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A MULTILEVEL METHOD FOR SOLVING THE HELMHOLTZ EQUATION : THE ANALYSIS OF THE ONE-DIMENSIONAL CASE

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Abstract. In this paper we apply and discuss a multilevel method to solve a scattering problem. The multilevel method belongs to the class of incremental unknowns method as in [10]; in this work, the best performance was obtained with a coarsest grid having roughly two points per linear wavelength. We analyze this method for a simple model problem following H. Yserentant [17]. In this case, the main limitation to multilevel methods is closely linked to the indefiniteness of the Helmholtz problem.

Key words. hierarchical basis, indefinite problem, Helmholtz equation, iterative methods

1. Introduction

In this paper we are interested in applying the strategy introduced by H. Yserentant [17] to solve an indefinite elliptic boundary value problem that comes from acoustics [7], [6]. This problem, leading to a non-coercive bilinear form, reads as follows

$$-u_{xx} - k^2 u = f, \quad \text{in }]0,1[\tag{1.1}$$

$$u(0) = 0, \tag{1.2}$$

$$u_x(1) = \imath k u(1). \tag{1.3}$$

Here we adopt the notations $\imath = \sqrt{-1}$, while the wavenumber $k = \frac{\omega L}{c}$ is a positive real number (supposed larger than 1 in the sequel). Here some scaling has been performed; this problem occurs when one considers a solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ that moves from the left to the right boundary, whose frequency is ω and that satisfies some Sommerfeld radiation condition at $+\infty$. For numerics, one tracks this solution on a box $[0, L]$, and after scaling in space, this condition (1.3) replaces $u(x) \sim e^{\imath k x}$ at $+\infty$.

This one-dimensional problem belongs to exterior boundary value problems of the form

$$-\Delta u - k^2 u = f \quad \text{in } \Omega \tag{1.4}$$

$$u = g \quad \text{on } \Gamma \subset \partial\Omega \tag{1.5}$$

$$\mathcal{F}u = 0 \quad \text{on } \partial\Omega \tag{1.6}$$

where the operator $\mathcal{F}$ corresponds to the chosen absorbing boundary condition (ABC), while the second equation depends of the (acoustic) properties of the scatterer. The Helmholtz problem at hand is expected to produce a solution with an oscillatory behavior on the wavelength ($\lambda = 2\pi/k$) scale. The analysis conducted hereafter should be extended to the two-dimensional or three-dimensional problem with an approximation of first order ABC without any other difficulties than technical ones. Otherwise, if one considers the problem with Sommerfeld radiation

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condition at infinity, one must use a different framework as weighted Sobolev spaces, but working without the assumptions needed to validate the Poincaré inequality might be hard.

Multilevel methods, like hierarchical basis [15, 16] for finite element approximation or incremental unknowns [12] in finite difference context, are effective for the numerical solution of many partial differential equations. They seem being almost so robust and powerful as classical multigrid methods for solving elliptic partial differential equations [3]. Nevertheless, for large scales problems, multilevel approaches do not apply straightforwardly and more involved method have to be considered [9, 1, 8]. For some classes of problems like ours, it has been proven that the effectiveness of classical multigrid methods often fails [13, 4, 5]. In particular, the indefiniteness of the discrete problems is certainly the main reason for which the coarsest grid must be not too coarse.

The multigrid methods by combining interpolation and pre and post-smoothing catch step by step the harmonics of the solution whereas the multilevel methods involve the projection of the solution onto a Krylov space in the multilevel basis. Even if the approach of the latter looks quite far away from the classical multigrid methods, we will prove that they have a similar limitation onto the sparsity of the coarsest level of grid for indefinite discrete problems (this result has been pointed out in [10]).

For the Helmholtz problem under consideration, despite the fact that the associated bilinear form is not positive definite, one can exhibit a large subspace W of the energy space with a finite co-dimension (which varies as k^4), and such that the bilinear form restricted to W becomes coercive. Hence, dealing with finite element multilevel approximation of the equation, we develop the strategy introduced by H. Yserentant to trap the bad behavior of the bilinear form on a finite element space corresponding to a coarse grid approximation of the equation, and then to proceed to multilevel analysis on finer grids. The significant drawback of our method is that it does not apply to very high frequency problems since the magnitude of the coarse grid behaves as k^4.

The outline of this paper is as follows. In section 2, we introduce the one-dimensional model problem and study its properties. The section 3 is devoted to its approximation by multilevel finite element (which is similar to the incremental unknowns in finite differences). One shows in particular the influence of the indefiniteness of the problem onto the discrete problem. Computations of the condition number of the stiffness matrix for the hierarchical basis are given in section 4, in agreement with the analysis.

