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CONVERGENCE AND STABILITY OF THE SEMI-IMPLICIT EULER METHOD WITH VARIABLE STEPSIZE FOR A LINEAR STOCHASTIC PANTOGRAPH DIFFERENTIAL EQUATION

YU XIAO, MINGHUI SONG AND MINGZHU LIU

Abstract. The paper deals with convergence and stability of the semi-implicit Euler method with variable stepsize for a linear stochastic pantograph differential equation(SPDE). It is proved that the semi-implicit Euler method with variable stepsize is convergent with strong order $p = \frac{1}{2}$. The conditions under which the method is mean square stability are determined and the numerical experiments are given.

Key Words. Stochastic pantograph differential equation, mean square stability, semi-implicit Euler method with variable stepsize.

1. Introduction

The importance of stochastic differential delay equations (SDDEs) derives from the fact that many of the phenomena witnessed around us do not have an immediate effect from the moment of their occurrence. A patient, for example, shows symptoms of an illness days (or even weeks) after the day in which he or she was infected. In general, we can find many "systems", in almost any area of science (medicine, physics, ecology, economics, etc.), for which the principle of causality, i.e., the future state of a system is independent of the past states and is determined solely by the present, does not apply. In order to incorporate this time lag (between the moment an action takes place and the moment its effect is observed) to our models, it is necessary to include an extra term which is called time delay. The SDDEs can be regarded as a generalization of stochastic differential equations (SDEs) and delay differential equations (DDEs). During the last few decades, many authors have studied SDDEs. some important results are given, for example, conditions which guarantee the existence and uniqueness of an analytical solution [13, 14, 15] and stability conditions for both exact solutions and numerical solutions, etc. [2, 6, 11, 16].

It is well known that in the deterministic situation there is a very special delay differential equation: the pantograph equation

$$\begin{aligned} y'(t) &= \bar{a}y(t) + \bar{b}y(qt), \quad 0 \leqslant t \leqslant t_f, \\ y(0) &= y_0. \end{aligned} \tag{1.1}$$

where $q \in (0, 1)$. It arises in quite different fields of pure and applied mathematics such as number theory, dynamical systems, probability, quantum mechanics and electrodynamics. In particular, it is used by Ockendon and Taylor[17] to study how the electric current is collected by the pantograph of an electric locomotive,

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from where it gets its name. In [17] the coefficients $\bar{a}, \bar{b}$ of Eq.(1.1) are constants. If we take into account the estimation error for system parameters as well as the environmental noise, it is better to estimate parameters $\bar{a}, \bar{b}$ as point estimator plus an error. By the central limit theorem, the error may be described by a normally distributed random variable. Then, Eq.(1.1) becomes the differential form

$$\begin{aligned} &dX(t) = [aX(t) + bX(qt)]dt + [cX(t) + dX(qt)]dW(t), \quad t > 0, \\ &X(0) = x_0, \end{aligned} \tag{1.2}$$

which is a linear stochastic pantograph differential equation. In Eq.(1.2), $a, b, c, d \in \mathbb{R}$, $q \in (0, 1)$, $W(t)$ is a one-dimensional standard wiener process. The initial value x_0 is a real -valued random variable. The first term on the right hand side of Eq.(1.2) is usually called the *drift* function and the second term is called the *diffusion* function. The integral version of equation (1.2) is given by

$$\begin{aligned} X(t) = x_0 + \int_0^t [aX(s) + bX(qs)]ds \\ + \int_0^t [cX(s) + dX(qs)]dW(s), \end{aligned} \tag{1.3}$$

for $t > 0$. The second integral in Eq.(1.3) is a stochastic integral which is to be interpreted as the Itô sense [5].

The study for stochastic pantograph equation has just begun. Baker and Buckwar [3] give the necessary analytical theory for existence and uniqueness of a strong solution of the linear stochastic pantograph equation

$$\begin{aligned} &dX(t) = [aX(t) + bX(qt)]dt + [\sigma_1 + \sigma_2 X(t) + \sigma_3 X(qt)]dW(t), \\ &X(0) = X_0. \end{aligned} \tag{1.4}$$

They also prove that the numerical solution produced by the continuous θ-method converges to the true solution with order 1/2. Liu et al. [12] give stability conditions of the analytical solution of the nonlinear stochastic pantograph equation and provide results concerning convergence and stability of the semi-implicit Euler method with constant stepsize. Fan [4] give the sufficient conditions that guarantee the existence and uniqueness of a strong solution to the nonlinear stochastic pantograph equation and proved that the semi-implicit Euler method with constant stepsize applied to the nonlinear equation has strong order 1/2.

When the numerical method with a constant stepsize is applied to the pantograph equation, the most difficult problem is the limited computer memory as shown in [9, 10]. In this paper, we use the semi-implicit Euler method with variable stepsize for a scalar test equation (1.2) to avoid the storage problem and discuss the convergence and stability properties of the method. The other reason of applying a numerical method with a variable stepsize is that when using the numerical method with a constant stepsize to Eq. (1.2), the resulting difference equation is not of fixed order.

