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ON ERROR ESTIMATES OF THE PRESSURE-CORRECTION PROJECTION METHODS FOR THE TIME-DEPENDENT NAVIER-STOKES EQUATIONS

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Abstract. In this paper, we present a new pressure-correction projection scheme for solving the time-dependent Navier-Stokes equations, which is based on the Crank-Nicolson extrapolation method in the time discretization. Error estimates for the velocity and the pressure of semidiscretized scheme are derived by interpreting the projection scheme as second-order time discretization of a perturbed system which approximates the incompressible Navier-Stokes equations.

Key Words. Navier-Stokes equations, projection method, pressure-correction, Crank-Nicolson extrapolation scheme, error estimates.

1. Introduction

Let Ω be a bounded domain in R^2 assumed to have a sufficiently smooth boundary $\partial\Omega$. Now we consider the time-dependent Navier-Stokes problem

$$(1)\qquad \begin{cases} u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \ (x,t)\in\Omega\times(0,T], \\ \ \text{div}\ u = 0, \ (x,t)\in\Omega\times(0,T], \\ u(x,0) = u_0(x), \ x\in\Omega, \end{cases}$$

where $u = u(x,t)$ represents the velocity vector of a viscous incompressible fluid, $p = p(x,t)$ the pressure, $f = f(x,t)$ the prescribed body force. The problem (1) should be completed with an appropriate boundary condition for the velocity u. For the sake of convenience, we consider the homogeneous Dirichlet boundary condition, i.e. $u|_{\partial\Omega} = 0, \forall\ t\in(0,T]$.

It is well known that the numerical solution of problem (1) involves several major difficulties, and the crucial difficult is that the unknowns u and p are coupled through the incompressibility condition div $u = 0$. Generally, in order to overcome this difficulty, people often relax the incompressibility constraint in an appropriate way, resulting in a class of pseudo-compressibility methods, among which are the penalty method, the artificial compressibility method, the pressure stabilization method and the projection method, see for instance [1, 2, 4, 7, 12, 17, 20, 22]. The projection method is perhaps the most efficient and the easiest to implement for solving the time-dependent Navier-Stokes equations.

The original projection method was introduced by Chorin [4] and Temam [26] respectively in the late 60s. The original method is simple, but is not satisfactory

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since its convergence rate is irreducibly limited to $O(\delta t)$. In order to solve these problems, many literatures are put into the construction, analysis and implementation of projection-type schemes, (see for instance [6, 8, 10, 11, 16, 18, 20, 25, 27]). An important class of projection methods is the so-called pressure correction methods introduced in [3, 8, 28]. These schemes consist of two substeps per time step: the pressure is treated explicitly in the first substep and corrected in the second substep by projecting the intermediate velocity onto the space of divergence-free fields. These schemes are widely used in practice and have been rigorously analyzed in [5, 9, 24].

The goal of this paper is to present a rigorous error analysis for the standard incremental pressure-correction scheme, which is based on the Crank-Nicolson extrapolation method in the time discretization. We prove the stability and second order convergence in the L^2-norm of the velocity, and first order convergence in the L^∞-norm of the pressure. Our results are consistent with the reference [23], it appear to be the best possible under the general context considered in this paper.

The remainder of this paper is organized as follows. In Section 2, we introduce some notations and recall important results which are used repeatedly in the core of this paper. In Section 3, we give the new pressure-correction projection scheme for solving the incompressible time-dependent Navier-Stokes equations, and we prove the stability of the scheme. In Section 4, we derive some additional a priori estimates for (u^n, p^n) and perform some error analysis.

2. Preliminaries

For the mathematical setting of problem (1), we introduce the following Hilbert spaces:

$$X = H_0^1(\Omega)^2,\ Y = L^2(\Omega)^2,\ W = L_0^2(\Omega) = \left\{ q \in L^2(\Omega); \int_\Omega q(x)dx = 0 \right\}.$$

The space Y is equipped with the usual L^2-scalar product $(\cdot,\cdot)$ and L^2-norm $\|\cdot\|_0$. Denote by $\|\cdot\|_r$ the norm on Sobolev spaces $H^r(\Omega)^2$, where $r = 1, 2$. We recall that if Ω is bounded in some direction then the Poincar*é* inequality holds:

$$\|v\|_0 \le c(\Omega)\|\nabla v\|_0, \quad \forall v \in X.$$

The quotient space $H^1(\Omega)/R$ is defined as follows: the element of the quotient space is equivalence classes. That is $\forall v \in H^1(\Omega)$, the equivalence class of v is often denoted

$$\hat{v} = \{u | u \in H^1(\Omega), u - v \in R\}.$$

Next, let the closed subset V of X be given by

$$V = \{v \in X;\ \text{div}\ v = 0 \text{ in } \Omega\},$$

and we denote by H the closed subset of Y, one can show that

$$H = \{v \in Y;\ \text{div}\ v = 0 \text{ in } \Omega \text{ and } v \cdot \vec{n}|_{\partial\Omega} = 0\}.$$

We refer reader to [7, 13-15] for details on these spaces. And P_H is the orthogonal projector in Y onto H, i.e.

$$(u - P_H u, v) = 0, \quad \forall u \in Y,\ v \in H.$$

The following inequalities (cf. [27])

$$\|P_H v\|_i \le c(\Omega)\|v\|_i, \quad \forall v \in H^1(\Omega)^2,\ i = 0, 1 \tag{2}$$

hold. In the following, we use c or C as a generic positive constant which depends only on Ω, ν, T and constants from various Sobolev inequalities. And we denote N as a generic positive constant which may additionally depends on u_0 and/or f.