Let us complete this introduction by some notations. To treat the absorbing boundary condition (1.3), we need to use complex valued functions. Furthermore, to adapt the guidelines introduced in [17] to complex valued functions, we consider $L^2(0,1)$ the real-Hilbert space whose scalar product is

$$(u,v) = \mathrm{Re}\int_0^1 u(x)\overline{v(x)}dx = \mathrm{Re}\int u\bar{v}dx. \tag{1.7}$$

Note that one omits to write generic constant that may vary from one line to another one, but that is independent of k and of h_0, h, denote respectively the mesh size of the coarse and fine grid approximation of the problem. We also use the space $V = \{u \in L^2(0,1); u_x \in L^2(0,1) \text{ and } u(0) = 0\}$. For the sake of conciseness, we denote respectively $\|u\| = (u,u)^{\frac{1}{2}}$, $((u,v)) = (u_x, v_x)$ and $|u| = \|u_x\|$. It is well-known that V equipped with the Poincaré semi-norm $|.|$ is an Hilbert space and that the following estimates are valid for any u in V

$$\|u\| + \|u\|_{L^\infty(0,1)} \leq |u| \,. \tag{1.8}$$

Let us recall that the constants are still replaced by one, because the use of the best constants in the Poincaré inequality and the Sobolev embedding is not of the main interest here.

2. Model problem analysis

It is a standard procedure to prove that (1.1)-(1.2)-(1.3) is equivalent to solve the following variational formulation:
find u in V such that, for all v in V,

$$\int u_x \overline{v_x} dx - \imath k u(1)\overline{v}(1) - k^2 \int u\overline{v} dx = \int f\overline{v} dx. \tag{2.1}$$

or, equivalently (by using v or $\imath v$ as test functions),

$$b(u,v) \equiv ((u,v)) + \mathrm{Im}(ku(1)\overline{v}(1)) - k^2(u,v) = (f,v), \tag{2.2}$$

since we chose to deal with a real-Hilbert space.

We now state and prove

Proposition 2.1. *The bilinear form $b(.,.)$ defined by (2.2) is continuous on $V \times V$.*

Proof. Due to inequality (1.8), we have, for all $u, v \in V$,

$$\begin{aligned} b(u,v) &\leq |u|\,|v| + k\,\|u\|_{L^\infty(0,1)}\,\|v\|_{L^\infty(0,1)} + k^2\,\|u\|\,\|v\| \\ &\leq (1+k+k^2)\,|u|\,|v|\,. \end{aligned} \tag{2.3}$$

□

Actually, b is not coercive but satisfies the following Gärding inequality (that turns out to be an equality here). For all u in V

$$b(u,u) = |u|^2 - k^2\,\|u\|^2\,. \tag{2.4}$$

Last not least, we prove

Proposition 2.2. *For $f \in L^2(0,1)$ and $k > 0$, the problem (2.2) has a unique solution.*

Proof. We give a short proof that relies on Fredholm alternative (see [2]). Set $\mathcal{A}$ the unbounded operator defined as $(\mathcal{A}u, v) = ((u,v))$ for any u, v in V. $\mathcal{A}$ is onto and has a compact inverse. Set $(\mathcal{B}u, v) = -k\mathrm{Im}u(1)\overline{v}(1) + k^2(u,v)$. Then $\mathcal{B}$ is a compact perturbation of $\mathcal{A}$, i.e $\mathcal{A}^{-1}\mathcal{B}$ is a compact operator. Hence either the equation $\mathcal{A}u - \mathcal{B}u = f$ has a unique solution for any f or the homogeneous equation $\mathcal{A}u - \mathcal{B}u = 0$ has nonzero solutions. Let us check that the latter assertion is not valid. If $b(u,v) = 0$ for all v then chosing $v = \imath u$ leads to $u(1) = 0$. Hence u satisfies $-u_{xx} - k^2 u = 0$ in $D'(0,1)$ supplemented with $u(0) = u(1) = 0$. This operator does not have negative eigenvalues. Then $u = 0$. Then we know that there exists a unique $u \in V$ solving the equation. □

Remark 2.3. *Following* [7], *we know that the solution to* (2.2) *reads*

$$u(x) = \int_0^1 K(x,s) f(s) ds, \tag{2.5}$$

where $K(x,s) = \frac{1}{k} e^{\imath k \max(x,s)} \sin(k \min(x,s))$. *As a consequence for* u, *the solution to* (2.2), *we have*

$$k\,\|u\| + |u| \le |f|. \tag{2.6}$$

We also have

Proposition 2.4. *There exists a unique solution to the following adjoint problem*

$$\textit{Find } u \in V \textit{ such that, } \forall v \in V,\ b(v,u) = (f,v). \tag{2.7}$$