The paper is organized as follows. In Section 2, we will introduce some notations and recall some properties of its analytical solution. In Section 3, we will prove that the semi-implicit Euler method with a variable stepsize is convergent to the true solution with order $\frac{1}{2}$ and mean-square stability if $\theta \in \left(\frac{|a|+|b|}{2|a|}, 1\right]$. We will provide some numerical examples in Section 4.

2. Analysis of exact solution

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, which satisfies the usual conditions, i.e. the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ is right-continuous and each $\{\mathcal{F}_t\}, t \geqslant 0$, contains all P-null sets in $\mathcal{F}$. Let $W(t), t \geqslant 0$ in Eq. (1.2) be $\mathcal{F}_t$-adapted and independent of $\mathcal{F}_0$. $|\cdot|$ is the Euclidean norm in $\mathcal{R}$. Moreover, we assume x_0 to be $\mathcal{F}_0$-measurable and $E|\ x_0\ |^2 < \infty$. In this paper, Eq.(1.2) is interpreted in the Itô sense.

Definition 2.1. *An $\mathbb{R}$-valued stochastic process $X(t) : [0,T] \times \Omega \to \mathbb{R}$ is called a strong solution of Eq.*(1.2)*, if it is a measurable, sample-continuous process such that $X|_{[0,T]}$ is $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$-adapted, and X satisfies Eq.* (1.2)*, almost surely, and satisfies the initial condition $X(0) = x_0$. A solution $X(t)$ is said to be path-wise unique if any other solution $\hat{X}(t)$ is stochastically indistinguishable from it, i.e.*

$$P\big\{X(t) = \hat{X}(t), \quad \textit{for all} \quad 0 \leqslant t \leqslant T\big\} = 1.$$

Theorem 2.2. [3, 7]*If $0 < q < 1$ and $E|\ x_0\ |^2 < \infty$, then there exists a pathwise unique strong solution to problem* (1.2)*.*

Lemma 2.3. [12]*If the coefficients of Eq.*(1.2) *are satisfied*

$$a < -|b| - \frac{1}{2}\big(|c| + |d|\big)^2. \tag{2.1}$$

Then, the solution is mean square stable, that is

$$\lim_{t\to\infty} E|X(t)|^2 = 0. \tag{2.2}$$

In a similar way as [3], we can get the following two lemmas. The detailed proofs are omitted here.

Lemma 2.4. *The solution of Eq.*(1.2) *has the property*

$$E\Big(\sup_{0\leqslant t\leqslant T} |X(t)|^2\Big) \leqslant C_1(T), \tag{2.3}$$

with

$$C_1(T) := \big(\frac{1}{2} + 3E|X(0)|^2\big) \exp\big(6K(T+4)T\big),$$

$$K := \max\{|a|, |b|, |c|, |d|\}.$$

Moreover, for any $0 < s < t < T$ with $t - s < 1$,

$$E|X(t) - X(s)|^2 \leqslant C_2(T)(t-s), \tag{2.4}$$

where $C_2(T) = 16KC_1(T)$.

Lemma 2.5. *The solution of Eq.*(1.2) *has the following estimate for all $t \in [0,T]$,*

$$E|aX(t) + bX(qt)| \leqslant \sqrt{2LC_1(T)}, \tag{2.5}$$

with

$$L = 2\max\{|a|, |b|\}.$$

3. The semi-implicit Euler method with the variable step-size

In this paper, the semi-implicit Euler method with variable step-size is defined as follows:

The mesh $H = \{m; t_0, t_1, \cdots, t_n, \cdots.\}$ is introduced as follows. Let $T_0 > 0$ be given, $t_0 = T_0$ and $t_m = q^{-1}T_0$. We choose $m-1$ grid points $t_1 < t_2 < \cdots < t_{m-1}$ in (t_0, t_m) and define other mesh points by

$$t_{km+i} = q^{-k}t_i, \text{ for } k = 1, 2, \cdots, \quad i = 0, 1, \cdots, m-1.$$

It is easy to see that the grid points t_n such that $qt_n = t_{n-m}$ for $n \geqslant 0$ and the step-size $h_n = t_{n+1} - t_n$ satisfies

$$h_n = q^{-1}h_{n-m}, \quad \text{for all } \ n \geqslant 1, \text{ and } \lim_{n\to\infty} h_n = \infty. \tag{3.1}$$

Furthermore, we suppose to have the numerical solution available until T_0 which is called initial data.

The semi-implicit Euler method of Eq.(1.2) with variable step-size has the form

$$\begin{aligned} X_{n+1} = X_n &+ [\theta(aX_{n+1} + bX_{n-m+1}) + (1-\theta)(aX_n \\ &+ bX_{n-m})]h_n + [cX_n + dX_{n-m}]\Delta W_n, \end{aligned} \tag{3.2}$$

where θ is parameter with $0 \leqslant \theta \leqslant 1$, X_n is an approximation to $X(t_n)$ and the increments $\Delta W_n := W(t_{n+1}) - W(t_n)$ are independent $N(0, h_n)$-distributed Gaussian random variables. Moreover, we assume that X_n is $\mathcal{F}_{t_n}$-measurable at the mesh-point t_n.

