied and } \mathcal{J}_\beta(u^h, p^h) < \infty\} \tag{3.8}$$

$$\begin{aligned}\mathcal{W}_{ad}^h =&\{(u_1^h, u_2^h, p_1^h, p_2^h, g_1^h, g_2^h) \in X_1^h \times X_2^h \times P_1^h \times P_2^h \times G_1^h \times G_2^h \\ &\text{such that (3.4) satisfied and } \mathcal{E}_{\beta\delta\sigma}(u_1^h, u_2^h, p_1^h, p_2^h, g_1^h, g_2^h) < \infty\}\end{aligned} \tag{3.9}$$

Then we have the following theorems without proofs from the same arguments in section 2.2.

Theorem 3.1. *There is a unique optimal solution in* $\mathcal{U}_{ad}^h$.

Theorem 3.2. *There is a unique optimal solution in* $\mathcal{W}_{ad}^h$.

3.3. Convergence of the discrete optimal solution. We state the finite versions of the convergence theorem and corollary without proofs. In fact, every step in proofs is parallel to that of the continuous case.

Theorem 3.3. *For each* $\beta, \delta, \sigma > 0$, *let* $(u_1^h, u_2^h, p_1^h, p_2^h, g_1^h, g_2^h)^{\beta\delta\sigma} \in \mathcal{W}_{ad}^h$ *denote an optimal solution of* $\mathcal{E}_{\beta\delta\sigma}$. *Let* $(u^h, p^h)^\beta$ *be the optimal solution of* $\mathcal{J}_\beta$ *and let* $u^{h\beta}|_{\Omega_i} = u_i^{h\beta}$ *and* $p^{h\beta}|_{\Omega_i} = p_i^{h\beta}$ *for* $i = 1, 2$. *Then* $(u_1^h, u_2^h, p_1^h, p_2^h)^{\beta\delta\sigma}$ *converges weakly to* $(u_1^h, u_2^h, p_1^h, p_2^h)^\beta$ *as* $\delta, \sigma \to 0$.

Corollary 3.4. *Under the same hypothesis in above theorem, for a given* U, *assume that* $\mathcal{J}_\beta(u^h, p^h)^\beta = 0$. *Then we have the weak convergence of the subdomain optimal solution to whole domain optimal solution as* $\delta \to 0$ *and* $\sigma \to \infty$.

3.4. The discrete Lagrange multiplier and optimality system. Define the Lagrangian of the finite element version

$$\begin{aligned}\mathcal{L}(u_1^h, u_2^h, &p_1^h, p_2^h, g_1^h, g_2^h, \xi_1^h, \xi_2^h) = \mathcal{E}_{\beta\delta\sigma}(u_1^h, u_2^h, p_1^h, p_2^h, g_1^h, g_2^h) \\ &- \sum_{i=1}^{2}\left(a_i[u_i^h, \xi_i^h] + \lambda^{-1}[u_i^h, \xi_i^h]_{\Gamma_0} - [f_i, \xi_i^h]_{\Omega_i} - [p_i^h, \xi_i^h]_{\Gamma_i} - \lambda^{-1}[g_i^h, \xi_i^h]_{\Gamma_0}\right).\end{aligned}$$

Then the finite element approximations of solutions of the optimality system (2.21)-(2.23) for $i = 1, 2$ are defined as follows:

(3.10)

$$[u_i^h - U, v_i^h]_{\Omega_i} + \frac{1}{\sigma}[2u_i^h - g_1^h - g_2^h, v_i^h]_{\Gamma_0} = a_i[\xi_i^h, v_i^h] + \lambda^{-1}[\xi_i^h, v_i^h]_{\Gamma_0} \quad \forall v_i^h \in X_i^h.$$

$$\beta[p_i^h, w_i^h]_{\Gamma_i} = -[\xi_i^h, w_i^h]_{\Gamma_i} \quad \forall w_i^h \in P_i^h. \tag{3.11}$$

$$\delta[g_i^h, z_i^h]_{\Gamma_0} + \frac{1}{\sigma}[g_1^h + g_2^h - u_1^h - u_2^h, z_i^h]_{\Gamma_0} = -\lambda^{-1}[\xi_i^h, z_i^h]_{\Gamma_0} \quad \forall z_i^h \in G_i^h. \tag{3.12}$$

3.5. The finite element version of the gradient method. The finite element version of the gradient method is described as follows: choose initial guess $\alpha_0, p_i^{h(0)} \in P_i^h$, and $g_i^{h(0)} \in G_i^h$; for $k = 1, 2, \cdots$ solve for $u_i^{h(k)} \in X_i^h$ from

$$a_i[u_i^{h(k)}, v_i^h] + \lambda^{-1}[u_i^{h(k)}, v_i^h]_{\Gamma_0} = [f_i, v_i^h]_{\Omega_i} + [p_i^{h(k)}, v_i^h]_{\Gamma_i} + \lambda^{-1}[g_i^{h(k)}, v_i^h]_{\Gamma_0} \quad \forall v_i^h \in X_i^h, \tag{3.13}$$

solve for $\xi_i^{h(k)} \in X_i^h$ from

$$a_i[\xi_i^{h(k)}, v_i^h] + \lambda^{-1}[\xi_i^{h(k)}, v_i^h]_{\Gamma_0} = [u_i^{h(k)} - U, v_i^h]_{\Gamma_i} + \frac{1}{\sigma}[2u_i^{h(k)} - g_1^{h(k)} - g_2^{h(k)}, v_i^h]_{\Gamma_0} \tag{3.14}$$