We also introduce the trilinear forms $b(\cdot,\cdot,\cdot)$ and $\tilde{b}(\cdot,\cdot,\cdot)$

$$b(u,v,w) = ((u\cdot\nabla)v,w), \quad \tilde{b}(u,v,w) = b(u,v,w) + \frac{1}{2}((\nabla\cdot u)v,w).$$

We note that

$$\tilde{b}(u,v,v) = 0, \quad \forall u \in H^1(\Omega)^2, v \in X. \tag{3}$$

Thus, we define $\tilde{B}(u,v)$ such that $\langle \tilde{B}(u,v), w\rangle = \tilde{b}(u,v,w),\ \forall w \in X$.

Moreover, we will use the inequality below

$$\begin{aligned} \tilde{b}(u,v,w) \quad &\le \quad c\|u\|_0^{\frac{1}{2}}\|u\|_1^{\frac{1}{2}}\|\nabla v\|_0^{\frac{1}{2}}\|\Delta v\|_0^{\frac{1}{2}}\|w\|_0, \\ &\qquad \forall u \in H^1(\Omega)^2 \cap H, v \in H^2(\Omega)^2 \cap X, w \in Y. \end{aligned} \tag{4}$$

In most cases, the following inequality is sufficient for our purposes. They can be proved by using a combination of integration by parts, Hölder's inequality and Sobolev inequalities.

(5)

$$\tilde{b}(u,v,w) \le \begin{cases} c\|u\|_0\|\Delta v\|_0\|\nabla w\|_0,\ \forall v \in H^2(\Omega)^2 \cap X,\ u \in H^1(\Omega)^2,\ w \in X, \\ c\|\Delta u\|_0\|\nabla v\|_0\|w\|_0,\ \forall u \in H^2(\Omega)^2 \cap X,\ v \in X,\ w \in Y, \\ c\|\nabla u\|_0\|\Delta v\|_0\|w\|_0,\ \forall\ v \in H^2(\Omega)^2 \cap X,\ u \in X,\ w \in Y. \end{cases}$$

The following Lemma of Gronwall type will be repeatedly used (see, for instance, [15] for a proof).

Lemma 2.1.(Discrete Gronwall Lemma). Let B, k be nonnegative numbers and a_n, b_n, c_n, d_n be nonnegative sequences satisfying

$$a_m + k\sum_{n=0}^{m} b_n \le k\sum_{n=0}^{m} a_n d_n + k\sum_{n=0}^{m} c_n + B, \quad \forall\ 0 \le m \le \frac{T}{k},$$

where $k\sum_{n=0}^{[T/k]} d_n \le M$. Assume $kd_n < 1$ and let $\sigma = \max_{0\le n\le \frac{T}{k}} (1-kd_n)^{-1}$. Then

$$a_m + k\sum_{n=0}^{m} b_n \le \exp(\sigma M)\left(k\sum_{n=0}^{m} c_n + B\right), \quad \forall\ m \le \frac{T}{k}.$$

Now, we give some assumptions about the data and the solutions of problem (1), which will be used throughout the rest of the paper.

We assume that u_0 and f are sufficiently smooth, more precisely

$$u_0 \in H^2(\Omega)^2 \cap V, \quad f \in C([0,T];Y). \tag{6}$$

Under the above hypotheses, it is proved in [9] that

$$\|\Delta u(t)\|_0 + \|u_t(t)\|_0 + \|\nabla p(t)\|_0 \le N,\ \forall\ t \in [0,T]. \tag{7}$$

We note that higher regularity at $t = 0$ requires that the data u_0 and $f(0)$ satisfy certain nonlocal compatibility conditions, but the smoothing property of the Navier-Stokes equations makes the solution become as smooth as the data allows for $t > 0$.

In particular, we have the following regularity result, which is sufficient for our error analysis (see, for instance, Theorem 2.4 in [14]).
Lemma 2.2. In addition to (6), we assume that

$$f_t,\ f_{tt} \in C([0,T];Y). \tag{8}$$

Then for any $t_0 \in (0,T)$, the solution of problem (1) satisfies

$$\begin{aligned} &\|u_{tt}(t)\|_0^2 + \|\Delta u_t(t)\|_0^2 + \|\nabla p_t(t)\|_0^2 \\ &\quad + \int_{t_0}^t (\|u_{ttt}(s)\|_0^2 + \|\Delta u_{tt}(s)\|_0^2 + \|\nabla p_{tt}(s)\|_0^2) ds \le N,\ \forall\, t \in [t_0, T]. \end{aligned} \tag{9}$$

3. The pressure correction scheme

In this section, there are two focuses. First, we consider the following version of the pressure correction scheme. And then, we will give the stability of the scheme.

Let $u^0 = u(t_0) \in H^2(\Omega)^2 \cap X$, and $p^0 = p(t_0) \in W$ (Which can be obtained by solving $\int_\Omega \nabla p \cdot \nabla q dx = \int_\Omega (f + \nu\Delta u - (u\cdot\nabla)u)\cdot\nabla q dx,\ \forall q \in H^1(\Omega)$ at $t = t_0$) be given, set (u^n, p^n) are the nth order approximation to $(u(t_0+nk), p(t_0+nk))$.

Note that we will need the $(u^1, \tilde{u}^1, p^1)$ to start the scheme, so we solve $(u^1, \tilde{u}^1, p^1)$ from the following pressure-correction projection scheme:

$$\begin{cases} \dfrac{1}{k}\left(\tilde{u}^1 - u^0\right) - \nu\Delta\tilde{u}^{\frac12} + \tilde{B}(\tilde{u}^{\frac12}, \tilde{u}^{\frac12}) + \nabla p^0 = f(t_{\frac12}), \\ (\tilde{u}^1 + u^0)|_{\partial\Omega} = 0, \end{cases} \tag{10}$$

and

$$\begin{cases} u^1 - \tilde{u}^1 + \beta k \nabla(p^1 - p^0) = 0, \\ \text{div } u^1 = 0, \\ u^1 \cdot \vec{n}|_{\partial\Omega} = 0. \end{cases} \tag{11}$$

