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A PRIORI ERROR ESTIMATES FOR SEMIDISCRETE FINITE ELEMENT APPROXIMATIONS TO EQUATIONS OF MOTION ARISING IN OLDROYD FLUIDS OF ORDER ONE

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Abstract. In this paper, a semidiscrete finite element Galerkin method for the equations of motion arising in the 2D Oldroyd model of viscoelastic fluids of order one with the forcing term independent of time or in L^∞ in time, is analyzed. A step-by-step proof of the estimate in the Dirichlet norm for the velocity term which is uniform in time is derived for the nonsmooth initial data. Further, new regularity results are obtained which reflect the behavior of solutions as $t \to 0$ and $t \to \infty$. Optimal $L^\infty(\mathbf{L}^2)$ error estimates for the velocity which is of order $O(t^{-1/2}h^2)$ and for the pressure term which is of order $O(t^{-1/2}h)$ are proved for the spatial discretization using conforming elements, when the initial data is divergence free and in H_0^1. Moreover, compared to the results available in the literature even for the Navier-Stokes equations, the singular behavior of the pressure estimate as $t \to 0$, is improved by an order $1/2$, from t^{-1} to $t^{-1/2}$, when conforming elements are used. Finally, under the uniqueness condition, error estimates are shown to be uniform in time.

Key Words. Viscoelastic fluids, Oldroyd fluid of order one, uniform *a priori* bound in Dirichlet norm, uniform in time and optimal error estimates, nonsmooth initial data.

1. Introduction

In this paper, we consider semi-discrete Galerkin approximations to the following system of equations of motion arising in the Oldroyd fluids (see, J. G. Oldroyd ([23])) of order one:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} - \int_0^t \beta(t-\tau)\Delta \mathbf{u}(x,\tau)\, d\tau + \nabla p = \mathbf{f}(x,t), \tag{1.1}$$

with $x \in \Omega,\ t > 0$ and incompressibility condition

$$\nabla \cdot \mathbf{u} = 0,\ x \in \Omega,\ t > 0, \tag{1.2}$$

and initial and boundary conditions

$$\mathbf{u}(x,0) = \mathbf{u}_0 \text{ in } \Omega,\ \mathbf{u} = 0, \text{ on } \partial\Omega,\ t \geq 0. \tag{1.3}$$

Here, Ω is a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$, $\mu = 2\kappa\lambda^{-1} > 0$ and the kernel $\beta(t) = \gamma \exp(-\delta t)$, where $\gamma = 2\lambda^{-1}(\nu - \kappa\lambda^{-1}) > 0$ and $\delta = \lambda^{-1} > 0$. We note that ν is the kinematic coefficient of viscosity. λ is the relaxation time, and is characterized by the fact that after instantaneous cessation of motion, the stresses

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in the fluid do not vanish instantaneously, but die out like $exp(-\lambda^{-1}t)$. Moreover, the velocities of the flow, after instantaneous removal of the stresses, die out like $exp(-\kappa^{-1}t)$, where κ is the retardation time. For further details of the physical background and its mathematical modeling, we refer to [14], [23] and [24].

There is considerable amount of literature devoted to Oldroyd model by Russian mathematicians such as A.P.Oskolkov, Kotsiolis, Karzeeva and Sobolevskii etc, see [24, 1, 8, 15, 18] and references, therein. Based on the analysis of Ladyzhenskaya [19] for the Navier-Stokes equations, Oskolkov [24] has proved existence of a unique 'almost' classical solution in finite time interval $[0, T]$ for the $2D$ problem (1.1)-(1.3). In the proof, the constant appeared in *a priori* bounds depends exponentially on T and therefore, it is not possible to extend the results for large time. Subsequently, Agranovich and Sobolevskii [1] have extended the analysis of Oskolkov and have derived global existence of solutions for all $t \geq 0$ when $f \in L^2(\mathbf{L}^2)$ with smallness conditions on data for $3D$ problem. The solvability on the semi-axis $t \geq 0$, for the problem (1.1)-(1.3), is discussed in [18, 8] when $\mathbf{f}, \mathbf{f}_t \in L_\infty(\mathbb{R}^+; \mathbf{L}^2(\Omega))$ in [18] and $\mathbf{f}, \mathbf{f}_t \in S^2(\mathbb{R}^+; \mathbf{L}^2(\Omega))$ in [8], where S^2 is a subspace of L^2_{loc}. We observe that results in [18, 8] hold true only for finite time ($T < \infty$), that is, for $\mathbf{f}, \mathbf{f}_t \in L_\infty(0, T; \mathbf{L}^2(\Omega))$, with estimate depending on T, but there seems to have some difficulties in extending these results for all $t \geq 0$, when $\mathbf{f}, \mathbf{f}_t \in L_\infty(\mathbb{R}^+; \mathbf{L}^2(\Omega))$. For example, in [18], it is difficult to derive the estimate (20) from (17) on page 2780 by applying integral version of the Gronwall's Lemma and the estimate (12). Unfortunately, this is further carried over to subsequent articles, see Theorem 2 of [8]. In the context of dynamical system generated by Oldroyd model when $\mathbf{f} \in L^\infty(\mathbf{L}^2)$, see, [15] and [17], it is not quite clear that the conclusion of Theorem 1.2 of [15] for $s = 1$ or Theorem 1 of [17] for $\ell = 1$ holds true. In fact a more careful observation in both these articles demands an estimate of $\int_0^1 \|\boldsymbol{\phi}\|^2_{E_1}$, which is difficult to establish prior to this result. In the context of 2D Navier-Stokes equations, a standard tool for deriving uniform Dirichlet norm for the velocity term is to apply uniform Gronwall's Lemma. Due to the presence of the integral term in (1.1), it is difficult to apply uniform Gronwall's Lemma (see Remark 3.3(i)). To be more precise, on the right-hand side of (17) on page 2780 of [18], the estimate of $\int_0^t \sum_{l=1}^L \beta_l \|\tilde{\Delta}\mathbf{u}_l\|^2_{2,\Omega_t} d\tau$ is not available from (12) (for notations, see [18]), which is crucial in applying uniform Gronwall's Lemma. This is exactly a similar problem faced in article [15] and [17].

In [30], Sobolevskii has examined the behavior of the solution as $t \to \infty$ under some stabilization conditions like positivity of the first eigenvalue of a selfadjoint spectral problem introduced therein and Hölder continuity of the function $\Phi = e^{\delta_0 t}(\mathbf{f}(x,t) - \mathbf{f}_\infty(x))$, where $\mathbf{f}_\infty = \overline{\lim}_{t\to\infty} \mathbf{f}$ and $\delta_0 > 0$, using energy arguments and positivity of the integral operator, see also Kotsiolis and Oskolkov [16]. Recently, He *et al.* [10] have proved similar results under milder conditions on $\mathbf{f}$ and weaker regularity assumptions on the initial data $\mathbf{u}_0$. In fact, in their analysis, they have shown both the power and exponential convergence of the solutions to a steady state solution, when $\Phi \in L^\infty(\mathbf{L}^2)$ only.

For the numerical approximations to the problem (1.1)-(1.3), we refer to Akhmatov and Oskolkov [2], Cannon *et al.* [5], He *et al.* [11] and Pani *et al.* [28]. In [2], stable finite difference schemes are discussed without any discussion on convergence. Cannon *et al.* [5] have proposed a modified nonlinear Galerkin scheme for (1.1)-(1.3) with periodic boundary condition using a spectral Galerkin method and have discussed convergence analysis while keeping time variable continuous. In [11], local optimal error estimates for the velocity in $L^\infty(\mathbf{H}^1)$-norm and the pressure in $L^\infty(L^2)$-norm are established. Moreover, these estimates are shown to be uniform provided the given data satisfy the uniqueness condition. In [28], optimal error bounds in $L^\infty(\mathbf{L}^2)$ as well as in the $L^\infty(\mathbf{H}^1)$-norms for the velocity and for the pressure in the $L^\infty(L^2)$-norm are derived which are valid uniformly in time $t > 0$ under the condition that $\mathbf{f} \equiv 0$. In fact, Pani *et al.* [28] have obtained new regularity results, which are valid for all time $t > 0$, without nonlocal compatibility conditions. Based on Stokes-Volterra projection and duality arguments, they have proved optimal error estimates, when the initial data $\mathbf{u}_0 \in \mathbf{H}^2 \cap \mathbf{J}_1$. Subsequently in [29], a backward Euler method is used to discretize in temporal direction and semi-group theoretic approach is then employed to establish *a priori* error estimates.

Our present investigation is a continuation of [28]. In this paper, we obtain regularity results which are uniform in time under realistically assumed regularity on the exact solution, when $\mathbf{f} \not\equiv 0$ with $\mathbf{f}, \mathbf{f}_t \in L^\infty(\mathbf{L}^2)$. As is pointed out in [12] and [28], some of the regularity results depend on the non-local compatibility conditions on the data at $t = 0$, which are either very hard to verify or difficult to meet in practice. We have, in this article, obtained new regularity results under realistically assumed conditions on initial data so that we can avoid non-local compatibility conditions. At this point, we would like to stress that for $2D$ Oldroyd fluids of order one, a step-by-step proof of the Dirichlet norm estimate which is uniform in time is missing in the literature. Following the analysis of $2D$ Navier-Stokes equations, it is hard to apply the uniform Gronwall's Lemma [32] or the proof techniques of Ladyzhenskaya [19] for deriving uniform estimate in the Dirichlet norm for the velocity term. Hence, we hope that, our present analysis will also fill this missing link.

Under the uniqueness condition (see, Section 5), we have shown uniform (in time) optimal error estimates for both velocity and pressure terms. This is an improvement over the results obtained in [11], where the uniform error estimate for velocity is not optimal in $L^\infty(\mathbf{L}^2)$. We have also improved the error estimation of the pressure term, in the sense that, the estimate now reads $O(t^{-1/2}h)$ instead of $O(t^{-1}h)$, which is again an improvement over the results observed in [11] for the nonsmooth data, i.e., $\mathbf{u}_0 \in \mathbf{J}_1$ and [28], when $\mathbf{u}_0 \in \mathbf{J}_1 \cap \mathbf{H}^2$. In [12], Heywood and Rannacher have noted that this singular behavior of the pressure estimate for the Navier-Stokes problem is due to a difficulty which appears technical, but which may be inherent to the problem. Therefore, our present analysis will improve the result of Heywood and Rannacher [12] for the Navier-Stokes equations using conforming elements.

The main contributions of the present article are as follows :

(i) step-by-step proof of uniform $\mathbf{H}^1$-bound for the velocity.
(ii) proof of regularity results for the solution which reflect the behavior as $t \to 0$ and as $t \to \infty$ when $\mathbf{f}, \mathbf{f}_t \in L^\infty(\mathbf{L}^2)$ and $\mathbf{u}_0 \in \mathbf{J}_1$ (see, for definition, Section 2).
(iii) proof of optimal error $L^\infty(\mathbf{L}^2)$ estimates for semidiscrete Galerkin approximations to the velocity and pressure for the nonsmooth initial data $\mathbf{u}_0$, that is, $\mathbf{u}_0 \in \mathbf{J}_1$.
(iv) proof of uniform optimal error estimates for both velocity and pressure terms under the assumption of the uniqueness condition.
(v) improvement in the singular behavior (as $t \to 0$) of the pressure estimate, i.e., $\|(p - p_h)(t)\| \le Kht^{-1/2}$ instead of Kht^{-1}.

For related papers on finite element approximations to parabolic partial integro-differential equations, we may refer to [7, 20, 22, 25, 33] for smooth solutions and

[21, 26, 27, 33] for the nonsmooth initial data. The smoothing properties proved via energy argument in [27] will be useful for deriving the regularity results for the present problem without nonlocal compatibility assumptions on the data at $t = 0$.

The remaining part of this paper is organized as follows. In Section 2, we discuss some notations, weak formulation, basic assumptions and statement of positivity and Gronwall's Lemma. Section 3 focuses on uniform estimates in $L^\infty(\mathbf{L}^2)$ and $L^\infty(\mathbf{H}^1)$-norms and new regularity results without nonlocal compatibility conditions. In Section 4, a semidiscrete Galerkin method is discussed. Section 5 is devoted to optimal $L^\infty(\mathbf{L}^2)$ error estimates of the velocity term, for the nonsmooth initial data. In Section 6, optimal error bound for the pressure term is derived. It is also shown that under the uniqueness assumption, uniform estimates in time $t > 0$ are also established. Finally, we summarize our results in the Section 7.

2. Preliminaries

For our subsequent use, we denote by bold face letters the $\mathbb{R}^2$-valued function space such as

$$\mathbf{H}_0^1 = [H_0^1(\Omega)]^2, \ \mathbf{L}^2 = [L^2(\Omega)]^2 \text{ and } \mathbf{H}^m = [H^m(\Omega)]^2,$$

where $H^m(\Omega)$ is the standard Hilbert Sobolev space of order m. Note that $\mathbf{H}_0^1$ is equipped with a norm

$$\|\nabla \mathbf{v}\| = \left(\sum_{i,j=1}^{2} (\partial_j v_i, \partial_j v_i) \right)^{1/2} = \left(\sum_{i=1}^{2} (\nabla v_i, \nabla v_i) \right)^{1/2}.$$

Further, we introduce some more function spaces for our future use:

$$\begin{aligned} \mathbf{J}_1 &= \{\boldsymbol{\phi} \in \mathbf{H}_0^1 : \nabla \cdot \boldsymbol{\phi} = 0\} \\ \mathbf{J} = \{\boldsymbol{\phi} \in \mathbf{L}^2 : \nabla \cdot \boldsymbol{\phi} &= 0 \text{ in } \Omega, \boldsymbol{\phi} \cdot \mathbf{n}|_{\partial\Omega} = 0 \text{ holds weakly}\}, \end{aligned}$$

where $\mathbf{n}$ is the outward normal to the boundary $\partial\Omega$ and $\boldsymbol{\phi} \cdot \mathbf{n}|_{\partial\Omega} = 0$ should be understood in the sense of trace in $\mathbf{H}^{-1/2}(\partial\Omega)$, see [31]. Let $H^m/\mathbb{R}$ be the quotient space consisting of equivalence classes of elements of H^m differing by constants, which is equipped with norm $\|p\|_{H^m/\mathbb{R}} = \|p + c\|_m$. For any Banach space X, let $L^p(0, T; X)$ denote the space of measurable X -valued functions $\boldsymbol{\phi}$ on $(0, T)$ such that

$$\int_0^T \|\boldsymbol{\phi}(t)\|_X^p \, dt < \infty \text{ if } 1 \le p < \infty,$$

and for $p = \infty$

$$ess \sup_{0<t<T} \|\boldsymbol{\phi}(t)\|_X < \infty \text{ if } p = \infty.$$

Further, let P be the orthogonal projection of $\mathbf{L}^2$ onto $\mathbf{J}$. Through out this paper, we make the following assumptions:

(**A1**). For $\mathbf{g} \in \mathbf{L}^2$, let the unique pair of solutions $\{\mathbf{v} \in \mathbf{J}_1, q \in L^2/R\}$ for the steady state Stokes problem

$$\begin{gathered} -\Delta \mathbf{v} + \nabla q = \mathbf{g}, \\ \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \ \mathbf{v}|_{\partial\Omega} = 0, \end{gathered}$$

satisfy the following regularity result

$$\|\mathbf{v}\|_2 + \|q\|_{H^1/R} \le C\|\mathbf{g}\|.$$

(**A2**). The initial velocity $\mathbf{u}_0$ and the external force $\mathbf{f}$ satisfy for positive constant

M_0, and for T with $0 < T \leq \infty$

$$\mathbf{u}_0 \in \mathbf{J}_1,\ \mathbf{f}, \mathbf{f}_t \in L^\infty(0,T;\mathbf{L}^2) \text{ with } \|\mathbf{u}_0\|_1 \leq M_0,\ \sup_{0<t<T} \Big\{ \|\mathbf{f}(.,t)\|, \|\mathbf{f}_t(.,t)\| \Big\} \leq M_0.$$

Setting

$$-\tilde{\Delta} = -P\Delta : \mathbf{J}_1 \cap \mathbf{H}^2 \subset \mathbf{J} \to \mathbf{J}$$

as the Stokes operator, the condition (**A1**) implies

$$\begin{aligned}
\|\mathbf{v}\|_2 &\leq C\|\tilde{\Delta}\mathbf{v}\| \ \forall \mathbf{v} \in \mathbf{J}_1 \cap \mathbf{H}^2, \\
\|\mathbf{v}\|^2 &\leq \lambda_1^{-1}\|\nabla \mathbf{v}\|^2 \ \forall \mathbf{v} \in \mathbf{J}_1,\ \|\nabla \mathbf{v}\|^2 \leq \lambda_1^{-1}\|\tilde{\Delta}\mathbf{v}\|^2 \ \forall \mathbf{v} \in \mathbf{J}_1 \cap \mathbf{H}^2,
\end{aligned}$$

where λ_1 is the least positive eigenvalue of the Stokes operator $-\tilde{\Delta}$, see [12]. Before going into the details, let us introduce the weak formulation of (1.1)-(1.3). Find a pair of functions $\{\mathbf{u}(t), p(t)\}$, $t > 0$, such that

$$\begin{aligned}
(\mathbf{u}_t, \boldsymbol{\phi}) + \mu(\nabla \mathbf{u}, \nabla \boldsymbol{\phi}) &+ (\mathbf{u}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}) + \int_0^t \beta(t-s)(\nabla \mathbf{u}(s), \nabla \boldsymbol{\phi})\, ds \\
&= (p, \nabla\cdot\boldsymbol{\phi}) + (\mathbf{f}, \boldsymbol{\phi}) \ \forall \boldsymbol{\phi} \in \mathbf{H}_0^1, \\
(\nabla\cdot\mathbf{u}, \chi) &= 0 \quad \forall \chi \in L^2.
\end{aligned} \tag{2.1}$$

Equivalently, find $\mathbf{u}(t) \in \mathbf{J}_1$ such that

$$\begin{aligned}
(\mathbf{u}_t, \boldsymbol{\phi}) + \mu(\nabla \mathbf{u}, \nabla \boldsymbol{\phi}) + (\mathbf{u}\cdot\nabla\mathbf{u}, \boldsymbol{\phi}) &+ \int_0^t \beta(t-s)(\nabla \mathbf{u}(s), \nabla \boldsymbol{\phi})\, ds \\
&= (\mathbf{f}, \boldsymbol{\phi}),\ \forall \boldsymbol{\phi} \in \mathbf{J}_1,\ t > 0.
\end{aligned} \tag{2.2}$$

For our subsequent analysis, we use the positive property (see [22] for a definition) of the kernel β associated with the integral operator in (1.1). This can be seen as a consequence of the following lemma. For a proof, we refer the reader to Sobolevskii ([30], p.1601), McLean and Thomeé [22].