for all $v_i^h \in X_i^h$, and update $p_i^{h(k+1)} \in P_i^h$ and $g_i^{h(k+1)} \in G_i^h$ from

$$\begin{aligned}(p_1^{h(k+1)}, p_2^{h(k+1)}, g_1^{h(k+1)}, g_2^{h(k+1)}) &= (p_1^{h(k)}, p_2^{h(k)}, g_1^{h(k)}, g_2^{h(k)}) \\ &- \alpha_k(\beta p_1^{h(k)} + \xi_1^{h(k)}, \beta p_2^{h(k)} + \xi_2^{h(k)}, \\ &\delta g_1^{h(k)} + \frac{1}{\sigma}(g_1^{h(k)} + g_2^{h(k)} - u_1^{h(k)} - u_2^{h(k)}) + \lambda^{-1}\xi_1^{h(k)}, \\ &\delta g_2^{h(k)} + \frac{1}{\sigma}(g_1^{h(k)} + g_2^{h(k)} - u_1^{h(k)} - u_2^{h(k)}) + \lambda^{-1}\xi_2^{h(k)}).\end{aligned} \tag{3.15}$$

3.6. Finite element approximations. In general, we cannot make $\mathcal{J}_\beta(u, p) = 0$. However in some situation, we have that $\mathcal{J}_\beta(u, p) = 0$. For instance, if we consider (4.1) with $p = 0$, then in fact, for a given target solution $U = 1$, we have the exact solution $u = 1$.

$$\begin{gathered}\mathcal{J}_\beta(u, p) = \frac{1}{2}\int_\Omega |u - U|^2 d\Omega + \frac{\beta}{2}\int_\Gamma p^2 d\Gamma, \\ -\Delta u + u = 1 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \mathbf{n}} = p \quad \text{on } \partial\Omega.\end{gathered} \tag{3.16}$$

I.e., we can minimize our functional, $\mathcal{J}_\beta(u, p)$, perfectly in some example in a continuous version. Here is our question. Then what if we consider this in a finite element version under assumption, $\mathcal{J}_\beta(u, p) = 0$. Later we shall show the finite element approximations of our optimization problem. Before we see that, let us introduce some definitions and theorem.

Definition 3.5. *T is star-shaped with respect to B_T if for all $x \in T$, the closed convex hull of $\{x\} \cup B_T$ is a subset of T.*

Definition 3.6. *Let Ω be a given domain and let $\{\mathcal{T}^h\}$ be a family of subdivisions such that*

$$\max\{diam\ T \ :\ T \in \mathcal{T}^h\} \le h \cdot diam\ \Omega. \tag{3.17}$$

$\{\mathcal{T}^h\}$ is said to be non-degenerate if there is a positive ρ such that for all $T \in \mathcal{T}^h$ and for all $h \in (0, 1]$,

$$diam\ B_T \ge \rho \cdot diam\ T, \tag{3.18}$$

where B_T is the largest ball contained in T such that T is star-shaped with respect to B_T.

Definition 3.7. *Let $K \subseteq \mathbb{R}^n$ be a bounded closed set with nonempty interior and piecewise smooth boundary, $\mathcal{P}$ be a finite-dimensional space of functions on K and $\mathcal{N} = \{\mathcal{N}_1, \mathcal{N}_2, \cdots, \mathcal{N}_K\}$ be a basis for $\mathcal{P}'$. Then $(K, \mathcal{P}, \mathcal{N})$ is called a finite element.*

Let $\mathcal{I}^h : C^l(\overline{\Omega}) \to L^1(\Omega)$ be the global interpolation operator defined by

$$\mathcal{I}^h u|_T := \mathcal{I}^h_T u \quad \text{for} \quad T \in \mathcal{T}^h,\ h \in (0,1], \tag{3.19}$$

where $\mathcal{I}^h_T$ is the interpolation operator for the affine-equivalent element $(T, \mathcal{P}_T, \mathcal{N}_T)$.

Consider W^m_p as a usual Sobolev's space and $|v|_{W^m_p(\Omega)} = \left(\sum_{\alpha=k} \|D^\alpha_w v\|^p_{p,\Omega}\right)^{1/p}$, where $D^\alpha_w v$ as a weak derivative of v.

Theorem 3.8. *Assume that $\{\mathcal{T}^h\}, 0 < h \le 1$, is non-degenerate in $\mathbb{R}^n$ and that $(K, \mathcal{P}, \mathcal{N})$ satisfies that K is star-shaped, $\mathcal{P}_{m-1} \subseteq \mathcal{P} \subseteq W^m_\infty(K)$ and $\mathcal{N} \subseteq (C^l(\overline{K}))'$. Then there exists a positive constant C depending on n, m, p and ρ such that for $0 \le s \le m$,*

$$\left(\sum_{T \in \mathcal{T}^h} \|v - \mathcal{I}^h v\|^p_{W^s_p(T)}\right)^{1/p} \le C h^{m-s} |v|_{W^m_p(\Omega)} \quad \forall v \in W^m_p(\Omega). \tag{3.20}$$

Proof: see [2].

Now we are ready to think of the finite element approximations of our optimization problem.

Theorem 3.9. *Consider the same assumption in above theorem. Then for $m = 3$, there is a constant $C > 0$ such that*

$$\|u^h - u\|_{1,\Omega} \le C h^2 |u|_{3,\Omega}. \tag{3.21}$$

Proof: By coercivity of our bilinear form, we have

$$\begin{aligned} C_1 \|u - u^h\|^2_{1,\Omega} &\le a[u - u^h, u - u^h] = a[u - u^h, u - v^h + v^h - u^h] \\ &= a[u - u^h, u - v^h] + a[u - u^h, v^h - u^h] \quad \forall v^h \in X^h, \end{aligned} \tag{3.22}$$

where C_1 is a coercive constant.