For $n \ge 1$, using problem (1), the first substep accounting for viscous convection-diffusion equations is

$$\begin{cases} \dfrac{1}{k}\left(\tilde{u}^{n+1} - u^n\right) - \nu\Delta\tilde{u}^{n+\frac12} + \tilde{B}(\phi(u^{n+1}), \tilde{u}^{n+\frac12}) + \nabla p^n = f(t_{n+\frac12}), \\ (\tilde{u}^{n+1} + u^n)|_{\partial\Omega} = 0, \end{cases} \tag{12}$$

and the second substep accounting for incompressibility is

$$\begin{cases} u^{n+1} - \tilde{u}^{n+1} + \beta k \nabla(p^{n+1} - p^n) = 0, \\ \text{div } u^{n+1} = 0, \\ u^{n+1} \cdot \vec{n}|_{\partial\Omega} = 0, \end{cases} \tag{13}$$

where $\tilde{u}^{n+\frac12} = \frac12(\tilde{u}^{n+1} + u^n),\ \phi(u^{n+1}) = (\frac32 u^n - \frac12 u^{n-1})$ and β is a constant to be determined.

From (13), we infer that $u^{n+1} = P_H\tilde{u}^{n+1}$, which explains why we call (12)-(13) a projection scheme. Actually, we observe that $\nabla(p^{n+1} - p^n)\cdot\vec{n}|_{\partial\Omega} = 0$ which implies that

$$\nabla p^{n+1} \cdot \vec{n}|_{\partial\Omega} = \nabla p^n \cdot \vec{n}|_{\partial\Omega} = \cdots = \nabla p^0 \cdot \vec{n}|_{\partial\Omega}.$$

As a major deficiency of projection methods, they often suffer from reduced accuracy for pressure iterations caused by nonphysical boundary data.

To simplify the notation, we denote $t_n = t_0 + nk$ and for any function $\omega(t)$ and any series a^n and $\tilde{a}^n$, we denote

$$\tilde{\omega}(t_{n+\frac12}) = \frac12(\omega(t_{n+1}) + \omega(t_n)),\ a^{n+\frac12} = \frac12(a^{n+1} + a^n),\ \tilde{a}^{n+\frac12} = \frac12(\tilde{a}^{n+1} + a^n).$$

We also denote

$$e^{n+1}=u(t_{n+1})-u^{n+1},\ \tilde{e}^{n+1}=u(t_{n+1})-\tilde{u}^{n+1},\ q^{n+1}=p(t_{n+1})-p^{n+1}.$$

It is noticed that u^{n+1} can be eliminated from (12)-(13). Replacing u^n in (12) by $\tilde{u}^n-\beta k\nabla(p^n-p^{n-1})$ (obtained from (13)), we have

(14)

$$\begin{cases}\frac{1}{k}\left(\tilde{u}^{n+1}-\tilde{u}^n\right)-\frac{\nu}{2}\Delta(\tilde{u}^{n+1}+P_H\tilde{u}^n)+\tilde{B}(\frac{3}{2}P_H\tilde{u}^n-\frac{1}{2}P_H\tilde{u}^{n-1},\frac{1}{2}(\tilde{u}^{n+1}+P_H\tilde{u}^n)) \\ \qquad +(1+\beta)\nabla p^n-\beta\nabla p^{n-1}=f(t_{n+\frac{1}{2}}), \\ (\tilde{u}^{n+1}+P_H\tilde{u}^n)|_{\partial\Omega}=0.\end{cases}$$

We also derive from (13) that

$$\text{div } \tilde{u}^{n+1}-\beta k\Delta(p^{n+1}-p^n)=0,\ \frac{\partial p^{n+1}}{\partial\vec{n}}|_{\partial\Omega}=\frac{\partial p^n}{\partial\vec{n}}|_{\partial\Omega}. \tag{15}$$

From above we can find that the scheme (14)-(15) with a decoupled system for $(\tilde{u}^{n+1},p^{n+1})$ is a second-order time discretization, this is the advantage of the scheme, to the perturbed system (see similar interpretations in [19] and [21]):

$$u^\epsilon_t-\nu\Delta u^\epsilon+\tilde{B}(u^\epsilon,u^\epsilon)+\nabla p^\epsilon=f,\quad u^\epsilon|_{\partial\Omega}=0, \tag{16}$$

$$\text{div}\, u^\epsilon-\epsilon\Delta\, p^\epsilon_t\ =0,\quad \frac{\partial p^\epsilon_t}{\partial\vec{n}}|_{\partial\Omega}=0, \tag{17}$$

with $\epsilon\sim\frac{1}{2}k^2$. On the other hand, when $\epsilon<<1$, the perturbed system (16)-(17) can be regarded as an approximation to the problem (1). In reference [23], the following theorem has been proved.

Theorem 3.1. Let $f,\ f_t,\ f_{tt}\in C([0,T];Y),u_0\in H^2(\Omega)^2\cap X$ and div $u_0=0$. Then for $t_0\in(0,T)$ sufficiently small, let (u,p) be the unique strong solution of problem (1) in $[0,T]$ and (u^ϵ,p^ϵ) be the solution of (16)-(17) with the initial data $(u^\epsilon(t_0),p^\epsilon(t_0))=(u(t_0),p(t_0))$. Then for all $t\in[t_0,T]$, we have

$$\begin{aligned}\int_{t_0}^t\|u(s)-u^\epsilon(s)\|_0^2ds \quad &+\quad \sqrt{\epsilon}\|u(t)-u^\epsilon(t)\|_0^2 \\ &+\quad \epsilon(\|u(t)-u^\epsilon(t)\|_1^2+\|p(t)-p^\epsilon(t)\|_0^2)\le N\epsilon^2,\end{aligned}$$

where N is a constant depending on the data u_0, f and t_0.