Lemma 2.1. *For arbitrary $\alpha > 0$, $t^* > 0$ and $\phi \in L^2(0,t^*)$, the following positive definite property holds*

$$\int_0^{t^*} \left(\int_0^t \exp\left[-\alpha(t-s)\right]\phi(s)\, ds \right) \phi(t)\, dt \geq 0.$$

In order to deal with the integral term, we present the following Lemma. See [28].

Lemma 2.2. *Let $g \in L^1(0,t^*)$ and $\phi \in L^2(0,t^*)$ for some $t^* > 0$. Then the following estimate holds*

$$\left(\int_0^{t^*} \left(\int_0^s g(s-\tau)\phi(\tau)\, d\tau \right)^2 ds \right)^{1/2} \leq \left(\int_0^{t^*} |g(s)|\, ds \right) \left(\int_0^{t^*} |\phi(s)|^2\, ds \right)^{1/2}.$$

Lemma 2.3 (Gronwall's lemma)**.** *Let g, h, y be three locally integrable non-negative functions on the time interval $[0,\infty)$ such that for all $t \geq 0$*

$$y(t) + G(t) \leq C + \int_0^t h(s)\ ds + \int_0^t g(s)y(s)\ ds,$$

where $G(t)$ is a non-negative function on $[0,\infty)$ and $C \geq 0$ is a constant. Then,

$$y(t) + G(t) \leq \Big(C + \int_0^t h(s)\ ds \Big) exp\Big(\int_0^t g(s)\ ds \Big).$$

3. A Priori Estimates

In this Section, we discuss *a priori* bounds for the solution $\{\mathbf{u}, p\}$ of (1.1)-(1.3). Here, we present a step-by-step proof of uniform estimate (in time) in H^1, when $d = 2$. Below, we derive *a priori* bounds following the proof techniques of [28].

Lemma 3.1. *Let $0 < \alpha < \min(\delta, \lambda_1\mu)$, and let the assumption (**A2**) hold. Then, there is a positive constant $K_0 = K_0(M_0, \mu, \delta, \lambda_1)$ such that the solution $\mathbf{u}$ of (2.2) satisfies, for $t > 0$*

$$\begin{aligned} \|\mathbf{u}(t)\|^2 + \mu\int_0^t \|\nabla\mathbf{u}(s)\|^2 ds &\leq \|\mathbf{u}_0\|^2 + \frac{1}{\mu\lambda_1}\int_0^t \|\mathbf{f}(s)\|^2 ds \\ &\leq M_0^2\Big(1 + \frac{t}{\lambda_1\mu}\Big), \end{aligned} \tag{3.1}$$

$$\begin{aligned} \|\mathbf{u}(t)\|^2 + (\mu - \frac{\alpha}{\lambda_1})e^{-2\alpha t}\int_0^t e^{2\alpha\tau}\|\nabla\mathbf{u}(\tau)\|^2\,d\tau &\leq e^{-2\alpha t}\|\mathbf{u}_0\|^2 \\ &+ \frac{(1-e^{-2\alpha t})}{2\alpha(\lambda_1\mu - \alpha)}\|\mathbf{f}\|^2_{L^\infty(\mathbf{L}^2)} = K_0. \end{aligned} \tag{3.2}$$

Moreover,

$$\overline{\lim_{t\to\infty}}\,\|\nabla\mathbf{u}(t)\| \leq \frac{\|\mathbf{f}\|^2_{L^\infty(\mathbf{L}^2)}}{\lambda_1\mu^2} = K_{01}^2. \tag{3.3}$$

Proof. We easily modify the proof of Lemma 4.1 in [28](pg 758) to derive estimates (3.1)-(3.2). For the estimate (3.3), we again modify the technique of [28] to obtain

$$\begin{aligned} \|\mathbf{u}(t)\|^2 + \mu e^{-2\alpha t}\int_0^t e^{2\alpha\tau}\|\nabla\mathbf{u}(\tau)\|^2\,d\tau &\leq e^{-2\alpha t}\|\mathbf{u}_0\|^2 + 2e^{-2\alpha t}\alpha\int_0^t e^{2\alpha\tau}\|\mathbf{u}(\tau)\|^2\,d\tau \\ &+ \frac{\|\mathbf{f}\|^2_{L^\infty(\mathbf{L}^2)}}{2\alpha\lambda_1\mu}(1 - e^{-2\alpha t}). \end{aligned} \tag{3.4}$$

Now, taking limit supremum as $t \to \infty$, the second term on the left-hand side (3.4) becomes

$$\begin{aligned} \overline{\lim_{t\to\infty}}\,\mu e^{-2\alpha t}\int_0^t e^{2\alpha\tau}\|\nabla\mathbf{u}(\tau)\|^2\,d\tau &= \mu\,\overline{\lim_{t\to\infty}}\,\frac{\int_0^t e^{2\alpha\tau}\|\nabla\mathbf{u}(\tau)\|^2\,d\tau}{e^{2\alpha t}} \\ &= \frac{\mu}{2\alpha}\,\overline{\lim_{t\to\infty}}\,\|\nabla\mathbf{u}(t)\|^2, \end{aligned} \tag{3.5}$$

and therefore, we find that

$$\frac{\mu}{2\alpha}\,\overline{\lim_{t\to\infty}}\,\|\nabla\mathbf{u}(t)\|^2 \leq \frac{\|\mathbf{f}\|^2_{L^\infty(\mathbf{L}^2)}}{2\alpha\lambda_1\mu}.$$

This completes the rest of the proof. □

Remark 3.1. *We obtain the final result (3.3) by using L'Hospital rule in (3.5), under the assumption that $\int_0^t e^{2\alpha t}\|\nabla\mathbf{u}\|^2 ds$ is not bounded as $t \to \infty$. This allows us to claim the uniform (in time) Dirichlet norm of $\mathbf{u}$ in the next lemma. On the contrary, if $\int_0^t e^{2\alpha t}\|\nabla\mathbf{u}\|^2 ds$ is bounded for all time, then we can obtain uniform (in time) Dirichlet norm of $\mathbf{u}$ directly using Gronwall's Lemma in (3.8)(see, Lemma 3.2).*

Lemma 3.2. *Let $0 < \alpha < \min(\delta, \lambda_1\mu)$ and let assumption (**A2**) hold. Then there is a positive constant $M_3 = M_3(\alpha, \mu, \lambda_1, M_0)$ such that for all $t > 0$*

$$\|\nabla \mathbf{u}(t)\|^2 + e^{-2\alpha t}\int_0^t e^{2\alpha\tau}\|\tilde{\Delta}\mathbf{u}(\tau)\|^2 \, d\tau \le M_3^2.$$

Proof. Setting $\hat{\mathbf{u}} = e^{\alpha t}\mathbf{u}$ and using the Stokes operator $\tilde{\Delta}$, we rewrite (2.2) as

$$\begin{aligned}(\hat{\mathbf{u}}_t, \boldsymbol{\phi}) - \alpha(\hat{\mathbf{u}}, \boldsymbol{\phi}) - \mu(\tilde{\Delta}\hat{\mathbf{u}}, \boldsymbol{\phi}) \quad &- \quad \int_0^t \beta(t-\tau)e^{\alpha(t-\tau)}(\tilde{\Delta}\hat{\mathbf{u}}(\tau), \boldsymbol{\phi})\, d\tau \\ &= \quad -e^{-\alpha t}(\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}}, \boldsymbol{\phi}) + (\hat{\mathbf{f}}, \boldsymbol{\phi}),\end{aligned} \tag{3.6}$$

With $\boldsymbol{\phi} = -\tilde{\Delta}\hat{\mathbf{u}}$ in (3.6), we note that

$$-(\hat{\mathbf{u}}_t, \tilde{\Delta}\hat{\mathbf{u}}) = \frac{1}{2}\frac{d}{dt}\|\nabla\hat{\mathbf{u}}\|^2.$$

Thus,

$$\begin{aligned}\frac{d}{dt}\|\nabla\hat{\mathbf{u}}\|^2 \quad &+ \quad 2\mu\|\tilde{\Delta}\hat{\mathbf{u}}\|^2 + 2\int_0^t \beta(t-\tau)e^{\alpha(t-\tau)}(\tilde{\Delta}\hat{\mathbf{u}}(\tau), \tilde{\Delta}\hat{\mathbf{u}}(t))\, d\tau \\ &= \quad -2\alpha(\hat{\mathbf{u}}, \tilde{\Delta}\hat{\mathbf{u}}) + 2e^{-\alpha t}(\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}}, \tilde{\Delta}\hat{\mathbf{u}}) - 2(\hat{\mathbf{f}}, \tilde{\Delta}\hat{\mathbf{u}}).\end{aligned}$$

On integration with respect to time and using Lemma 2.1 with definition of β, it follows for $0 < \alpha < \min(\delta, \lambda_1\mu)$ that

$$\begin{aligned}\|\nabla\hat{\mathbf{u}}(t)\|^2 \quad &+ \quad 2\mu\int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}(\tau)\|^2 d\tau \le \|\nabla\mathbf{u}_0\|^2 - 2\alpha\int_0^t (\hat{\mathbf{u}}, \tilde{\Delta}\hat{\mathbf{u}})\, d\tau \\ &+ \quad 2\int_0^t e^{-\alpha\tau}(\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}}, \tilde{\Delta}\hat{\mathbf{u}})\, d\tau - 2\int_0^t (\hat{\mathbf{f}}, \tilde{\Delta}\hat{\mathbf{u}})\, d\tau \\ &= \quad \|\nabla\mathbf{u}_0\|^2 + I_1 + I_2 + I_3.\end{aligned} \tag{3.7}$$

To estimate $|I_1|$ and $|I_3|$, we apply Cauchy-Schwarz inequality with $ab \le \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2$, $a, b \ge 0$, $\epsilon > 0$ to obtain

$$|I_1| + |I_3| \le \frac{\alpha^2}{\epsilon}\int_0^t \|\hat{\mathbf{u}}(\tau)\|^2 d\tau + \frac{1}{\epsilon}\int_0^t \|\hat{\mathbf{f}}(\tau)\|^2 d\tau + 2\epsilon\int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}(\tau)\|^2 d\tau.$$

To estimate I_2, a use of Hölder's inequality shows that

$$|(\hat{\mathbf{u}}\cdot\nabla\hat{\mathbf{u}}, \tilde{\Delta}\hat{\mathbf{u}})| \le \|\hat{\mathbf{u}}\|_{L^4(\Omega)}\|\nabla\hat{\mathbf{u}}\|_{L^4(\Omega)}\|\tilde{\Delta}\hat{\mathbf{u}}\|.$$

Now, we appeal to the following Sobolev inequality($d = 2$)(see [31])

$$\|\boldsymbol{\phi}\|_{L^4(\Omega)} \le 2^{1/4}\|\boldsymbol{\phi}\|^{1/2}\|\nabla\boldsymbol{\phi}\|^{1/2}, \ \phi \in H_0^1(\Omega),$$

and, therefore,

$$\begin{aligned}|I_2| \quad &\le \quad C\int_0^t e^{-\alpha\tau}\|\hat{\mathbf{u}}\|^{1/2}\|\nabla\hat{\mathbf{u}}\|\,\|\tilde{\Delta}\hat{\mathbf{u}}\|^{3/2}\, d\tau \\ &\le \quad C(\epsilon)\int_0^t e^{-4\alpha\tau}\|\hat{\mathbf{u}}\|^2\|\nabla\hat{\mathbf{u}}\|^4\, d\tau + \epsilon\int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}\|^2\, d\tau.\end{aligned}$$

Substituting the estimates of I_1, I_2 and I_3 in (3.7), we find that

$$\begin{aligned}\|\nabla\hat{\mathbf{u}}(t)\|^2 + \mu\int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}(\tau)\|^2 d\tau \quad &\le \quad \|\nabla\mathbf{u}_0\|^2 + C(\alpha, \mu)\int_0^t (\|\hat{\mathbf{u}}(\tau)\|^2 + \|\hat{\mathbf{f}}(\tau)\|^2)\, d\tau \\ &+ \quad C(\mu)\int_0^t \|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2\|\nabla\hat{\mathbf{u}}\|^2\, d\tau.\end{aligned} \tag{3.8}$$

Using Gronwall's lemma, we arrive at

$$\|\nabla\hat{\mathbf{u}}(t)\|^2+\mu\int_0^t\|\tilde{\Delta}\hat{\mathbf{u}}(\tau)\|^2\,d\tau \le \Big\{\|\nabla\mathbf{u}_0\|^2+C(\alpha,\mu)\int_0^t(\|\hat{\mathbf{u}}(\tau)\|^2+\|\hat{\mathbf{f}}(\tau)\|^2)\,d\tau\Big\} \times\exp\Big(C(\mu)\int_0^t\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2\,d\tau\Big).$$

Finally, we use Lemma 3.1 with condition (**A2**) to obtain the following integral inequality

$$\|\nabla\mathbf{u}(t)\|^2+\mu e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\tilde{\Delta}\mathbf{u}(\tau)\|^2 d\tau \le \Big(e^{-2\alpha t}M_0^2+C(\alpha,\mu)\{K_0+M_0^2\}\Big) \times\exp\Big(C(\mu,M_0,K_0)(1+\frac{t}{\lambda_1\mu})\Big).$$

This inequality along with $\limsup_{t\to\infty}\|\nabla\mathbf{u}\|\le K_{01}$ would lead us to the following conclusion

$$\|\nabla\mathbf{u}(t)\|<M_3\;\; t>0,$$

for some positive constant M_3. Now (3.8) along with this estimate of $\|\nabla\mathbf{u}(t)\|$ would provide us the desired result. □

Remark 3.2. *Although Lemma* 3.2 *provides a uniform Dirichlet norm estimate in time, it is difficult to obtain a precise bound for the estimate $\|\nabla\mathbf{u}\|$ for all $t>0$. We present below another proof of the uniform estimate in $L^\infty(\mathbf{H}_0^1)$ norm which enables us to obtain a precise bound depending on the initial data, forcing term and the smoothness of the domain.*

Lemma 3.3. (Uniform estimate in $\mathbf{L}^2$). *Let the assumption (**A2**) hold. Then, for $0<\alpha<\min(2\delta,\lambda_1\mu)$, the solution $\mathbf{u}$ of (2.2) satisfies the following estimates for $t>0$*

$$\|\mathbf{u}(t)\|^2+\frac{1}{\gamma}\|\nabla\tilde{\mathbf{u}}_\beta\|^2\le e^{-\alpha t}\|\mathbf{u}_0\|^2+\frac{\|\mathbf{f}\|^2_{L^\infty(\mathbf{L}^2)}}{\lambda_1\mu\alpha}(1-e^{-\alpha t})=M_1, \tag{3.9}$$

and for fixed $T_0>0$

$$\int_t^{t+T_0}\Big(\mu\|\nabla\mathbf{u}(s)\|^2+\frac{2\delta}{\gamma}\|\nabla\tilde{\mathbf{u}}_\beta\|^2\Big)\,ds\le M_1+\frac{M_0^2T_0}{\lambda_1\mu}=M_2, \tag{3.10}$$

where $\tilde{\mathbf{u}}_\beta(t)=\int_0^t\beta(t-s)\mathbf{u}(s)\,ds$.

