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AN ELASTO-VISCOPLASTIC CONTACT PROBLEM: AN A POSTERIORI ERROR ANALYSIS AND COMPUTATIONAL EXPERIMENTS

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Abstract. In this paper, we reconsider a contact problem between an elasto-viscoplastic body and a deformable obstacle. The contact is modeled by the classical normal compliance contact condition. Then, fully discrete approximations are obtained by using the finite element method to approximate the spatial variable and the forward Euler scheme to discretize time derivatives. An a posteriori error analysis is provided and upper and lower error bounds are obtained. Finally, some two-dimensional numerical simulations are presented to demonstrate the accurary and the behavior of the error estimators.

Key words. Elasto-viscoplasticity, normal compliance contact, fully discrete approximations, a posteriori error estimates, finite elements, numerical simulations.

1. Introduction

During the past twenty years many problems have been studied dealing with elasto-viscoplastic materials modeled using the constitutive law introduced in [9] (see the monograph [19] and its references). Then, numerous nonlinear problems including this kind of materials (as, for instance, contact problems) were considered (see, e.g., [1, 2, 5, 6, 10, 13, 16, 24, 25, 26], the well-written monograph [17] and the large number of references cited therein). We note that, as it was justified in [9], this law is mechanically correct and it can be used for the modeling of some types of metals or rocks since it allows both creep and relaxation phenomena.

In this work, we revisite the contact problem between an elasto-viscoplastic body and a deformable obstacle. The contact is modeled using the classical normal compliance contact law described, for example, in [20, 21]. This problem was already studied in [14] (see also the paper [11] where internal variables were also considered). A priori error estimates were proved there (see Section 3 where they are recalled) and numerical simulations were provided in order to show the accuracy of the algorithm and the behavior of the solution. However, even if many other papers were published since then, only a priori error estimates were obtained. Recently, an a posteriori error analysis was presented in [12] in the case without contact, extending some arguments already applied in the study of the heat equation (see, e.g., [22, 23, 28]), some parabolic equations ([3]) or the Stokes equation ([4]). Hence, this work continues the above referenced work by Fernández and Hild [12], extending the analysis presented there to the case including the contact with a deformable obstacle and also the previous paper [14], where the a priori error analysis was conducted. Moreover, here we also perform several two-dimensional numerical simulations in order to demonstrate the accuracy of the algorithm and the behavior of the error estimators.

The paper is outlined as follows. In Section 2 the mechanical model and its variational formulation are briefly described following the notation and assumptions

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introduced in [14]. Then, fully discrete approximations are provided in Section 3, by using the finite element method to approximate the spatial variable and the forward Euler scheme to discretize the time derivatives. An a priori error analysis obtained in [14] is recalled. Then, by using some results obtained in the study of the heat equation, an a posteriori error analysis is done in Section 4, providing an upper bound for the error, Theorem 4.1, and a lower bound, Theorem 4.2. Finally, some two-dimensional numerical simulations are presented in Section 5 in order to demonstrate the accuracy and the behavior of the error estimators introduced in the previous section.

2. Mechanical and variational formulations

In this section, we present a brief description of the contact problem between an elasto-viscoplastic body and a deformable obstacle (further details can be found in [14, 17]).

Denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ and by "·" and $|\cdot|$ the inner product and the Euclidean norms on $\mathbb{R}^d$ and $\mathbb{S}^d$.

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, denote a domain occupied by an elasto-viscoplastic body with a smooth boundary $\Gamma = \partial\Omega$ decomposed into three disjoint parts Γ_D, Γ_F and Γ_C such that meas $(\Gamma_D) > 0$ and meas $(\Gamma_C) > 0$. Moreover, let $[0, T]$, $T > 0$, be the time interval of interest and denote by $\boldsymbol{\nu}$ the unit outer normal vector to Γ. The body is being acted upon by a volume force of density $\boldsymbol{f}_0$, it is clamped on Γ_D and surface tractions with density $\boldsymbol{f}_F$ are applied on Γ_F. Finally, we assume that the body may come in contact with a deformable obstacle, on the boundary part Γ_C, which is located at a distance g measured along the outward unit normal vector $\boldsymbol{\nu}$ (see FIGURE 1).

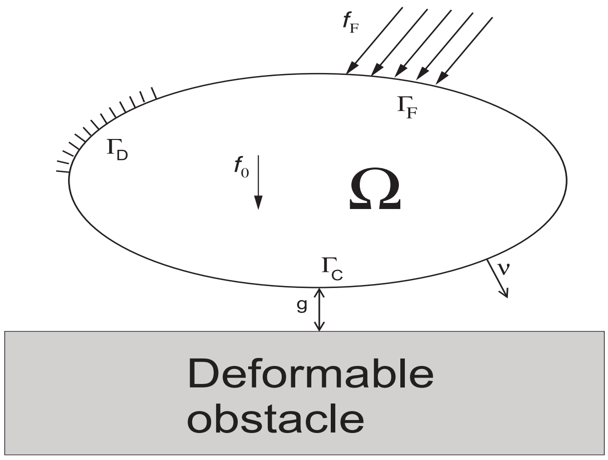

FIGURE 1. Physical setting: an elasto-viscoplastic body in contact with a deformable obstacle.

Let $\boldsymbol{x} \in \Omega$ and $t \in [0, T]$ be the spatial and time variables, respectively, and, in order to simplify the writing, we do not indicate the dependence of the functions on $\boldsymbol{x}$ and t. Moreover, a dot above a variable represents the derivative with respect to the time variable.

Let us denote by $\boldsymbol{u} = (u_i)_{i=1}^d$, $\boldsymbol{\sigma} = (\sigma_{ij})_{i,j=1}^d$ and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))_{i,j=1}^d$ the displacement field, the stress tensor and the linearized strain tensor, respectively.

We recall that
$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

The body is assumed elasto-viscoplastic and satisfying the following rate-type constitutive law (see [9, 14]),

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\varepsilon(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})), \tag{1}$$

where $\mathcal{E}$ and $\mathcal{G}$ denote the fourth-order elastic tensor and the viscoplastic function, respectively.

We turn now to describe the boundary conditions.

On the boundary part Γ_D we assume that the body is clamped and thus the displacement field vanishes there (and so $\boldsymbol{u} = \boldsymbol{0}$ on $\Gamma_D \times (0,T)$). Moreover, we assume that a density of traction forces, denoted by $\boldsymbol{f}_F$, is applied on the boundary part Γ_F; i.e.
$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_F \quad \text{on} \quad \Gamma_F \times (0,T).$$

Finally, since the contact is assumed with a deformable obstacle, the well-known normal compliance contact condition is employed (see [20, 21]); that is, the normal stress $\sigma_\nu = \boldsymbol{\sigma\nu}\cdot\boldsymbol{\nu}$ on Γ_C is given by
$$-\sigma_\nu = p(u_\nu - g) \quad \text{on} \quad \Gamma_C \times (0,T),$$
where $u_\nu = \boldsymbol{u}\cdot\boldsymbol{\nu}$ denotes the normal displacement in such a way that, when $u_\nu > g$, the difference $u_\nu - g$ represents the interpenetration of the body's asperities into those of the obstacle. The normal compliance function p is prescribed and satisfies $p(r) = 0$ for $r \le 0$, since then there is no contact. As an example, we use in Section 5 the following function,

$$p(r) = \mu\, r_+, \tag{2}$$

where $\mu > 0$ represents a deformability constant (that is, it denotes the stiffness of the obstacle), and $r_+ = \max\{0, r\}$. Moreover, we also assume that the contact is frictionless, i.e. the tangential component of the stress field, denoted by $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu\boldsymbol{\nu}$, vanishes on the contact surface.

Therefore, the mechanical formulation of the quasistatic contact problem between an elasto-viscoplastic body and a deformable obstacle, within the small displacements theory, is written as follows.

Problem P. *Find a displacement field* $\boldsymbol{u} : \Omega \times (0,T) \to \mathbb{R}^d$ *and a stress field* $\boldsymbol{\sigma} : \Omega \times (0,T) \to \mathbb{S}^d$ *such that,*

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})) \quad \textit{in} \quad \Omega \times (0,T), \tag{3}$$
$$-\mathrm{Div}\,\boldsymbol{\sigma} = \boldsymbol{f}_0 \quad \textit{in} \quad \Omega \times (0,T), \tag{4}$$
$$\boldsymbol{u} = \boldsymbol{0} \quad \textit{on} \quad \Gamma_D \times (0,T), \tag{5}$$
$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_F \quad \textit{on} \quad \Gamma_F \times (0,T), \tag{6}$$
$$\boldsymbol{\sigma}_\tau = \boldsymbol{0}, \quad -\sigma_\nu = p(u_\nu - g) \quad \textit{on} \quad \Gamma_C \times (0,T), \tag{7}$$
$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0 \quad \textit{in} \quad \Omega. \tag{8}$$

Here, $\boldsymbol{u}_0$ and $\boldsymbol{\sigma}_0$ represent initial conditions for the displacement field and the stress tensor, respectively. Moreover, we notice that equilibrium equation (4) does not include the acceleration term because the problem is assumed quasistatic.

In order to obtain the variational formulation of Problem P, let $H = [L^2(\Omega)]^d$ and define the following variational spaces:
$$V = \{\boldsymbol{w} \in [H^1(\Omega)]^d\,;\, \boldsymbol{w} = \boldsymbol{0} \quad \text{on} \quad \Gamma_D\},$$
$$Q = \{\boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^d \in [L^2(\Omega)]^{d\times d} \;;\; \tau_{ij} = \tau_{ji},\ i,j = 1,\dots,d\}.$$

The following assumptions are required on the problem data.