From Theorem 3.1, we can speculate that the convergence rate of the scheme (14)-(15) is second-order in $L^2([t_0,T];Y)$ for the velocity and first order in $L^\infty([t_0,T];L^2(\Omega))$ for the pressure. In view of Theorem 3.1, we expect to prove the following error estimates for (12)-(13):

$$k\sum_{n=1}^m\|u(t_n)-u^n\|_0^2+k^2\|\nabla(u(t_m)-u^m)\|_0^2+k^2\|p(t_m)-p^m\|_0^2\le Nk^4, \tag{18}$$

for all $1\le m\le M$, where $M=[\frac{T-t_0}{k}]$ denotes the integer part of $\frac{T-t_0}{k}$.

We first establish a stability result for the scheme (12)-(13). The techniques used here will be repeatedly used later in different circumstances.

Lemma 3.2. Let $\beta>\frac{1}{2}$, if u^{n+1} and $\tilde{u}^{n+1}$ are the solutions of problems (12) and (13), then for $m=1,2,\cdots,M-1$, the following inequality holds:

$$\begin{aligned}(1-\frac{1}{2\beta})\|u^{m+1}\|_0^2 \quad &+\quad \frac{\nu k}{4}\sum_{n=1}^m\|\nabla(\tilde{u}^{n+1}+P_H\tilde{u}^n)\|_0^2 \\ &\le\quad C(\|\tilde{u}^1\|_0^2+\frac{\beta k^2}{2}\|\nabla p^1\|_0^2+\|f\|^2_{C([0,T],H^{-1}(\Omega)^2)}).\end{aligned}$$

Proof. Taking the inner product of (14) with $k(\tilde{u}^{n+1}+P_H\tilde{u}^n)$ and of (15) with $k((1+\beta)p^n-\beta p^{n-1})$ and summing up the two relations, since div $P_H\tilde{u}^n=0$ and (3), we find that:

$$\begin{aligned}(\tilde{u}^{n+1}-\tilde{u}^n,\tilde{u}^{n+1}+P_H\tilde{u}^n)&+\frac{\nu k}{2}\|\nabla(\tilde{u}^{n+1}+P_H\tilde{u}^n)\|_0^2\\&+\beta k^2((1+\beta)\nabla p^n-\beta\nabla p^{n-1},\nabla(p^{n+1}-p^n))\\&=k<f(t_{n+\frac{1}{2}}),\tilde{u}^{n+1}+P_H\tilde{u}^n>\\&\le Ck\|f(t_{n+\frac{1}{2}})\|_{-1}^2+\frac{\nu k}{4}\|\nabla(\tilde{u}^{n+1}+P_H\tilde{u}^n)\|_0^2. \quad (19)\end{aligned}$$

We derive from (13) and (15) that

$$\begin{aligned}P_H\tilde{u}^n-\tilde{u}^n&=-\beta k\nabla(p^n-p^{n-1}),\\(\tilde{u}^{n+1}-\tilde{u}^n,\nabla\gamma)&=\beta k(\nabla(p^{n+1}-2p^n+p^{n-1}),\nabla\gamma),\quad\forall\,\gamma\in H^1(\Omega)/R,\end{aligned}$$

therefore,

$$\begin{aligned}(\tilde{u}^{n+1}-\tilde{u}^n,\tilde{u}^{n+1}+P_H\tilde{u}^n)&=\|\tilde{u}^{n+1}\|_0^2-\|\tilde{u}^n\|_0^2-\beta k(\tilde{u}^{n+1}-\tilde{u}^n,\nabla(p^n-p^{n-1}))\\&=\|\tilde{u}^{n+1}\|_0^2-\|\tilde{u}^n\|_0^2-\beta^2k^2(\nabla(p^{n+1}-2p^n+p^{n-1}),\nabla(p^n-p^{n-1}))\\&=\|\tilde{u}^{n+1}\|_0^2-\|\tilde{u}^n\|_0^2-\frac{\beta^2k^2}{2}(\|\nabla(p^{n+1}-p^n)\|_0^2-\|\nabla(p^n-p^{n-1})\|_0^2)\\&\quad+\frac{\beta^2k^2}{2}\|\nabla(p^{n+1}-2p^n+p^{n-1})\|_0^2.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\beta k^2((1+\beta)\nabla p^n-\beta\nabla p^{n-1},\nabla(p^{n+1}-p^n))\\=\;&\beta k^2(\nabla p^n,\nabla(p^{n+1}-p^n))+\beta^2k^2(\nabla(p^n-p^{n-1}),\nabla(p^{n+1}-p^n))\\=\;&\frac{\beta k^2}{2}(\|\nabla p^{n+1}\|_0^2-\|\nabla p^n\|_0^2-\|\nabla(p^{n+1}-p^n)\|_0^2)\\&+\frac{\beta^2k^2}{2}(\|\nabla(p^{n+1}-p^n)\|_0^2+\|\nabla(p^n-p^{n-1})\|_0^2)-\frac{\beta^2k^2}{2}\|\nabla(p^{n+1}-2p^n+p^{n-1})\|_0^2.\end{aligned}$$

We derive from (15) that

$$\beta^2k^2\|\nabla(p^{m+1}-p^m)\|_0^2\le\|\tilde{u}^{m+1}\|_0^2.$$

Summing up (19) for $n=1,\cdots,m$ and collecting the above inequalities, since $\beta>\frac{1}{2}$, we arrive at

$$\begin{aligned}(1-\frac{1}{2\beta})\|\tilde{u}^{m+1}\|_0^2&+\frac{\nu k}{4}\sum_{n=1}^{m}\|\nabla(\tilde{u}^{n+1}+P_H\tilde{u}^n)\|_0^2\\&\le\|\tilde{u}^1\|_0^2+\frac{\beta k^2}{2}\|\nabla p^1\|_0^2+C\|f\|_{C([0,T];H^{-1}(\Omega)^2)}^2.\end{aligned}$$

From (6) and the below lemma, we obtain the scheme is stable.