Note that since $a[u, v] = F(v) \quad \forall v \in H^1(\Omega)$ and $a[u^h, v^h] = F(v^h) \quad \forall v^h \in X^h \subset H^1(\Omega)$, then $a[u - u^h, v^h] = 0 \quad \forall v^h \in X^h$. Because $v^h - u^h \in X^h$, by above note and by continuity of our bilinear form, we have

$$C_1 \|u - u^h\|^2_{1,\Omega} \le a[u - u^h, u - v^h] \le C_2 \|u - u^h\|_{1,\Omega} \|u - v^h\|_{1,\Omega} \tag{3.23}$$

for some positive constant C_2.

Thus for any $v^h \in X^h$, we have

$$\|u - u^h\|_{1,\Omega} \le C_2/C_1 \|u - v^h\|_{1,\Omega}. \tag{3.24}$$

In particular, we have

$$\|u - u^h\|_{1,\Omega} \le C_2/C_1 \|u - \mathcal{I}^h u\|_{1,\Omega}. \tag{3.25}$$

Hence, by previous theorem, there is a constant $C > 0$ such that

$$\|u - u^h\|_{1,\Omega} \le C h^2 |u|_{3,\Omega}. \quad \square \tag{3.26}$$

Remark 3.10. *In above theorem, if we assume that $u = U$ in Ω, then we have*

$$\|u^h - U\|_{0,\Omega} \le C h^2 |u|_{3,\Omega}. \tag{3.27}$$

In discrete case, we know that a control p^h is updated every step and hence, even though we start running our code with $p^h = 0$, there is no guarantee to have $p^h = 0$ at other steps. I.e., there is no guarantee to have that $\|u - U\|_{0,\Omega} = 0$ except the first step. Thus if we could not assume $u = U$ in Ω, then we would have

$$\|u^h - U\|_{0,\Omega} \to \|u - U\|_{0,\Omega} \neq 0 \quad \text{as} \quad h \to 0. \tag{3.28}$$

Also we need to think of quadrature formula errors.

4. Numerical experiments

In this section, we present numerical results from both a serial code and a MPI parallel code and compare the results.

Let $\Omega = \{(x, y) : 0 < x < 2, 0 < y < 1\}$ and then consider subdomains $\Omega_1 = \{(x, y) : 0 < x < 1, 0 < y < 1\}$ and $\Omega_2 = \{(x, y) : 1 < x < 2, 0 < y < 1\}$. Then we have $\Gamma_0 = \{(x, y) : x = 1, 0 < y < 1\}$ as the interface of two subdomains; see Figure 4.2.

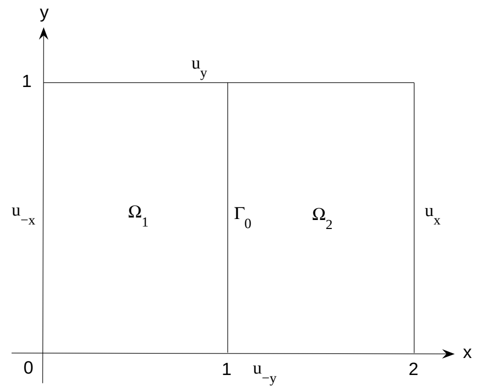

FIGURE 4.2. A subdivision of the rectangular domain

In this paper, we develop codes for the following example:

$$\mathcal{J}_\beta(u, p) = \frac{1}{2}\int_\Omega |u - 1|^2 d\Omega + \frac{\beta}{2}\int_\Gamma p^2 d\Gamma,$$

$$-\Delta u + u = 1 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \mathbf{n}} = p \quad \text{on } \partial\Omega. \tag{4.1}$$

Let $p_1^{(0)} = 0$, $p_2^{(0)} = 0.0001$, $g_1^{(0)} = 1$, and $g_2^{(0)} = 1$ that are initial guesses for our iterative approximation method.

Recall that there are $\alpha, \beta, \delta, \sigma$, and λ as parameters. First we fix that $\lambda = 1$ and $\beta = 1$. Since solutions of $\mathcal{E}_{\beta\delta\sigma}$ converges weakly to the optimal solution of $\mathcal{J}_\beta$ as $\delta, \sigma \to 0$ theoretically, we put $\delta = \sigma = 10^{-5}$ at the beginning of our experiment in solving our example. In fact, for $\delta = 0$ (no additional δ term in $\mathcal{E}_{\beta\delta\sigma}$), our iterative method is also convergent numerically and the result of this case is almost same as that of $\delta = 10^{-5}$ case. Note that we also have the convergence for a small δ and a large σ in Corollary 2.4. Thus we run codes with $\delta = 10^{-5}$ and $\sigma = 200001$, for instance. Note that we have α as a step size in our method that must be strictly less than $\frac{2m}{M^2}$ and be strictly greater than 0 from Theorem 2.5. So we may guess that we need to start putting a small α as a step size in our method. In our codes, we use $\sum_{i=1}^{2}(\|p_i^{(k+1)} - p_i^{(k)}\|_{0,\Gamma_i} + \|g_i^{(k+1)} - g_i^{(k)}\|_{0,\Gamma_0})$ as a stopping criterion and 10^{-5} as a tolerance. Also we use the same mesh size in two subdomains for convenience.