The elastic tensor $\mathcal{E}(\boldsymbol{x}) = (e_{ijkl}(\boldsymbol{x}))_{i,j,k,l=1}^{d} : \boldsymbol{\tau} \in \mathbb{S}^d \to \mathcal{E}(\boldsymbol{x})(\boldsymbol{\tau}) \in \mathbb{S}^d$ satisfies:

(9)
$$\begin{array}{l} \text{(a) } e_{ijkl} = e_{klij} = e_{jikl} \quad \text{for} \quad i,j,k,l = 1,\dots,d. \\ \text{(b) } e_{ijkl} \in L^\infty(\Omega) \quad \text{for} \quad i,j,k,l = 1,\dots,d. \\ \text{(c) There exists } m_{\mathcal{E}} > 0 \text{ such that } \mathcal{E}(\boldsymbol{x})\boldsymbol{\tau}\cdot\boldsymbol{\tau} \geq m_{\mathcal{E}}\,|\boldsymbol{\tau}|^2 \\ \quad \forall\, \boldsymbol{\tau} \in \mathbb{S}^d, \ \text{a.e. } \boldsymbol{x} \in \Omega. \end{array}$$

The viscoplastic function $\mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathcal{G}(\boldsymbol{x})(\boldsymbol{\tau},\boldsymbol{\varepsilon}) \in \mathbb{S}^d$ satisfies:

(10)
$$\begin{array}{l} \text{(a) There exists } L_{\mathcal{G}} > 0 \text{ such that} \\ \quad |\mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_1,\boldsymbol{\varepsilon}_1) - \mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2)| \leq L_{\mathcal{G}}\,(|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| + |\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2|) \\ \quad \text{for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2,\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2 \in \mathbb{S}^d, \quad \text{a.e.} \quad \boldsymbol{x} \in \Omega. \\ \text{(b) The function } \ \boldsymbol{x} \to \mathcal{G}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) \ \text{ is measurable.} \\ \text{(c) The mapping } \boldsymbol{x} \to \mathcal{G}(\boldsymbol{x},\boldsymbol{0},\boldsymbol{0}) \ \text{ belongs to } Q. \end{array}$$

The normal compliance function $p : \Gamma_C \times \mathbb{R} \longrightarrow \mathbb{R}^+$ satisfies:

(11)
$$\begin{array}{l} \text{(a) There exists } L_\nu > 0 \text{ such that} \\ \quad |p(\boldsymbol{x},r_1) - p(\boldsymbol{x},r_2)| \leq L_\nu\,|r_1 - r_2| \quad \forall\, r_1,r_2 \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_C. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto p(\boldsymbol{x},r) \text{ is Lebesgue measurable on } \Gamma_C,\ \forall r \in \mathbb{R}. \\ \text{(c) } (p(\boldsymbol{x},r_1) - p(\boldsymbol{x},r_2))\cdot(r_1 - r_2) \geq 0 \quad \forall\, r_1,r_2 \in \mathbb{R}, \ \text{a.e. } \boldsymbol{x} \in \Gamma_C. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto p(\boldsymbol{x},r) = 0 \quad \text{for all } r \leq 0. \end{array}$$

The following regularity is assumed on the density of volume forces and tractions:

$$\boldsymbol{f}_0 \in C^1([0,T];H), \quad \boldsymbol{f}_F \in C^1([0,T];[L^2(\Gamma_F)]^d). \tag{12}$$

Using Riesz' theorem, from (12) we can define the element $\boldsymbol{f}(t) \in V$ given by

$$(\boldsymbol{f}(t),\boldsymbol{w})_V = \int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{w}\,d\boldsymbol{x} + \int_{\Gamma_F} \boldsymbol{f}_F(t)\cdot\boldsymbol{w}\,d\gamma(\boldsymbol{x}) \quad \forall \boldsymbol{w} \in V,$$

and then $\boldsymbol{f} \in C^1([0,T];V)$.

Let us define the contact functional $j : V \times V \to \mathbb{R}$ as,

$$j(\boldsymbol{u},\boldsymbol{v}) = \int_{\Gamma_C} p(u_\nu - g)\,v_\nu\,d\gamma(\boldsymbol{x}) \quad \forall \boldsymbol{u},\boldsymbol{v} \in V,$$

where we let $v_\nu = \boldsymbol{v}\cdot\boldsymbol{\nu}$ for all $\boldsymbol{v} \in V$.

Finally, we assume that the initial displacement and stress fields satisfy the following regularity and compatibility conditions,

$$\begin{array}{l} \boldsymbol{u}_0 \in V, \quad \boldsymbol{\sigma}_0 \in Q, \\ (\boldsymbol{\sigma}_0,\boldsymbol{\varepsilon}(\boldsymbol{u}_0))_Q + j(\boldsymbol{u}_0,\boldsymbol{u}_0) = (\boldsymbol{f}(0),\boldsymbol{u}_0)_V. \end{array} \tag{13}$$

Using the previous boundary conditions and applying Green's formula, we obtain the following variational formulation of Problem P.

Problem VP. *Find a displacement field* $\boldsymbol{u} : [0,T] \to V$ *and a stress field* $\boldsymbol{\sigma} : [0,T] \to Q$ *such that* $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0$ *and for a.e.* $t \in (0,T)$,

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)) + \mathcal{G}(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{u}(t))), \tag{14}$$

$$(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\boldsymbol{w}))_Q + j(\boldsymbol{u}(t),\boldsymbol{w}) = (\boldsymbol{f}(t),\boldsymbol{w})_V \quad \forall \boldsymbol{w} \in V. \tag{15}$$

The existence of a unique weak solution to Problem VP has been considered in [14]. The following theorem, which establishes the existence of a unique solution to Problem VP, was proved there by using Banach fixed point theorem and well-known results on nonlinear variational equations.

Theorem 2.1. *Let assumptions (9)-(13) hold. Therefore, there exists a unique solution to Problem VP such that* $\boldsymbol{u} \in C^1([0,T];V)$ *and* $\boldsymbol{\sigma} \in C^1([0,T];Q)$.

3. Fully discrete approximations

In this section, we introduce a finite element algorithm to approximate solutions to Problem VP and we recall an a priori error estimates result proved in [14].

The discretization of Problem VP is done as follows. First, we assume that Ω is a polyhedral domain and we consider the finite dimensional spaces $V^h \subset V$ and $Q^h \subset Q$, approximating variational spaces V and Q, respectively, and given by

$$V^h = \{\boldsymbol{w}^h \in [C(\overline{\Omega})]^d \,;\, \boldsymbol{w}^h_{|T} \in [P_1(T)]^d \;\; T \in \mathcal{T}^h, \quad \boldsymbol{w}^h = \boldsymbol{0} \text{ on } \Gamma_D\}, \tag{16}$$

$$Q^h = \{\boldsymbol{\tau}^h \in Q \,;\, \boldsymbol{\tau}^h_{|T} \in [P_0(T)]^{d\times d} \;\; T \in \mathcal{T}^h\}, \tag{17}$$

where $P_q(T)$, $q = 0, 1$, represents the space of polynomials of global degree less or equal to q in T and we denote by $\mathcal{T}^h$ a triangulation of $\overline{\Omega}$ compatible with the partition of the boundary $\Gamma = \partial\Omega$ into Γ_D, Γ_F and Γ_C; i.e. the finite element space V^h is composed of continuous and piecewise affine functions and the finite element space Q^h is made of piecewise constant functions. Here, $h > 0$ is the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by $\boldsymbol{u}^h_0$ and $\boldsymbol{\sigma}^h_0$, are given by

$$\boldsymbol{u}^h_0 = \Pi_{V^h}\boldsymbol{u}_0, \quad \boldsymbol{\sigma}^h_0 = \Pi_{Q^h}\boldsymbol{\sigma}_0, \tag{18}$$

where $\Pi_{V^h} : [C(\overline{\Omega})]^d \to V^h$ and $\Pi_{Q^h} : Q \to Q^h$ are the standard finite element L^2-projection operators onto V^h and Q^h, respectively (see, e.g., [7]).

Let us denote by $0 = t_0 < t_1 < \ldots < t_N = T$ a uniform partition of the time interval $[0, T]$, and let k be the time step size, $k = T/N$. For a continuous function $f(t)$, let $f_n = f(t_n)$ and for a sequence $\{w_n\}_{n=0}^N$ we let $\delta w_n = (w_n - w_{n-1})/k$ be its corresponding divided differences.

In order to simplify the writing and the calculations, we assume, without loss of generality, that $\mathcal{G}(Q^h, Q^h) \subset Q^h$. It is straightforward to extend the results presented in the next section to more general situations by using the operator Π_{Q^h} (see [11]).

Therefore, using the classical forward Euler scheme, we obtain the following fully discrete approximation of Problem VP.