4. Error estimate for the time discrete case

In this section, we consider error estimate of the time discrete scheme. Now we begin with a preliminary lemma for the truncation error which is defined by

$$\begin{aligned}R^n&=\frac{1}{k}\big(u(t_{n+1})-u(t_n)\big)-\nu\Delta\tilde{u}(t_{n+\frac{1}{2}})\\&+(\tilde{u}(t_{n+\frac{1}{2}})\cdot\nabla)\tilde{u}(t_{n+\frac{1}{2}})+\nabla\tilde{p}(t_{n+\frac{1}{2}})-f(t_{n+\frac{1}{2}}). \quad (20)\end{aligned}$$

Lemma 4.1. Under the assumptions of (6) and (8), there holds:

$$k\sum_{n=0}^{m}\|R^n\|_{-1}^2 \le Nk^4\int_{t_0}^{T}(\|u_{ttt}(s)\|_{-1}^2+\|\nabla u_{tt}(s)\|_0^2+\|p_{tt}(s)\|_0^2)\mathrm{d}s, \tag{21}$$

$$\|R^m\|_0 \le Nk(\max_{t\in[t_0,T]}\|u_{tt}(t)\|_0+\max_{t\in[t_0,T]}\|\Delta u_t(t)\|_0+\max_{t\in[t_0,T]}\|\nabla p_t(t)\|_0), \quad \forall\ 0\le m\le M-1, \tag{22}$$

which was proved in [20, 24].

Lemma 4.2. Let $\beta > \frac{1}{2}$ and under the assumptions of (6) and (8), then there holds the following inequality

$$\|u^1-u(t_1)\|_0^2+k\nu\|\nabla(u^1-u(t_1))\|_0^2+\frac{\beta k^2}{2}\|\nabla(p^1-p(t_1))\|_0^2 \le Nk^4, \tag{23}$$

where (u^1,p^1) is the solution of (10)-(11).

Proof. When $n=0$, R^0 is the truncation error defined by

$$R^0 =\frac{1}{k}\big(u(t_1)-u(t_0)\big)-\nu\Delta\tilde{u}(t_{\frac{1}{2}})+B(\tilde{u}(t_{\frac{1}{2}}),\tilde{u}(t_{\frac{1}{2}}))+\nabla\tilde{p}(t_{\frac{1}{2}})-f(t_{\frac{1}{2}}). \tag{24}$$

Subtracting (10) from (24), we obtain the error equations

$$\frac{\tilde{e}^1-e^0}{k}-\nu\Delta\tilde{e}^{\frac{1}{2}}+\nabla(\tilde{p}(t_{\frac{1}{2}})-p^0)=Q^0+R^0, \tag{25}$$

where

$$Q^0=\tilde{B}(\tilde{u}^{\frac{1}{2}},\tilde{u}^{\frac{1}{2}})-B(\tilde{u}(t_{\frac{1}{2}}),\tilde{u}(t_{\frac{1}{2}}))=-\tilde{B}(\tilde{u}^{\frac{1}{2}},\tilde{e}^{\frac{1}{2}})-\tilde{B}(\tilde{e}^{\frac{1}{2}},\tilde{u}(t_{\frac{1}{2}})).$$

We derived from (11) that

$$\frac{e^1-\tilde{e}^1}{k}=\beta\nabla(p^1-p^0). \tag{26}$$

Let $\delta=\beta-\frac{1}{2}>0$, using the Lagrange mean value theorem, we have $p(t_1)-p(t_0)=kp_t(\xi_0)$, which parameter ξ_0 is from t_0 to t_1. Taking the inner product of (25) with $2k\tilde{e}^{\frac{1}{2}}$, we derived from (3) and (5) that

$$\begin{aligned}
&\|\tilde{e}^1\|_0^2-\|e^0\|_0^2+2k\nu\|\nabla\tilde{e}^{\frac{1}{2}}\|_0^2 \\
&= 2k(Q^0+R^0,\tilde{e}^{\frac{1}{2}})+2k(\nabla(p^0-\tilde{p}(t_{\frac{1}{2}})),\tilde{e}^{\frac{1}{2}}) \\
&\le -2k\tilde{b}(\tilde{u}^{\frac{1}{2}},\tilde{e}^{\frac{1}{2}},\tilde{e}^{\frac{1}{2}})-2k\tilde{b}(\tilde{e}^{\frac{1}{2}},\tilde{u}(t_{\frac{1}{2}}),\tilde{e}^{\frac{1}{2}})+2k(R^0,\tilde{e}^{\frac{1}{2}})+2k(\nabla(p^0-\tilde{p}(t_{\frac{1}{2}})),\tilde{e}^{\frac{1}{2}}) \\
&\le \frac{\delta}{2\delta+1}\|\tilde{e}^1\|_0^2+c\|e^0\|_0^2+Nk^2(\|R^0\|_0^2+\|\nabla\tilde{e}^{\frac{1}{2}}\|_0^2)+2k(\nabla(p^0-\tilde{p}(t_{\frac{1}{2}})),\tilde{e}^{\frac{1}{2}}).
\end{aligned} \tag{27}$$

On the other hand, we have

$$\|e^1\|_0^2-\|\tilde{e}^1\|_0^2+\frac{2\beta-1}{2\beta}\|e^1-\tilde{e}^1\|_0^2=\frac{k}{2}(\nabla(p^1-p^0),\tilde{e}^1). \tag{28}$$

Adding (27) and (28), we arrive to

$$\begin{aligned}
\|e^1\|_0^2 \quad - &\quad \|e^0\|_0^2+\frac{2\beta-1}{2\beta}\|e^1-\tilde{e}^1\|_0^2+2k\nu\|\nabla\tilde{e}^{\frac{1}{2}}\|_0^2 \\
\le &\quad \frac{\delta}{2\delta+1}\|\tilde{e}^1\|_0^2+c\|e^0\|_0^2+Nk^2(\|R^0\|_0^2+\|\nabla\tilde{e}^{\frac{1}{2}}\|_0^2)-\frac{k}{2}(\nabla(q^1+q^0),\tilde{e}^1).
\end{aligned}$$