4.1. Numerical results in a serial code. As you see in tables for a serial code, calculations are performed with various values for α. The number of iterations are presented and also L^2 distances between u^h, a numerical solution, and U, a target solution, are presented in each table. Also L^2 norms for transmission conditions are presented. Note that these norms are σ terms in $\mathcal{E}_{\beta\delta\sigma}$. Throughout this and next sections, σ term in each table means $\|u_1 - u_2\|_{0,\Gamma_0} + \|g_1 + g_2 - u_1 - u_2\|_{0,\Gamma_0}$. The first table was computed for a mesh size $h = 1/4$, the second is for $h = 1/8$ and the last is for $h = 1/16$. For three mesh sizes, we can see what good choices of α's are.

Even though we do not put implementation results without δ term (this term was added into our functional in (2.10) for the convergence result in Theorem 2.3, we calculated it for each mesh size and got almost same results as in experiments with a δ term.

α	time (second)	Iter.	$\|u^h - U\|_{0,\Omega}$	σ term
$9 \cdot 10^{-7}$	2.466893196105E-002	13	1.997418312946E-004	6.873888100872E-005
$1 \cdot 10^{-6}$	2.277278900146E-002	12	1.970068380445E-004	6.514413959851E-005
$3 \cdot 10^{-6}$	1.336503028869E-002	7	1.475700922115E-004	1.177203692602E-005
$5 \cdot 10^{-6}$	9.537935256958E-003	5	1.296742127975E-004	6.155711830622E-006
$7 \cdot 10^{-6}$	9.530067443847E-003	5	1.210168405531E-004	5.278757929596E-006
$9 \cdot 10^{-6}$	2.660298347473E-002	14	1.114873040307E-004	5.676295535439E-006
$1 \cdot 10^{-5}$	0.100439071655E-000	53	1.165247077840E-004	4.420551398658E-006
$3 \cdot 10^{-5}$	diverges	NA	NA	NA

TABLE 4.1. Implementation results in a serial code with different α's for h=1/4 and $\delta = \sigma = 10^{-5}$

α	time (second)	Iter.	$\|u^h - U\|_{0,\Omega}$	σ term
$9 \cdot 10^{-7}$	0.155728101730E-000	13	1.997123469270E-004	6.873901052020E-005
$1 \cdot 10^{-6}$	0.144021034240E-000	12	1.969762156995E-004	6.514123245076E-005
$3 \cdot 10^{-6}$	8.402109146118E-002	7	1.475029653610E-004	1.147974522672E-005
$5 \cdot 10^{-6}$	5.961108207702E-002	5	1.295854457792E-004	5.780054152812E-006
$7 \cdot 10^{-6}$	6.064605712890E-002	5	1.209086213115E-004	4.886699478088E-006
$9 \cdot 10^{-6}$	0.166739940643E-000	14	1.112109560517E-004	5.170742061253E-006
$1 \cdot 10^{-5}$	0.634408950805E-000	53	1.157274136336E-004	4.293188302406E-006
$3 \cdot 10^{-5}$	diverges	NA	NA	NA

TABLE 4.2. Implementation results in a serial code with different α's for h=1/8 and $\delta = \sigma = 10^{-5}$

α	time (second)	Iter.	$\|u^h - U\|_{0,\Omega}$	σ term
$9 \cdot 10^{-7}$	8.111814022064E-000	13	1.997083970627E-004	6.873996279215E-005
$1 \cdot 10^{-6}$	7.453678846359E-000	12	1.969721354970E-004	6.514199781864E-005
$3 \cdot 10^{-6}$	4.346349954605E-000	7	1.474953627227E-004	1.146496919158E-005
$5 \cdot 10^{-6}$	3.121901035308E-000	5	1.295758087626E-004	5.754274120000E-006
$7 \cdot 10^{-6}$	3.086591958999E-000	5	1.208967704205E-004	4.857833634353E-006
$9 \cdot 10^{-6}$	8.573842048645E-000	14	1.111709164906E-004	5.107731508622E-006
$1 \cdot 10^{-5}$	32.46032500267E-000	53	1.155629962598E-004	4.242764921427E-006
$3 \cdot 10^{-5}$	diverges	NA	NA	NA

TABLE 4.3. Implementation results in a serial code with different α's for h=1/16 and $\delta = \sigma = 10^{-5}$

α	time (second)	Iter.	$\|u^h - U\|_{0,\Omega}$	σ term
$9 \cdot 10^{-7}$	616.8868517875E-000	13	1.997079825138E-004	6.874012942710E-005
$1 \cdot 10^{-6}$	570.0024251937E-000	12	1.969717083257E-004	6.514215634468E-005
$3 \cdot 10^{-6}$	332.0776438713E-000	7	1.474946319267E-004	1.146490442299E-005
$5 \cdot 10^{-6}$	237.3855371475E-000	5	1.295749074365E-004	5.753232889565E-006
$7 \cdot 10^{-6}$	237.6106598377E-000	5	1.208956524997E-004	4.856562041823E-006
$9 \cdot 10^{-6}$	665.3056440353E-000	14	1.111669543600E-004	5.103249238829E-006
$1 \cdot 10^{-5}$	3108.938740015E-000	53	1.155446509444E-004	4.235361636009E-006
$3 \cdot 10^{-5}$	diverges	NA	NA	NA

TABLE 4.4. Implementation results in a serial code with different α's for h=1/32 and $\delta = \sigma = 10^{-5}$