Proof. With

$$\tilde{\mathbf{u}}_\beta(t)=\int_0^t\beta(t-s)\mathbf{u}(s)\,ds,$$

we rewrite the equation (2.2) as

$$(\mathbf{u}_t,\boldsymbol{\phi})+\mu(\nabla\mathbf{u},\nabla\boldsymbol{\phi})+(\mathbf{u}\cdot\nabla\mathbf{u},\boldsymbol{\phi})+(\nabla\tilde{\mathbf{u}}_\beta,\nabla\boldsymbol{\phi})=(\mathbf{f},\boldsymbol{\phi}),\ \forall\boldsymbol{\phi}\in\mathbf{J}_1,\ t>0. \tag{3.11}$$

Take $\boldsymbol{\phi}=\mathbf{u}$ in (3.11) and use $\tilde{\mathbf{u}}_{\beta,t}+\delta\tilde{\mathbf{u}}_\beta=\gamma\mathbf{u}$, to obtain

$$\frac{d}{dt}(\|\mathbf{u}\|^2+\frac{1}{\gamma}\|\nabla\tilde{\mathbf{u}}_\beta\|^2)+\mu\|\nabla\mathbf{u}\|^2+\frac{2\delta}{\gamma}\|\nabla\tilde{\mathbf{u}}_\beta\|^2\le\frac{1}{\mu\lambda_1}\|\mathbf{f}\|^2. \tag{3.12}$$

We use the Poincaré inequality $\lambda_1\|\mathbf{u}\|^2\le\|\nabla\mathbf{u}\|^2$ and multiply (3.12) by $e^{\alpha t}$ to find that

$$\frac{d}{dt}\Big(e^{\alpha t}(\|\mathbf{u}\|^2+\frac{1}{\gamma}\|\nabla\tilde{\mathbf{u}}_\beta\|^2)\Big)\le\frac{1}{\mu\lambda_1}e^{\alpha t}\|\mathbf{f}\|^2.$$

Now, integrate with respect to time and multiply by $e^{-\alpha t}$ to conclude (3.9). Next, we integrate (3.12) from t to $t+T_0$, for fixed $T_0>0$ and use (3.9) to obtain (3.10). This completes the rest of the proof. □

Lemma 3.4. (Uniform estimates in $\mathbf{H}^1$). *Let $0<\alpha<\min(\delta,\lambda_1\mu)$. Under assumption (**A2**), there exists a positive constant $M_3=M_3(\alpha,\mu,\lambda_1,M_1)$ such that for $T_0>0$ with $t\in[T_0,\infty)$, the following estimates hold:*

$$\|\nabla\mathbf{u}(t)\|^2+\frac{1}{\gamma}\|\tilde{\Delta}\tilde{\mathbf{u}}_\beta\|^2\le M_3^2,\ t\ge T_0,\ \forall\ T_0>0.$$

and

$$\|\nabla\mathbf{u}(t)\|^2+e^{-2\alpha t}\int_0^t e^{2\alpha\tau}\|\tilde{\Delta}\mathbf{u}(\tau)\|^2\,d\tau\le M_3^2,\ t>0.$$

Proof. Put $\boldsymbol{\phi}=-\tilde{\Delta}\mathbf{u}$ in (3.11) and use $\tilde{\mathbf{u}}_{\beta,t}+\delta\tilde{\mathbf{u}}_\beta=\gamma\mathbf{u}$, to obtain

$$\frac{1}{2}\frac{d}{dt}(\|\nabla\mathbf{u}\|^2+\frac{1}{\gamma}\|\tilde{\Delta}\tilde{\mathbf{u}}_\beta\|^2)+\mu\|\tilde{\Delta}\mathbf{u}\|^2+\frac{\delta}{\gamma}\|\tilde{\Delta}\tilde{\mathbf{u}}_\beta\|^2\le\|\mathbf{f}\|\|\tilde{\Delta}\mathbf{u}\|+|(\mathbf{u}\cdot\nabla\mathbf{u},-\tilde{\Delta}\mathbf{u})|. \tag{3.13}$$

Recollecting the estimates of I_2,I_3 of (3.7), we find that

$$\|\mathbf{f}\|\|\tilde{\Delta}\mathbf{u}\|\le\frac{\mu}{6}\|\tilde{\Delta}\mathbf{u}\|^2+\frac{3}{\mu}\|\mathbf{f}\|^2, \tag{3.14}$$

and

$$|(\mathbf{u}\cdot\nabla\mathbf{u},-\tilde{\Delta}\mathbf{u})|\le\frac{\mu}{6}\|\tilde{\Delta}\mathbf{u}\|^2+(\frac{9}{2\mu})^3\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^4. \tag{3.15}$$

Using (3.14)-(3.15) in (3.13), we obtain

$$\begin{aligned}\frac{d}{dt}\left(\|\nabla\mathbf{u}\|^2+\frac{1}{\gamma}\|\tilde{\Delta}\tilde{\mathbf{u}}_\beta\|^2\right) &+ \frac{4\mu}{3}\|\tilde{\Delta}\mathbf{u}\|^2+\frac{2\delta}{\gamma}\|\tilde{\Delta}\tilde{\mathbf{u}}_\beta\|^2\\ &\le \frac{3}{\mu}\|\mathbf{f}\|^2+(\frac{9/2}{\mu})^3\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^4.\end{aligned} \tag{3.16}$$

Note that

$$\alpha_0\|\nabla\mathbf{u}\|^2=\alpha_0(\mathbf{u},-\tilde{\Delta}\mathbf{u})\le\alpha_0\|\mathbf{u}\|\|\tilde{\Delta}\mathbf{u}\|,$$

where $\alpha_0>0$ is a constant to be chosen precisely at a later stage and then, we find that

$$\alpha_0\|\nabla\mathbf{u}\|^2\le\frac{\mu}{3}\|\tilde{\Delta}\mathbf{u}\|^2+\frac{3}{4\mu}\alpha_0^2\|\mathbf{u}\|^2. \tag{3.17}$$

Now, add (3.17) to (3.16) and use the inequality $\|\tilde{\Delta}\mathbf{u}\|^2\ge\lambda_1\|\nabla\mathbf{u}\|^2$ to arrive at

$$\begin{aligned}\frac{d}{dt}\left(\|\nabla\mathbf{u}\|^2+\frac{1}{\gamma}\|\tilde{\Delta}\tilde{\mathbf{u}}_\beta\|^2\right) &+ \left(\alpha_0+\mu\lambda_1-(\frac{9/2}{\mu})^3\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2\right)\|\nabla\mathbf{u}\|^2+\frac{2\delta}{\gamma}\|\tilde{\Delta}\tilde{\mathbf{u}}_\beta\|^2\\ &\le \frac{3}{\mu}\|f\|^2+\frac{3}{4\mu}\alpha_0^2\|\mathbf{u}\|^2.\end{aligned}$$

As $\|\mathbf{u}\|^2\le M_1$ from Lemma 3.1, and $\|f\|_{L^\infty(L^2)}\le M_0$, we obtain

$$\begin{aligned}\frac{d}{dt}\left(\|\nabla\mathbf{u}\|^2+\frac{1}{\gamma}\|\tilde{\Delta}\tilde{\mathbf{u}}_\beta\|^2\right) &+ \left(\alpha_0+\mu\lambda_1-(\frac{9}{2\mu})^3\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2\right)\|\nabla\mathbf{u}\|^2+\frac{2\delta}{\gamma}\|\tilde{\Delta}\tilde{\mathbf{u}}_\beta\|^2\\ &\le K_1(M_0,\mu,\alpha_0,M_1)=K_1.\end{aligned} \tag{3.18}$$

Setting

$$h(t)=\min\left\{\alpha_0+\mu\lambda_1-(\frac{9}{2\mu})^3\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2,\ 2\delta\right\} \tag{3.19}$$

we obtain from (3.18) with $E(t) = \|\nabla \mathbf{u}\|^2 + \frac{1}{\gamma}\|\tilde{\Delta}\tilde{\mathbf{u}}_\beta\|^2$,

$$\frac{d}{dt}E(t) + h(t)E(t) \le K_1.$$

Hence, for $0 \le s < t$, we find that

$$\frac{d}{dt}\Big(e^{\int_s^t h(\tau)\ d\tau}E(t)\Big) \le K_1 e^{\int_s^t h(\tau)\ d\tau}.$$

On integrating with respect to time over 0 to t, we arrive at

$$E(t) \le e^{-\int_0^t h(\tau)\,d\tau}\|\nabla \mathbf{u}_0\|^2 + K_1\int_0^t e^{-\int_s^t h(\tau)\,d\tau}\,ds. \tag{3.20}$$

Now, from (3.19), we note, for some $T_0 > 0$ that

$$\int_t^{t+T_0} h(s)\,ds = \min\Big\{(\alpha_0 + \mu\lambda_1)T_0 - (\frac{9}{2\mu})^3\int_t^{t+T_0}\|\mathbf{u}(s)\|^2\|\nabla\mathbf{u}(s)\|^2\,ds,\ 2\delta T_0\Big\}.$$

A use of Lemma 3.1 yields

$$(\frac{9}{2\mu})^3\int_t^{t+T_0}\|\mathbf{u}(s)\|^2\|\nabla\mathbf{u}(s)\|^2\,ds \le \frac{CM_1}{\mu^3}\int_t^{t+T_0}\|\nabla\mathbf{u}(s)\|^2\,ds \le \frac{CK_0}{\mu^4}M_2 = K_2.$$

Therefore, we obtain

$$\int_t^{t+T_0} h(s)\,ds \ge \min\Big\{(\alpha_0 + \mu\lambda_1)T_0 - K_2,\ 2\delta T_0\Big\}.$$

Now, choose $\alpha_0 T_0 = K_2$ to arrive at

$$\int_t^{t+T_0} h(s)\,ds\ \ge\ T_0\ \min\{\mu\lambda_1,\ 2\delta\} \ge \alpha T_0. \tag{3.21}$$

Given s and t with $0 \le s < t$, we choose two positive integers k and l such that

$$kT_0 \le s \le (k+1)T_0,\ lT_0 \le t \le (l+1)T_0.$$

Then, from (3.21) and $-h(t) \ge -2\delta,\ t > 0$ using (3.19), we find that

$$\begin{aligned}\int_s^t h(\tau)\,d\tau &= \int_{kT_0}^{(l+1)T_0} h(\tau)\,d\tau - \int_{kT_0}^{s} h(\tau)\,d\tau - \int_t^{(l+1)T_0} h(\tau)\,d\tau\\ &\ge (l+1-k)\alpha T_0 - 2\delta T_0 - 2\delta T_0\\ &\ge (t-s)\alpha - 4\delta T_0.\end{aligned}$$

Hence,

$$-\int_s^t h(\tau)\,d\tau \le 4\delta T_0 - (t-s)\alpha.$$

Now, without loss of generality, we have assumed that $s \le (k+1)T_0 \le lT_0 \le t$ (with one of these $\le$ is strict inequality to preserve the fact that $0 \le s < t$). All other possible cases will simplify the present situation and hence, we skip the related analysis.

From (3.20), we obtain

$$\begin{aligned}E(t) &\le e^{-t\alpha}e^{4\delta T_0}\|\nabla\mathbf{u}_0\|^2 + K_1\int_0^t e^{-(t-s)\alpha}\,ds\ e^{4\delta T_0}\\ &\le \Big(e^{-t\alpha}\|\nabla\mathbf{u}_0\|^2 + \frac{K_1}{\alpha}(1 - e^{-t\alpha})\Big)e^{4\delta T_0}\\ &\le e^{4\delta T_0}\Big(\|\nabla\mathbf{u}_0\|^2 + \frac{K_1}{\alpha}\Big) = M_3^2.\end{aligned}$$

This establishes the first estimate of Lemma 3.4. Use it in (3.8) to obtain the second estimate, and this completes the rest of the proof. ☐

Remark 3.3. *(i) It is difficult to apply the uniform Gronwall's Lemma to (3.16) to obtain the desired result as we do not have an estimate of*

$$\int_t^{t+T_0} (\|\nabla \mathbf{u}(s)\|^2 + \frac{1}{\gamma}\|\tilde{\Delta}\tilde{\mathbf{u}}_\beta(s)\|^2)\, ds \leq M_4,\ T_0 > 0, \tag{3.22}$$

where M_4 is some positive constant independent of t. Note that from Lemma 3.1, *we only obtain the estimate*

$$\mu \int_t^{t+T_0} \|\nabla \mathbf{u}(s)\|^2\, ds \leq M_2.$$

*(ii) Instead of the assumption (**A2**), if we make the following assumption*

$$\int_0^t e^{2\alpha t}\|\mathbf{f}(\tau)\|^2\, d\tau \leq M_1, \text{ and } \|\nabla \mathbf{u}_0\|^2 \leq M_1,$$

then a simple modification of the above Lemmas yields

$$\|\nabla \mathbf{u}(t)\| \leq C(\alpha, \mu, \delta, \lambda_1, M_1)e^{-\alpha t}\ \forall t > 0,$$

and

$$\int_0^t e^{2\alpha\tau}\|\tilde{\Delta}\mathbf{u}(\tau)\|^2\, d\tau \leq C(\alpha, \mu, \delta, \lambda_1, M_1)\ \forall t > 0.$$

Note that we have the exponential decay property for the gradient of $\mathbf{u}(t)$ in $L^\infty(\mathbf{L}^2)$-norm.
(iii) Following the arguments of Sobolevskii [30], *it is possible to obtain similar asymptotic behavior for some $0 < \delta_0 < \min(\delta, \lambda_1\mu)$ provided*

$$\sup_{0<t<\infty} (e^{\delta_0 t}\|\mathbf{f}(t)\|) \leq M,$$

for some positive constant M. With some changes in the proof of the above Lemmas like setting $\alpha = \delta_0 \pm \alpha_0$ with $0 < \alpha_0 < \min\{\delta - \delta_0, \mu\lambda_1 - \delta_0\}$, it is easy to derive the exponential decay proper for the solution now replacing α by δ_0. In fact, the above asymptotic behavior holds true, when $\mathbf{f} \equiv 0$, see Pani et al. [28].
(iv) A priori bounds in above Lemmas are useful for proving existence of a unique global strong solutions to (1.1)-(1.3) by employing Faedo-Galerkin method, see Temam [31], *Ladyzhenskaya* [19] *for similar analysis in case of Navier-Stokes equations.*

We present below, regularity results for the nonsmooth $\mathbf{u}_0$, i.e., when $\mathbf{u}_0 \in \mathbf{J}_1$.

Theorem 3.1. *Suppose the assumptions (**A1**) and (**A2**) hold. Then, there is a constant $K = K(M_0, M_3, \alpha, \mu, \delta, \gamma) > 0$ such that for $0 < \alpha < \min(\delta, \lambda_1\mu)$ the following estimate*

$$\sup_{0<t<\infty} (\tau^*)^{1/2}(t)\{\|\mathbf{u}\|_2 + \|\mathbf{u}_t\| + \|p\|_{H^1/R}\} \leq K$$

holds, where $\tau^*(t) = min\{t, 1\}$.