We infer from (26) that

$$\begin{aligned} -\frac{k}{2}(\nabla(q^1+q^0),\tilde{e}^1) &= \frac{\beta k^2}{2}(\nabla(q^1+q^0),\nabla(p^1-p^0)) \\ &\leq -\frac{\beta k^2}{4}\|\nabla q^1\|_0^2+\frac{3\beta k^2}{4}\|\nabla q^0\|_0^2+\frac{\beta k^4}{2}\|\nabla p_t(\xi_0)\|_0^2, \end{aligned}$$

since $\|\tilde{e}^1\|_0^2 = \|e^1\|_0^2 + \|e^1 - \tilde{e}^1\|_0^2$, $e^0 = 0$, $q^0 = 0$ and Lemma 4.1, that for k sufficiently small,

$$\frac{\delta}{2\delta+1}\|\tilde{e}^1\|_0^2 + k\nu\|\nabla\tilde{e}^{\frac{1}{2}}\|_0^2 + \frac{\beta k^2}{4}\|\nabla q^1\|_0^2 \leq Nk^4.$$

By using (2), we have

$$\|e^1\|_0^2 + k\nu\|\nabla e^1\|_0^2 + \frac{\beta k^2}{2}\|\nabla q^1\|_0^2 \leq Nk^4.$$

Lemma 4.3. Let $\beta > \frac{1}{2}$, and under the assumptions of (6) and (8), then there holds inequality

$$\|\nabla u^{m+1}\|_0^2 + \|\nabla\tilde{u}^{m+1}\|_0^2 + \|\Delta(\tilde{u}^{m+1}+u^m)\|_0^2 + \|\nabla p^{m+1}\|_0^2 \leq N,\ \forall\ 1\leq m\leq M-1.$$

Proof. Subtracting (12) from (20), we get the following error equations:

$$\begin{cases} \frac{1}{k}\left(\tilde{e}^{n+1}-e^n\right) - \nu\Delta\tilde{e}^{n+\frac{1}{2}} + \nabla(\tilde{p}(t_{n+\frac{1}{2}}) - p^n) = Q^n + R^n, \\ \tilde{e}^{n+\frac{1}{2}}|_{\partial\Omega} = 0, \end{cases} \tag{29}$$

on the other hand, we derive from the equations (13) that

$$\begin{cases} \frac{1}{k}\left(e^{n+1}-\tilde{e}^{n+1}\right) = \beta\nabla(p^{n+1}-p^n), \\ \operatorname{div} e^{n+1} = 0, \\ e^{n+1}\cdot\vec{n}|_{\partial\Omega} = 0. \end{cases} \tag{30}$$

Rewrite the equations (30) as

$$\begin{aligned} -(\tilde{e}^{n+1},\nabla\gamma) \quad &+ \quad \beta k(\nabla(q^{n+1}-q^n),\nabla\gamma) \\ &= \quad \beta k(\nabla(p(t_{n+1})-p(t_n)),\nabla\gamma), \quad \forall\ \gamma\in H^1(\Omega)/R, \end{aligned} \tag{31}$$

since $\operatorname{div}\ \tilde{u}(t_{n+\frac{1}{2}}) = 0$, we can rearrange the nonlinear term on the right-hand side as

$$\begin{aligned} Q^n &= \tilde{B}(\phi(u^{n+1}),\tilde{u}^{n+\frac{1}{2}}) - (\tilde{u}(t_{n+\frac{1}{2}})\cdot\nabla)\tilde{u}(t_{n+\frac{1}{2}}) \\ &= \tilde{B}(\phi(u^{n+1}),\tilde{u}^{n+\frac{1}{2}}) - \tilde{B}(\tilde{u}(t_{n+\frac{1}{2}}),\tilde{u}(t_{n+\frac{1}{2}})) \\ &= -\tilde{B}(\phi(u(t_{n+1}))-\phi(u^{n+1}),\tilde{u}(t_{n+\frac{1}{2}})) + \tilde{B}(\phi(u(t_{n+1}))-\phi(u^{n+1}),\tilde{e}^{n+\frac{1}{2}}) \\ &\quad -\tilde{B}(\phi(u(t_{n+1})),\tilde{e}^{n+\frac{1}{2}}) - \tilde{B}(\tilde{u}(t_{n+\frac{1}{2}})-\phi(u(t_{n+1})),\tilde{u}(t_{n+\frac{1}{2}})), \end{aligned} \tag{32}$$

where

$$\phi(u(t_{n+1})) = \frac{3}{2}u(t_n) - \frac{1}{2}u(t_{n-1}),$$

$$\phi(u(t_{n+1})) - \phi(u^{n+1}) = \frac{3}{2}e^n - \frac{1}{2}e^{n-1}.$$

Let us denote

$$\begin{aligned} E_u^n &= \tilde{u}(t_{n+\frac{1}{2}}) - \phi(u(t_{n+1})) \\ &= \frac{1}{2}(u(t_{n+1}) - 2u(t_n) + u(t_{n-1})) \\ &= \frac{1}{2}\int_{t_{n-1}}^{t_n}(s - t_{n-1})u_{tt}(s)\mathrm{d}s + \frac{1}{2}\int_{t_n}^{t_{n+1}}(t_{n+1} - s)u_{tt}(s)\mathrm{d}s, \end{aligned}$$

so we can derive the following results by Schwarz inequality

$$\|E_u^n\|_0^2 \le ck^3\int_{t_{n-1}}^{t_{n+1}}\|u_{tt}(s)\|_0^2\mathrm{d}s,$$

and

$$\|\nabla E_u^n\|_0^2 \le ck^3\int_{t_{n-1}}^{t_{n+1}}\|\nabla u_{tt}(s)\|_0^2\mathrm{d}s.$$

Now we consider the error estimates about the nonlinear terms.