Problem VP^{hk}. *Find a discrete displacement field $\boldsymbol{u}^{hk} = \{\boldsymbol{u}^{hk}_n\}_{n=0}^N \subset V^h$ and a discrete stress field $\boldsymbol{\sigma}^{hk} = \{\boldsymbol{\sigma}^{hk}_n\}_{n=0}^N \subset Q^h$ such that $\boldsymbol{u}^{hk}_0 = \boldsymbol{u}^h_0$, $\boldsymbol{\sigma}^{hk}_0 = \boldsymbol{\sigma}^h_0$ and for all $n = 1, \ldots, N$,*

$$\delta\boldsymbol{\sigma}^{hk}_n = \mathcal{E}\boldsymbol{\varepsilon}(\delta\boldsymbol{u}^{hk}_n) + \mathcal{G}(\boldsymbol{\sigma}^{hk}_{n-1}, \boldsymbol{\varepsilon}(\boldsymbol{u}^{hk}_{n-1})), \tag{19}$$

$$(\boldsymbol{\sigma}^{hk}_n, \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q + j(\boldsymbol{u}^{hk}_n, \boldsymbol{w}^h) = (\boldsymbol{f}_n, \boldsymbol{w}^h)_V \quad \forall \boldsymbol{w}^h \in V^h. \tag{20}$$

Using well-known results on nonlinear variational equations (see [15]), it is easy to obtain the following theorem which states the existence of a unique discrete solution $\boldsymbol{u}^{hk} \subset V^h$ and $\boldsymbol{\sigma}^{hk} \subset Q^h$ to Problem VP^{hk}.

Theorem 3.1. *Let assumptions (9)-(13) hold. Therefore, there exists a unique solution to Problem VP^{hk}.*

We recall now an a priori error estimates for Problem VP^{hk}, which were proved in [14] for the case of an implicit time scheme. Since the modifications are minor, proceeding in a similar way we have the following.

Theorem 3.2. *Let assumptions (9)-(13) hold. Let us denote by $(\boldsymbol{u}, \boldsymbol{\sigma})$ and $(\boldsymbol{u}^{hk}, \boldsymbol{\sigma}^{hk})$ the respective solutions to problems VP and VP^{hk}. Therefore, there exists a positive constant $c > 0$, independent of the discretization parameters h and k, such that* *for all* $\{\boldsymbol{w}_n^h\}_{n=0}^N \subset V^h$,

$$\begin{aligned}\max_{0\le n\le N}\left\{\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V^2+\|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\|_Q^2\right\}\le c\Big(&\max_{0\le n\le N}\|\boldsymbol{u}_n-\boldsymbol{w}_n^h\|_V^2\\&+\max_{0\le n\le N}I_{\mathcal{G}_n}+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V^2+\|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0^h\|_Q^2\Big),\end{aligned}\tag{21}$$

where the integration error $I_{\mathcal{G}_n}$ *is given by*

$$I_{\mathcal{G}_n}=\left\|\int_0^{t_n}\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds-\sum_{j=1}^n k\mathcal{G}(\boldsymbol{\sigma}_{j-1},\boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}))\right\|_Q^2.$$

These error estimates are the basis for the analysis of the convergence rate of the algorithm. Hence, under additional regularity assumptions, we obtain the linear convergence of the algorithm that we state in the following (see again [14]).

Corollary 3.3. *Let the assumptions of Theorem 3.2 hold. Under the additional regularity conditions*

$$\boldsymbol{u}\in C([0,T];[H^2(\Omega)]^d),\quad \boldsymbol{\sigma}_0\in[H^1(\Omega)]^{d\times d},$$

there exists a positive constant $c>0$*, independent of the discretization parameters* h *and* k*, such that*

$$\max_{0\le n\le N}\left\{\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\|_V+\|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_n^{hk}\|_Q\right\}\le c(h+k).\tag{22}$$

4. An a posteriori error analysis

In this section, we will use the finite element spaces and the notations introduced in the previous two sections. Moreover, throughout this section, we will assume that the mesh of the domain Ω may change during the time, and so, for any $0<h<1$ and for any $n=0,1,\dots,N$, let $\mathcal{T}^{hn}$ be a mesh of $\overline{\Omega}$ composed of finite elements T with diameter less than h. We will also assume that, for each $n=1,\dots,N$, the mesh $\{(t_{n-1},t_n)\times T\,;\,T\in\mathcal{T}^{hn}\}$ is regular in the sense of [7] and, to simplify the calculations, that $\mathcal{T}^{hn}\subset T^{h(n-1)}$. Thus, for any $n=1,\dots,N$ and for any $T\in\mathcal{T}^{hn}$, let h_T^n (respectively ρ_T^n) be the diameter of the smallest (resp. largest) ball containing (resp. contained in) $(t_{n-1},t_n)\times T$. Therefore, there exists a positive constant β such that

$$\frac{h_T^n}{\rho_T^n}\le\beta\quad\forall T\in\mathcal{T}^{hn},\ n=1,\dots,N.$$

In order to simplify the writing and the calculations, in this section we assume that $\boldsymbol{f}_F=\mathbf{0}$ and so $(\boldsymbol{f},\boldsymbol{w})_V=(\boldsymbol{f},\boldsymbol{w})_H$, where $\boldsymbol{f}=\boldsymbol{f}_0\in C([0,T];H)$. It is straightforward to extend the results presented below to more general situations.

Moreover, for an element $T\in\mathcal{T}^{hn}$, we denote by $\mathcal{E}_T^{hn}$ its set of interior edges or faces, and for the triangulation $\mathcal{T}^{hn}$, let us define as $\mathcal{E}^{hn}$, $\mathcal{E}_{int}^{hn}$ and $\mathcal{E}_C^{hn}$ its set of edges or faces, its set of interior edges or faces and its set of edges or faces that belong to Γ_C (i.e., $\mathcal{E}_C^{hn}=\{E\in\mathcal{E}^{hn}\,;\,E\subset\Gamma_C\}$), respectively.

Finally, the notation $a\lesssim b$ means that there exists a positive constant c independent of a and b (and of the discretization parameters) such that $a\le cb$. The notation $a\sim b$ means that $a\lesssim b$ and $b\lesssim a$ hold simultaneously.

Let us define the continuous and piecewise linear approximations in time given by

$$\boldsymbol{u}^{h\tau}(\boldsymbol{x},t)=\frac{t-t_{n-1}}{k}\boldsymbol{u}_n^{hk}(\boldsymbol{x})+\frac{t_n-t}{k}\boldsymbol{u}_{n-1}^{hk}(\boldsymbol{x})\quad t_{n-1}\le t\le t_n,\quad \boldsymbol{x}\in\overline{\Omega},$$
$$\boldsymbol{\sigma}^{h\tau}(\boldsymbol{x},t)=\frac{t-t_{n-1}}{k}\boldsymbol{\sigma}_n^{hk}(\boldsymbol{x})+\frac{t_n-t}{k}\boldsymbol{\sigma}_{n-1}^{hk}(\boldsymbol{x})\quad t_{n-1}\le t\le t_n,\quad \boldsymbol{x}\in\overline{\Omega}.$$

Since $\dot{\boldsymbol{u}}^{h\tau}=\delta\boldsymbol{u}_n^{hk}$ and $\dot{\boldsymbol{\sigma}}^{h\tau}=\delta\boldsymbol{\sigma}_n^{hk}$, we can write discrete problem VP^{hk} in the following more general form, for $n=1,\dots,N$,

(23) $\quad \dot{\boldsymbol{\sigma}}^{h\tau}=\mathcal{E}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}^{h\tau})+\mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk},\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk})),$

(24) $\quad (\boldsymbol{\sigma}^{h\tau}(t),\boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q+j(\boldsymbol{u}^{h\tau},\boldsymbol{w}^h)=(\boldsymbol{f}(t),\boldsymbol{w}^h)_H\quad \forall\boldsymbol{w}^h\in V^h,\ t_{n-1}\le t\le t_n.$

We have the following theorem which provides an upper bound for the numerical errors.

Theorem 4.1. *Let assumptions (9)-(13) hold. Denote by $(\boldsymbol{u},\boldsymbol{\sigma})$ the solution to Problem VP and by $(\boldsymbol{u}^{h\tau},\boldsymbol{\sigma}^{h\tau})$ the continuous piecewise linear approximation of the solution to Problem VP^{hk}. Then*

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\|_{C([0,T];V)}&+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{h\tau}\|_{C([0,T];Q)}\lesssim\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V+\|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0^h\|_Q\\&+\sum_{n=1}^N k\eta_1^n+\max_{1\le n\le N}\max_{t\in[t_{n-1},t_n]}\eta_2^n(t)+\max_{1\le n\le N}\max_{t\in[t_{n-1},t_n]}\eta_3^n(t)\\&+\max_{1\le n\le N}\max_{t\in[t_{n-1},t_n]}\eta_4^n(t),\end{aligned}$$

where the error estimators η_1^n, η_2^n, η_3^n and η_4^n are given by

(25) $$\eta_1^n=\|\boldsymbol{\sigma}_n^{hk}-\boldsymbol{\sigma}_{n-1}^{hk}\|_Q+\|\boldsymbol{u}_n^{hk}-\boldsymbol{u}_{n-1}^{hk}\|_V,$$

(26) $$\eta_2^n(t)=\left(\sum_{T\in\mathcal{T}^{hn}}|T|^2\|\boldsymbol{f}(t)\|^2_{[L^2(T)]^d}\right)^{1/2},$$

(27) $$\eta_3^n(t)=\left(\sum_{T\in\mathcal{T}^{hn}}\sum_{E\in\mathcal{E}_T^{hn}}|E|\|[\boldsymbol{\sigma}^{h\tau}(t)\boldsymbol{\nu}]\|^2_{[L^2(E)]^d}\right)^{1/2}$$

$$\eta_4^n(t)=\left(\sum_{E\in\mathcal{E}_C^{hn}}|E|\|\boldsymbol{\sigma}_\tau^{h\tau}(t)\|^2_{[L^2(E)]^d}\right)^{1/2}$$

(28) $$+\left(\sum_{E\in\mathcal{E}_C^{hn}}|E|\|p(u_\nu^{h\tau}(t))+\sigma_\nu^{h\tau}(t)\|^2_{L^2(E)}\right)^{1/2},$$

and $[\boldsymbol{\tau\nu}]$ denotes the jump of $\boldsymbol{\tau\nu}$ across the edge or face E. Moreover, $|T|$ and $|E|$ represent the size of the T and E, respectively. Note that, since the mesh is assumed regular, we have $|T|\sim|E|\sim h$.