α	time (second)	Iter.	$\|u^h - U\|_{0,\Omega}$	σ term
$7 \cdot 10^{-2}$	2.467918395996E-002	13	4.283766643331E-005	8.765582549459E-005
$9 \cdot 10^{-2}$	2.279996871948E-002	12	2.340487392811E-005	7.279437100208E-005
$1 \cdot 10^{-1}$	2.093696594238E-002	11	2.159322893891E-005	7.269508001866E-005
$3 \cdot 10^{-1}$	1.143503189086E-002	6	2.693443356321E-005	8.619221783624E-005
$5 \cdot 10^{-1}$	2.280807495117E-002	12	4.551292995465E-005	1.265001510155E-004
$7 \cdot 10^{-1}$	2.472615242004E-002	13	5.456908051142E-005	1.475156284943E-004
$9 \cdot 10^{-1}$	0.214687108993E-000	113	7.874654504747E-005	2.077719440108E-004
1	diverges	NA	NA	NA

TABLE 4.5. Implementation results in a serial code with different α's for h=1/4, $\delta = 10^{-5}$, and $\sigma = 200001$

α	time (second)	Iter.	$\|u^h - U\|_{0,\Omega}$	σ term
$7 \cdot 10^{-2}$	0.155239105224E-000	13	4.279631393326E-005	8.761766246796E-005
$9 \cdot 10^{-2}$	0.143465995788E-000	12	2.333948142878E-005	7.262218012300E-005
$9 \cdot 10^{-2}$	0.132026910781E-000	11	2.152352850818E-005	7.255782254128E-005
$1 \cdot 10^{-1}$	7.171988487243E-002	6	2.690300713257E-005	8.621443899831E-005
$3 \cdot 10^{-1}$	0.143271207809E-000	12	4.549396319482E-005	1.265517100226E-004
$5 \cdot 10^{-1}$	0.155797004699E-000	13	5.454707929014E-005	1.475832128376E-004
$7 \cdot 10^{-1}$	1.332403898239E-000	112	8.658460891622E-005	1.990408592694E-004
1	diverges	NA	NA	NA

TABLE 4.6. Implementation results in a serial code with different α's for h=1/8, $\delta = 10^{-5}$, and $\sigma = 200001$

α	time (second)	Iter.	$\|u^h - U\|_{0,\Omega}$	σ term
$7 \cdot 10^{-2}$	8.079663991928E-000	13	4.279336352971E-005	8.761188238319E-005
$9 \cdot 10^{-2}$	7.491341114044E-000	12	2.333486130043E-005	7.259820684824E-005
$1 \cdot 10^{-1}$	6.872836112976E-000	11	2.151860644615E-005	7.253871250498E-005
$3 \cdot 10^{-1}$	3.702414989471E-000	6	2.690085175715E-005	8.621562436734E-005
$5 \cdot 10^{-1}$	7.402709960937E-000	12	4.549278661156E-005	1.265551911652E-004
$7 \cdot 10^{-1}$	8.076992034912E-000	13	5.454569025552E-005	1.475878239405E-004
$9 \cdot 10^{-1}$	68.33642888069E-000	112	8.658059650897E-005	1.990429763932E-004
1	diverges	NA	NA	NA

TABLE 4.7. Implementation results in a serial code with different α's for h=1/16, $\delta = 10^{-5}$, and $\sigma = 200001$

α	time (second)	Iter.	$\|u^h - U\|_{0,\Omega}$	σ term
$7 \cdot 10^{-2}$	618.3429238796E-000	13	4.279316795496E-005	8.761122623580E-005
$9 \cdot 10^{-2}$	569.8975131511E-000	12	2.333455691543E-005	7.259566184451E-005
$1 \cdot 10^{-1}$	522.8556110858E-000	11	2.151828231612E-005	7.253667650150E-005
$3 \cdot 10^{-1}$	288.0468401908E-000	6	2.690071225905E-005	8.621567463279E-005
$5 \cdot 10^{-1}$	576.2458240985E-000	12	4.549271468689E-005	1.265554146979E-004
$7 \cdot 10^{-1}$	616.5302550792E-000	13	5.454560472599E-005	1.475881208438E-004
$9 \cdot 10^{-1}$	6558.536365032E-000	112	8.658033906986E-005	1.990436566988E-004
1	diverges	NA	NA	NA

TABLE 4.8. Implementation results in a serial code with different α's for h=1/32, $\delta = 10^{-5}$, and $\sigma = 200001$

4.2. Numerical results in a parallel code. The tables in this section show the results in a MPI parallel code. In this experiment, we partition the whole domain into two subdomains as we did theoretically; i.e., we use only two processors in computation. We calculate u_1 and ξ_1 using one processor and u_2 and ξ_2 using the other processor at each iteration step in a MPI program. That is, we solve the first subdomain problem in one processor and the second in the other processor. We here may expect that if it could take 2 hours for solving our optimization problem by using our serial code, then it would take ideally only one hour for solving the same problem in our MPI program. In fact, we obtain almost ideal times on the MPI program as you see in the tables in this section. In the MPI program, to measure the time elapsed, we use MPI-BARRIER and MPI-WTIME and for the communication, we use MPI-SEND and MPI-RECV; see [12].