Moreover the solution of (2.7) satisfies

$$k\,\|u\| + |u| \le |f|. \tag{2.8}$$

Proof. The adjoint problem shares the same properties as the original problem. It is related to $b^*(u,v) = ((u,v)) - k\mathrm{Im}(u(1)\bar{v}(1)) - k^2(u,v) = (f,v)$. The solution is given by $u(x) = \int_0^1 \overline{K(x,s)} f(s) ds$. The proof of the Proposition follows readily (see [6] section 4.2). □

3. A multilevel finite element approximation

Following preliminary definitions of the previous section, we introduce the Galerkin finite element approximation of our model problem with piecewise linear functions. Then we introduce a multilevel decomposition using hierarchical basis of our problem and we prove some results concerning our indefinite elliptic boundary value problems.

Let us recall for reader convenience some results from [7]. Let the stepsize $h = 1/(n+1)$ for n an integer, and let on the interval $[0,1]$ a uniform mesh of $n+2$ nodes $\{x_j = jh,\ j = 0, \dots, n+1\}$. We define the space $V^h \subset V$ as the set of functions of V such that their restriction to each interval $[x_{j-1}, x_j]$ is a linear function.

The approximate model problem is then:

$$\text{find } u_h \in V^h \text{ such that } b(u_h, v_h) = (f_h, v_h) \text{ for all } v_h \in V^h. \tag{3.1}$$

We follow here the strategy in [17] (see also [14, 9]). Let us introduce now more precisely the multilevel finite element approximation by using linear basis functions in 1D. Let P_1 denote the space spanned by these functions. For a more general setting in higher dimension, one can consider a family of nested uniform triangulations [15]).

In the case of the interval $[0,1]$, we start with an initial coarsest triangulation τ_0 whose mesh size is h_0 gathering the subintervals $K_j^0 = [x_{j-1}^0, x_j^0]$ of the nodes $\{0 = x_0^0, \dots, x_{n^0}^0 = 1\}$. Each interval K_j^0 is divided into two subintervals K_l^1 and K_{l+1}^1 which compose the τ_1 and the process is recursively applied until the last level d. Then we introduce the space

$$V_0^h = \{v_h \in \mathcal{C}^0([0,1]), v_h(0) = 0, \text{ and } v_{h/K_j^0} \in P_1,\ j = 0, \dots, n^0\},$$

and if K_j^l denotes the $(j+1)$-th subinterval of the triangulation τ_l, we introduce the space

$$V_l^h = \{v_h \in \mathcal{C}^0([0,1]), v_h(0) = 0, \text{ and } v_{h/K_j^l} \in P_1,\ j = 0, \dots, n^l\}.$$

Owing to the construction of the nested subintervals, one recovers that $h_0 = 2^d h$ and that

$$V_0^h \subset V_1^h \subset \cdots \subset V_{d-1}^h \subset V_d^h.$$

One defines the interpolation operator relative to the j-th level of subdivision by

$$\forall u \in \mathcal{C}^0([0,1]),\ I_j u \in V_j^h\ I_j u(x) = u(x),\ \text{for all}\, x \in \{x_0^j, \dots, x_{n^j}^j\}.$$

Thus, we obtain the multilevel decomposition of the approximation space $V^h = V_d^h$,

$$\forall u_h \in V^h, \quad u_h = I_0 u_h + \sum_{j=1}^{d} (I_j u_h - I_{j-1} u_h).$$

Furthermore, as for all $u_h \in V^h$,

$$\forall x \in \{x_0^l, \dots, x_{n^l}^l\}, \quad (I_j u_h - I_{j-1} u_h)(x) = 0,\ \text{for}\ l \le j-1,$$

one obtains straight the decomposition of V_d^h in a direct sum

$$V^h = V_0^h \oplus W_1^h \oplus W_2^h \oplus \cdots \oplus W_d^h, \quad \text{where} \quad W_j^h = V_{j+1}^h \setminus V_j^h.$$