In the following, we choose a finite interval $[0, T]$ with $T_0 = q^sT \leqslant 1$, s is a positive integer. Let $h = \max\limits_{0\leqslant n\leqslant N-1}\{h_n\}$ with $N = sm$. It is easy to see that $h_n \leqslant h$ for all $n \in [0, N]$.

Definition 3.1. *1) The local truncation error for semi-implicit Euler method is defined as follows*

$$\begin{aligned} \delta_{n+1} =& X(t_{n+1}) - \big\{X(t_n) + \theta[aX(t_{n+1}) + bX(t_{n-m+1})]h_n \\ &+ (1-\theta)[aX(t_n) + bX(t_{n-m})]h_n + [cX(t_n) + dX(t_{n-m})]\Delta W_n\big\}. \end{aligned} \tag{3.3}$$

2) The global error for semi-implicit Euler method is defined by

$$\varepsilon_n = X(t_n) - X_n. \tag{3.4}$$

3) The semi-implicit Euler method is said to be consistent with order p_1 in the mean sense and with order p_2 in the mean-square sense if the following estimates hold with $p_2 \geqslant \frac{1}{2}$ and $p_1 \geqslant p_2 + \frac{1}{2}$

$$\max_{0\leqslant n\leqslant N}\big|E(\delta_n)\big| \leqslant Ch^{p_1}, \qquad \textit{as} \quad h \to 0,$$

$$\max_{0\leqslant n\leqslant N}\big(E(\delta_n)^2\big)^{\frac{1}{2}} \leqslant Ch^{p_2}, \qquad \textit{as} \quad h \to 0,$$

where the constant C does not depend on h, but may depend on T and the initial data.

4) For fixed $T < \infty$, the approximation X_n are convergent in the mean-square sense on mesh-points, with order p if

$$\max_{1\leqslant n\leqslant N}\big(E(\varepsilon_n)^2\big)^{\frac{1}{2}} \leqslant Ch^p, \qquad \textit{as} \quad h \to 0, \tag{3.5}$$

where C is a positive constant.

Lemma 3.2. *If $0 < q < 1$, the semi-implicit Euler method for Eq.(1.2) is consistent with order 2 in the mean sense and with order 1 in the mean-square sense, that is*

$$\max_{0\leqslant n\leqslant N}\left|E(\delta_n)\right| \leqslant C_3(T)h^2, \qquad as \quad h\to 0, \tag{3.6}$$

$$\max_{0\leqslant n\leqslant N}\left(E(\delta_n)^2\right)^{\frac{1}{2}} \leqslant C_4(T)h, \qquad as \quad h\to 0, \tag{3.7}$$

where $C_3(T), C_4(T)$ are positive constants and independent of h.

Proof. In view of

$$\begin{aligned}\delta_{n+1} =& a\int_{t_n}^{t_{n+1}} \left[X(s)-(\theta X(t_{n+1})+(1-\theta)X(t_n))\right]ds\\ &+ b\int_{t_n}^{t_{n+1}} \left[X(qs)-(\theta X(qt_{n+1})+(1-\theta)X(qt_n))\right]ds\\ &+\int_{t_n}^{t_{n+1}} c[X(s)-X(t_n)]+d[X(qs)-X(qt_n)]dW(s),\end{aligned} \tag{3.8}$$

for $n=0,1,2,\cdots,N-1$, Lemma2.4 and 2.5, we can obtain

$$\max_{0\leqslant n\leqslant N}\left|E(\delta_n)\right| \leqslant C_3(T)h^2$$

and

$$\max_{0\leqslant n\leqslant N}\left(E(\delta_n)^2\right)^{\frac{1}{2}} \leqslant C_4(T)h,$$

where $C_3(T) := \frac{3}{2}(1+q)L\sqrt{2LC_1(T)}$, $C_4(T) := \sqrt{14C_2(T)H(1+q)}$ and $H = \max\{|a|^2,|b|^2,|c|^2,|d|^2\}$. □

The detailed process is absent here since that of the proof as well as style of analysis borrows heavily from theorem 3.2 in [3]. In the following, we state the first main theorem of this article.