α	time (second)	Iter.	$\|u^h - U\|_{0,\Omega}$	σ term
$9 \cdot 10^{-7}$	1.272201538085E-002	13	1.997418312946E-004	6.873888100872E-005
$1 \cdot 10^{-6}$	1.174283027648E-002	12	1.970068380445E-004	6.514413959851E-005
$3 \cdot 10^{-6}$	6.839036941528E-003	7	1.475700922115E-004	1.177203692602E-005
$5 \cdot 10^{-6}$	4.961967468261E-003	5	1.296742127975E-004	6.155711830622E-006
$7 \cdot 10^{-6}$	4.964113235473E-003	5	1.210168405531E-004	5.278757929596E-006
$9 \cdot 10^{-6}$	1.365613937377E-002	14	1.114873040307E-004	5.676295535439E-006
$1 \cdot 10^{-5}$	5.149793624877E-002	53	1.165247077840E-004	4.420551398658E-006
$3 \cdot 10^{-5}$	diverges	NA	NA	NA

TABLE 4.9. Implementation results in a parallel code with different α's for h=1/4 and $\delta = \sigma = 10^{-5}$

α	time (second)	Iter.	$\|u^h - U\|_{0,\Omega}$	σ term
$9 \cdot 10^{-7}$	7.812285423278E-002	13	1.997123469270E-004	6.873901052020E-005
$1 \cdot 10^{-6}$	7.233309745788E-002	12	1.969762156995E-004	6.514123245076E-005
$3 \cdot 10^{-6}$	4.236793518066E-002	7	1.475029653610E-004	1.147974522672E-005
$5 \cdot 10^{-6}$	3.028607368469E-002	5	1.295854457792E-004	5.780054152812E-006
$7 \cdot 10^{-6}$	3.030300140380E-002	5	1.209086213115E-004	4.886699478088E-006
$9 \cdot 10^{-6}$	8.396100997924E-002	14	1.112109560517E-004	5.170742061253E-006
$1 \cdot 10^{-5}$	0.318790912628E-000	53	1.157274136336E-004	4.293188302406E-006
$3 \cdot 10^{-5}$	diverges	NA	NA	NA

TABLE 4.10. Implementation results in a parallel code with different α's for h=1/8 and $\delta = \sigma = 10^{-5}$

α	time (second)	Iter.	$\|u^h - U\|_{0,\Omega}$	σ term
$9 \cdot 10^{-7}$	4.075603008270E-000	13	1.997083970627E-004	6.873996279215E-005
$1 \cdot 10^{-6}$	3.746003866195E-000	12	1.969721354970E-004	6.514199781864E-005
$3 \cdot 10^{-6}$	2.184069156646E-000	7	1.474953627227E-004	1.146496919158E-005
$5 \cdot 10^{-6}$	1.563800096511E-000	5	1.295758087626E-004	5.754274120000E-006
$7 \cdot 10^{-6}$	1.552680015563E-000	5	1.208967704205E-004	4.857833634353E-006
$9 \cdot 10^{-6}$	4.397615194320E-000	14	1.111709164906E-004	5.107731508622E-006
$1 \cdot 10^{-5}$	16.46376609802E-000	53	1.155629962598E-004	4.242764921427E-006
$3 \cdot 10^{-5}$	diverges	NA	NA	NA

Table 4.11. Implementation results in a parallel code with different α's for h=1/16 and $\delta = \sigma = 10^{-5}$

α	time (second)	Iter.	$\|u^h - U\|_{0,\Omega}$	σ term
$9 \cdot 10^{-7}$	308.8284487724E-000	13	1.997079825138E-004	6.874012942710E-005
$1 \cdot 10^{-6}$	285.1444430351E-000	12	1.969717083257E-004	6.514215634468E-005
$3 \cdot 10^{-6}$	166.3281209468E-000	7	1.474946319267E-004	1.146490442299E-005
$5 \cdot 10^{-6}$	118.9667358398E-000	5	1.295749074365E-004	5.753232889565E-006
$7 \cdot 10^{-6}$	118.7424390316E-000	5	1.208956524997E-004	4.856562041823E-006
$9 \cdot 10^{-6}$	332.2033951282E-000	14	1.111669543600E-004	5.103249238829E-006
$1 \cdot 10^{-5}$	1553.100791931E-000	53	1.155446509444E-004	4.235361636009E-006
$3 \cdot 10^{-5}$	diverges	NA	NA	NA

Table 4.12. Implementation results in a parallel code with different α's for h=1/32 and $\delta = \sigma = 10^{-5}$

α	time (second)	Iter.	$\|u^h - U\|_{0,\Omega}$	σ term
$7 \cdot 10^{-2}$	1.269698143005E-002	13	4.283766643331E-005	8.765582549459E-005
$9 \cdot 10^{-2}$	1.172995567321E-002	12	2.340487392811E-005	7.279437100208E-005
$1 \cdot 10^{-1}$	1.075410842895E-002	11	2.159322893891E-005	7.269508001866E-005
$3 \cdot 10^{-1}$	5.949974060058E-003	6	2.693443356321E-005	8.619221783624E-005
$5 \cdot 10^{-1}$	1.172280311584E-002	12	4.551292995465E-005	1.265001510155E-004
$7 \cdot 10^{-1}$	1.269412040710E-002	13	5.456908051142E-005	1.475156284943E-004
$9 \cdot 10^{-1}$	0.109573125839E-000	113	7.874654504747E-005	2.077719440108E-004
1	diverges	NA	NA	NA

Table 4.13. Implementation results in a parallel code with different α's for h=1/4, $\delta = 10^{-5}$, and $\sigma = 200001$

α	time (second)	Iter.	$\|u^h - U\|_{0,\Omega}$	σ term
$7 \cdot 10^{-2}$	7.789492607116E-002	13	4.279631393326E-005	8.761766246796E-005
$9 \cdot 10^{-2}$	7.297086715698E-002	12	2.333948142878E-005	7.262218012300E-005
$9 \cdot 10^{-2}$	6.613898277282E-002	11	2.152352850818E-005	7.255782254128E-005
$1 \cdot 10^{-1}$	3.648304939270E-002	6	2.690300713257E-005	8.621443899831E-005
$3 \cdot 10^{-1}$	7.209014892578E-002	12	4.549396319482E-005	1.265517100226E-004
$5 \cdot 10^{-1}$	7.831001281738E-002	13	5.454707929014E-005	1.475832128376E-004
$7 \cdot 10^{-1}$	0.673575878143E-000	112	8.658460891622E-005	1.990408592694E-004
1	diverges	NA	NA	NA