Proof. Following the proof of Theorem 2.1 in ([28], page 760), we arrive at

$$e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\mathbf{u}_t(s)\|^2\, ds \leq C(M_3). \tag{3.23}$$

Next, we differentiating the equation (2.2) with respect to time, to obtain

$$
\begin{aligned}
&(\mathbf{u}_{tt},\boldsymbol{\phi})-\mu(\nabla\mathbf{u}_t,\nabla\boldsymbol{\phi})-\beta(0)(\nabla\mathbf{u},\nabla\boldsymbol{\phi})-\int_0^t\beta_t(t-s)(\nabla\mathbf{u}(s),\nabla\boldsymbol{\phi})\,ds\\
&=-(\mathbf{u}_t\cdot\nabla\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u}_t,\boldsymbol{\phi})+(\mathbf{f}_t,\boldsymbol{\phi}),\ \forall\ \boldsymbol{\phi}\in\mathbf{J}_1.
\end{aligned}
\tag{3.24}
$$

Setting $\boldsymbol{\phi}=\sigma(t)\mathbf{u}_t$ in (3.24), where $\sigma(t)=\tau^*(t)e^{2\alpha t}$, we rewrite the resulting equation as

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\sigma(t)\|\mathbf{u}_t\|^2)-\Big(\frac{e^{2\alpha t}}{2}+\alpha\sigma(t)\Big)\|\mathbf{u}_t\|^2\\
&+\mu\sigma(t)\|\nabla\mathbf{u}_t\|^2-\sigma(t)\int_0^t\beta_t(t-s)(\tilde{\Delta}\mathbf{u}(s),\mathbf{u}_t)ds\\
=\;&\beta(0)\sigma(t)(\tilde{\Delta}\mathbf{u},\mathbf{u}_t)-\tau^*(t)e^{-\alpha t}(e^{\alpha t}\mathbf{u}_t\cdot\nabla\hat{\mathbf{u}},e^{\alpha t}\mathbf{u}_t)+\tau^*(t)(e^{\alpha t}\mathbf{f}_t,e^{\alpha t}\mathbf{u}_t).
\end{aligned}
$$

Observe that $\|\mathbf{u}_t\|^2\le\frac{1}{\lambda_1}\|\nabla\mathbf{u}_t\|^2$. Now we use Hölder's inequality for the nonlinear term on the right-hand side to obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\sigma(t)\|\mathbf{u}_t\|^2)\quad&+\quad(\mu-\frac{\alpha}{\lambda_1})\sigma(t)\|\nabla\mathbf{u}_t\|^2\le\frac{1}{2}e^{2\alpha t}\|\mathbf{u}_t\|^2+\gamma\sigma(t)\|\tilde{\Delta}\mathbf{u}\|\|\mathbf{u}_t\|\\
&+\quad\tau^*(t)e^{-\alpha t}\|e^{\alpha t}\mathbf{u}_t\|^2_{\mathbf{L}^4(\Omega)}.\|\nabla\hat{\mathbf{u}}\|+\sigma(t)\|\mathbf{f}_t\|\|\mathbf{u}_t\|\\
&+\quad\frac{\tau^*(t)}{\delta}\int_0^t\beta(t-s)e^{\alpha(t-s)}\|\tilde{\Delta}\hat{\mathbf{u}}(s)\|\|e^{\alpha t}\mathbf{u}_t\|\;ds.
\end{aligned}
$$

As usual, we can estimate the third term (say I_1) on the right-hand side as follows

$$
I_1\le\epsilon_0.\sigma(t)\|\nabla\mathbf{u}_t\|^2+\frac{1}{2\epsilon_0}\sigma(t)e^{-2\alpha t}\|\mathbf{u}_t\|^2\|\nabla\hat{\mathbf{u}}\|^2,
$$

for some positive ϵ_0.

$$
\begin{aligned}
\frac{d}{dt}(\sigma(t)\|\mathbf{u}_t\|^2)\quad&+\quad2(\mu-\frac{\alpha}{\lambda_1})\sigma(t)\|\nabla\mathbf{u}_t\|^2\le e^{2\alpha t}\|\mathbf{u}_t\|^2+\gamma^2\sigma(t)\|\tilde{\Delta}\mathbf{u}\|^2\\
&+\quad2\epsilon_0\;\sigma(t)\|\nabla\mathbf{u}_t\|^2+\frac{1}{\epsilon_0}\sigma(t)\|\mathbf{u}_t\|^2\Big(\epsilon_0+e^{-2\alpha t}\|\nabla\hat{\mathbf{u}}\|^2\Big)\\
&+\quad\sigma(t)\|\mathbf{f}_t\|^2+\frac{\tau^*(t)}{\delta^2}\Big(\int_0^t\beta(t-s)e^{\alpha(t-s)}\|\tilde{\Delta}\hat{\mathbf{u}}(s)\|\;ds\Big)^2.
\end{aligned}
$$

With $\epsilon_0=\frac{1}{2}(\mu-\frac{\alpha}{\lambda_1})$, and $\sigma(t)\le e^{2\alpha t}$ for all $t>0$, we integrate with respect to time in the interval $0<\varepsilon<t\le\infty$ and use (**A2**) to obtain

$$
\begin{aligned}
&\sigma(t)\|\mathbf{u}_t(t)\|^2+(\mu-\frac{\alpha}{\lambda_1})\int_\varepsilon^t\sigma(s)\|\nabla\mathbf{u}_t(s)\|^2\;ds \qquad (3.25)\\
\le\;&\sigma(\varepsilon)\|\mathbf{u}_t(\varepsilon)\|^2+C(\mu,\alpha,\lambda_1,M_1)\int_\varepsilon^t e^{2\alpha s}\|\mathbf{u}_s(s)\|^2\;ds+\int_\varepsilon^t e^{2\alpha s}\|\tilde{\Delta}\mathbf{u}(s)\|^2\;ds\\
+\;&\int_\varepsilon^t\sigma(s)\|\mathbf{f}_t(s)\|^2\;ds+\frac{1}{\delta^2}\int_\varepsilon^t\Big(\int_0^s\beta(s-\tau)e^{\alpha(s-\tau)}\|\tilde{\Delta}\hat{\mathbf{u}}(\tau)\|\;d\tau\Big)^2ds.
\end{aligned}
$$

Note that the first term on the right-hand side may not be finite as $\varepsilon\to0$ and hence, there can be some problem in integrating directly from 0 to t. Now, by (3.23), we find that

$$
\int_0^t e^{2\alpha s}\|\mathbf{u}_t(s)\|^2\,ds\le C(M_1)e^{2\alpha t}.
$$

Therefore, there exists a sequence of positive real numbers $\varepsilon_n\to0$ such that

$$
\varepsilon_n\{e^{2\alpha\varepsilon_n}\|\mathbf{u}_t(\varepsilon_n)\|^2\}\to0,\ \text{as}\ n\to\infty.
$$

Choosing $\varepsilon = \varepsilon_n$ in (3.25) and passing the limit as $n \to \infty$, and using Lemma 3.1, (**A2**), (3.23) and the estimate (for a proof see [28])

$$
\begin{aligned}
I &= \int_0^t (\int_0^s \beta(s-\tau) e^{\alpha(s-\tau)} \|\tilde{\Delta}\hat{\mathbf{u}}(\tau)\| \, d\tau)^2 \, ds \\
&\leq (\frac{\gamma}{\delta - \alpha})^2 \int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}(s)\|^2 \, ds, \qquad (3.26)
\end{aligned}
$$

we conclude that

$$
\tau^*(t)\|\mathbf{u}_t(t)\|^2 + (\mu - \frac{\alpha}{\lambda_1}) e^{-2\alpha t} \int_0^t \sigma(s) \|\nabla \mathbf{u}_t(s)\|^2 \, ds \leq K. \qquad (3.27)
$$

To estimate $\|\tilde{\Delta}\mathbf{u}(t)\|$, we proceed as in the proof of Theorem 2.1 in [28] to obtain

$$
\begin{aligned}
\mu \|\tilde{\Delta}\hat{\mathbf{u}}(t)\|^2 &\leq C(\mu) \Big\{ e^{2\alpha t} \|\mathbf{u}_t\|^2 + e^{2\alpha t} \|\mathbf{u}\|^2 \|\nabla \mathbf{u}\|^4 + \|\hat{\mathbf{f}}\|^2 \Big\} \\
&\quad + C(\gamma, \mu) \Big(\int_0^t e^{-(\delta-\alpha)(t-s)} \|\tilde{\Delta}\hat{\mathbf{u}}(s)\| \, ds \Big)^2.
\end{aligned}
$$

The integral term can be estimated as

$$
\leq C(\gamma, \mu) (\int_0^t e^{-2(\delta-\alpha)(t-s)} \, ds) (\int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}(s)\|^2 \, ds) \leq C(\gamma, \mu, \delta) \int_0^t \|\tilde{\Delta}\hat{\mathbf{u}}(s)\|^2 \, ds.
$$

Note that if we majorize the exponential term by 1, then the right-hand side depends on time. For the sake of brevity, in the rest of the paper, similar integrals are estimated by constants independent of time, skipping the explicit calculations. Using the assumption (**A2**) and Lemma 3.2, we now arrive at

$$
\|\tilde{\Delta}\mathbf{u}(t)\|^2 \leq C(\gamma, \alpha, \delta, \mu) \{ \|\mathbf{u}_t\|^2 + M_3^6 + M_0^2 + M_3^2 \}
$$

Now, multiply by $\tau^*(t)$ and use the fact $\tau^*(t) \leq 1$ and (3.27) to find that

$$
\tau^*(t) \|\tilde{\Delta}\mathbf{u}\|^2 \leq K. \qquad (3.28)
$$

For the pressure term, we again appeal to the equation (1.1) and with the help of the results in Theorem 3.1, namely; (3.27) and (3.28), we complete the rest of the proof. □

Theorem 3.2. *Under the assumptions of the Theorem 2.1, there is a constant $K > 0$ such that the pair of solutions $\{\mathbf{u}, p\}$ satisfies the following estimates for $0 < \alpha < \min\{\delta, \lambda\mu\}$*

$$
\sup_{0<t<\infty} e^{-2\alpha t} \int_0^t \sigma(s) \|\mathbf{u}_t\|_1^2 \, ds \leq K, \quad \sup_{0<t<\infty} \tau^*(t) \|\mathbf{u}_t\|_1 \leq K,
$$

where $\tau^*(t) = \min\{t, 1\}$. Moreover,

$$
\sup_{0<t<\infty} e^{-2\alpha t} \int_0^t \sigma_1(s) \Big(\|\mathbf{u}_t\|_2^2 + \|\mathbf{u}_{tt}\|^2 + \|p_t\|_{H^1/R}^2 \Big) \, ds \leq K,
$$

where $\sigma_1(t) = (\tau^*)^2(t) e^{2\alpha t}$.

Proof. First estimate clearly follows from (3.27). For the second one, we follow the proof of Theorem 2.2 in ([28], page 763), except that, we use $\sigma_1(t)$ in place of $\sigma(t)$.

We obtain as earlier

$$
\begin{aligned}
\frac{d}{dt}(\sigma_1(t)\|\nabla \mathbf{u}_t\|^2) \quad &+ \quad \mu\sigma_1(t)\|\tilde{\Delta}\mathbf{u}_t\|^2 \le C(\mu,\gamma)\sigma_1(t)\Big(\|\tilde{\Delta}\mathbf{u}\|^2 + \|\mathbf{f}_t\|^2\Big) \\
&+ \quad \sigma_{1,t}(t)\|\nabla \mathbf{u}_t\|^2 + C(\mu)\Big(\sup_{t>0}\tau^*(t)\|\tilde{\Delta}\mathbf{u}(t)\|^2\Big)\sigma(t)\|\nabla \mathbf{u}_t\|^2 \\
&+ \quad \frac{\gamma^2}{2\epsilon\delta^2}(\tau^*(t))^2\Big(\int_0^t e^{-(\delta-\alpha)(t-s)}\|(\tilde{\Delta}\hat{\mathbf{u}}(s)\|\ ds\Big)^2.
\end{aligned}
$$

We integrate with respect to time in the interval $0 < \epsilon < t \le \infty$ and using (3.27), we can pass the limit as $\epsilon \to 0$ and then using the estimates (3.27), (3.26) and Lemma 3.2, we arrive at

$$
(\tau^*(t))^2\|\nabla \mathbf{u}_t\|^2 + \mu e^{-2\alpha t}\int_0^t \sigma_1(s)\|\tilde{\Delta}\mathbf{u}_t\|^2 \ \le\ K.
$$

Next, form an inner product between (3.24) and $\sigma_1(t)\mathbf{u}_{tt}(t)$ and proceed in a similar fashion to obtain

$$
\int_0^t \sigma_1(s)\|\mathbf{u}_{tt}\|^2\ ds \le C(\gamma,\mu)\int_0^t \sigma_1(s)\Big(\|\tilde{\Delta}\mathbf{u}\|^2 + \|\tilde{\Delta}\mathbf{u}_t\|^2 + \|\mathbf{f}_t\|^2 + \|\nabla \mathbf{u}_t\|^2\Big)\ ds,
$$

and the required estimate for $\mathbf{u}_{tt}$ now follows by multiplying $e^{-2\alpha t}$ and by using previously obtained estimates. Similar analysis, using the equation (1.1), would result in the estimate of the pressure term. $\square$

Remark 3.4. *As in* [28], *we can easily modify our analysis to derive regularity results, when* $u_0 \in \mathbf{J}_1 \cap \mathbf{H}^2$.

4. Semidiscrete Galerkin Approximations

From now on, we denote h with $0 < h < 1$ by a real positive discretization parameter tending to zero. Let $\mathbf{H}_h$ and L_h, $0 < h < 1$ be two family of finite dimensional subspaces of $\mathbf{H}_0^1$ and L^2, respectively, approximating velocity vector and the pressure. Assume that the following approximation properties are satisfied for the spaces $\mathbf{H}_h$ and L_h:

(**B1**) For each $\mathbf{w} \in \mathbf{H}_0^1 \cap \mathbf{H}^2$ and $q \in H^1/\mathbb{R}$ there exist approximations $i_h w \in \mathbf{H}_h$ and $j_h q \in L_h$ such that

$$
\|\mathbf{w} - i_h\mathbf{w}\| + h\|\nabla(\mathbf{w} - i_h\mathbf{w})\| \le K_0 h^2\|\mathbf{w}\|_2,\ \ \|q - j_h q\|_{L^2/\mathbb{R}} \le K_0 h\|q\|_{H^1/\mathbb{R}}.
$$

Further, suppose that the following inverse hypothesis holds for $\mathbf{w}_h \in \mathbf{H}_h$

$$
\|\nabla \mathbf{w}_h\| \le K_0 h^{-1}\|\mathbf{w}_h\|.
$$

For defining the Galerkin approximations, set for $\mathbf{v}, \mathbf{w}, \boldsymbol{\phi} \in \mathbf{H}_0^1$,

$$
a(\mathbf{v}, \boldsymbol{\phi}) = (\nabla \mathbf{v}, \nabla \boldsymbol{\phi})
$$

and

$$
b(\mathbf{v}, \mathbf{w}, \boldsymbol{\phi}) = \frac{1}{2}(\mathbf{v}\cdot\nabla \mathbf{w}, \boldsymbol{\phi}) - \frac{1}{2}(\mathbf{v}\cdot\nabla \boldsymbol{\phi}, \mathbf{w}).
$$

Note that the operator $b(\cdot,\cdot,\cdot)$ preserves the antisymmetric property of the original nonlinear term, that is,

$$
b(\mathbf{v}_h, \mathbf{w}_h, \mathbf{w}_h) = 0 \quad \forall \mathbf{v}_h, \mathbf{w}_h \in \mathbf{H}_h.
$$

The discrete analogue of the weak formulation (2.1) now reads as: Find $\mathbf{u}_h(t) \in \mathbf{H}_h$ and $p_h(t) \in L_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and for $t > 0$

$$\begin{aligned}(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \mu a(\mathbf{u}_h, \boldsymbol{\phi}_h) &+ b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) - (p_h, \nabla \cdot \boldsymbol{\phi}_h) = (\mathbf{f}, \boldsymbol{\phi}_h) \\ &- \int_0^t \beta(t-s) a(\mathbf{u}_h(s), \boldsymbol{\phi}_h) ds \ \forall \boldsymbol{\phi}_h \in \mathbf{H}_h, \\ (\nabla \cdot \mathbf{u}_h, \chi_h) &= 0 \quad \forall \chi_h \in L_h,\end{aligned} \tag{4.1}$$

where $\mathbf{u}_{0h} \in \mathbf{H}_h$ is a suitable approximation of $\mathbf{u}_0 \in \mathbf{J}_1$.
In order to consider a discrete space analogous to $\mathbf{J}_1$, we impose the discrete incompressibility condition on $\mathbf{H}_h$ and call it as $\mathbf{J}_h$. Thus, we define $\mathbf{J}_h$, as

$$\mathbf{J}_h = \{v_h \in \mathbf{H}_h : (\chi_h, \nabla \cdot v_h) = 0 \ \forall \chi_h \in L_h\}.$$

Note that $\mathbf{J}_h$ is not a subspace of $\mathbf{J}_1$. With $\mathbf{J}_h$ as above, we now introduce an equivalent Galerkin formulation as: Find $\mathbf{u}_h(t) \in \mathbf{J}_h$ such that $\mathbf{u}_h(0) = \mathbf{u}_{0h}$ and for $t > 0$

$$\begin{aligned}(\mathbf{u}_{ht}, \boldsymbol{\phi}_h) + \mu a(\mathbf{u}_h, \boldsymbol{\phi}_h) &+ \int_0^t \beta(t-s) a(\mathbf{u}_h(s), \boldsymbol{\phi}_h)\, ds \\ &= -b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) + (\mathbf{f}, \boldsymbol{\phi}_h) \ \forall \boldsymbol{\phi}_h \in \mathbf{J}_h.\end{aligned} \tag{4.2}$$