Theorem 3.3. *Suppose that $0 < q < 1$, $E(\varepsilon_0^2|\mathcal{F}_{t_0}) = \bar{K}h$. Then the numerical solution produced by the semi-implicit Euler method* (3.2) *is convergent to the exact solution of Eq.*(1.2) *on the mesh-point in the mean-square sense on* $[0,T]$ *with order $\frac{1}{2}$, i.e. there exists a positive constant C_0, such that*

$$\max_{1\leqslant n\leqslant N}\left(E(\varepsilon_n)^2\right)^{\frac{1}{2}} \leqslant C_0 h^{\frac{1}{2}}, \qquad as\ h\to 0\ . \tag{3.9}$$

Proof. It is easy to obtain from (3.2), (3.3)and (3.4) that

$$\varepsilon_{n+1} = \varepsilon_n + u_n + \delta_{n+1}, \tag{3.10}$$

where

$$\begin{aligned}u_n :=& ah_n\theta(X(t_{n+1})-X_{n+1}) + [ah_n(1-\theta)\\ &+c\Delta W_n](X(t_n)-X_n)+bh_n\theta(X(t_{n-m+1})-X_{n-m+1})\\ &+[bh_n(1-\theta)+d\Delta W_n](X(t_{n-m})-X_{n-m}).\end{aligned} \tag{3.11}$$

Thus

$$\begin{aligned}E(\varepsilon_{n+1}^2|\mathcal{F}_{t_0}) \leqslant& E(\varepsilon_n^2|\mathcal{F}_{t_0})+E(u_n^2|\mathcal{F}_{t_0})+E(\delta_{n+1}^2|\mathcal{F}_{t_0})+2|E(\delta_{n+1}u_n|\mathcal{F}_{t_0})|\\ &+2|E(\delta_{n+1}\varepsilon_n|\mathcal{F}_{t_0})|+2|E(\varepsilon_n u_n|\mathcal{F}_{t_0})|.\end{aligned} \tag{3.12}$$

Using (3.11), Hölder inequality and Lemma 3.2, we can obtain that

$$
\begin{aligned}
E(\delta_{n+1}^2|\mathcal{F}_{t_0}) &= E(E(\delta_{n+1}^2|\mathcal{F}_{t_n})|\mathcal{F}_{t_0}) \leqslant C_4^2 h_n^2,\\
E(u_n^2|\mathcal{F}_{t_0}) &\leqslant C_{u_1} h_n \big[E(\varepsilon_{n+1}^2|\mathcal{F}_{t_0}) + E(\varepsilon_n^2|\mathcal{F}_{t_0})\\
&\quad + E(\varepsilon_{n-m+1}^2|\mathcal{F}_{t_0}) + E(\varepsilon_{n-m}^2|\mathcal{F}_{t_0})\big],\\
2|E\delta_{n+1}u_n|\mathcal{F}_{t_0}| &\leqslant C_4^2 h_n^2 + C_u^1 h_n \big[E(\varepsilon_{n+1}^2|\mathcal{F}_{t_0}) + E(\varepsilon_n^2|\mathcal{F}_{t_0})\\
&\quad + E(\varepsilon_{n-m+1}^2|\mathcal{F}_{t_0}) + E(\varepsilon_{n-m}^2|\mathcal{F}_{t_0})\big],\\
2|E\delta_{n+1}\varepsilon_n|\mathcal{F}_{t_0}| &\leqslant C_3^2 h_n^2 + h_n E(\varepsilon_n^2|\mathcal{F}_{t_0}),\\
2|E\varepsilon_n u_n|\mathcal{F}_{t_0}| &\leqslant 5C_{u_2} h_n E(\varepsilon_n^2|\mathcal{F}_{t_0}) + C_{u_2} h_n \big[E(\varepsilon_{n+1}^2|\mathcal{F}_{t_0})\\
&\quad + E(\varepsilon_{n-m+1}^2|\mathcal{F}_{t_0}) + E(\varepsilon_{n-m}^2|\mathcal{F}_{t_0})\big],
\end{aligned}
$$

where $C_{u_1} = \max\{4(a^2+c^2), 4(b^2+d^2)\}, C_{u_2} = \max\{|a|,|b|\}$. Thus (3.12) becomes

$$
\begin{aligned}
[1 - h_n(2C_{u_1} + C_{u_2})]E(\varepsilon_{n+1}^2|\mathcal{F}_{t_0}) &\leqslant [1 + h_n(2C_{u_1} + 5C_{u_2} + 1)]E(\varepsilon_n^2|\mathcal{F}_{t_0})\\
&\quad + h_n(2C_{u_1} + C_{u_2})E(\varepsilon_{n-m+1}^2|\mathcal{F}_{t_0})\\
&\quad + h_n(2C_{u_1} + C_{u_2})E(\varepsilon_{n-m}^2|\mathcal{F}_{t_0})\\
&\quad + h_n^2(2C_4^2 + C_3^2).
\end{aligned} \tag{3.13}
$$

Let $C_5' = 2C_{u_1} + 5C_{u_2} + 1$, $C_6' = 2C_{u_1} + C_{u_2}$, $C_7' = (2C_4^2 + C_3^2)$ and $E_n = \max\limits_{0\leqslant i\leqslant n}\{E(\varepsilon_i^2|\mathcal{F}_{t_0})\}$. Then

$$
(1 - h_n C_6')E_{n+1} \leqslant (1 + h_n C_5' + 2h_n C_6')E_n + h_n^2 C_7'. \tag{3.14}
$$