We now state

Theorem 3.1. *Assume that the coarse mesh-size h_0 satisfies $h_0 k^4$ is small enough. Introduce $(V_0^h)^\perp$ the orthogonal complementary of V_0^h with respect to b, i.e $(V_0^h)^\perp = \{v \in V; b(u_0^h, v) = 0,\ \text{for all}\ u_0^h \in V_0^h\}$. Then $b/(V_0^h)^\perp$ is coercive. Moreover if A_h denotes the stiffness matrix defined by $b(u_h, v_h) = (A_h u_h, v_h)$ associated to the problem under consideration, then preconditioning the underlined linear system with the hierarchical basis associated to the multilevel finite element splitting leads to a new system of size $n = \frac{1}{h}$ that separates into a well-conditioned elliptic system and an invertible (non elliptic system) of size $\frac{1}{h_0}$.*

Proof. The proof of this Theorem is based on some results of [17]. The assumption requires that the size of the coarse grid must be sufficient to catch the oscillations characteristic of the solution of the Helmholtz problem. Throughout the proof we assume without loss of generality that $k \ge 1$.

The first step is devoted to prove that $(V_0^h)^\perp$ satisfies a enhanced Poincaré inequality and intensively uses the so-called Aubin-Nitsche trick. Consider u in $(V_0^h)^\perp$. Consider φ that solves the adjoint problem, where the right hand side f is replaced by u,

$$\forall v \in V,\ b(v, \varphi) = (u, v). \tag{3.2}$$

Specifying $v = u$ in the previous equality and using the orthogonality property leads to

$$\forall \varphi_0^h \in V_0^h, \quad \|u\|^2 = b(u, \varphi - \varphi_0^h). \tag{3.3}$$

By standard finite element theory we know that there exists a constant c_* that does not depend on h such that

$$\inf_{\varphi_0^h \in V_0^h} \left|\varphi - \varphi_0^h\right| \le c_* h_0 \left\|\varphi_{xx}\right\|. \tag{3.4}$$

Then, using the continuity assumption on b we obtain

$$\|u\|^2 \le (1 + k + k^2)\, |u| \left|\varphi - \varphi_0^h\right| \le c_* h_0 (1 + k + k^2)\, |u| \left\|\varphi_{xx}\right\|. \tag{3.5}$$

Hence, since $\|\varphi_{xx}\| \le \|u\| + k^2 \|\varphi\| \le (1 + k)\, \|u\|$, due to the estimate (2.8), we have that

$$\|u\| \le 6 c_* h_0 k^3\, |u|. \tag{3.6}$$

Consequently, for u in $(V_0^h)^\perp$,

$$b(u,u) = |u|^2 - k^2 \|u\|^2 \geq (1 - 36c_*^2 h_0^2 k^8) |u|^2 . \tag{3.7}$$

Hence the assumption reads $6c_* h_0 k^4 \leq \sqrt{\frac{1}{2}}$ where c_* is the best constant involved in the P^1 finite element result.

We now move to the proof of the second part of the Theorem. Consider the hierarchical basis associated to the multilevel decomposition. We denote such basis as $\psi_1, ...\psi_m, ...\psi_n$ where $\psi_1, ...\psi_m$ is a basis for the coarse space V_0^h; hence $m \sim \dfrac{1}{2^d h}$, $n \sim \dfrac{1}{h}$.

Solving the approximate problem $b(u_h, v_h) = (f, v_h)$, for all v_h in V^h, amounts to solve the linear system $AU = F$ where A is the matrix whose entries are $b(\psi_k, \psi_j)$ and where F is the vector $F_j = (f, \psi_j)$. We split this matrix into blocks as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} is a $m \times m$ matrix corresponding to the restriction to the bilinear space to the coarse space. Hence solving $AU = F$ requires to solve (with obvious notations) the system

$$\begin{aligned} A_{11}u_1 + A_{12}u_2 &= f_1, \\ A_{21}u_1 + A_{22}u_2 &= f_2. \end{aligned} \tag{3.8}$$

To begin with we prove that the $m \times m$ matrix A_{11} is invertible. This is equivalent to prove that the discrete problem

$$\forall v_{h_0} \in V_0^h, \quad b(u_{h_0}, v_{h_0}) = (f, v_{h_0}), \tag{3.9}$$

has a unique solution. By denoting w_0 the difference between two solutions u_{h_0} and $\tilde{u}_{h_0}$ of the problem (3.9), w_0 belongs to $(V_0^h)^\perp$.
Since $0 = b(w_0, w_0) \geq \frac{1}{2} |w_0|^2$, then $w_0 = 0$. Therefore A_{11} has null kernel and thus is invertible.
Solving $AU = F$ is then equivalent to solve

$$\begin{aligned} (A_{22} - A_{21}A_{11}^{-1}A_{12})u_2 &= f_2 - A_{21}A_{11}^{-1}f_1, \\ A_{11}u_1 &= f_1 - A_{12}u_2. \end{aligned} \tag{3.10}$$

Hence the second equation in (3.10) has small size with respect to the problem under consideration, while the first one requires to invert the Schur matrix $R = A_{22} - A_{21}A_{11}^{-1}A_{12}$ which is symmetric and positive definite. Following [17] we now compute its condition number.