Since $\mathbf{J}_h$ is finite dimensional, the problem (4.2) leads to a system of nonlinear integro-differential equations. For global existence of a solution pair of (4.2), we refer to [28]. Uniqueness is obtained on the quotient space L_h/N_h, where

$$N_h = \{q_h \in L_h : (q_h, \nabla \cdot \boldsymbol{\phi}_h) = 0, \forall \boldsymbol{\phi}_h \in \mathbf{H}_h\}.$$

The norm on L_h/N_h is given by

$$\|q_h\|_{L^2/N_h} = \inf_{\chi_h \in N_h} \|q_h + \chi_h\|.$$

For continuous dependence of the discrete pressure $p_h(t) \in L_h/N_h$ on the discrete velocity $u_h(t) \in \mathbf{J}_h$, we assume the following discrete inf-sup (LBB) condition for the finite dimensional spaces $\mathbf{H}_h$ and L_h:
(**B2′**) For every $q_h \in L_h$, there exists a non-trivial function $\boldsymbol{\phi}_h \in \mathbf{H}_h$ and a positive constant K_0, independent of h, such that

$$|(q_h, \nabla \cdot \boldsymbol{\phi}_h)| \geq K_0 \|\nabla \boldsymbol{\phi}_h\| \|q_h\|_{L^2/N_h}.$$

Moreover, we also assume that the following approximation property holds true for $\mathbf{J}_h$.
(**B2**) For every $\mathbf{w} \in \mathbf{J}_1 \cap \mathbf{H}^2$, there exists an approximation $r_h\mathbf{w} \in \mathbf{J_h}$ such that

$$\|\mathbf{w} - r_h\mathbf{w}\| + h\|\nabla(\mathbf{w} - r_h\mathbf{w})\| \leq K_5 h^2 \|\mathbf{w}\|_2.$$

This is a less restrictive condition than (**B2′**) and it has been used to derive the following properties of the L^2 projection $P_h : \mathbf{L}^2 \mapsto \mathbf{J}_h$. We now state, without proof, these results. For a proof, see [12]. For $\boldsymbol{\phi} \in \mathbf{J}_h$, note that

$$\|\boldsymbol{\phi} - P_h\boldsymbol{\phi}\| + h\|\nabla P_h\boldsymbol{\phi}\| \leq Ch\|\nabla\boldsymbol{\phi}\|, \tag{4.3}$$

and for $\boldsymbol{\phi} \in \mathbf{J}_1 \cap \mathbf{H}^2$,

$$\|\boldsymbol{\phi} - P_h\boldsymbol{\phi}\| + h\|\nabla(\boldsymbol{\phi} - P_h\boldsymbol{\phi})\| \leq Ch^2\|\tilde{\Delta}\boldsymbol{\phi}\|. \tag{4.4}$$

We now define the discrete operator $\Delta_h : \mathbf{H}_h \mapsto \mathbf{H}_h$ through the bilinear form $a(\cdot,\cdot)$ as

$$a(\mathbf{v}_h, \boldsymbol{\phi}_h) = (-\Delta_h\mathbf{v}_h, \boldsymbol{\phi}) \ \forall \mathbf{v}_h, \boldsymbol{\phi}_h \in \mathbf{H}_h. \tag{4.5}$$

Set the discrete analogue of the Stokes operator $\tilde{\Delta} = P\Delta$ as $\tilde{\Delta}_h = P_h\Delta_h$. The restriction of $\tilde{\Delta}_h$ to $\mathbf{J}_h$ is invertible and its inverse is denoted as $\tilde{\Delta}_h^{-1}$. We recall the 'discrete' Sobolev norms on $\mathbf{J}_h$ (see [13]): For $r \in \mathbb{R}$, set

$$\|\mathbf{v}_h\|_r := \|(-\tilde{\Delta}_h)^{r/2}\mathbf{v}_h\|, \ \mathbf{v}_h \in \mathbf{J}_h.$$

We note that $\|\mathbf{v}_h\|_0 = \|\mathbf{v}_h\|$ and $\|\mathbf{v}_h\|_1 = \|\nabla \mathbf{v}_h\|$ for $\mathbf{v}_h \in \mathbf{J}_h$. The norms $\|\tilde{\Delta}_h(\cdot)\|$ and $\|\cdot\|_2$ are equivalent in $\mathbf{J}_h$, with constants independent of h.
Using Sobolev embedding and Sobolev inequality, it is easy to prove the following properties for the nonlinear term:

Lemma 4.1. *Suppose conditions (***A1***),(***B1***) and (***B2***) are satisfied. Then there exists a positive constant C such that for $\mathbf{v} \in \mathbf{J}_1$ and $\boldsymbol{\phi}, \boldsymbol{\xi} \in \mathbf{J}_h$*

$$|b(\mathbf{v}, \boldsymbol{\phi}, \boldsymbol{\xi})| \leq C\|\mathbf{v}\|^{1/2}\|\nabla\mathbf{v}\|^{1/2}\|\boldsymbol{\phi}\|\|\nabla\boldsymbol{\xi}\|^{1/2}\|\tilde{\Delta}_h\boldsymbol{\xi}\|^{1/2} \tag{4.6}$$

$$|b(\mathbf{v}, \boldsymbol{\phi}, \boldsymbol{\xi})| \leq C\|\nabla\mathbf{v}\|^{1/2}\|\tilde{\Delta}\mathbf{v}\|^{1/2}\|\boldsymbol{\phi}\|\|\nabla\boldsymbol{\xi}\| \tag{4.7}$$

and for $\boldsymbol{\phi}_h, \ \boldsymbol{\xi}, \ \boldsymbol{\chi} \in \mathbf{J}_h$

$$|b(\boldsymbol{\phi}_h, \boldsymbol{\xi}, \boldsymbol{\chi})| \leq C\|\boldsymbol{\phi}_h\|\|\nabla\boldsymbol{\xi}\|^{1/2}\|\tilde{\Delta}_h\boldsymbol{\xi}_h\|^{1/2}(\|\boldsymbol{\chi}\|^{1/2}\|\nabla\boldsymbol{\chi}\|^{1/2} + \|\nabla\boldsymbol{\chi}\|) \tag{4.8}$$

$$|b(\boldsymbol{\phi}_h, \boldsymbol{\xi}, \boldsymbol{\chi})| \leq C\|\boldsymbol{\phi}_h\|(\|\nabla\boldsymbol{\xi}\| + \|\boldsymbol{\xi}\|^{1/2}\|\nabla\boldsymbol{\xi}\|^{1/2})\|\nabla\boldsymbol{\chi}\|^{1/2}\|\tilde{\Delta}_h\boldsymbol{\chi}\|^{1/2} \tag{4.9}$$

Examples of subspaces $\mathbf{H}_h$ and L_h satisfying assumptions (**B1**), (**B2′**), and (**B2**) can be found in [9, 4, 3]. For nonconforming finite elements, we would like to refer to [12]. The error estimate, presented in this paper, would also go through for nonconforming finite elements with appropriate incorporation of the boundary terms. These terms along with their estimates can be found in [12] in the context of Navier-Stokes equations.
Before proceeding to the next section, we state without proof some estimates of $\mathbf{u}_h$. The proof proceeds along the lines of proof of Lemmas 3.1 − 3.4 and Theorem 3.1 using the definition of discrete Stokes operator (see (4.5)).

Lemma 4.2. *The semi-discrete Galerkin approximation $\mathbf{u}_h$ of the velocity $\mathbf{u}$ satisfies, for $t > 0$,*

$$\|\mathbf{u}_h(t)\| + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\nabla\mathbf{u}_h(t)\|^2 \, ds \leq K, \tag{4.10}$$

$$\|\nabla\mathbf{u}_h(t)\| + e^{-2\alpha t}\int_0^t e^{2\alpha s}\|\tilde{\Delta}_h\mathbf{u}_h(t)\|^2 \, ds \leq K, \tag{4.11}$$

$$(\tau^*(t))^{1/2}\|\tilde{\Delta}_h\mathbf{u}_h(t)\| \leq K, \tag{4.12}$$

where K depends only on the given data. In particular, K is independent of h.

5. *A* priori Error Estimates for the Velocity

Since $\mathbf{J}_h$ is not a subspace of $\mathbf{J}_1$, the weak solution $\mathbf{u}$ satisfies

$$\begin{aligned}(\mathbf{u}_t, \boldsymbol{\phi}_h) + \mu a(\mathbf{u}, \boldsymbol{\phi}_h) \ &+ \ \int_0^t \beta(t-s)a(\mathbf{u}(s), \boldsymbol{\phi}_h)\, ds = -b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h) \\ &+ \ (\mathbf{f}, \boldsymbol{\phi}_h) + (p, \nabla\cdot\boldsymbol{\phi}_h) \ \forall \boldsymbol{\phi}_h \in \mathbf{J}_h.\end{aligned} \tag{5.1}$$

In this Section, we discuss optimal error estimates for the error $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$. By introducing an intermediate solution $\mathbf{v}_h$ which is a finite element Galerkin approximation to a linearized Oldroyd equation, that is, $\mathbf{v}_h$ satisfies

$$
\begin{aligned}
(5.2)\qquad (\mathbf{v}_{ht}, \boldsymbol{\phi}_h) + \mu a(\mathbf{v}_h, \boldsymbol{\phi}_h) &+ \int_0^t \beta(t-s) a(\mathbf{v}_h(s), \boldsymbol{\phi}_h)\, ds \\
&= (\mathbf{f}, \boldsymbol{\phi}_h) - b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h) \;\; \forall \boldsymbol{\phi}_h \in \mathbf{J}_h,
\end{aligned}
$$

we split $\mathbf{e}$ as

$$\mathbf{e} := \mathbf{u} - \mathbf{u}_h = (\mathbf{u} - \mathbf{v}_h) + (\mathbf{v}_h - \mathbf{u}_h) = \boldsymbol{\xi} + \boldsymbol{\eta}.$$

Note that $\boldsymbol{\xi}$ is the error committed by approximating a linearized Oldroyd equation and $\boldsymbol{\eta}$ represents the error due to the presence of non-linearity in the equation. Below, we derive some estimates of $\boldsymbol{\xi}$. Subtracting (5.2) from (5.1), the equation in $\boldsymbol{\xi}$ is written as

$$(5.3)\quad (\boldsymbol{\xi}_t, \boldsymbol{\phi}_h) + \mu a(\boldsymbol{\xi}, \boldsymbol{\phi}_h) + \int_0^t \beta(t-s) a(\boldsymbol{\xi}(s), \boldsymbol{\phi}_h)\, ds = (p, \nabla\cdot \boldsymbol{\phi}_h), \quad \boldsymbol{\phi}_h \in \mathbf{J}_h.$$

Lemma 5.1. *Let $\mathbf{v}_h(t) \in \mathbf{J}_h$ be a solution of (5.2) with initial condition $\mathbf{v}_h(0) = P_h\mathbf{u}_0$ and $\mathbf{u}$ be a weak solution of (1.1) with initial condition $\mathbf{u}_0 \in \mathbf{J}_1$. Then, $\boldsymbol{\xi}$ satisfies*

$$\int_0^t e^{2\alpha\tau} \|\boldsymbol{\xi}(\tau)\|^2\, d\tau \le C e^{2\alpha t} h^4, \;\; t > 0.$$

For a proof, see [28]. Note that the estimate involving the pressure term can easily be obtained using the equation (1.1), (3.23) and Lemma 3.2 in Section 3. For optimal error estimates of $\boldsymbol{\xi}$ in $L^\infty(L^2)$ and $L^\infty(H^1)$-norms, we recall the Stokes-Volterra projection $V_h\mathbf{u} : [0,\infty) \to \mathbf{J}_h$, which is introduced in [28], satisfying,

$$(5.4)\; \mu a(\mathbf{u} - V_h\mathbf{u}, \boldsymbol{\phi}_h) + \int_0^t \beta(t-s) a(\mathbf{u}(s) - V_h\mathbf{u}(s), \boldsymbol{\phi}_h)\, ds = (p, \nabla\cdot\boldsymbol{\phi}_h),$$

for $\boldsymbol{\phi}_h \in \mathbf{J}_h$. Now, we decompose the error $\boldsymbol{\xi}$ as follows:

$$\boldsymbol{\xi} = (\mathbf{u} - V_h\mathbf{u}) + (V_h\mathbf{u} - \mathbf{v}_h) = \boldsymbol{\zeta} + \boldsymbol{\theta}.$$

First of all, we derive optimal error bounds for the error $\boldsymbol{\zeta} = \mathbf{u} - V_h\mathbf{u}$.

Lemma 5.2. *Assume that the conditions (**A1**), (**B1**) and (**B2**) are satisfied. Suppose $\mathbf{u}$ is a weak solution of (1.1) with initial condition $\mathbf{u}_0 \in \mathbf{J}_1$. Then there is a positive constant C such that*

$$\|(\mathbf{u} - V_h\mathbf{u})(t)\|^2 + h^2\|\nabla(\mathbf{u} - V_h\mathbf{u})(t)\|^2 \le Ch^4\Big(\mathcal{K}^2(t) + e^{-2\alpha t}\int_0^t e^{2\alpha s}\mathcal{K}^2(s)\, ds\Big)$$

where

$$\mathcal{K}(t) := \|\tilde{\Delta}\mathbf{u}(t)\| + \|\nabla p(t)\|.$$

Moreover, the following estimate holds:

$$\|(\mathbf{u} - V_h\mathbf{u})_t(t)\|^2 + h^2\|\nabla(\mathbf{u} - V_h\mathbf{u})_t(t)\|^2 \le Ch^4\Big(\mathcal{K}^2(t) + \mathcal{K}_t^2(t) + e^{-2\alpha t}\int_0^t e^{2\alpha s}\mathcal{K}^2(s)\, ds\Big),$$

where

$$\mathcal{K}_t(t) := \|\tilde{\Delta}\mathbf{u}_t(t)\| + \|\nabla p_t(t)\|.$$

Proof. We again refer to [28] for a proof of the first estimate involving $\boldsymbol{\zeta} = \mathbf{u} - V_h\mathbf{u}$. For estimate involving $\boldsymbol{\zeta}_t$, we differentiate (5.4) with respect to the temporal variable t to find that, for $\boldsymbol{\phi}_h \in J_h$,

$$\mu a(\boldsymbol{\zeta}_t, \boldsymbol{\phi}_h) + \beta(0) a(\boldsymbol{\zeta}(t), \boldsymbol{\phi}_h) + \int_0^t \beta_t(t-s) a(\boldsymbol{\zeta}(s), \boldsymbol{\phi}_h)\, ds = (p_t, \nabla \cdot \boldsymbol{\phi}_h). \tag{5.5}$$

Choose $\boldsymbol{\phi}_h = e^{2\alpha t} P_h \boldsymbol{\zeta}_t$ in (5.5), use discrete incompressibility condition, $\mathbf{H}_0^1$-stability of P_h for the term on the right-hand side and approximation properties, and finally observe that

$$\boldsymbol{\phi}_h = e^{2\alpha t}\boldsymbol{\zeta}_t - e^{2\alpha t}(\boldsymbol{\zeta}_t - P_h\boldsymbol{\zeta}_t) \text{ and } \boldsymbol{\zeta}_t - P_h\boldsymbol{\zeta}_t = \mathbf{u}_t - P_h\mathbf{u}_t.$$

Then, we obtain

$$\begin{aligned}
\mu\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\|^2 &= \mu a(e^{\alpha t}\boldsymbol{\zeta}_t, e^{\alpha t}(\mathbf{u}_t - P_h\mathbf{u}_t)) - \frac{1}{\gamma} a(\hat{\boldsymbol{\zeta}}, P_h(e^{\alpha t}\boldsymbol{\zeta}_t)) \\
&\quad + \delta \int_0^t \beta(t-s) e^{\alpha(t-s)} a(\hat{\boldsymbol{\zeta}}(s), P_h(e^{\alpha t}\boldsymbol{\zeta}_t))\, ds \\
&\quad + (e^{\alpha t} p_t - j_h(e^{\alpha t} p_t), \nabla \cdot P_h(e^{\alpha t}\boldsymbol{\zeta}_t)) \\
&\le C\|e^{\alpha t}\nabla\boldsymbol{\zeta}_t\| \Big[h\|e^{\alpha t}\Delta\hat{\mathbf{u}}_t\| + \|\nabla\hat{\boldsymbol{\zeta}}\| + \int_0^t \|\nabla\hat{\boldsymbol{\zeta}}(s)\|\, ds + h\|\nabla p_t\| \Big].
\end{aligned}$$

Now, with the help of estimate of $\|\nabla\zeta(t)\|$ we complete the proof of the estimate $\|\nabla\boldsymbol{\zeta}_t\|$.