Assume $1 - C_6' h_n \geqslant \frac{1}{2}$(Due to $h_n \to 0$, the assumption is reasonable). Then

$$
\begin{aligned}
E_{n+1} &\leqslant \Big(1 + h_n \frac{C_5' + C_6'}{1 - h_n C_6'} + 4h_n C_6'\Big)E_n + 2h_n^2 C_7'\\
&\leqslant \prod_{i=0}^{n}(1 + h_i C_5)E_0 + \sum_{i=1}^{n}\prod_{j=i}^{n}(1 + h_j C_5)h_{i-1}^2 C_7 + h_n^2 C_7\\
&\leqslant e^{\sum\limits_{i=0}^{n} h_i C_5}\bar{K}h + h\sum_{i=1}^{n} e^{\sum\limits_{j=i}^{n} h_j C_5} h_{i-1}C_7 + hTC_7\\
&\leqslant h\big[\bar{K}e^{TC_5} + TC_7(e^{TC_5} + 1)\big]\\
&\leqslant hC_8,
\end{aligned} \tag{3.15}
$$

where $C_5 = 2C_5' + 6C_6'$, $C_7 = 2C_7'$ and $C_8 = \big\{\bar{K}e^{TC_5} + TC_7 e^{TC_5} + 1\big\}$. This implies

$$
(E_{n+1})^{\frac{1}{2}} \leqslant C_0 h^{\frac{1}{2}},
$$

i.e.,

$$
\max_{1\leqslant n\leqslant N}\big(E(\varepsilon_n)^2\big)^{\frac{1}{2}} \leqslant C_0 h^{\frac{1}{2}},
$$

with $C_0 = \sqrt{C_8}$. The theorem is completed. □

In the following, we state the second main theorem of this article. To be precise, we state the definition of mean-square stability from [1] at first:

Definition 3.4. *A numerical method is said to be mean-square stable (with respect to a given SPDE) if*

$$
\lim_{n\to\infty} E|X_n|^2 = 0. \tag{3.16}
$$

Theorem 3.5. *Under the condition* (2.1), *if* $\theta \in \left(\frac{|a|+|b|}{2|a|}, 1\right]$, *then the semi-implicit Euler method of Eq.*(1.2) *is mean-square stable, that is*

$$\lim_{n\to\infty} E|X_n|^2 = 0.$$

Proof. From (3.2), we can have

$$\begin{aligned}(1 - ah_n\theta)X_{n+1} = {} & [1 + ah_n(1-\theta) + c\Delta W_n]X_n \\ & + bh_n\theta X_{n-m+1} + [bh_n(1-\theta) + d\Delta W_n]X_{n-m}.\end{aligned}$$

Squaring both side of the above equality, we get

$$\begin{aligned}(1 - ah_n\theta)^2 X_{n+1}^2 \leqslant {} & [1 + ah_n(1-\theta) + c\Delta W_n]^2 X_n^2 \\ & + (bh_n\theta)^2 X_{n-m+1}^2 + [bh_n(1-\theta) + d\Delta W_n]^2 X_{n-m}^2 \\ & + 2[1 + ah_n(1-\theta) + c\Delta W_n] bh_n\theta X_n X_{n-m+1} \\ & + 2[1 + ah_n(1-\theta) + c\Delta W_n][bh_n(1-\theta) + d\Delta W_n] X_n X_{n-m} \\ & + 2[bh_n(1-\theta) + d\Delta W_n] bh_n\theta X_{n-m+1} X_{n-m}.\end{aligned}$$

It follows from $2\beta\gamma xy \leqslant |\beta\gamma|(x^2 + y^2)$, where $\beta, \gamma \in \mathcal{R}$, that

$$\begin{aligned}(1 - ah_n\theta)^2 X_{n+1}^2 \leqslant {} & [1 + ah_n(1-\theta) + c\Delta W_n]^2 X_n^2 \\ & + (bh_n\theta)^2 X_{n-m+1}^2 + [bh_n(1-\theta) + d\Delta W_n]^2 X_{n-m}^2 \\ & + |1 + ah_n(1-\theta)||bh_n\theta|(X_n^2 + X_{n-m+1}^2) \\ & + \big[|1 + ah_n(1-\theta)||bh_n(1-\theta)| + |cd|(\Delta W_n)^2\big](X_n^2 + X_{n-m}^2) \\ & + b^2h_n^2\theta(1-\theta)(X_{n-m+1}^2 + X_{n-m}^2) \\ & + 2\Delta W_n\big[d(1 + ah_n(1-\theta)) + bch_n(1-\theta)\big] X_n X_{n-m} \\ & + 2bch_n\theta\Delta W_n X_n X_{n-m+1} + 2bdh_n\theta\Delta W_n X_{n-m+1} X_{n-m}.\end{aligned} \tag{3.17}$$