Proof. Proceeding as in [12], let us estimate the error on the stress field. Therefore, integrate (14) and (23) between t_{n-1} and $t\in(t_{n-1},t_n]$ to obtain

$$\boldsymbol{\sigma}(t)=\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t))+\boldsymbol{\sigma}_{n-1}-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1})+\int_{t_{n-1}}^t\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))\,ds,$$

$$\boldsymbol{\sigma}^{h\tau}(t)=\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}^{h\tau}(t))+\boldsymbol{\sigma}_{n-1}^{hk}-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk})+\int_{t_{n-1}}^t\mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk},\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}))\,ds,$$

and therefore, by induction it follows that

$$\boldsymbol{\sigma}(t) = \mathcal{E}\varepsilon(\boldsymbol{u}(t)) + \boldsymbol{\sigma}_0 - \mathcal{E}\varepsilon(\boldsymbol{u}_0) + \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \mathcal{G}(\boldsymbol{\sigma}(s), \varepsilon(\boldsymbol{u}(s)))\, ds$$

$$+ \int_{t_{n-1}}^{t} \mathcal{G}(\boldsymbol{\sigma}(s), \varepsilon(\boldsymbol{u}(s)))\, ds, \tag{29}$$

$$\boldsymbol{\sigma}^{h\tau}(t) = \mathcal{E}\varepsilon(\boldsymbol{u}^{h\tau}(t)) + \boldsymbol{\sigma}_0^h - \mathcal{E}\varepsilon(\boldsymbol{u}_0^h) + \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} \mathcal{G}(\boldsymbol{\sigma}_{j-1}^{hk}, \varepsilon(\boldsymbol{u}_{j-1}^{hk}))\, ds$$

$$+ \int_{t_{n-1}}^{t} \mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk}, \varepsilon(\boldsymbol{u}_{n-1}^{hk}))\, ds. \tag{30}$$

By subtracting now (29) and (30), we find that

$$\begin{aligned}\|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}^{h\tau}(t)\|_Q \lesssim{}& \|\boldsymbol{u}(t) - \boldsymbol{u}^{h\tau}(t)\|_V + \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h\|_Q + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V \\ &+ \sum_{j=1}^{n-1} \int_{t_{j-1}}^{t_j} [\|\boldsymbol{\sigma}(s) - \boldsymbol{\sigma}_{j-1}^{hk}\|_Q + \|\boldsymbol{u}(s) - \boldsymbol{u}_{j-1}^{hk}\|_V]\, ds \\ &+ \int_{t_{n-1}}^{t} [\|\boldsymbol{\sigma}(s) - \boldsymbol{\sigma}_{n-1}^{hk}\|_Q + \|\boldsymbol{u}(s) - \boldsymbol{u}_{n-1}^{hk}\|_V]\, ds \quad \forall t \in (t_{n-1}, t_n],\end{aligned}$$

and we immediately get (see [12] for details),

$$\begin{aligned}\|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}^{h\tau}(t)\|_Q \lesssim{}& \|\boldsymbol{u}(t) - \boldsymbol{u}^{h\tau}(t)\|_V + \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h\|_Q + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V \\ &+ \int_0^t [\|\boldsymbol{\sigma}(s) - \boldsymbol{\sigma}^{h\tau}(s)\|_Q + \|\boldsymbol{u}(s) - \boldsymbol{u}^{h\tau}(s)\|_V]\, ds \\ &+ \sum_{j=1}^{n} k[\|\boldsymbol{\sigma}_j^{hk} - \boldsymbol{\sigma}_{j-1}^{hk}\|_Q + \|\boldsymbol{u}_j^{hk} - \boldsymbol{u}_{j-1}^{hk}\|_V] \quad \forall t \in (t_{n-1}, t_n].\end{aligned}$$

Next, we estimate the numerical errors on the displacement field. Then, we subtract equation (15) for $\boldsymbol{w} = \boldsymbol{w}^h \in V^h \subset V$ and equation (24) to obtain

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{h\tau}, \varepsilon(\boldsymbol{w}^h))_Q + j(\boldsymbol{u}, \boldsymbol{w}^h) - j(\boldsymbol{u}^{h\tau}, \boldsymbol{w}^h) = 0 \quad \forall \boldsymbol{w}^h \in V^h.$$

Therefore, since $\boldsymbol{u}^{h\tau} \in V^h$, we have

$$\begin{aligned}&(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{h\tau}, \varepsilon(\boldsymbol{u} - \boldsymbol{u}^{h\tau}))_Q + j(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}^{h\tau}) - j(\boldsymbol{u}^{h\tau}, \boldsymbol{u} - \boldsymbol{u}^{h\tau}) \\ &\quad = (\boldsymbol{\sigma} - \boldsymbol{\sigma}^{h\tau}, \varepsilon(\boldsymbol{u} - \boldsymbol{w}^h))_Q + j(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{w}^h) - j(\boldsymbol{u}^{h\tau}, \boldsymbol{u} - \boldsymbol{w}^h) \quad \forall \boldsymbol{w}^h \in V^h.\end{aligned} \tag{31}$$

We consider the left-hand side of the previous equation. Using again equations (29) and (30) it leads to the following,

$$\begin{aligned}&(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{h\tau}, \varepsilon(\boldsymbol{u} - \boldsymbol{u}^{h\tau}))_Q + j(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}^{h\tau}) - j(\boldsymbol{u}^{h\tau}, \boldsymbol{u} - \boldsymbol{u}^{h\tau}) \\ &\quad = (\mathcal{E}\varepsilon(\boldsymbol{u} - \boldsymbol{u}^{h\tau}), \varepsilon(\boldsymbol{u} - \boldsymbol{u}^{h\tau}))_Q + j(\boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}^{h\tau}) - j(\boldsymbol{u}^{h\tau}, \boldsymbol{u} - \boldsymbol{u}^{h\tau}) \\ &\qquad + (\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h - \mathcal{E}\varepsilon(\boldsymbol{u}_0 - \boldsymbol{u}_0^h), \varepsilon(\boldsymbol{u} - \boldsymbol{u}^{h\tau}))_Q \\ &\qquad + (\int_{t_{n-1}}^{t} [\mathcal{G}(\boldsymbol{\sigma}(s), \varepsilon(\boldsymbol{u}(s))) - \mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk}, \varepsilon(\boldsymbol{u}_{n-1}^{hk}))]\, ds, \varepsilon(\boldsymbol{u} - \boldsymbol{u}^{h\tau}))_Q \\ &\qquad + \sum_{j=1}^{n-1} (\int_{t_{j-1}}^{t_j} [\mathcal{G}(\boldsymbol{\sigma}(s), \varepsilon(\boldsymbol{u}(s))) - \mathcal{G}(\boldsymbol{\sigma}_{j-1}^{hk}, \varepsilon(\boldsymbol{u}_{j-1}^{hk}))]\, ds, \varepsilon(\boldsymbol{u} - \boldsymbol{u}^{h\tau}))_Q,\end{aligned}$$

and taking into account properties (9)–(11) and the previous algebra, we have

$$(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}^{h\tau}),\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}^{h\tau}))_Q \geq m_{\mathcal{E}}\|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\|_V^2,$$

$$j(\boldsymbol{u},\boldsymbol{u}-\boldsymbol{u}^{h\tau})-j(\boldsymbol{u}^{h\tau},\boldsymbol{u}-\boldsymbol{u}^{h\tau})\geq 0,$$

$$\begin{aligned}|(\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0^h-\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_0-\boldsymbol{u}_0^h),\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}^{h\tau}))_Q|&\\ \lesssim (\|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0^h\|_Q+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V)\|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\|_V,&\end{aligned}$$

$$\begin{aligned}&\Big|\Big(\int_{t_{n-1}}^t[\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))-\mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk},\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}))]\,ds,\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}^{h\tau})\Big)_Q\Big|\\
&\qquad+\Big|\Big(\sum_{j=1}^{n-1}\int_{t_{j-1}}^{t_j}[\mathcal{G}(\boldsymbol{\sigma}(s),\boldsymbol{\varepsilon}(\boldsymbol{u}(s)))-\mathcal{G}(\boldsymbol{\sigma}_{j-1}^{hk},\boldsymbol{\varepsilon}(\boldsymbol{u}_{j-1}^{hk}))]\,ds,\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}^{h\tau})\Big)_Q\Big|\\
&\quad\lesssim\Big(\int_0^t[\|\boldsymbol{u}(s)-\boldsymbol{u}^{h\tau}(s)\|_V+\|\boldsymbol{\sigma}(s)-\boldsymbol{\sigma}^{h\tau}(s)\|_Q]\,ds\\
&\qquad\qquad+\sum_{j=1}^n k[\|\boldsymbol{\sigma}_j^{hk}-\boldsymbol{\sigma}_{j-1}^{hk}\|_Q+\|\boldsymbol{u}_j^{hk}-\boldsymbol{u}_{j-1}^{hk}\|_V]\Big)\|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\|_V,\end{aligned}$$

and therefore, for all $\boldsymbol{w}^h\in V^h$,

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\|_V\lesssim\|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0^h\|_Q&+\sum_{j=1}^n k[\|\boldsymbol{\sigma}_j^{hk}-\boldsymbol{\sigma}_{j-1}^{hk}\|_Q+\|\boldsymbol{u}_j^{hk}-\boldsymbol{u}_{j-1}^{hk}\|_V]\\
&+\int_0^t\|\boldsymbol{u}(s)-\boldsymbol{u}^{h\tau}(s)\|_V+\|\boldsymbol{\sigma}(s)-\boldsymbol{\sigma}^{h\tau}(s)\|_Q\,ds+\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V\\
&+(\boldsymbol{\sigma}-\boldsymbol{\sigma}^{h\tau},\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{w}^h))_Q+j(\boldsymbol{u},\boldsymbol{u}-\boldsymbol{w}^h)-j(\boldsymbol{u}^{h\tau},\boldsymbol{u}-\boldsymbol{w}^h).\end{aligned}$$

Let $\boldsymbol{w}\in V$ and denote by Π_C^h the Clément's interpolant on the triangulation $\mathcal{T}^{hn}$ (see [8]). We recall that this operator satisfies:

$$\|\boldsymbol{w}-\Pi_C^h\boldsymbol{w}\|_{[L^2(T)]^d}\lesssim|T|\|\boldsymbol{w}\|_{[H^1(\Delta T)]^d},$$

$$\|\boldsymbol{w}-\Pi_C^h\boldsymbol{w}\|_{[L^2(E)]^d}\lesssim|E|^{1/2}\|\boldsymbol{w}\|_{[H^1(\Delta T)]^d},$$

where ΔT denotes the set of elements having a common edge or face with T, and E being an edge or a face of T.