Finally, for the estimation of $\boldsymbol{\zeta}_t$ in $\mathbf{L}^2$-norm, we appeal to Aubin-Nitsche duality argument. For fixed h, let $\{\mathbf{w}, q\}$ be a pair of unique solution of the following steady state Stokes system

$$\begin{aligned}
-\mu\Delta\mathbf{w} + \nabla q &= e^{\alpha t}\boldsymbol{\zeta}_t \text{ in } \Omega \\
\nabla \cdot \mathbf{w} &= 0 \text{ in } \Omega \\
\mathbf{w}|_{\partial\Omega} &= 0.
\end{aligned} \tag{5.6}$$

From assumption (**A1**), the following regularity result for the problem (5.6)

$$\|\mathbf{w}\|_2 + \|q\|_{H^1/R} \le C\|e^{\alpha t}\boldsymbol{\zeta}_t\|, \tag{5.7}$$

holds. Form $\mathbf{L}^2$-inner-product with $e^{\alpha t}\boldsymbol{\zeta}_t$ to obtain

$$\|e^{\alpha t}\boldsymbol{\zeta}_t\|^2 = \mu a(e^{\alpha t}\boldsymbol{\zeta}_t, \mathbf{w} - P_h\mathbf{w}) - (e^{\alpha t}\nabla \cdot \boldsymbol{\zeta}_t, q) + \mu a(e^{\alpha t}\boldsymbol{\zeta}_t, P_h\mathbf{w}).$$

From (5.5) with $\boldsymbol{\phi}_h = P_h\mathbf{w}$ and a use of discrete incompressibility condition now leads to

$$\begin{aligned}
\|e^{\alpha t}\boldsymbol{\zeta}_t\|^2 &= \mu a(e^{\alpha t}\boldsymbol{\zeta}_t, \mathbf{w} - P_h\mathbf{w}) - (e^{\alpha t}(\mathbf{u}_t - P_h\mathbf{u}_t), \nabla q) - (\nabla \cdot P_h(e^{\alpha t}\boldsymbol{\zeta}_t), q - j_h q) \\
&\quad + (e^{\alpha t}(p_t - j_h p_t), \nabla \cdot (P_h\mathbf{w} - \mathbf{w})) - \beta(0) a(\hat{\boldsymbol{\zeta}}, P_h\mathbf{w} - \mathbf{w}) + \beta(0)(\hat{\boldsymbol{\zeta}}, -\Delta\mathbf{w}) \\
&\quad + \int_0^t \beta(t-s) e^{\alpha(t-s)} a(\hat{\boldsymbol{\zeta}}(s), P_h\mathbf{w} - \mathbf{w})\, ds - \int_0^t \beta(t-s) e^{\alpha(t-s)} (\hat{\boldsymbol{\zeta}}(s), -\Delta\mathbf{w})\, ds.
\end{aligned}$$

Using Cauchy-Schwarz inequality, properties of P_h and regularity results, we note that

$$\begin{aligned}
\|\boldsymbol{\zeta}_t\| &\le C(\mu, \gamma, \delta) \Big[h^2 \Big(\|\tilde{\Delta}\mathbf{u}_t\| + \|\nabla p_t\| \Big) + h(\|\nabla\boldsymbol{\zeta}_t\| + \|\nabla\boldsymbol{\zeta}\| \\
&\quad + e^{-\alpha t} \left(\int_0^t \|\nabla\hat{\boldsymbol{\zeta}}(s)\|^2\, ds \right)^{1/2}) \Big] + C \left(\|\boldsymbol{\zeta}\| + e^{-\alpha t} \left(\int_0^t \|\hat{\boldsymbol{\zeta}}(s)\|^2\, ds \right)^{1/2} \right).
\end{aligned}$$

On substituting various known estimates of $\boldsymbol{\zeta}$, we obtain the required result for $\boldsymbol{\zeta}_t$ in $\mathbf{L}^2$-norm and this completes the rest of the proof. $\square$

Remark 5.1. *Lemmas* 5.1 *and* 5.2 *are still valid for* $\mathbf{u}_0 \in \mathbf{J}_1 \cap \mathbf{H}^2$.

Now we are in a position to estimate $\boldsymbol{\xi}$ in $L^\infty(\mathbf{L}^2)$ and $L^\infty(\mathbf{H}_0^1)$-norms. Since $\boldsymbol{\xi} = \boldsymbol{\zeta} + \boldsymbol{\theta}$ and estimates of $\boldsymbol{\zeta}$ are known from the previous Lemma, it is sufficient to estimate $\boldsymbol{\theta}$. From (5.3) and (5.4), the equation in $\boldsymbol{\theta}$ becomes

$$(5.8)\; (\boldsymbol{\theta}_t, \boldsymbol{\phi}_h) + \mu a(\boldsymbol{\theta}, \boldsymbol{\phi}_h) + \int_0^t \beta(t-s) a(\boldsymbol{\theta}(s), \boldsymbol{\phi}_h)\, ds = -(\boldsymbol{\zeta}_t, \boldsymbol{\phi}_h) \;\forall \boldsymbol{\phi}_h \in \mathbf{J}_h.$$

Note that for estimation of $\|\boldsymbol{\zeta}_t\|$ in (5.8), it is essential to introduce $\sigma(t)$ term, as in [28], so that we can avoid nonlocal compatibility conditions. However, a direct use of $\sigma(t)$ as in Heywood and Rannacher [12] forced us to apply Gronwall's Lemma. This is mainly due to the presence of the integral term in (5.8). Therefore, as in Pani and Sinha [27], we first introduce

$$\tilde{\boldsymbol{\theta}}(t) = \int_0^t \boldsymbol{\theta}(s)\, ds,$$

and shall derive an improved estimate for

$$\int_0^t e^{\alpha t} \|\nabla \tilde{\boldsymbol{\theta}}(s)\|^2\, ds.$$

This, in turn, helps us to introduce $\sigma(t)$ and then we derive estimates of $\boldsymbol{\theta}$ without using Gronwall's Lemma.

Lemma 5.3. *There is a positive constant* K *such that for* $\mathbf{u}_0 \in \mathbf{J}_1$, $\boldsymbol{\xi}$ *satisfies, for* $t > 0$, *the following estimate*

$$\|\boldsymbol{\xi}(t)\| + h\|\nabla \boldsymbol{\xi}(s)\| \;\le\; Ch^2 t^{-1/2}.$$

Proof. Following the proof of Lemma 5.3 in [28] (page 772), we obtain

$$\begin{aligned} e^{2\alpha t}\|\tilde{\boldsymbol{\theta}}(t)\|^2 + (\mu - \frac{\alpha}{\lambda_1}) \int_0^t e^{2\alpha s} \|\nabla \tilde{\boldsymbol{\theta}}(s)\|^2\, ds \;&\le\; \frac{1}{(\mu\lambda_1 - \alpha)} \int_0^t \|\hat{\boldsymbol{\zeta}}(s)\|^2\, ds \\ &\le\; Ch^4 \int_0^t e^{2\alpha s} \mathcal{K}^2(s)\, ds. \end{aligned} \tag{5.9}$$

Now, choosing $\boldsymbol{\phi}_h = \sigma(t)\boldsymbol{\theta}$ in (5.8), it now follows that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\sigma(t)\|\boldsymbol{\theta}\|^2\right) + \mu\sigma(t)\|\nabla\boldsymbol{\theta}\|^2 \;&=\; -\sigma(t)(\boldsymbol{\zeta}_t, \boldsymbol{\theta}) + \frac{1}{2}\sigma_t(t)\|\boldsymbol{\theta}\|^2 \\ &\quad -\sigma(t)\int_0^t \beta(t-\tau) a(\boldsymbol{\theta}(\tau), \boldsymbol{\theta})\, d\tau \\ &\le\; \frac{1}{2}\sigma_1(t)\|\boldsymbol{\zeta}_t\|^2 + \frac{1}{2}(e^{2\alpha t} + \sigma_t(t))\|\boldsymbol{\theta}(t)\|^2 + I_1. \end{aligned} \tag{5.10}$$

For I_1, we follow the arguments of [28] to obtain

$$\int_0^t |I_1(s)|\, ds \;\le\; C(\mu,\delta,\gamma) \int_0^t e^{2\alpha s} \|\nabla \tilde{\boldsymbol{\theta}}(s)\|^2\, ds + \frac{\mu}{2}\int_0^t \sigma(s)\|\nabla\boldsymbol{\theta}(s)\|^2\, ds. \tag{5.11}$$

On integrating (5.10) with respect to time and substituting the above estimate in the resulting equation, it follows that

$$\begin{aligned}
\sigma(t)\|\boldsymbol{\theta}(t)\|^2 \quad &+ \quad \mu\int_0^t \sigma(s)\|\nabla\boldsymbol{\theta}(s)\|^2\,ds \;\le\; C\int_0^t \sigma_1(s)\|\boldsymbol{\zeta}_t\|^2\,ds \\
&+ \quad C\int_0^t e^{2\alpha s}(\|\boldsymbol{\theta}(s)\|^2 + \|\nabla\tilde{\boldsymbol{\theta}}(s)\|^2)\,ds \\
&\le \quad C\int_0^t (\sigma_1(s)\|\boldsymbol{\zeta}_t\|^2 + e^{2\alpha s}(\|\boldsymbol{\xi}\|^2 + \|\boldsymbol{\zeta}\|^2) + e^{2\alpha s}\|\nabla\tilde{\boldsymbol{\theta}}(s)\|^2)\,ds \\
&\le \quad Ch^4\Big\{e^{2\alpha t} + \int_0^t (e^{2\alpha s}\mathcal{K}^2(s) + \sigma_1(s)\mathcal{K}_s^2(s))\,ds\Big\},
\end{aligned}$$

where we have used Lemmas 5.1-5.2 and the estimate (5.9). Now, Theorem 3.2 yields

$$\|\boldsymbol{\theta}(t)\|^2 + \sigma^{-1}(t)\int_0^t \sigma(s)\|\nabla\boldsymbol{\theta}(s)\|^2\,ds \;\le\; Ch^4t^{-1}.$$

An use of triangle inequality and inverse hypothesis finally completes the rest of the proof. □

We require a couple of Lemmas before we can establish our main result.

Lemma 5.4. *Suppose the assumptions (***A1***)-(***A2***) and, (***B1***) and (***B2***) hold. Let* $\mathbf{u}_h(t)$ *be a solution of (4.2) with initial condition* $\mathbf{u}_{0h} = P_h\mathbf{u}_0$*, where* $\mathbf{u}_0 \in \mathbf{J}_1$*. Then, there exists a positive constant* C *such that for* $0 < T < \infty$ *with* $t \in (0,T]$

$$\int_0^T e^{2\alpha s}\|\mathbf{e}(s)\|^2\;ds \le Ce^{2\alpha t}h^4.$$

Proof. In view of the Lemma 5.1, we only need to prove the estimate for $\boldsymbol{\eta}$. From (4.2) and (5.2), the equation in $\boldsymbol{\eta}$ becomes

$$\begin{aligned}
(\boldsymbol{\eta}_t, \boldsymbol{\phi}_h) + \mu a(\boldsymbol{\eta}, \boldsymbol{\phi}_h) \quad &+ \quad \int_0^t \beta(t-s)a(\boldsymbol{\eta}(s), \boldsymbol{\phi}_h)\,ds \\
&= \quad b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) - b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h),\;\; \boldsymbol{\phi}_h \in \mathbf{J}_h.
\end{aligned} \tag{5.12}$$

Choose $\boldsymbol{\phi}_h = e^{2\alpha t}(\tilde{\Delta}_h^{-1}\boldsymbol{\eta})$ to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\hat{\boldsymbol{\eta}}\|_{-1}^2 - \alpha\|\hat{\boldsymbol{\eta}}\|_{-1}^2 + \mu\|\hat{\boldsymbol{\eta}}\|^2 \quad &+ \quad \int_0^t \beta(t-s)e^{\alpha(t-s)}a(\hat{\boldsymbol{\eta}}(s), \hat{\boldsymbol{\eta}})\;ds \\
&= \quad e^{\alpha t}\Lambda_h(\tilde{\Delta}_h^{-1}\hat{\boldsymbol{\eta}}),
\end{aligned} \tag{5.13}$$

where

$$\begin{aligned}
\Lambda_h(\boldsymbol{\phi}_h) =& b(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\phi}_h) - b(\mathbf{u}, \mathbf{u}, \boldsymbol{\phi}_h) \\
=& -b(\mathbf{e}, \mathbf{u}_h, \boldsymbol{\phi}_h) - b(\mathbf{u}, \mathbf{e}, \boldsymbol{\phi}_h).
\end{aligned} \tag{5.14}$$

Using $\mathbf{e} = \boldsymbol{\xi} + \boldsymbol{\eta}$ in (5.14) along with (4.6) and (4.9), we find that

$$\begin{aligned}
|e^{\alpha t}\Lambda_h(\tilde{\Delta}_h^{-1}\hat{\boldsymbol{\eta}})| \quad &\le \quad C\Big(\|\nabla\mathbf{u}_h\| + +\|\mathbf{u}_h\|^{1/2}\|\nabla\mathbf{u}_h\|^{1/2} + \|\mathbf{u}\|^{1/2}\|\nabla\mathbf{u}\|^{1/2}\Big) \\
&\qquad \times\Big(\|\hat{\boldsymbol{\eta}}\|_{-1}^{1/2}\|\hat{\boldsymbol{\eta}}\|^{3/2} + \|\hat{\boldsymbol{\eta}}\|\|\hat{\boldsymbol{\xi}}\|\Big) \\
&\le \quad \varepsilon\|\hat{\boldsymbol{\eta}}\|^2 + C(\varepsilon)\Big(\|\nabla\mathbf{u}_h\|^2 + \|\mathbf{u}_h\|\|\nabla\mathbf{u}_h\| + \|\mathbf{u}\|\|\nabla\mathbf{u}\|\Big)\|\hat{\boldsymbol{\xi}}\|^2 \\
&+ \quad C(\varepsilon)\|\hat{\boldsymbol{\eta}}\|_{-1}^2\Big(\|\nabla\mathbf{u}_h\|^4 + \|\mathbf{u}_h\|^2\|\nabla\mathbf{u}_h\|^2 + \|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2\Big)
\end{aligned} \tag{5.15}$$

Put $\varepsilon = \mu/2$ in (5.15), substitute it in (5.13) and use Lemmas of Section 3 and 4 to estimate $\mathbf{u}$ and $\mathbf{u}_h$ to yield

$$\begin{aligned}(5.16)\ \frac{d}{dt}\|\hat{\boldsymbol{\eta}}\|_{-1}^2 + \mu\|\hat{\boldsymbol{\eta}}\|^2 \quad &+ \quad \int_0^t \beta(t-s)e^{\alpha(t-s)}a(\hat{\boldsymbol{\eta}}(s),\hat{\boldsymbol{\eta}})\ ds \le C(K,\mu)\|\hat{\boldsymbol{\xi}}\|^2 \\ &+ \quad (C(K,\mu)+2\alpha)\|\hat{\boldsymbol{\eta}}\|_{-1}^2.\end{aligned}$$

Integrate (5.16). Observe that the double integral is positive and $\boldsymbol{\eta}(0) = 0$.

$$\|\hat{\boldsymbol{\eta}}\|_{-1}^2 + \mu\int_0^t \|\hat{\boldsymbol{\eta}}\|^2 \le C(K,\mu)\int_0^t \|\hat{\boldsymbol{\xi}}\|^2 ds + (C(K,\mu)+2\alpha)\int_0^t \|\hat{\boldsymbol{\eta}}\|_{-1}^2 ds.$$

Apply Gronwall's Lemma, use Lemma 5.1 and now, a use of triangular inequality completes the rest of the proof. □

Lemma 5.5. *Suppose the assumptions (**A1**)-(**A2**) and, (**B1**) and (**B2**) hold. Let $\mathbf{u}_h(t)$ and $\mathbf{v}_h(t)$, both in $\mathbf{J}_h$, be solutions of (4.2) and (5.2), respectively, with initial conditions $\mathbf{u}_h(0) = \mathbf{v_h}(0) = P_h\mathbf{u}_0$. Then, $\boldsymbol{\eta} = \mathbf{v_h} - \mathbf{u}_h$ satisfies*

$$\int_0^t e^{2\alpha s}\|\nabla\tilde{\boldsymbol{\eta}}(s)\|^2 ds \le C(t)e^{Ct}h^4,$$

where $\tilde{\boldsymbol{\eta}}(t) = \int_0^t \boldsymbol{\eta}(s)\ ds$.