Taking the inner product of (29) with $2k\tilde{e}^{n+\frac{1}{2}}$, we obtain

$$\|\tilde{e}^{n+1}\|_0^2 - \|e^n\|_0^2 + 2k\nu\|\nabla\tilde{e}^{n+\frac{1}{2}}\|_0^2 = 2k(Q^n + R^n + \nabla(p^n - \tilde{p}(t_{n+\frac{1}{2}})), \tilde{e}^{n+\frac{1}{2}}). \tag{33}$$

The terms on the right-hand side of (33) can be handled as follows:

$$2k < R^n, \tilde{e}^{n+\frac{1}{2}} > \le \frac{\nu k}{2}\|\nabla\tilde{e}^{n+\frac{1}{2}}\|_0^2 + \frac{2k}{\nu}\|R^n\|_{-1}^2.$$

By using (3), (5), (7) and Young inequality, we get

$$\begin{aligned} 2k < Q^n, \tilde{e}^{n+\frac{1}{2}} > &= -2k\tilde{b}(\phi(u(t_{n+1})) - \phi(u^{n+1}), \tilde{u}(t_{n+\frac{1}{2}}), \tilde{e}^{n+\frac{1}{2}}) \\ &\quad -2k\tilde{b}(\tilde{u}(t_{n+\frac{1}{2}}) - \phi(u(t_{n+1})), \tilde{u}(t_{n+\frac{1}{2}}), \tilde{e}^{n+\frac{1}{2}}) \\ &\le ck\|\phi(u(t_{n+1})) - \phi(u^{n+1})\|_0\|\Delta\tilde{u}(t_{n+\frac{1}{2}})\|_0\|\nabla\tilde{e}^{n+\frac{1}{2}}\|_0 \\ &\quad +ck\|E_u^n\|_0\|\Delta\tilde{u}(t_{n+\frac{1}{2}})\|_0\|\nabla\tilde{e}^{n+\frac{1}{2}}\|_0 \\ &\le \frac{\nu k}{2}\|\nabla\tilde{e}^{n+\frac{1}{2}}\|_0^2 + ck\|\Delta\tilde{u}(t_{n+\frac{1}{2}})\|_0^2\|\phi(u(t_{n+1})) - \phi(u^{n+1})\|_0^2 \\ &\quad +ck\|\Delta\tilde{u}(t_{n+\frac{1}{2}})\|_0^2\|E_u^n\|_0^2 \\ &\le \frac{\nu k}{2}\|\nabla\tilde{e}^{n+\frac{1}{2}}\|_0^2 + Nk\|\frac{3}{2}e^n - \frac{1}{2}e^{n-1}\|_0^2 + Nk\|E_u^n\|_0^2. \end{aligned}$$

Taking the inner product of (30) with $\frac{2\beta-1}{2\beta}ke^{n+1}$, since

$$(\nabla p, v) = 0, \ \forall\ p \in H^1(\Omega), \ v \in H,$$

we obtain

$$\frac{2\beta-1}{2\beta}\{\|e^{n+1}\|_0^2 - \|\tilde{e}^{n+1}\|_0^2 + \|e^{n+1} - \tilde{e}^{n+1}\|_0^2\} = 0. \tag{34}$$

Now, taking the inner product of (30) with $\frac{k}{2\beta}(e^{n+1} + \tilde{e}^{n+1})$, we get

$$\frac{1}{2\beta}\{\|e^{n+1}\|_0^2 - \|\tilde{e}^{n+1}\|_0^2\} = \frac{k}{2}(\nabla(p^{n+1} - p^n), \tilde{e}^{n+1}). \tag{35}$$

Adding (33), (34) and (35), we arrive to

$$\begin{aligned}
&\|e^{n+1}\|_0^2-\|e^n\|_0^2+\frac{2\beta-1}{2\beta}\|e^{n+1}-\tilde{e}^{n+1}\|_0^2+k\nu\|\nabla\tilde{e}^{n+\frac{1}{2}}\|_0^2\\
&\le Nk(\|R^n\|_{-1}^2+\|\frac{3}{2}e^n-\frac{1}{2}e^{n-1}\|_0^2+\|E_u^n\|_0^2)\\
&\quad+\frac{k}{2}(\nabla(p^{n+1}+p^n-2\tilde{p}(t_{n+\frac{1}{2}})),\tilde{e}^{n+1})\\
&= Nk(\|R^n\|_{-1}^2+\|\frac{3}{2}e^n-\frac{1}{2}e^{n-1}\|_0^2+\|E_u^n\|_0^2)-\frac{k}{2}(\nabla(q^{n+1}+q^n),\tilde{e}^{n+1}).
\end{aligned}\tag{36}$$

Using (31), we get

$$\begin{aligned}
&-\frac{k}{2}(\nabla(q^{n+1}+q^n),\tilde{e}^{n+1})\\
&=\frac{\beta k^2}{2}(\nabla(p(t_{n+1})-p(t_n))-\nabla(q^{n+1}-q^n),\nabla(q^{n+1}+q^n))\\
&=-\frac{\beta k^2}{2}(\|\nabla q^{n+1}\|_0^2-\|\nabla q^n\|_0^2)+\frac{\beta k^2}{2}I_p^n,
\end{aligned}\tag{37}$$

where

$$\begin{aligned}
I_p^n &= (\nabla(q^{n+1}+q^n),\nabla(p(t_{n+1})-p(t_n)))\\
&= (\nabla(q^{n+1}+q^n),\int_{t_n}^{t_{n+1}}\nabla p_t(s)\mathrm{d}s)\\
&\le k(\|\nabla q^{n+1}\|_0^2+\|\nabla q^n\|_0^2)+\int_{t_n}^{t_{n+1}}\|\nabla p_t(s)\|_0^2\mathrm{d}s.
\end{aligned}\tag{38}$$