Note that $E(\Delta W_n) = 0$, $E[(\Delta W_n)^2] = h_n$ and X_n, X_{n-m+1}, X_{n-m} are $\mathcal{F}_{t_n}$-measurable, hence

$$\begin{aligned}E(\Delta W_n X_i X_j) &= E\big[X_i X_j E(\Delta W_n|\mathcal{F}_{t_n})\big] = 0, \\ E[(\Delta W_n)^2 X_i^2] &= E\big[X_i^2 E((\Delta W_n)^2|\mathcal{F}_{t_n})\big] = h_n E(X_i)^2,\end{aligned} \tag{3.18}$$

where $i, j \in \{n, n-m+1, n-m\}$.
Let $Y_n = E|X_n|^2$, using (3.18), it is follows from (3.17) that

$$\begin{aligned}(1 - ah_n\theta)^2 Y_{n+1} \leqslant {} & P(a,b,c,d,h_n,\theta)Y_n + Q(a,b,h_n,\theta)Y_{n-m+1} \\ & + R(a,b,c,d,h_n,\theta)Y_{n-m},\end{aligned}$$

where

$$\begin{aligned}P(a,b,c,d,h_n,\theta) &= [1 + ah_n(1-\theta)]^2 + |1 + ah_n(1-\theta)||bh_n| \\ &\quad + |cdh_n| + c^2h_n, \\ Q(a,b,h_n,\theta) &= b^2h_n^2\theta^2 + |bh_n\theta|[|1 + ah_n(1-\theta)| + |bh|\theta(1-\theta)], \\ R(a,b,c,d,h_n,\theta) &= b^2h_n^2(1-\theta)^2 + d^2h_n + |cdh_n| \\ &\quad |bh_n(1-\theta)|[|1 + ah_n(1-\theta)| + |bh_n\theta|].\end{aligned} \tag{3.19}$$

Note that (2.1) implies $1 - ah_n\theta \neq 0$, then

$$Y_{n+1} \leqslant \frac{1}{(1-ah_n\theta)^2}\big[P(a,b,c,d,h_n,\theta) + Q(a,b,h_n,\theta) + R(a,b,c,d,h_n,\theta)\big] \max\big\{Y_n, Y_{n-m+1}, Y_{n-m}\big\}. \tag{3.20}$$

Let

$$\begin{aligned} A_n &= \frac{1}{(1-ah_n\theta)^2}\big[P(a,b,c,d,h_n,\theta) + Q(a,b,h_n,\theta) + R(a,b,c,d,h_n,\theta)\big], \\ A &= \frac{(|a|+|b|)(|a|(1-2\theta)+|b|)}{a^2\theta^2} + 1. \end{aligned} \tag{3.21}$$

Replacing (3.19), then

$$\begin{aligned} A_n &= \frac{1}{(1-ah_n\theta)^2}\big[a^2h_n^2 + 2ah_n(1-ah_n\theta) \\ &\quad + 2|1+ah_n(1-\theta)||bh_n| + 2|cdh_n| + b^2h_n^2 + (c^2+d^2)h_n\big] + 1 \\ &\leqslant \frac{1}{(1-ah_n\theta)^2}\big[a^2h_n^2 + 2ah_n(1-ah_n\theta) \\ &\quad + 2(1+|a|h_n(1-\theta))||bh_n| + 2|cdh_n| + b^2h_n^2 + (c^2+d^2)h_n\big] + 1 \\ &\leqslant \frac{1}{(1-ah_n\theta)^2}\big[(|a|+|b|)(|a|(1-2\theta)+|b|)h_n^2 \\ &\quad + (2a+2|b|+(|c|+|d|)^2)h_n\big] + 1 \\ &\leqslant \frac{(|a|+|b|)(|a|(1-2\theta)+|b|)}{a^2\theta^2} + 1 \\ &= A. \end{aligned} \tag{3.22}$$

If $\frac{|a|+|b|}{2|a|} < \theta \leqslant 1$, then

$$(|a|+|b|)(|a|(1-2\theta)+|b|) < 0. \tag{3.23}$$

It implies $A_n < A < 1$ for all n. Therefore

$$Y_{n+1} \leqslant A \max\{Y_n, Y_{n-m+1}, Y_{n-m}\},$$

hence

$$Y_n \leqslant A^{\frac{n-2}{m}+1} \max_{-m\leqslant i\leqslant 0}\{Y_i\},$$

i.e.,

$$\lim_{n\to\infty} E|X_n|^2 = 0.$$

□

4. Numerical experiments

In this section, we consider the following equation

$$dX(t) = [aX(t) + bX(qt)]dt + [cX(t) + dX(qt)]dW(t), \; t \in [0,T], \tag{4.1}$$

with initial value $X(0) = 2$.