Proof. We integrate (5.12) to arrive at the following

$$(5.17)\quad (\boldsymbol{\eta},\boldsymbol{\phi}_h) + \mu a(\tilde{\boldsymbol{\eta}},\boldsymbol{\phi}_h) + \int_0^t\int_0^s \beta(s-\tau)a(\boldsymbol{\eta}(\tau),\boldsymbol{\phi}_h)\ d\tau\ ds = \int_0^t \Lambda_h(\boldsymbol{\phi}_h)\ ds.$$

As in the proof of Lemma 5.3 in [28] (page 772), we find

$$(5.18)\qquad \int_0^t\int_0^s \beta(s-\tau)a(\boldsymbol{\eta}(\tau),\boldsymbol{\phi}_h)\ ds = \int_0^t \beta(t-s)a(\tilde{\boldsymbol{\eta}}(s),\boldsymbol{\phi}_h)\ ds.$$

Combining (5.17)-(5.18) yields

$$(5.19)\qquad (\boldsymbol{\eta},\boldsymbol{\phi}_h) + \mu a(\tilde{\boldsymbol{\eta}},\boldsymbol{\phi}_h) + \int_0^t \beta(t-s)a(\tilde{\boldsymbol{\eta}}(s),\boldsymbol{\phi}_h)\ ds = \int_0^t \Lambda_h(\boldsymbol{\phi}_h)\ ds.$$

Put $\boldsymbol{\phi}_h = e^{2\alpha t}\tilde{\boldsymbol{\eta}}$ in (5.19) to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(e^{2\alpha t}\|\tilde{\boldsymbol{\eta}}\|^2) - \alpha e^{2\alpha t}\|\tilde{\boldsymbol{\eta}}\|^2 + \mu e^{2\alpha t}\|\nabla\tilde{\boldsymbol{\eta}}\|^2 \\ (5.20)\qquad &+ \int_0^t \beta(t-s)e^{\alpha(t-s)}a(e^{\alpha s}\tilde{\boldsymbol{\eta}}(s), e^{\alpha t}\tilde{\boldsymbol{\eta}})\ ds = e^{2\alpha t}\int_0^t \Lambda_h(\tilde{\boldsymbol{\eta}})\ ds\end{aligned}$$

Integrate (5.20) and use the positivity of the double integral term to drop it. This results in the following

$$(5.21)\ \ e^{2\alpha t}\|\tilde{\boldsymbol{\eta}}\|^2 + 2(\mu - \frac{\alpha}{\lambda_1})\int_0^t e^{2\alpha s}\|\nabla\tilde{\boldsymbol{\eta}}(s)\|^2\ ds \le 2\int_0^t e^{2\alpha s}|\int_0^s \Lambda_h(\tilde{\boldsymbol{\eta}}(s))\ d\tau|\ ds.$$

It remains to estimate the nonlinear term in (5.21). Similar to (5.14), we find that

$$(5.22)\qquad \Lambda_h(\tilde{\boldsymbol{\eta}}) = -b(\mathbf{e},\mathbf{u}_h,\tilde{\boldsymbol{\eta}}) - b((\mathbf{u},\mathbf{e},\tilde{\boldsymbol{\eta}}).$$

Use (4.7) and (4.8) to estimate the nonlinear terms in (5.22):

$$\begin{aligned}|\Lambda_h(\tilde{\boldsymbol{\eta}})| \quad &\le \quad C\|\mathbf{e}\|\|\nabla\mathbf{u}_h\|^{1/2}\|\tilde{\Delta}_h\mathbf{u}_h\|^{1/2}\|\tilde{\boldsymbol{\eta}}\|^{1/2}\|\nabla\tilde{\boldsymbol{\eta}}\|^{1/2} \\ &+ \quad C\|\mathbf{e}\|\|\nabla\tilde{\boldsymbol{\eta}}\|\{\|\nabla\mathbf{u}_h\|^{1/2}\|\tilde{\Delta}_h\mathbf{u}_h\|^{1/2} + \|\nabla\mathbf{u}\|^{1/2}\|\tilde{\Delta}\mathbf{u}\|^{1/2}\}.\end{aligned}$$

Hence, we find that

$$\int_0^t |\Lambda_h(\tilde{\boldsymbol\eta})|\, ds \tag{5.23}$$
$$\leq C\Big(\int_0^t \|\mathbf{e}(s)\|\{\|\nabla\mathbf{u}_h(s)\|^{1/2}\|\tilde\Delta_h\mathbf{u}_h(s)\|^{1/2}+\|\nabla\mathbf{u}(s)\|^{1/2}\|\tilde\Delta\mathbf{u}(s)\|^{1/2}\}\, ds\Big)\|\nabla\tilde{\boldsymbol\eta}\|$$
$$\leq C\Big(\int_0^t \|\mathbf{e}(s)\|^2 ds\Big)^{1/2}\Big(\int_0^t (\|\nabla\mathbf{u}_h(s)\|\|\tilde\Delta_h\mathbf{u}_h(s)\|+\|\nabla\mathbf{u}(s)\|\|\tilde\Delta\mathbf{u}(s)\|)\, ds\Big)^{1/2}\|\nabla\tilde{\boldsymbol\eta}\|.$$

Use the uniform bound for the Dirichlet norm and the fact that $e^{\alpha t}\geq 1$ for α positive and for $t\geq 0$, to rewrite (5.23) as:

$$\int_0^t |\Lambda_h(\tilde{\boldsymbol\eta})|\, ds \;\leq\; C\Big(\int_0^t \|\hat{\mathbf{e}}(s)\|^2 ds\Big)^{1/2} \times \Big(\int_0^t (\|\tilde\Delta_h\mathbf{u}_h(s)\|+\|\tilde\Delta\mathbf{u}(s)\|)\, ds\Big)^{1/2}\|\nabla\tilde{\boldsymbol\eta}\|. \tag{5.24}$$

Use Lemmas 3.4 and 5.4 in (5.24) to arrive at

$$\int_0^t |\Lambda_h(\tilde{\boldsymbol\eta})|\, ds \leq Ch^2t^{1/4}e^{\alpha t}\|\nabla\tilde{\boldsymbol\eta}\|. \tag{5.25}$$

And hence, from (5.25), we find

$$2\int_0^t e^{2\alpha s}\int_0^s |\Lambda_h(\tilde{\boldsymbol\eta}(s))|\, d\tau\, ds \;\leq\; C(\mu,\lambda_1)h^4t^{1/2}e^{4\alpha t} + (\mu-\frac{\alpha}{\lambda_1})\int_0^t e^{2\alpha s}\|\nabla\tilde{\boldsymbol\eta}(s)\|^2\, ds. \tag{5.26}$$

Plug (5.26) in (5.21) and this completes the rest of the proof. □

Below, we discuss one of the main theorems of this Section.

Theorem 5.1. *Let Ω be a convex polygon and let the conditions (**A1**)-(**A2**) and, (**B1**) and (**B2**) be satisfied. Further, let the discrete initial velocity $\mathbf{u}_{0h}\in\mathbf{J}_h$ with $\mathbf{u}_{0h}=P_h\mathbf{u}_0$, where $\mathbf{u}_0\in\mathbf{J}_1$. Then, there exists a positive constant C such that for $0<T<\infty$ with $t\in(0,T]$*

$$\|(\mathbf{u}-\mathbf{u}_h)(t)\|+h\|\nabla(\mathbf{u}-\mathbf{u}_h)(t)\|\leq Ce^{Ct}h^2t^{-1/2}.$$

Proof. Since $\mathbf{e}=\mathbf{u}-\mathbf{u}_h=(\mathbf{u}-\mathbf{v_h})+(\mathbf{v_h}-\mathbf{u}_h)=\boldsymbol\xi+\boldsymbol\eta$ and the estimate of $\boldsymbol\xi$ is known, it is enough to estimate $\|\boldsymbol\eta\|$. Choose $\boldsymbol\phi_h=\sigma(t)\boldsymbol\eta$ in (5.12) to obtain

$$\frac{1}{2}\frac{d}{dt}(\sigma(t)\|\boldsymbol\eta\|^2)+\mu\sigma(t)\|\nabla\boldsymbol\eta\|^2 =\frac{1}{2}\sigma_t(t)\|\boldsymbol\eta\|^2-\sigma(t)\int_0^t\beta(t-s)a(\boldsymbol\eta(s),\boldsymbol\eta)\, ds+\sigma(t)\Lambda_h(\boldsymbol\eta)$$
$$=\frac{1}{2}\sigma_t(t)\|\boldsymbol\eta\|^2+I_2(t)+\sigma(t)\Lambda_h(\boldsymbol\eta). \tag{5.27}$$

Observe that we follow the proof of Lemma 5.3, the only difference with (5.10) being the involvement of the nonlinear term in this case. Similar to estimate (5.11) in Lemma 5.3, we find

$$\int_0^t |I_2(s)|\, ds \;\leq\; C(\mu,\delta,\gamma)\int_0^t e^{2\alpha s}\|\nabla\tilde{\boldsymbol\eta}(s)\|^2\, ds+\frac{\mu}{4}\int_0^t\sigma(s)\|\nabla\boldsymbol\eta(s)\|^2\, ds, \tag{5.28}$$

where $\tilde{\boldsymbol\eta}(t)=\int_0^t\boldsymbol\eta(s)\, ds$. We treat the nonlinear term, as has been done before, to arrive at the following:

$$\sigma(t)|\Lambda_h(\hat{\boldsymbol\eta})|\leq \frac{\mu}{4}\sigma(t)\|\nabla\boldsymbol\eta\|^2+C\sigma(t)\|\mathbf{e}\|^2\left(\|\nabla\mathbf{u}\|\|\tilde\Delta\mathbf{u}\|+\|\nabla\mathbf{u}_h\|\|\tilde\Delta_h\mathbf{u}_h\|\right). \tag{5.29}$$

Incorporate the estimate (5.29) in (5.27), integrate the resulting inequality and use (5.28) to yield

$$
\begin{aligned}
(5.30)\qquad & \sigma(t)\|\boldsymbol{\eta}\|^2+\mu\int_0^t \sigma(s)\|\nabla\boldsymbol{\eta}(s)\|^2 \\
\leq\ & 2(1+\alpha)\int_0^t \|\hat{\boldsymbol{\eta}}(s)\|^2 ds + C\int_0^t e^{2\alpha s}\|\nabla\tilde{\boldsymbol{\eta}}(s)\|^2 ds \\
+\ & C\int_0^t \tau^*(s)(\|\nabla\mathbf{u}(s)\|\|\tilde{\Delta}\mathbf{u}(s)\|+\|\nabla\mathbf{u}_h(s)\|\|\tilde{\Delta}_h\mathbf{u}_h(s)\|)\|\hat{\mathbf{e}}(s)\|^2.
\end{aligned}
$$

Apply Theorem 3.1 and Lemmas 3.4, 5.4 and 5.5. Multiply the resulting inequality by $e^{-2\alpha t}$ to yield

$$\tau^*(t)\|\boldsymbol{\eta}\|^2+e^{-2\alpha t}\mu\int_0^t \sigma(s)\|\nabla\boldsymbol{\eta}\|^2\, ds \ \leq\ C(K,\mu)e^{Ct}h^4.$$

Here, e^{Ct} is of the form $\exp(C(\mu,\alpha,\delta)\int_0^t(\|\nabla\mathbf{u}_h\|^4+\|\mathbf{u}\|^2\|\nabla\mathbf{u}\|^2)\, ds)$, contributed by Lemma 5.4.

Since $\boldsymbol{\eta}\in\mathbf{J}_h$, we use inverse hypothesis to obtain an estimate for $\|\nabla\boldsymbol{\eta}\|$. A use of triangle inequality with Lemma 5.3 completes the rest of the proof. $\square$

Remark 5.2. *(i) If $\mathbf{u}_0$ and $\mathbf{f}$ are sufficiently small with respect to the norms in the assumption (**A2**) such that*

$$\mu-C(\mu,\lambda_1,\delta,\gamma)(\|\nabla\mathbf{u}_h\|^4+\|\mathbf{u}_h\|^2\|\nabla\mathbf{u}_h\|^2+\|\nabla\mathbf{u}\|^2\|\tilde{\Delta}\mathbf{u}\|^2)\geq 0,$$

then, in (5.16), we can avoid Gronwall's Lemma so as to obtain

$$\int_0^t \|\hat{\boldsymbol{\eta}}\|^2 ds \leq Ch^2,$$

where C is independent of time. This means, the estimate of Lemma 5.4 is uniform in time and hence the estimate

$$\|\boldsymbol{\eta}\|\leq Ch^2$$

remains uniformly bounded as $t\to\infty$.

(ii) When $d=2$ and $\mathbf{f}\in\mathbf{H}^1(0,\infty,\mathbf{L}^2(\Omega))$, then it is straight forward to check that

$$\int_0^t \|\tilde{\Delta}\mathbf{u}\|^2\, ds<\infty.$$

Thus, the error estimates holds for all time $t>0$.

(iii) For $\mathbf{u}_0\in\mathbf{J}_1\cap\mathbf{H}^2$, based on the analysis of [28]*, we can easily obtain the following error estimate for the velocity term*

$$\|(\mathbf{u}-\mathbf{u}_h)(t)\|\leq Ch^2e^{Ct}, t\in[0,T].$$

Below, we derive uniform (in time) error estimate for the velocity term under the assumption of the uniqueness condition, that is,

$$(5.31)\qquad \frac{N}{\nu^2}\|\mathbf{f}\|_{L^\infty(0,\infty;\mathbf{L}^2(\Omega))}<1 \text{ and } N=\sup_{\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbf{H}_0^1(\Omega)}\frac{b(\mathbf{u},\mathbf{v},\mathbf{w})}{\|\nabla\mathbf{u}\|\|\nabla\mathbf{v}\|\|\nabla\mathbf{w}\|},$$

where $\nu=\mu+\frac{\gamma}{\delta}$.

Theorem 5.2. *Under the assumptions of Theorem* 5.1 *and the uniqueness condition (5.31), there exists a positive constant C, independent of time and h, such that for all $t>0$*

$$\|(\mathbf{u}-\mathbf{u}_h)(t)\|\leq Ch^2t^{-1/2}.$$

Proof. We note that, in Lemma 5.3, it is shown that estimates of $\boldsymbol{\xi}$ are uniform in time, for all time away from zero. As $\mathbf{e} = \boldsymbol{\xi} + \boldsymbol{\eta}$, uniform estimates of $\boldsymbol{\eta}$ would give us the desired result. In Theorem 5.1, we apply Gronwall's Lemma, and hence, the constant depends on exponential in time. In order to have estimates which are uniformly valid for all $t > 0$, we need a different estimate of the nonlinear term $\Lambda_h(\hat{\boldsymbol{\eta}})$ with the help of the uniqueness condition. We rewrite

$$\Lambda_h(\boldsymbol{\eta}) = -[b(\boldsymbol{\xi}, \mathbf{u}_h, \boldsymbol{\eta}) + b(\boldsymbol{\eta}, \mathbf{u}_h, \boldsymbol{\eta}) + b(\mathbf{u}, \boldsymbol{\xi}, \boldsymbol{\eta})]. \tag{5.32}$$

Using the uniqueness condition, we find

$$|b(\boldsymbol{\eta}, \mathbf{u}_h, \boldsymbol{\eta})| \le N\|\nabla\boldsymbol{\eta}\|^2\|\nabla\mathbf{u}_h\|. \tag{5.33}$$

Using (4.7) and (4.8), we find

(5.34)

$$|b(\mathbf{u}, \boldsymbol{\xi}, \boldsymbol{\eta})| + |b(\boldsymbol{\xi}, \mathbf{u}_h, \boldsymbol{\eta})| \le C(\|\nabla\mathbf{u}\|^{1/2}\|\Delta\mathbf{u}\|^{1/2} + \|\nabla\mathbf{u}_h\|^{1/2}\|\tilde{\Delta}_h\mathbf{u}_h\|^{1/2})\|\nabla\boldsymbol{\eta}\|\|\boldsymbol{\xi}\|.$$

Substitute (5.33)-(5.34) in (5.32) and use Lemmas 3.1, 3.2, Theorem 3.1 and Lemmas 4.2, 5.3 to find that

$$|\Lambda_h(\boldsymbol{\eta})| \le N\|\nabla\boldsymbol{\eta}\|^2\|\nabla\mathbf{u}_h\| + Ch^2[\tau*]^{-3/4}\|\nabla\boldsymbol{\eta}\|. \tag{5.35}$$

We now modify the proof of Theorem 5.1 as follows: We choose Choose $\boldsymbol{\phi}_h = e^{2\alpha t}\boldsymbol{\eta}$ in (5.12) and use the estimate (5.35) to obtain

$$\begin{aligned} \frac{d}{dt}\|\hat{\boldsymbol{\eta}}\|^2 + 2(\mu - N\|\nabla\mathbf{u}_h\|)\|\nabla\hat{\boldsymbol{\eta}}\|^2 + 2\int_0^t \beta(t-s)e^{\alpha(t-s)}a(\hat{\boldsymbol{\eta}}(s), \hat{\boldsymbol{\eta}}(t))\,ds \\ \le \quad 2\alpha\|\hat{\boldsymbol{\eta}}\|^2 + Ch^2[\tau*]^{-3/4}e^{2\alpha t}\|\nabla\boldsymbol{\eta}\|. \end{aligned} \tag{5.36}$$