We consider now the right-hand side of equation (31) which equals to

$$(\boldsymbol{f},\boldsymbol{u}-\boldsymbol{w}^h)_H-(\boldsymbol{\sigma}^{h\tau},\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{w}^h))_Q-j(\boldsymbol{u}^{h\tau},\boldsymbol{u}-\boldsymbol{w}^h).$$

Taking $\boldsymbol{w}^h=\boldsymbol{u}^{h\tau}+\Pi_C^h(\boldsymbol{u}-\boldsymbol{u}^{h\tau})$ in the previous expression, applying Green's formula on each finite element and using the approximation properties of Π_C^h, it follows that

$$
\begin{aligned}
&(\boldsymbol{f}, \boldsymbol{u}-\boldsymbol{w}^h)_H - (\boldsymbol{\sigma}^{h\tau}, \boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{w}^h))_Q - j(\boldsymbol{u}^{h\tau}, \boldsymbol{u}-\boldsymbol{w}^h) \\
&\quad = \sum_{T\in\mathcal{T}^{hn}} \int_T (\boldsymbol{f} + \mathrm{Div}\,(\boldsymbol{\sigma}^{h\tau}))\cdot(\boldsymbol{u}-\boldsymbol{u}^{h\tau} - \Pi_C^h(\boldsymbol{u}-\boldsymbol{u}^{h\tau}))\,d\boldsymbol{x} \\
&\qquad - \sum_{T\in\mathcal{T}^{hn}} \sum_{E\in\mathcal{E}_T^{hn}} \int_E \boldsymbol{\sigma}^{h\tau}\boldsymbol{\nu}\cdot(\boldsymbol{u}-\boldsymbol{u}^{h\tau} - \Pi_C^h(\boldsymbol{u}-\boldsymbol{u}^{h\tau}))\,d\gamma(\boldsymbol{x}) \\
&\qquad + \sum_{E\in\mathcal{E}_C^{hn}} \int_E \boldsymbol{\sigma}^{h\tau}\boldsymbol{\nu}\cdot(\boldsymbol{u}-\boldsymbol{u}^{h\tau} - \Pi_C^h(\boldsymbol{u}-\boldsymbol{u}^{h\tau}))\,d\gamma(\boldsymbol{x}) \\
&\qquad + \sum_{E\in\mathcal{E}_C^{hn}} \int_E p(u_\nu^{h\tau})(\boldsymbol{u}-\boldsymbol{u}^{h\tau} - \Pi_C^h(\boldsymbol{u}-\boldsymbol{u}^{h\tau}))_\nu\,d\gamma(\boldsymbol{x}) \\
&\quad \lesssim \sum_{T\in\mathcal{T}^{hn}} \|\boldsymbol{f} + \mathrm{Div}\,(\boldsymbol{\sigma}^{h\tau})\|_{[L^2(T)]^d} \|\boldsymbol{u}-\boldsymbol{u}^{h\tau} - \Pi_C^h(\boldsymbol{u}-\boldsymbol{u}^{h\tau})\|_{[L^2(T)]^d} \\
&\qquad + \sum_{T\in\mathcal{T}^{hn}} \sum_{E\in\mathcal{E}_T^{hn}} \|[\boldsymbol{\sigma}^{h\tau}\boldsymbol{\nu}]\|_{[L^2(E)]^d} \|\boldsymbol{u}-\boldsymbol{u}^{h\tau} - \Pi_C^h(\boldsymbol{u}-\boldsymbol{u}^{h\tau})\|_{[L^2(E)]^d} \\
&\qquad + \sum_{E\in\mathcal{E}_C^{hn}} \|\boldsymbol{\sigma}_\tau^{h\tau}\|_{[L^2(E)]^d} \|\boldsymbol{u}-\boldsymbol{u}^{h\tau} - \Pi_C^h(\boldsymbol{u}-\boldsymbol{u}^{h\tau})\|_{[L^2(E)]^d} \\
&\qquad + \sum_{E\in\mathcal{E}_C^{hn}} \|p(u_\nu^{h\tau}) + \sigma_\nu^{h\tau}\|_{L^2(E)} \|\boldsymbol{u}-\boldsymbol{u}^{h\tau} - \Pi_C^h(\boldsymbol{u}-\boldsymbol{u}^{h\tau})\|_{[L^2(E)]^d} \\
&\quad \lesssim \Big(\sum_{T\in\mathcal{T}^{hn}} |T|^2 \|\boldsymbol{f}\|^2_{[L^2(T)]^d}\Big)^{1/2} \Big(\sum_{T\in\mathcal{T}^{hn}} \|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\|^2_{[H^1(\Delta T)]^d}\Big)^{1/2} \\
&\qquad + \Big(\sum_{T\in\mathcal{T}^{hn}} \sum_{E\in\mathcal{E}_T^{hn}} |E| \|[\boldsymbol{\sigma}^{h\tau}\boldsymbol{\nu}]\|^2_{[L^2(E)]^d}\Big)^{1/2} \Big(\sum_{T\in\mathcal{T}^{hn}} \|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\|^2_{[H^1(\Delta T)]^d}\Big)^{1/2} \\
&\qquad + \Big(\sum_{E\in\mathcal{E}_C^{hn}} |E| \|\boldsymbol{\sigma}_\tau^{h\tau}\|^2_{[L^2(E)]^d}\Big)^{1/2} \Big(\sum_{T\in\mathcal{T}^{hn}} \|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\|^2_{[H^1(\Delta T)]^d}\Big)^{1/2} \\
&\qquad + \Big(\sum_{E\in\mathcal{E}_C^{hn}} |E| \|p(u_\nu^{h\tau}) + \sigma_\nu^{h\tau}\|^2_{L^2(E)}\Big)^{1/2} \Big(\sum_{T\in\mathcal{T}^{hn}} \|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\|^2_{[H^1(\Delta T)]^d}\Big)^{1/2} \\
&\quad \lesssim (\eta_2^n(t) + \eta_3^n(t) + \eta_4^n(t)) \|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\|_V,
\end{aligned}
$$

where we take into account that $\mathrm{Div}\,(\boldsymbol{\sigma}^{h\tau}) = \mathbf{0}$ in T, the decomposition

$$\boldsymbol{\sigma}^{h\tau}\boldsymbol{\nu}\cdot\boldsymbol{w} = \boldsymbol{\sigma}_\tau^{h\tau}\cdot\boldsymbol{w}_\tau + \sigma_\nu^{h\tau} w_\nu \quad \forall \boldsymbol{w}\in V,$$

and the notations

$$u_\nu^{h\tau} = \boldsymbol{u}^{h\tau}\cdot\boldsymbol{\nu}, \quad \sigma_\nu^{h\tau} = \boldsymbol{\sigma}^{h\tau}\boldsymbol{\nu}\cdot\boldsymbol{\nu}, \quad \boldsymbol{\sigma}_\tau^{h\tau} = \boldsymbol{\sigma}^{h\tau}\boldsymbol{\nu} - \sigma_\nu^{h\tau}\boldsymbol{\nu}.$$

Combining the previous estimates, we conclude that

$$
\begin{aligned}
&\|\boldsymbol{u}(t) - \boldsymbol{u}^{h\tau}(t)\|_V + \|\boldsymbol{\sigma}(t) - \boldsymbol{\sigma}^{h\tau}(t)\|_Q \lesssim \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h\|_Q + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V \\
&\qquad + \int_0^t [\|\boldsymbol{u}(s) - \boldsymbol{u}^{h\tau}(s)\|_V + \|\boldsymbol{\sigma}(s) - \boldsymbol{\sigma}^{h\tau}(s)\|_Q]\,ds \\
&\qquad + \sum_{j=1}^n k[\|\boldsymbol{\sigma}_j^{hk} - \boldsymbol{\sigma}_{j-1}^{hk}\|_Q + \|\boldsymbol{u}_j^{hk} - \boldsymbol{u}_{j-1}^{hk}\|_V] + \eta_2^n(t) + \eta_3^n(t) + \eta_4^n(t),
\end{aligned}
$$

for all $t \in (t_{n-1}, t_n]$. Using Gronwall's inequality we find that

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\|_{C([0,T];V)} + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{h\tau}\|_{C([0,T];Q)} &\lesssim \|\boldsymbol{u}_0-\boldsymbol{u}_0^h\|_V + \|\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_0^h\|_Q \\ &+ \sum_{n=1}^N k\eta_1^n + \max_{1\le n\le N}\max_{t\in[t_{n-1},t_n]}\eta_2^n(t) + \max_{1\le n\le N}\max_{t\in[t_{n-1},t_n]}\eta_3^n(t) \\ &+ \max_{1\le n\le N}\max_{t\in[t_{n-1},t_n]}\eta_4^n(t),\end{aligned}$$

which concludes the proof. □

Now, we prove a lower bound for these error estimators that we provide in the following.