Table 1 Errors of the method

r	1	2	3	4
ϵ_I	0.0021	0.0049	0.0090	0.0176
ϵ_{II}	0.0014	0.0026	0.0058	0.0124

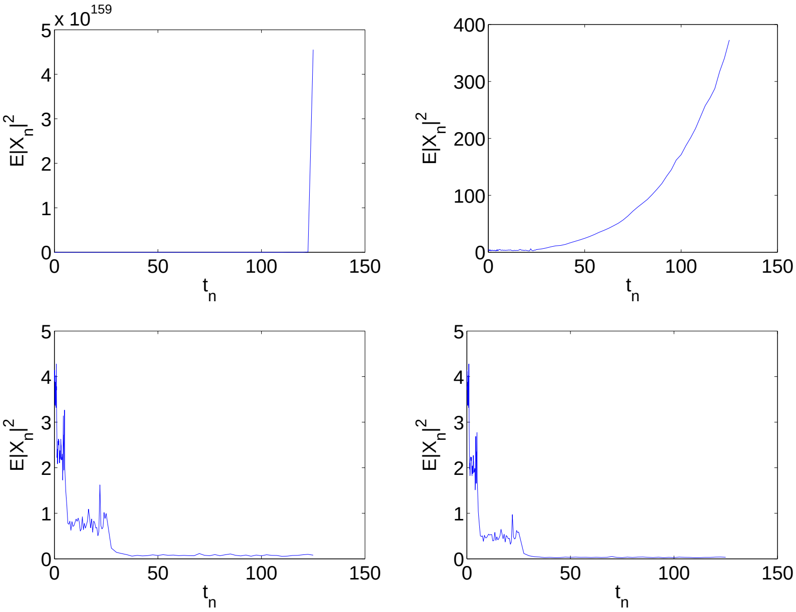

FIGURE 1. The influence of parameter θ on the stability of the numerical method.

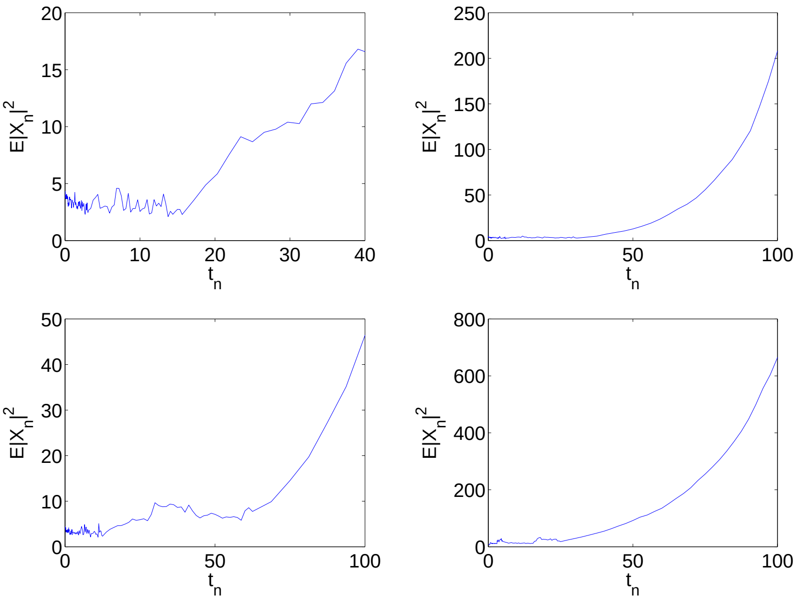

FIGURE 2. The influence of parameter h on the stability of the numerical method with fixed $\theta = 0.45$.

In a similar way as [2, 3, 4], we use discrete Brownian path over [0,1] with $\triangle t = 2^{-11}$. The solution of (4.1) can be written as a closed-form expression involving a stochastic integral. For simplicity, we take the numerical solution of Euler-Maruyama scheme with $h = \triangle t$ as good approximation of explicit solution.

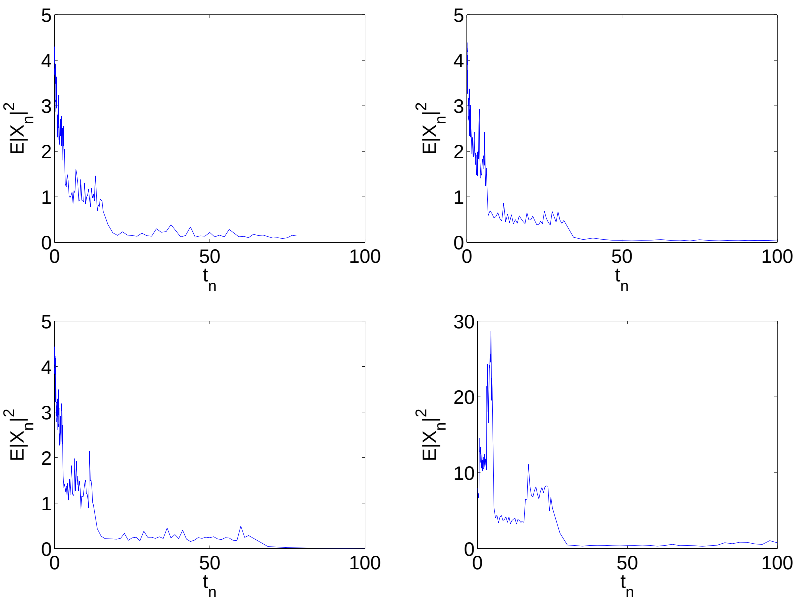

FIGURE 3. The influence of parameter h on the stability of the numerical method with fixed $\theta = 0.8$.