Integrate (5.36) with respect to time from 0 to t and multiply by $e^{-2\alpha t}$ to arrive at

$$\begin{aligned} \|\boldsymbol{\eta}(t)\|^2 \quad &+ \quad e^{-2\alpha t}\int_0^t e^{2\alpha s}\ 2(\mu - N\|\nabla\mathbf{u}_h(s)\|)\|\nabla\boldsymbol{\eta}(s)\|^2\,ds \\ &+ \quad 2e^{-2\alpha t}\int_0^t e^{2\alpha s}\int_0^s \beta(s-\tau)a(\boldsymbol{\eta}(\tau), \boldsymbol{\eta}(s))\ d\tau ds \\ &\le \quad e^{-2\alpha t}\Big[\|\boldsymbol{\eta}(0)\|^2 + 2\alpha\int_0^t e^{2\alpha s}\|\boldsymbol{\eta}(s)\|^2\,ds + Ch^2\int_0^t [\tau^*(s)]^{-3/4}e^{2\alpha s}\|\nabla\boldsymbol{\eta}(s)\|\,ds\Big]. \end{aligned}$$

Take $t \to \infty$ and use the following results from [11]

$$\overline{\lim_{t\to\infty}}\, 2e^{-2\alpha t}\int_0^t e^{2\alpha s}\int_0^s \beta(s-\tau)a(\boldsymbol{\phi}(\tau), \boldsymbol{\phi}(s))d\tau ds = \frac{\gamma}{\alpha\delta}\overline{\lim_{t\to\infty}}\,\|\nabla\boldsymbol{\phi}(t)\|^2,$$

$$\overline{\lim_{t\to\infty}}\,\|\nabla\mathbf{u}_h(t)\|^2 \le \nu^{-1}\|\mathbf{f}\|_{L^\infty(0,\infty;\mathbf{L}^2(\Omega))}$$

and

$$\overline{\lim_{t\to\infty}}\,(\tau^*(t))^{-3/4} = 1$$

to find that

$$\frac{1}{\alpha}\Big[(\mu - N\nu^{-1}\|\mathbf{f}\|_{L^\infty(0,\infty;\mathbf{L}^2(\Omega))}) + \frac{\gamma}{\delta}\Big]\overline{\lim_{t\to\infty}}\,\|\nabla\boldsymbol{\eta}(t)\|^2 \le \frac{Ch^2}{\alpha}\overline{\lim_{t\to\infty}}\,\|\nabla\boldsymbol{\eta}(t)\|.$$

Therefore, we obtain

$$\frac{1}{\nu}(1 - N\nu^{-2}\|\mathbf{f}\|_{L^\infty(0,\infty;\mathbf{L}^2(\Omega))})\,\overline{\lim_{t\to\infty}}\,\|\nabla\boldsymbol{\eta}(t)\| \le Ch^2.$$

From (5.31), we conclude that

$$\overline{\lim_{t\to\infty}}\,\|\nabla\boldsymbol{\eta}(t)\| \le Ch^2.$$

Clearly,

$$\overline{\lim_{t\to\infty}}\, \|\boldsymbol{\eta}(t)\| \le Ch^2.$$

This, with uniform estimate of $\boldsymbol{\xi}$ from Lemma 5.3, leads to

$$\overline{\lim_{t\to\infty}}\, \|\mathbf{e}(t)\| \le Ch^2 t^{-1/2}.$$

Note that the constant C is valid uniformly for all $t > 0$, and this completes the rest of the proof. □

6. *A* priori Error Estimates for the Pressure

In this Section, we derive optimal error estimates for the Galerkin approximation p_h of the pressure p. The main theorem of this Section is stated as follows.

Theorem 6.1. *In addition to the hypotheses of Theorem 5.1, assume that (*$\mathbf{B2'}$*) holds. Then, there exists a positive constant C such that for $0 < t < T$ and for $\mathbf{u}_0 \in \mathbf{J}_1$,*

$$\|(p - p_h)(t)\|_{L^2/N_h} \le Ce^{Ct} h t^{-1/2}.$$

Now, we prove Theorem 6.1 with the help of a series of Lemmas.
As in [28], we obtain the following result.

Lemma 6.1. *The semi-discrete Galerkin approximation p_h of the pressure satisfies for all $t > 0$*

(6.1)

$$\|(p-p_h)(t)\|_{L^2/N_h} \le \left[C|||\mathbf{e}_t|||_{-1;h} + (\tilde{K} + C\|\mathbf{e}\|_{\mathbf{L}^3})\|\nabla\mathbf{e}\| + \int_0^t \beta(t-s)\|\nabla\mathbf{e}(s)\|\, ds \right],$$

where

$$|||\mathbf{g}|||_{-1;h} = \sup\Big\{ \frac{<\mathbf{g}, \boldsymbol{\phi}_h>}{\|\nabla\boldsymbol{\phi}_h\|},\ \boldsymbol{\phi}_h \in \mathbf{H}_h,\ \boldsymbol{\phi}_h \ne 0 \Big\}.$$

From Theorem 5.1, the estimate $\|\nabla\mathbf{e}\|$ is known and using Sobolev embedding Theorem for $d = 2$, the estimate $\|\mathbf{e}\|_{\mathbf{L}^3} \le C\|\nabla\mathbf{e}\| \le K$. In order to complete the proof of Theorem 6.1, we need to estimate $|||\mathbf{e}_t|||_{-1;h}$ in (6.1). Since $\mathbf{H}_h \subset \mathbf{H}_0^1$, we note that

$$\begin{aligned} |||\mathbf{e}_t|||_{-1;h} &= \sup\Big\{ \frac{<\mathbf{e}_t, \boldsymbol{\phi}_h>}{\|\nabla\boldsymbol{\phi}_h\|},\ \boldsymbol{\phi}_h \in \mathbf{H}_h,\ \boldsymbol{\phi}_h \ne 0 \Big\} \\ &\le \sup\Big\{ \frac{<\mathbf{e}_t, \boldsymbol{\phi}>}{\|\nabla\boldsymbol{\phi}\|},\ \boldsymbol{\phi} \in \mathbf{H}_0^1,\ \boldsymbol{\phi} \ne 0 \Big\}, \end{aligned}$$

supremum being taken over a bigger set. Therefore, we obtain

$$|||\mathbf{e}_t|||_{-1;h} \le \|\mathbf{e}_t\|_{-1}.$$

Lemma 6.2. *The error $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ in approximating the velocity satisfies for $t > 0$*

$$\|\mathbf{e}_t(t)\|_{-1} \le Ce^{Ct} h (\tau^*(t))^{-1/2},$$

where $\tau^*(t) = \min\{t, 1\}$.

Proof. From (4.2) and (5.1), we write the equation in $\mathbf{e}$ as

$$\begin{aligned} (\mathbf{e}_t, \boldsymbol{\phi}_h) &+ \mu a(\mathbf{e}, \boldsymbol{\phi}_h) + \int_0^t \beta(t-s) a(\mathbf{e}(s), \boldsymbol{\phi}_h)\, ds \\ &= \Lambda_h(\boldsymbol{\phi}_h) + (p, \nabla\cdot\boldsymbol{\phi}_h)\, \forall \boldsymbol{\phi}_h \in \mathbf{J}_h, \end{aligned} \tag{6.2}$$

where, from (5.14), we find

$$\Lambda_h(\boldsymbol{\phi}_h) = -b(\mathbf{e}, \mathbf{u}_h, \boldsymbol{\phi}_h) - b(\mathbf{u}, \mathbf{e}, \boldsymbol{\phi}_h).$$

Since $P_h : \mathbf{L}^2 \to \mathbf{J}_h$, then, for any $\boldsymbol{\psi} \in \mathbf{H}_0^1$, we obtain using (6.2),

$$\begin{aligned}(\mathbf{e}_t, \boldsymbol{\psi}) &= (\mathbf{e}_t, \boldsymbol{\psi} - P_h\boldsymbol{\psi}) + (\mathbf{e}_t, P_h\boldsymbol{\psi}) \\ &= (\mathbf{e}_t, \boldsymbol{\psi} - P_h\boldsymbol{\psi}) - \mu a(\mathbf{e}, P_h\boldsymbol{\psi}) - \int_0^t \beta(t-s) a(\mathbf{e}(s), P_h\boldsymbol{\psi})\, ds \\ &\quad + \Lambda_h(P_h\boldsymbol{\psi}) + (p, \nabla\cdot P_h\boldsymbol{\psi}).\end{aligned} \tag{6.3}$$

Using discrete incompressibility condition, we write

$$(p, \nabla\cdot P_h\boldsymbol{\psi}) = (p - j_h p, \nabla\cdot P_h\boldsymbol{\psi})$$

and hence, H^1-stability of P_h yields

$$|(p, \nabla\cdot P_h\boldsymbol{\psi})| \le Ch\|\nabla p\|\|\nabla\boldsymbol{\psi}\|. \tag{6.4}$$

Also, using the Cauchy-Schwarz inequality and H^1-stability of P_h, we obtain

$$|\Lambda_h(P_h\boldsymbol{\psi})| \le \|\nabla\mathbf{e}\|(\|\nabla\mathbf{u}_h\| + \|\nabla\mathbf{u}\|)\|\nabla\boldsymbol{\psi}\|. \tag{6.5}$$

Using approximation property of P_h, we find that

$$(\mathbf{e}_t, \boldsymbol{\psi} - P_h\boldsymbol{\psi}) = (\mathbf{u}_t, \boldsymbol{\psi} - P_h\boldsymbol{\psi}) \le Ch\|\mathbf{u}_t\|\|\nabla\boldsymbol{\psi}\|. \tag{6.6}$$

Substitute (6.4)-(6.6) in (6.3) and use the boundedness of $a(\cdot,\cdot)$ to obtain

$$\begin{aligned}(\mathbf{e}_t, \boldsymbol{\psi}) \le &\Big\{Ch\|\mathbf{u}_t\| + C\|\nabla\mathbf{e}\| + \int_0^t \beta(t-s)\|\nabla\mathbf{e}(s)\|\, ds \\ &+ \|\nabla\mathbf{e}\|(\|\nabla\mathbf{u}_h\| + \|\nabla\mathbf{u}\|) + Ch\|\nabla p\|\Big\}\|\nabla\boldsymbol{\psi}\|,\end{aligned}$$

and therefore,

$$\begin{aligned}\|\mathbf{e}_t\|_{-1} &= \sup\Big\{\frac{< \mathbf{e}_t, \boldsymbol{\phi} >}{\|\nabla\boldsymbol{\phi}\|},\ \boldsymbol{\phi} \in \mathbf{H}_0^1,\ \boldsymbol{\phi} \ne 0\Big\} \\ &\le Ch\|\mathbf{u}_t\| + C\|\nabla\mathbf{e}\| + \int_0^t \beta(t-s)\|\nabla\mathbf{e}(s)\|\, ds \\ &\quad + \|\nabla\mathbf{e}\|(\|\nabla\mathbf{u}_h\| + \|\nabla\mathbf{u}\|) + Ch\|\nabla p\|.\end{aligned} \tag{6.7}$$

From Lemma 4.2, we find that

$$\|\nabla\mathbf{u}_h(t)\| \le K,\ t > 0.$$

Now, using Theorem 3.1 and 5.1, we obtain from (6.7)

$$\begin{aligned}\|\mathbf{e}_t\|_{-1} &\le Ce^{Ct}h(\tau^*(t))^{-1/2} + \int_0^t \beta(t-s)\|\nabla\mathbf{e}(s)\|\, ds \\ &\le Ce^{Ct}h(\tau^*(t))^{-1/2}\Big\{1 + \int_0^t (\tau^*(s))^{-1/2}\, ds\Big\} \\ &\le Ce^{Ct}h(\tau^*(t))^{-1/2},\end{aligned} \tag{6.8}$$

and this completes the rest of the proof. □

Proof of Theorem 6.1. Use (6.8) in (6.1) to obtain

$$\begin{aligned}\|(p - p_h)(t)\|_{L^2/N_h} &\le Ce^{Ct}h(\tau^*(t))^{-1/2} + (\tilde{K} + C\|\mathbf{e}\|_{\mathbf{L}^3})\|\nabla\mathbf{e}\| \\ &+ \int_0^t \beta(t-s)\|\nabla\mathbf{e}(s)\|\, ds.\end{aligned} \tag{6.9}$$

The last term involving integral can be estimated as in (6.8). From Theorem 5.1, we find that

$$\|\nabla \mathbf{e}\| \leq Ce^{Ct}h(\tau^*(t))^{-1/2}$$
$$\|\mathbf{e}\|_{\mathbf{L}^3} \leq C\|\nabla \mathbf{e}\| \leq Ce^{Ct}h(\tau^*(t))^{-1/2}.$$

Now, plugging these estimates in (6.9), we obtain

$$\|(p-p_h)(t)\|_{L^2/N_h} \leq Ce^{Ct}h(\tau^*(t))^{-1/2},$$

and this completes the rest of the proof. □

Remark 6.1. *Under the uniqueness condition (5.31), we establish error estimate for pressure, which is valid for all time $t > 0$. From the proof of Theorem 5.2 and using the inverse inequality along with the triangle inequality, we easily arrive at*

$$\|\nabla \mathbf{e}\| \leq Ch(\tau^*(t))^{-1/2},$$

where generic constant C is independent of time. As the estimates in Theorem 3.1 *are uniform in time, we obtain from Lemma* 6.2

$$\|\mathbf{e}_t\|_{-1} \leq Ch(\tau^*(t))^{-1/2}.$$

Finally,

$$\begin{aligned}\int_0^t \beta(t-s)\|\nabla \mathbf{e}(s)\|\, ds \leq& Che^{-\delta t}\int_0^t e^{\delta s}(\tau^*(s))^{-1/2}\, ds\\ =&Che^{-\delta t}\Big\{\int_0^1 e^{\delta s}s^{-1/2}\, ds + \int_1^t e^{\delta s}\, ds\Big\}\\ \leq& Ch\{2+\frac{1}{\delta}\}. \end{aligned} \tag{6.10}$$

Note that the second integral on right-hand side of (6.10) vanishes for $t \leq 1$. Now, an appeal to (6.1) yields

$$\|(p-p_h)(t)\|_{L^2/N_h} \leq Ch(\tau^*(t))^{-1/2},$$

and we obtain optimal error estimate for the pressure term, which is uniform in time.

7. Conclusion

In this paper, we have discussed optimal error estimates of the velocity and the pressure terms in $L^\infty(\mathbf{L}^2)$ and $L^\infty(L^2)$, respectively, for the semidiscrete finite element approximations to the equations of motion arising in the $2D$ Oldroyd fluids of order one. We have also established uniform in time error estimates under the uniqueness assumption. All these results are proved when forcing term $\mathbf{f}$ in $L^\infty(\mathbf{L}^2)$ and initial velocity $\mathbf{u}_0$ in $\mathbf{J}_1$. Optimal error estimates are also proved in [28], when $\mathbf{f} \equiv 0$ and $\mathbf{u}_0$ in $\mathbf{J}^1 \cap \mathbf{H}^2$. The main difficulty we encounter here, in proving *a priori* estimate in $\mathbf{H}^1_0$, which is uniform in time, is due to nonzero $\mathbf{f}$, with $\mathbf{f}, \mathbf{f}_t \in L^\infty(\mathbf{L}^2)$. Although a vast amount of literature is devoted to the problem (1.1)-(1.3) (mainly Oskolkov and his pupil, Sobolevskii and recently Lin et.al), to the best of our knowledge, a direct proof of uniform estimate in Dirichlet norm is missing in the literature. Therefore, in the first part of this article, we have discussed a step-by-step proof of this estimate, which is uniform in time, in Section 3. All the regularity results are obtained without the nonlocal compatibility conditions. In [11], authors have considered assumptions (**A1**)-(**A2**) and derived error estimate which is optimal in $L^\infty(\mathbf{H}^1)$, but suboptimal in $L^\infty(\mathbf{L}^2)$ of the velocity. We, in this paper, have improved their results in [11] and obtain optimal error estimate for the velocity in $L^\infty(\mathbf{L}^2)$-norm. Under the uniqueness condition, uniform error estimates in time, for both velocity and pressure are also obtained.

We have also managed to improve the pressure estimate in the following sense. For $\mathbf{u}_0 \in \mathbf{J}_1$, we obtain $\|p-p_h\| \leq Kht^{-1/2}$, which exhibits similar singular behavior (as $t \to 0$) as that of velocity estimate. In earlier articles (e.g. [28], [11]), this singularity is of order 1, i.e. $\|p-p_h\| \leq Kht^{-1}$.

Finally, we would like to comment that, although Lemmas 3.1 and 3.2 yield estimates in Dirichlet norm which is uniform in time, they do not provide us with a concrete bound for the norm. Lemma 3.4 gives us a concrete bound, which in turn will be useful in the study of global attractors for the Oldroyd model of order one.

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