Theorem 4.2. *Let assumptions (9)-(13) hold. For all elements $T \in \mathcal{T}^{hn}$, the following local lower error bounds are obtained for $n = 1,\dots,N$:*

$$\begin{aligned}\eta_{1T}^{hn} &\lesssim \|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{h\tau}\|_{C([t_{n-1},t_n];[L^2(T)]^{d\times d})} + \|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\|_{C([t_{n-1},t_n];[H^1(T)]^d)} \\ &\quad + \|\boldsymbol{u}_n-\boldsymbol{u}_{n-1}\|_{[H^1(T)]^d} + \|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_{n-1}\|_{[L^2(T)]^{d\times d}}, \\ \eta_{2T}^{hn}(t) &\lesssim \|\boldsymbol{\sigma}(t)-\boldsymbol{\sigma}^{h\tau}(t)\|_{[L^2(T)]^{d\times d}} \quad t\in(t_{n-1},t_n], \\ \eta_{3T}^{hn}(t) &\lesssim \|\boldsymbol{\sigma}(t)-\boldsymbol{\sigma}^{h\tau}(t)\|_{[L^2(\Delta T)]^{d\times d}} \quad t\in(t_{n-1},t_n], \\ \eta_{4T}^{hn}(t) &\lesssim \|\boldsymbol{\sigma}(t)-\boldsymbol{\sigma}^{h\tau}(t)\|_{[L^2(T)]^{d\times d}} + \|\boldsymbol{u}(t)-\boldsymbol{u}^{h\tau}(t)\|_{[H^1(T)]^d} \\ &\quad + \|\sigma_\nu^{h\tau}(t)-\sigma_\nu(t)\|_{L^2(E)} \quad t\in(t_{n-1},t_n],\end{aligned}$$

where we denote by η_{1T}^{hn}, η_{2T}^{hn}, η_{3T}^{hn} and η_{4T}^{hn} the local errors given by

$$\begin{aligned}\eta_{1T}^{hn} &= \|\boldsymbol{\sigma}_n^{hk}-\boldsymbol{\sigma}_{n-1}^{hk}\|_{[L^2(T)]^{d\times d}} + \|\boldsymbol{u}_n^{hk}-\boldsymbol{u}_{n-1}^{hk}\|_{[H^1(T)]^d}, \\ \eta_{2T}^{hn}(t) &= |T|\|\boldsymbol{f}(t)\|_{[L^2(T)]^d}, \\ \eta_{3T}^{hn}(t) &= \Big(\sum_{E\in\mathcal{E}_T^{hn}} |E|\|[\boldsymbol{\sigma}^{h\tau}(t)\boldsymbol{\nu}]\|^2_{[L^2(E)]^d}\Big)^{1/2}, \\ \eta_{4T}^{hn}(t) &= \Big(\sum_{E\in\mathcal{E}_T^C} |E|\left[\|[\boldsymbol{\sigma}_\tau^{h\tau}(t)]\|_{[L^2(E)]^d} + \|p(u_\nu^{h\tau}(t)) + \sigma_\nu^{h\tau}(t)\|_{L^2(E)}\right]^2\Big)^{1/2}.\end{aligned}$$

Here, E_T^C represents the set of edges or faces of T that belong to Γ_C.

Obviously, we have

$$\begin{aligned}\eta_1^n &\sim \Big(\sum_{T\in\mathcal{T}^{hn}}(\eta_{1T}^{hn})^2\Big)^{1/2}, \\ \eta_2^n &= \Big(\sum_{T\in\mathcal{T}^{hn}}(\eta_{2T}^{hn})^2\Big)^{1/2}, \\ \eta_3^n &= \Big(\sum_{T\in\mathcal{T}^{hn}}(\eta_{3T}^{hn})^2\Big)^{1/2}, \\ \eta_4^n &\sim \Big(\sum_{T\in\mathcal{T}^{hn}}(\eta_{4T}^{hn})^2\Big)^{1/2}.\end{aligned}$$

Proof. First, error estimator η_1^n was bounded in [12]. We proved there that

$$\begin{aligned}\eta_1^n &= \|\boldsymbol{\sigma}_n^{hk}-\boldsymbol{\sigma}_{n-1}^{hk}\|_Q + \|\boldsymbol{u}_n^{hk}-\boldsymbol{u}_{n-1}^{hk}\|_V \\ &\le \|\boldsymbol{\sigma}^{h\tau}-\boldsymbol{\sigma}\|_{C([t_{n-1},t_n];Q)} + \|\boldsymbol{u}^{h\tau}-\boldsymbol{u}\|_{C([t_{n-1},t_n];V)} \\ &\quad + \|\boldsymbol{u}_n-\boldsymbol{u}_{n-1}\|_V + \|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_{n-1}\|_Q,\end{aligned}$$

and therefore,

$$\begin{aligned}(\eta_1^n)^2 \lesssim \sum_{T\in\mathcal{T}^{hn}} \Big(&\|\boldsymbol{\sigma}^{h\tau}-\boldsymbol{\sigma}\|^2_{C([t_{n-1},t_n];[L^2(T)]^{d\times d})} + \|\boldsymbol{u}^{h\tau}-\boldsymbol{u}\|^2_{C([t_{n-1},t_n];[H^1(T)]^d)} \\ &+ \|\boldsymbol{u}_n-\boldsymbol{u}_{n-1}\|^2_{[H^1(T)]^d} + \|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_{n-1}\|^2_{[L^2(T)]^{d\times d}}\Big).\end{aligned}$$

Proceeding in a similar way we also obtain that

$$\begin{aligned}\eta_{1T}^{hn} \lesssim \|\boldsymbol{\sigma}^{h\tau}-\boldsymbol{\sigma}\|^2_{C([t_{n-1},t_n];[L^2(T)]^{d\times d})}+\|\boldsymbol{u}^{h\tau}-\boldsymbol{u}\|^2_{C([t_{n-1},t_n];[H^1(T)]^d)}\\ +\|\boldsymbol{u}_n-\boldsymbol{u}_{n-1}\|^2_{[H^1(T)]^d}+\|\boldsymbol{\sigma}_n-\boldsymbol{\sigma}_{n-1}\|^2_{[L^2(T)]^{d\times d}}.\end{aligned}$$

We estimate now η_2^n (see again [12]). Let w_T be the bubble function associated with the element T and define the function $\boldsymbol{w}_T=(w_i)_{i=1}^d\in[H_0^1(T)]^d$ which is constructed as $w_i=w_T$ for $i=1,\dots,d$.

It is easy to check that function $\boldsymbol{\psi}_T=\boldsymbol{w}_T\cdot\boldsymbol{f}$ satisfies (see [27]),

$$\|\boldsymbol{f}\|^2_{[L^2(T)]^d}\lesssim\int_T(\boldsymbol{\sigma}-\boldsymbol{\sigma}^{h\tau})\cdot\boldsymbol{\varepsilon}(\boldsymbol{\psi}_T)\,d\boldsymbol{x}.$$

Using the inverse inequality, we find that

$$\|\boldsymbol{\varepsilon}(\boldsymbol{\psi}_T)\|_{[L^2(T)]^{d\times d}}\lesssim|T|^{-1}\|\boldsymbol{\psi}_T\|_{[L^2(T)]^d},$$

and therefore,

$$\|\boldsymbol{f}\|_{[L^2(T)]^d}\lesssim|T|^{-1}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{h\tau}\|_{[L^2(T)]^{d\times d}}. \tag{32}$$

Estimate η_3^n is bounded now proceeding like in the previous estimate. Thus, let us consider the bubble function w_E associated with the edge or face E. Hence, taking now $\boldsymbol{w}_E=[w_E]^d$ we deduce that (see again [27]),

$$\|[\boldsymbol{\sigma}^{h\tau}\boldsymbol{\nu}]\|^2_{[L^2(E)]^d}\lesssim|E|^{-1}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{h\tau}\|_{[L^2(\Delta E)]^{d\times d}}\|\boldsymbol{\psi}_E\|_{[L^2(\Delta E)]^d},$$

where $\boldsymbol{\psi}_E=\boldsymbol{w}_E\cdot[\boldsymbol{\sigma}^{h\tau}\boldsymbol{\nu}]$ and ΔE stands for the set of elements of $\mathcal{T}^{hn}$ sharing the common edge or face E. From the definition of $\boldsymbol{\psi}_E$, it follows that $\|\boldsymbol{\psi}_E\|_{[L^2(\Delta E)]^d}\lesssim|E|^{1/2}\|[\boldsymbol{\sigma}^{h\tau}\boldsymbol{\nu}]\|_{[L^2(E)]^d}$, and we conclude that

$$\|[\boldsymbol{\sigma}^{h\tau}\boldsymbol{\nu}]\|_{[L^2(E)]^d}\lesssim|E|^{-1/2}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{h\tau}\|_{[L^2(\Delta E)]^{d\times d}},$$

which implies, for all $T\in\mathcal{T}^{hn}$,

$$\Big(\sum_{E\in\mathcal{E}_T^{hn}}|E|\|[\boldsymbol{\sigma}^{h\tau}\boldsymbol{\nu}]\|^2_{[L^2(E)]^d}\Big)^{1/2}\lesssim\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{h\tau}\|_{[L^2(\Delta T)]^{d\times d}}.$$