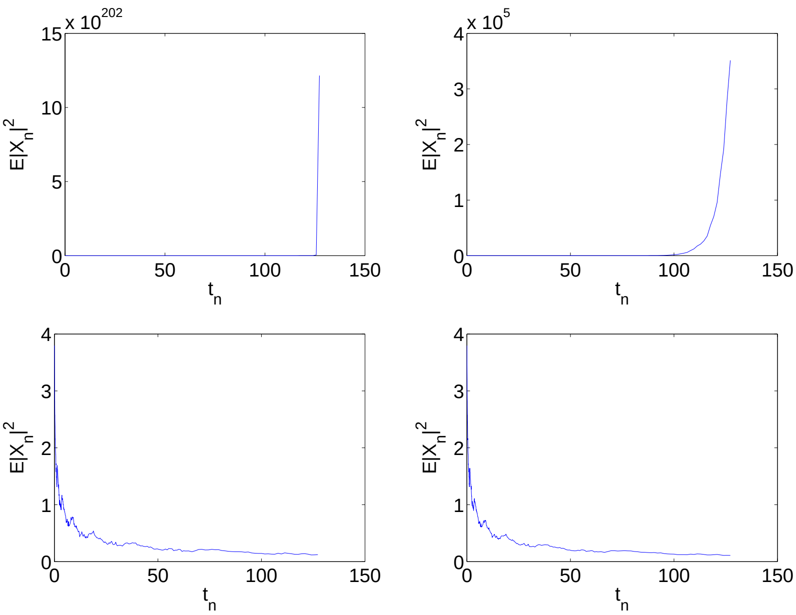

FIGURE 4. upper left: $\theta = 0$, upper right $\theta = 0.3$, lower left $\theta = 0.8$, lower right $\theta = 1$.

One of our tests illustrate the theoretical order of convergence. In this case the mean-square error $\epsilon|X(T) - X_N|^2$ at the final time T was estimated in the following way. A set of 20 blocks each containing 100 outcomes $(\omega_{i,j} : 1 \leqslant i \leqslant 20, 1 \leqslant j \leqslant 100)$, are simulated and for each block the estimator $\epsilon_i = 1/100 \sum_{j=1}^{100} |X(T, \omega_{i,j}) -$

$X_N(\omega_{i,j})|^2$ is formed. In Table 1 above ϵ denotes the mean of this estimator, which was itself estimated in the usual way: $\epsilon = 1/20 \sum_{i=1}^{20} \epsilon_i$.

We use the set of parameters I: $a = -4, b = 1, c = 0, d = 1, q = 0.5$ in (4.1) and method (3.2) with $\theta = 0.8, T_0 = 1, k = 2(i.e. T = 4)$, II: $a = -2, b = 0.5, c = 0.5, d = 0, q = 0.2$ in (4.1) and method (3.2) with $\theta = 0.8, T_0 = 1, k = 1(i.e. T = 5)$ and choose stepsize $h = 2^r \Delta t, r = 1, 2, 3, 4$ on the interval $[t_0, t_m]$. It is easy to see, the figures in the table are compatible with the results give in Theorem 4.3.

In the following tests, we use the coefficients $a = -10, b = 1, c = 0.5, d = 4$ and $q = 0.2$ (in the case $a < -|b| - \frac{1}{2}(|c| + |d|)^2$ and $\theta \in (0.55, 1]$) to show the influence of parameter θ and stepsize h on mean square stability of semi-implicit Euler method. The data used in all figures are obtained by the mean square of data by 100 trajectories, that is $\omega_i : 1 \leqslant i \leqslant 100, X_n = 1/100 \sum_{i=1}^{100} |X_n(\omega_i)|^2$. In all figures t_n denotes the mesh-point.

In Fig. 1, we fix the stepsize $h_1 = 0.1$ and vary the parameter θ in order to observe some stability behavior of the method. Using four parameter: $\theta = 0$(upper left), $\theta = 0.4$ (upper right), $\theta = 0.8$ (lower left) and $\theta = 1$ (lower right), we observe that the method is unstable on $\theta = 0$ and $\theta = 0.4$, but it is stable on $\theta = 0.8$ and $\theta = 1$.

We choose the fixed parameter $\theta = 0.45$ in Fig. 2 and $\theta = 0.8$ in Fig. 3 and vary the stepsize on interval $[t_0, t_m]$ to show the influence of the stepsize. Using the stepsizes: $h_{11} = 0.0005$ (upper left), $h_{12} = 0.005$ (upper right), $h_{13} = 0.05$ (lower left) and $h_{14} = 0.5$ (lower right), we observe that the stability behavior do not change when the stepsize is varied.

In Fig. 4 we choose another set of parameters $a = -10, b = 2, c = 0.5, d = 0.5, q = 0.5$ and let $h = 0.05$. It is also shown that the method is mean square stable for $\theta \in \left(\frac{|a|+|b|}{2|a|}, 1\right] = (0.6, 1]$, which is satisfied the condition of theorem.

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Department of mathematics, Harbin Institute of Technology, Harbin, 150001, P.R.China
E-mail: `xiaoyhit@126.com, songmh@lsec.cc.ac.cn and mzliu@hit.edu.cn`