TABLE 4.14. Implementation results in a parallel code with different α's for h=1/8, $\delta = 10^{-5}$, and $\sigma = 200001$

α	time (second)	Iter.	$\|u^h - U\|_{0,\Omega}$	σ term
$7 \cdot 10^{-2}$	4.067804098129E-000	13	4.279336352971E-005	8.761188238319E-005
$9 \cdot 10^{-2}$	3.742178201675E-000	12	2.333486130043E-005	7.259820684824E-005
$1 \cdot 10^{-1}$	3.435353994369E-000	11	2.151860644615E-005	7.253871250498E-005
$3 \cdot 10^{-1}$	1.877459049224E-000	6	2.690085175715E-005	8.621562436734E-005
$5 \cdot 10^{-1}$	3.749070167541E-000	12	4.549278661156E-005	1.265551911652E-004
$7 \cdot 10^{-1}$	4.042662143707E-000	13	5.454569025552E-005	1.475878239405E-004
$9 \cdot 10^{-1}$	35.26904892921E-000	112	8.658059650897E-005	1.990429763932E-004
1	diverges	NA	NA	NA

TABLE 4.15. Implementation results in a parallel code with different α's for h=1/16, $\delta = 10^{-5}$, and $\sigma = 200001$

α	time (second)	Iter.	$\|u^h - U\|_{0,\Omega}$	σ term
$7 \cdot 10^{-2}$	308.4915351867E-000	13	4.279316795496E-005	8.761122623580E-005
$9 \cdot 10^{-2}$	285.0998339653E-000	12	2.333455691543E-005	7.259566184451E-005
$1 \cdot 10^{-1}$	260.8818871974E-000	11	2.151828231612E-005	7.253667650150E-005
$3 \cdot 10^{-1}$	142.6557381153E-000	6	2.690071225905E-005	8.621567463279E-005
$5 \cdot 10^{-1}$	285.0065190792E-000	12	4.549271468689E-005	1.265554146979E-004
$7 \cdot 10^{-1}$	308.5888981819E-000	13	5.454560472599E-005	1.475881208438E-004
$9 \cdot 10^{-1}$	3292.318363904E-000	112	8.658033906986E-005	1.990436566988E-004
1	diverges	NA	NA	NA

TABLE 4.16. Implementation results in a parallel code with different α's for h=1/32, $\delta = 10^{-5}$, and $\sigma = 200001$

4.3. A comparison of the serial and parallel codes. In general, when we talk about a serial code and a parallel code, it it natural to consider speedup which is the ratio of runtime of a serial code and the runtime of a parallel code. We put the results about speedup in the following tables. In our cases, speedup is denoted by

$$\text{Speedup} = \frac{\text{runtime using one processor}}{\text{runtime using two processors}}.$$

We, from tables in this section, see that we have ideal times using parallel computers to solve the boundary control problem by the Robin type DDM.

α	speedup	α	speedup
$9 \cdot 10^{-7}$	1.93907421289355	$1 \cdot 10^{-6}$	1.93929303798753
$3 \cdot 10^{-6}$	1.95422694788217	$5 \cdot 10^{-6}$	1.92220834134153
$7 \cdot 10^{-6}$	1.91979251717016	$9 \cdot 10^{-6}$	1.94806033730228
$1 \cdot 10^{-5}$	1.95035139214251	$3 \cdot 10^{-5}$	NA

TABLE 4.17. Speedup with different α's for h=1/4 and $\delta = \sigma = 10^{-5}$

α	speedup	α	speedup
$9 \cdot 10^{-7}$	1.99337445181295	$1 \cdot 10^{-6}$	1.99108069890932
$3 \cdot 10^{-6}$	1.98312924863818	$5 \cdot 10^{-6}$	1.96826708861756
$7 \cdot 10^{-6}$	2.00132179386310	$9 \cdot 10^{-6}$	1.98592109223701
$1 \cdot 10^{-5}$	1.99004716155637	$3 \cdot 10^{-5}$	NA

TABLE 4.18. Speedup with different α's for h=1/8 and $\delta = \sigma = 10^{-5}$

α	speedup	α	speedup
$9 \cdot 10^{-7}$	1.99033468313857	$1 \cdot 10^{-6}$	1.98976806020464
$3 \cdot 10^{-6}$	1.99002396118179	$5 \cdot 10^{-6}$	1.99635557145216
$7 \cdot 10^{-6}$	1.98791246622603	$9 \cdot 10^{-6}$	1.94965718231048
$1 \cdot 10^{-5}$	1.97162209481154	$3 \cdot 10^{-5}$	NA

TABLE 4.19. Speedup with different α's for h=1/16 and $\delta = \sigma = 10^{-5}$

α	speedup	α	speedup
$9 \cdot 10^{-7}$	1.99750655821912	$1 \cdot 10^{-6}$	1.99899538327517
$3 \cdot 10^{-6}$	1.99652134576422	$5 \cdot 10^{-6}$	1.99539422067607
$7 \cdot 10^{-6}$	2.00105928238924	$9 \cdot 10^{-6}$	2.00270573327070
$1 \cdot 10^{-5}$	2.00176238153180	$3 \cdot 10^{-5}$	NA