Finally, it only remains to estimate η_{4T}^{hn}. Assume that $\boldsymbol{w}_E$ is constructed in such a way that $(\boldsymbol{w}_E)_\nu=0$ and $(\boldsymbol{w}_E)_\tau=w_E\boldsymbol{\sigma}_\tau^{h\tau}$. Hence,

$$\begin{aligned}\|\boldsymbol{\sigma}_\tau^{h\tau}\|^2_{[L^2(E)]^d}&\sim\int_E\boldsymbol{\sigma}_\tau^{h\tau}\cdot(\boldsymbol{w}_E)_\tau\,d\gamma(\boldsymbol{x})\\ &=\int_T\boldsymbol{\sigma}^{h\tau}\cdot\boldsymbol{\varepsilon}(\boldsymbol{w}_E)\,d\boldsymbol{x}\\ &=\int_T(\boldsymbol{\sigma}^{h\tau}-\boldsymbol{\sigma})\cdot\boldsymbol{\varepsilon}(\boldsymbol{w}_E)\,d\boldsymbol{x}+\int_T\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}(\boldsymbol{w}_E)\,d\boldsymbol{x}\\ &\lesssim\|\boldsymbol{f}\|_{[L^2(T)]^d}\|\boldsymbol{w}_E\|_{[L^2(T)]^d}+|T|^{-1}\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^{h\tau}\|_{[L^2(T)]^{d\times d}}\|\boldsymbol{w}_E\|_{[L^2(T)]^d}.\end{aligned}$$

We only need to estimate now the second term of η_{4T}^{hn}. Taking into account that

$$p(u_\nu(t))+\sigma_\nu(t)=0,$$

we find that

$$\begin{aligned}\|p(u_\nu^{h\tau}(t))+\sigma_\nu^{h\tau}(t)\|_{L^2(E)}&\le\|p(u_\nu^{h\tau}(t))-p(u_\nu(t))\|_{L^2(E)}+\|\sigma_\nu^{h\tau}(t)-\sigma_\nu(t)\|_{L^2(E)}\\ &\lesssim\|\boldsymbol{u}(t)-\boldsymbol{u}^{h\tau}(t)\|_{[H^1(T)]^d}+\|\sigma_\nu^{h\tau}(t)-\sigma_\nu(t)\|_{L^2(E)}.\end{aligned}$$

Combining all these results and taking into account the definitions (25), (26), (27) and (28), we obtain the desired lower error bounds. □

From Theorem 4.2, we can prove a similar convergence order than in the a priori error analysis that we state in the following.

Corollary 4.3. *Let assumptions (9)-(13) hold. If the continuous solution has the following additional regularity:*

$$\boldsymbol{u} \in C([0,T];[H^2(\Omega)]^d), \quad \boldsymbol{\sigma} \in C([0,T];[H^1(\Omega)]^{d\times d}),$$

there exists a positive constant $c > 0$*, depending on the given data and the continuous solution, such that*

$$\sum_{n=1}^{N} k\eta_1^n + \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h\|_Q + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V + \max_{1\le n\le N} \max_{t\in[t_{n-1},t_n]} \eta_2^n + \max_{1\le n\le N} \max_{t\in[t_{n-1},t_n]} \eta_3^n + \max_{1\le n\le N} \max_{t\in[t_{n-1},t_n]} \eta_4^n \le c(h+k).$$

Proof. Using estimates (22), under the required regularity we conclude that

$$\|\boldsymbol{u} - \boldsymbol{u}^{h\tau}\|_{C([0,T];V)} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{h\tau}\|_{C([0,T];Q)} \le c(h+k), \tag{33}$$

which implies that

$$\max_{1\le n\le N} \max_{t\in[t_{n-1},t_n]} \eta_2^n + \max_{1\le n\le N} \max_{t\in[t_{n-1},t_n]} \eta_3^n \le c(h+k).$$

From the regularity $\boldsymbol{u} \in C^1([0,T];V)$ and $\boldsymbol{\sigma} \in C^1([0,T];Q)$ (see Theorem 2.1), we easily find that

$$\sum_{n=1}^{N} k \left[\|\boldsymbol{u}_n - \boldsymbol{u}_{n-1}\|_V + \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_{n-1}\|_Q\right] \le ck,$$

and using again (33), it follows that

$$\sum_{n=1}^{N} k\eta_1^n \le c(h+k).$$

Next, we estimate the numerical error on the approximation of the initial conditions. From the definition of the finite element projection operators Π_{V^h} and Π_{Q^h} (see [7]), we have

$$\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V \le ch\|\boldsymbol{u}_0\|_{[H^2(\Omega)]^d}, \quad \|\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_0^h\|_Q \le ch\|\boldsymbol{\sigma}_0\|_{[H^1(\Omega)]^{d\times d}}.$$

Finally, we only have to bound the second part of estimator η_4^n. Without loss of generality, assume that $d = 2$ (i.e. the two-dimensional setting), and that Γ_C is a straight line segment parallel to the x-axis. Taking into account the following inequality for all $E \in \mathcal{E}_T^C$,

$$\|v\|_{L^2(E)} \le |E|^{-1/2}\|v\|_{L^2(\Delta E)} + |E|^{1/2}\|\nabla v\|_{[L^2(\Delta E)]^2} \quad \forall v \in H^1(\Delta E),$$

we find that

$$\begin{aligned}|E|^{1/2}\|\sigma_\nu^{h\tau}(t) - \sigma_\nu(t)\|_{L^2(E)} &= |E|^{1/2}\|\sigma_{yy}^{h\tau}(t) - \sigma_{yy}(t)\|_{L^2(E)} \\ &\lesssim \|\sigma_{yy}^{h\tau}(t) - \sigma_{yy}(t)\|_{L^2(\Delta E)} + |E|\|\nabla(\sigma_{yy}^{h\tau}(t) - \sigma_{yy}(t))\|_{[L^2(\Delta E)]^2} \\ &\lesssim \|\boldsymbol{\sigma}^{h\tau} - \boldsymbol{\sigma}\|_{C([0,T];Q))} + |E|\|\boldsymbol{\sigma}\|_{C([0,T];[H^1(\Omega)]^{2\times 2})}.\end{aligned}$$

Keeping in mind that $|E| \sim h$, this concludes the proof. □

5. Numerical results

5.1. Numerical scheme. First, we recall that the variational spaces V and Q are approximated by using the finite element spaces V^h and Q^h defined by (16) and (17), respectively.

Let $\boldsymbol{u}_{n-1}^{hk} \in V^h$ and $\boldsymbol{\sigma}_{n-1}^{hk} \in Q^h$ be known. For $n = 1, \dots, N$, the fully discrete problem VP^{hk} can be written in the following form,

$$\begin{aligned}&(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q + j(\boldsymbol{u}_n^{hk}, \boldsymbol{w}^h) = (\boldsymbol{f}_n, \boldsymbol{w}^h)_V \\ &\quad -(\boldsymbol{\sigma}_{n-1}^{hk} - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}) + k\mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk})), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q \quad \forall \boldsymbol{w}^h \in V^h.\end{aligned}$$

This leads to a nonlinear variational equation which was solved by using a penalty-duality algorithm (see, for instance, [29]), already applied in other contact problems. Then, the discrete stress field is updated from the equation:

$$\boldsymbol{\sigma}_n^{hk} = \boldsymbol{\sigma}_{n-1}^{hk} + \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}) - \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}) + k\mathcal{G}(\boldsymbol{\sigma}_{n-1}^{hk}, \boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk})).$$

The numerical scheme was implemented on a Intel Core2 Duo 2.4GHz PC using MATLAB, and a typical 2D run ($h = k = 0.05$) took about 5 seconds of CPU time.

5.2. A first 2D-example: error estimators with respect to the exact error. As a first two-dimensional example, the following problem is considered.

Problem T2D. *Find a displacement field* $\boldsymbol{u} : [0,1] \times [0,1] \times [0,1] \to \mathbb{R}^2$ *and a stress field* $\boldsymbol{\sigma} : [0,1] \times [0,1] \times [0,1] \to \mathbb{S}^2$ *such that,*

$$\begin{aligned}&\dot{\boldsymbol{\sigma}} = 2\mathcal{I}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{I}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad in \quad [0,1] \times [0,1] \times (0,1), \\ &-\mathrm{Div}\, \boldsymbol{\sigma} = \boldsymbol{0} \quad in \quad [0,1] \times [0,1] \times (0,1), \\ &\boldsymbol{u} = \boldsymbol{0} \quad on \quad \{0\} \times [0,1] \times (0,1), \\ &\boldsymbol{\sigma\nu} = \boldsymbol{f}_F \quad on \quad ([0,1] \times \{1\} \cup \{1\} \times [0,1]) \times (0,1), \\ &\boldsymbol{\sigma}_\tau = \boldsymbol{0}, \quad -\sigma_\nu = (u_\nu)_+ \quad on \quad [0,1] \times \{0\} \times (0,1), \\ &\boldsymbol{u}(0) = 0.1 \times (x(0.1\frac{y^2}{2} + 0.1y), -\frac{x^2}{2}(0.1y + 0.1)) \quad in \quad [0,1] \times [0,1], \\ &\boldsymbol{\sigma}(0) = \mathcal{I}\boldsymbol{\varepsilon}(\boldsymbol{u}(0)) \quad in \quad [0,1] \times [0,1],\end{aligned}$$

where traction forces $\boldsymbol{f}_F$ are given by

$$\boldsymbol{f}_F(x,y,t) = \begin{cases} 0.1 \times (0, -\dfrac{x^2}{2}\, e^{-t} 0.1) & \text{if} \quad x \in [0,1],\ y = 1, \\ 0.1 \times ((0.1\dfrac{y^2}{2} + 0.1y)e^{-t}, 0) & \text{if} \quad y \in [0,1],\ x = 1. \end{cases}$$