The vector whose entries are $(-A_{11}^{-1}A_{12}u_2, u_2)^T$ corresponds to a vector that belongs to $(V_0^h)^\perp$. Then, denoting the scalar product in $\mathbb{R}^n$ as $u.v$, we first obtain the equality

$$Ru_2.u_2 = A \begin{pmatrix} -A_{11}^{-1}A_{12}u_2 \\ u_2 \end{pmatrix} . \begin{pmatrix} -A_{11}^{-1}A_{12}u_2 \\ u_2 \end{pmatrix} = b(u - u_{h_0}, u - u_{h_0}). \tag{3.11}$$

Therefore, due to (2.3), (3.7) and the assumption $k \geq 1$, we recover that

$$\frac{1}{2} |u - u_{h_0}|^2 \leq Ru_2.u_2 \leq 3k^2 |u - u_{h_0}|^2 . \tag{3.12}$$

Finally, we obtain that the condition number $\kappa(R) = 6k^2\kappa(\tilde{A}_0)$ where $\tilde{A}_0$ is the matrix of the usual Laplacian preconditioned by the hierarchical basis. As we know from [15] that $\kappa(\tilde{A}_0) = O(d^2)$ where d is the number of refinement levels, the proof is complete. □

4. Numerical results

The analysis which has been given before, is confirmed by the following numerical results. It is well known that the number of iterations required for convergence is nearly proportional to the square root of the condition number of the stiffness matrix; hence we study this last quantity for the model problem.

At the left picture of Fig. 1, we first observe the behavior of the condition number as the discretization is refined while the wavelength is kept fixed. Then, at the right picture, we plot the condition number when the number of mesh points grows proportionally with the wavelength. The curves of Fig.1 (a) represent the variation of condition number of the stiffness matrix which is obtained after successive application of grid level of hierarchical basis (in other terms, setting a fixed value of the wavenumber, and growing the coarsest mesh h_0 by doubling its size). And the result is that the condition number is improved only in the begining and up to the limit of the coarse grid mesh reaches 0.3λ. Beyond this point, no further improvement is observed and the condition number is getting even worse.

Similar results for the second type of refinements is obtained Fig.1 (b). Hierarchical basis provide a preconditioner which performs well as long as the coarse mesh is less or equal than $\lambda/4$.

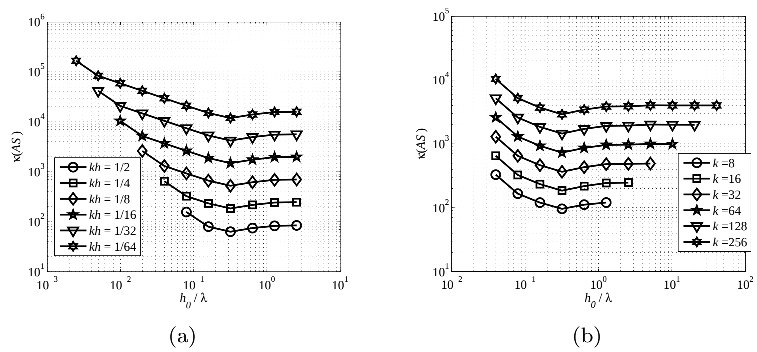

FIGURE 1. Condition number for the 1D Helmholtz equation on a $]0, 1[$ domain, vs the coarse mesh size for $k = 16$ and various fine mesh sizes (a), and using $kh = 4$ versus the coarse mesh size for various values of the wavenumber (b).

5. Concluding remarks

Numerical results of Helmholtz problems in 2D showed that incremental unknowns preconditionning is an adapted and robust technique, but only when the ratio between the coarse mesh and the wavelength is limited [10]. In this paper, we develop an analysis of a 1D Helmholtz model problem to check the behavior of our multilevel approach. The results which are obtained for this model problem are similar to these of the 2D acoustic scattering one, explaining that this limit is linked to the coercivity default of the bilinear form associated to the problem. More specifically, the dimension of the coarsest approximation space must be sufficient to catch the oscillations of the solutions, providing a bilinear form coercive on the orthogonal of the coarsest approximation space. Therefore we acknowledge that our method loses efficiency for very high frequency problems. An extension to this work shall treat the equation-based interpolation technique introduced recently in [11].

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