TABLE 4.20. Speedup with different α's for h=1/32 and $\delta = \sigma = 10^{-5}$

α	speedup	α	speedup
$7 \cdot 10^{-2}$	1.94370481644916	$9 \cdot 10^{-2}$	1.94373869387589
$1 \cdot 10^{-1}$	1.94688068106239	$3 \cdot 10^{-1}$	1.92186247796121
$5 \cdot 10^{-1}$	1.94561614025097	$7 \cdot 10^{-1}$	1.94784290892700
$9 \cdot 10^{-1}$	1.95930441291345	1	NA

TABLE 4.21. Speedup with different α's for h=1/4, $\delta = 10^{-5}$, and $\sigma = 200001$

α	speedup	α	speedup
$7 \cdot 10^{-2}$	1.99292961755658	$9 \cdot 10^{-2}$	1.96607223373042
$1 \cdot 10^{-1}$	1.99620413327709	$3 \cdot 10^{-1}$	1.96584128975761
$5 \cdot 10^{-1}$	1.98738953857551	$7 \cdot 10^{-1}$	1.98949022091239
$9 \cdot 10^{-1}$	1.97810512738649	1	NA

TABLE 4.22. Speedup with different α's for h=1/8, $\delta = 10^{-5}$, and $\sigma = 200001$

α	speedup	α	speedup
$7 \cdot 10^{-2}$	1.98624707508502	$9 \cdot 10^{-2}$	2.00186648265179
$1 \cdot 10^{-1}$	2.00061947742228	$3 \cdot 10^{-1}$	1.97203501775448
$5 \cdot 10^{-1}$	1.97454558867110	$7 \cdot 10^{-1}$	1.99793891940354
$9 \cdot 10^{-1}$	1.93757503974218	1	NA

TABLE 4.23. Speedup with different α's for h=1/16, $\delta = 10^{-5}$, and $\sigma = 200001$

References

[1] R. Adams, SOBOLEV SPACES, Academic, New York, 1975.

α	speedup	α	speedup
$7 \cdot 10^{-2}$	2.00440807396956	$9 \cdot 10^{-2}$	1.99894017904103
$1 \cdot 10^{-1}$	2.00418517629810	$3 \cdot 10^{-1}$	2.01917458068216
$5 \cdot 10^{-1}$	2.02186892412254	$7 \cdot 10^{-1}$	1.99790160537733
$9 \cdot 10^{-1}$	1.99207234541354	1	NA

TABLE 4.24. Speedup with different α's for h=1/32, $\delta = 10^{-5}$, and $\sigma = 200001$

[2] P. G. Ciarlet, INTRODUCTION TO NUMERICAL LINEAR ANALYSIS AND OPTIMISATION, Cambridge University Press, 1989.
[3] L. C. Evans, PARTIAL DIFFERENTIAL EQUATIONS, AMS, 1998.
[4] M. D. Gunzburger, L. S. Hou, and T. Svobdny, HEATING AND COOLING CONTROL OF TEMPERATURE DISTRIBUTIONS ALONG BOUNDARIES OF FLOW DOMAINS., J. Math. Sys. Estim. Cont., 3 (1993), pp. 147–172.
[5] M. D. Gunzburger, J. S. Peterson, and H. Kwon, AN OPTIMIZATION BASED DOMAIN DECOMPOSITION METHOD FOR PARTIAL DIFFERENTIAL EQUATIONS., Comput. Math. Appl., 37 (1999), no. 10, pp. 77–93.
[6] M.D. Gunzburger and J. Lee, A DOMAIN DECOMPOSTION METHOD FOR OPTIMIZATION PROBLEMS FOR PARTIAL DIFFERENTIAL EQUATIONS., Comput. Math. Appl., 40 (2000), no. 2-3, pp. 177–192.
[7] W. Guo and L. S. Hou, GENERALIZATIONS AND ACCELERATIONS OF LIONS' NONOVERLAPPING DOMAIN DECOMPOSITION METHOD FOR LINEAR ELLIPTIC PDE., SIAM J. Numer. Anal., 41 (2003), no. 6, pp. 2056–2080.
[8] L. S. Hou and S. S. Ravindran, A PENALIZED NEUMANN CONTROL APPROACH FOR SOLVING AN OPTIMAL DIRICHLET CONTROL PROBLEM FOR THE NAVIER-STOKES EQUATIONS, SIAM J. Control. Optim., Vol. 36, no. 5, pp. 1795–1814, September 1998.
[9] J. E. Lagness and G. Leugering, DOMAIN DECOMPOSITION METHODS IN OPTIMAL CONTROL OF PARTIAL DIFFERENTIAL EQUATIONS, Birkhauser, 2000.
[10] P. L. Lions, ON THE SCHWARZ ALTERNATING METHODS I, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Periaux, eds., SIAM, Philadelphia, PA, 1988, pp. 1-42.
[11] P. L. Lions, ON THE SCHWARZ ALTERNATING METHODS III: A VARIANT FOR NONOVERLAPPING SUBDOMAINS, in Third International Symposium on Domain Decomposition Methods for Partial Differential Equations, T. F. Chan, R. Glowinski, J. Periaux, and O. B. Wildlund, eds., SIAM, Philadelphia, PA, 1990, pp. 202-231.
[12] M. Snir, S. Otto, S. Huss-Lederman, David Walker, and Jack Dongarra, MPI-THE COMPLETE REFERENCE VOL. 1 AND VOL. 2, Second Edition, The MIT Press, 1996.

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