Problem $T2D$ corresponds to Problem P with the following data:

$$\begin{aligned}&T = 1, \quad \Omega = [0,1] \times [0,1], \quad \Gamma_D = \{0\} \times [0,1], \quad \Gamma_C = [0,1] \times \{0\}, \quad g = 0, \\ &\Gamma_F = [0,1] \times \{1\} \cup \{1\} \times [0,1], \quad \mathcal{E} = 2\mathcal{I}, \quad \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})) = \mathcal{I}\boldsymbol{\varepsilon}(\boldsymbol{u}), \quad \boldsymbol{f}_0 = \boldsymbol{0}, \\ &p(r) = r_+, \quad \boldsymbol{u}_0 = 0.1 \times (x(0.1\,\frac{y^2}{2} + 0.1y), -\frac{x^2}{2}(0.1y + 0.1)), \quad \boldsymbol{\sigma}_0 = \boldsymbol{\varepsilon}(\boldsymbol{u}_0).\end{aligned}$$

The exact solution to Problem $T2D$ can be easily obtained after some algebra and it has the following form:

$$\begin{aligned}&\boldsymbol{u}(x,y,t) = (x(0.1\frac{y^2}{2} + 0.1y)e^{-t}, -\frac{x^2}{2}(0.1y + 0.1)e^{-t}) \times 0.1, \\ &\sigma_{11}(x,y,t) = 0.1(0.1\frac{y^2}{2} + 0.1y)e^{-t}, \quad \sigma_{22} = -0.01\frac{x^2}{2}e^{-t}, \quad \sigma_{12} = \sigma_{21} = 0.\end{aligned}$$

In Table 1 the numerical results obtained for several discretization parameters h and k are shown where

$$\eta_1 = \sum_{n=1}^{N} k\eta_1^n,$$
$$\eta_2 = \max_{1\le n\le N} \max_{t\in(t_{n-1},t_n]} \eta_2^n(t) + \max_{1\le n\le N} \max_{t\in(t_{n-1},t_n]} \eta_3^n(t) + \max_{1\le n\le N} \max_{t\in(t_{n-1},t_n]} \eta_4^n(t),$$

and $\eta = \sqrt{\eta_1^2 + \eta_2^2}$. The exact (or true) error e is defined as

$$e = \max_{0\le n\le N} \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V + \max_{0\le n\le N} \|\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^{hk}\|_Q,$$

and e.i. denotes the so-called effectivity index and it equals to η/e.

h	k	η_1	η_2	η	e	e.i.
0.2	0.1	0.0331661	0.1686995	0.1719341	0.1108826	1.55
0.1	0.1	0.0349942	0.1310162	0.13560091	0.0997694	1.35
0.05	0.1	0.0358721	0.0895523	0.0964698	0.0969012	0.99
0.025	0.1	0.0362967	0.0574728	0.0679748	0.0730912	0.93
0.0125	0.1	0.0365051	0.0356754	0.0510426	0.0560903	0.91
0.2	0.05	0.0161195	0.1784489	0.1791744	0.1113638	1.61
0.1	0.05	0.0169719	0.1378073	0.1388614	0.0999072	1.38
0.05	0.05	0.0173855	0.0939912	0.0955855	0.0965511	0.99
0.025	0.05	0.0175858	0.0602631	0.0627765	0.0660805	0.95
0.0125	0.05	0.0176841	0.0383793	0.0431017	0.0463459	0.93
0.2	0.025	0.0079436	0.1834036	0.1835785	0.1119439	1.63
0.1	0.025	0.0083609	0.1412451	0.1414923	0.1005961	1.41
0.05	0.025	0.0085617	0.0962355	0.0966156	0.0973039	0.99
0.025	0.025	0.0086591	0.0616732	0.0622781	0.0648731	0.96
0.0125	0.025	0.0087067	0.0382554	0.0392337	0.0412986	0.95
0.2	0.0125	0.0039443	0.1858967	0.1859385	0.1123063	1.65
0.1	0.0125	0.0041499	0.1429723	0.14303254	0.1010041	1.42
0.05	0.0125	0.0042489	0.0973627	0.0974554	0.0976869	1
0.025	0.0125	0.0042968	0.0623815	0.0625928	0.0638702	0.98
0.0125	0.0125	0.0043204	0.0386904	0.0389309	0.0405531	0.96

Table 1. Example T2D: Numerical errors (x100) for some h and k.

As can be seen, the convergence of the discrete solution is clearly observed when the discretization parameters converge to zero. As it was also noticed in [23], the estimator due to the time discretization η_1 is not greater than the error due to the space discretization η_2 but they oscillate. Moreover, the estimator error η is not always greater than the exact error but it seems that, when the discretization parameters decrease, both errors become closer. Finally, the effectivity index seems to increase as parameters h and k tend to zero but it is greater than 0.9.

5.3. A second 2D-example: an elasto-viscoplastic body in contact with a deformable obstacle. As a second two-dimensional example, we consider a numerical example already simulated in other works. Hence, the elasto-viscoplastic body is assumed to occupy the domain $\Omega = (0, 6) \times (0, 1.2)$, no volume forces are supposed to act in the body and vertical constant tractions are applied on the boundary part $[0, 6] \times \{1.2\}$. Due to the symmetry conditions, the middle line $\{3\} \times [0, 1.2]$ has restricted its horizontal displacements. Finally, the body is supposed to be in contact with a deformable obstacle on the contact boundary $\Gamma_C = [2, 4] \times \{0\}$ (see FIGURE 2).

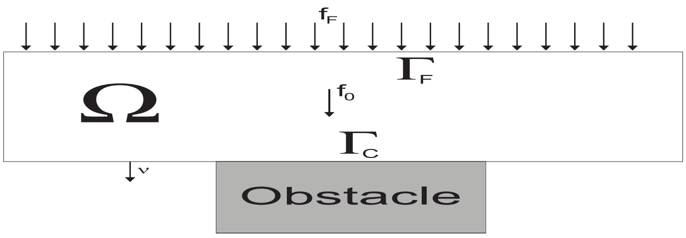

FIGURE 2. Example 2D-2: Physical setting.

The following data have been employed in the simulations:

$$T = 1, \quad \Omega = [0, 6] \times [0, 1.2], \quad \Gamma_D = \emptyset, \quad \Gamma_F = [0, 6] \times \{1.2\},$$
$$\Gamma_C = [2, 4] \times \{0\}, \quad \mathcal{E} = 2\mathcal{I}, \quad \mathcal{G}(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{u})) = \boldsymbol{\varepsilon}(\boldsymbol{u}), \quad \boldsymbol{f}_0 = \boldsymbol{0}, \quad \boldsymbol{f}_F = (0, -0.01),$$
$$g = 0, \quad p(r) = 1000 r_+, \quad \boldsymbol{u}_0 = \boldsymbol{0}, \quad \boldsymbol{\sigma}_0 = \boldsymbol{0}.$$

Taking $k = 0.01$ as the time discretization parameter, the displacement field at final time and the reference configuration are plotted in FIGURE 3. We observe

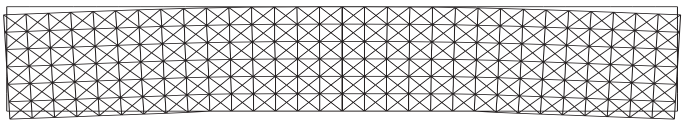

FIGURE 3. Example 2D-2: Reference configuration and displacement field at final time.

that no penetration into the obstacle has been produced because of the size of the deformability coefficient μ. Moreover, in FIGURE 4 the von Mises stress norm at final time is plotted over the deformed mesh. As expected, the highest stressed areas are located near the contact corners and also where body bends.

Finally, the error estimators η_1, η_2 and η have the following values:

$$\eta_1 = 8.74594358 \times 10^{-4}, \quad \eta_2 = 0.04147338, \quad \eta = 0.0414826.$$

We notice that, even if the exact solution is unknown (and, in fact, it can not be calculated using an analytical procedure), this estimate gives us an idea of the error approximation and this constitutes no doubt one of the main aspects of this a posteriori error analysis.

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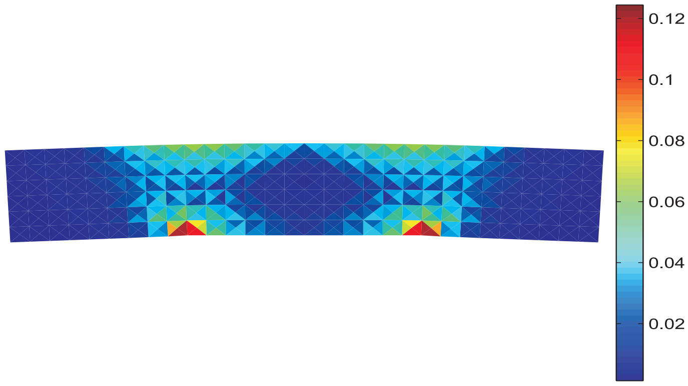

FIGURE 4. Example 2D-2: von Mises stress norm at final time over the deformed mesh.

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