Hence, adding (36) to (37), because of (38), we arrive to

$$\begin{aligned}
\|e^{n+1}\|_0^2 &- \|e^n\|_0^2+\frac{2\beta-1}{2\beta}\|e^{n+1}-\tilde{e}^{n+1}\|_0^2\\
&+ k\nu\|\nabla\tilde{e}^{n+\frac{1}{2}}\|_0^2+\frac{\beta k^2}{2}(\|\nabla q^{n+1}\|_0^2-\|\nabla q^n\|_0^2)\\
&\le Nk(\|R^n\|_{-1}^2+\|\frac{3}{2}e^n-\frac{1}{2}e^{n-1}\|_0^2+\|E_u^n\|_0^2)\\
&+ \frac{\beta k^3}{2}(\|\nabla q^{n+1}\|_0^2+\|\nabla q^n\|_0^2)+\frac{\beta k^2}{2}\int_{t_n}^{t_{n+1}}\|\nabla p_t(s)\|_0^2\mathrm{d}s.
\end{aligned}\tag{39}$$

Now, taking the sum of (39) for n from 1 to m, thanks to (9), we arrive to

$$\begin{aligned}
\|e^{m+1}\|_0^2 &+ \frac{\beta k^2}{2}\|\nabla q^{m+1}\|_0^2+\sum_{n=1}^m\{k\nu\|\nabla\tilde{e}^{n+\frac{1}{2}}\|_0^2+\frac{2\beta-1}{2\beta}\|e^{n+1}-\tilde{e}^{n+1}\|_0^2\}\\
&\le Nk\sum_{n=1}^m(\|e^n\|_0^2+\frac{\beta k^2}{2}\|\nabla q^{n+1}\|_0^2)+Nk\sum_{n=1}^m(\|R^n\|_{-1}^2+\|E_u^n\|_0^2)\\
&\quad+\|e^1\|_0^2+\frac{\beta k^2}{2}(\int_{t_0}^T\|\nabla p_t(s)\|_0^2\mathrm{d}s+\|\nabla q^1\|_0^2)\\
&\le Nk^2+Nk\sum_{n=1}^m(\|e^n\|_0^2+\frac{\beta k^2}{2}\|\nabla q^{n+1}\|_0^2).
\end{aligned}$$

Applying Lemma 2.1 with $a_n = \|e^n\|_0^2 + \frac{\beta k^2}{2}\|\nabla q^{n+1}\|_0^2$ to the above inequality, we obtain

$$\begin{aligned}\|e^{m+1}\|_0^2 \quad &+ \quad \sum_{n=1}^{m}\{k\nu\|\nabla\tilde{e}^{n+\frac{1}{2}}\|_0^2 + \frac{2\beta-1}{2\beta}\|e^{n+1}-\tilde{e}^{n+1}\|_0^2\}\\ &+ \quad \frac{\beta k^2}{2}\|\nabla q^{m+1}\|_0^2 \le Nk^2, \quad \forall\ 1 \le m \le M-1. \end{aligned} \tag{40}$$

In view of (7), the above inequality implies in particular that

$$\|\nabla(\tilde{u}^{n+1}+u^n)\|_0^2 + \|\nabla p^{n+1}\|_0^2 \le N, \quad \forall\ 1 \le n \le M-1. \tag{41}$$

We now consider the term ∇p^n in (12) as a source term and take the scalar product of (12) with $-2k\Delta(\tilde{u}^{n+1}+u^n)$. Denoting $g^n = f(t_{n+\frac{1}{2}}) - \nabla p^n$, and using (4), (40) and (41), we obtain

$$\begin{aligned}&2\|\nabla\tilde{u}^{n+1}\|_0^2 - 2\|\nabla u^n\|_0^2 + 4k\nu\|\Delta\tilde{u}^{n+\frac{1}{2}}\|_0^2\\ &= -2k(g^n, \Delta(\tilde{u}^{n+1}+u^n)) + 2k\tilde{b}(\phi(u^{n+1}), \tilde{u}^{n+\frac{1}{2}}, \Delta(\tilde{u}^{n+1}+u^n))\\ &\le k\nu\|\Delta\tilde{u}^{n+\frac{1}{2}}\|_0^2 + ck\|g^n\|_0^2 + ck\|\phi(u^{n+1})\|_0^{\frac{1}{2}}\|\phi(u^{n+1})\|_1^{\frac{1}{2}}\|\nabla\tilde{u}^{n+\frac{1}{2}}\|_0^{\frac{1}{2}}\|\Delta\tilde{u}^{n+\frac{1}{2}}\|_0^{\frac{3}{2}}\\ &\le 2k\nu\|\Delta\tilde{u}^{n+\frac{1}{2}}\|_0^2 + ck\|g^n\|_0^2 + Nk\|\nabla\phi(u^{n+1})\|_0^2 + Nk\|\phi(u^{n+1})\|_0^4.\end{aligned}$$

Since $\|u^n\|_0 \le \|\tilde{u}^n\|_0$, $\forall\ 1 \le n \le M$, we can rewrite the above inequality as

$$\|\nabla\tilde{u}^{n+1}\|_0^2 - \|\nabla\tilde{u}^n\|_0^2 + k\nu\|\Delta\tilde{u}^{n+\frac{1}{2}}\|_0^2 \le ck\|g^n\|_0^2 + Nk(\|\nabla\phi(u^{n+1})\|_0^2 + \|\phi(u^{n+1})\|_0^4).$$

Taking the sum of above inequality for $n = 1$ to m, we derive that

$$\|\nabla\tilde{u}^{m+1}\|_0^2 + k\nu\sum_{n=1}^{m}\|\Delta\tilde{u}^{n+\frac{1}{2}}\|_0^2 \le N, \quad \forall\ 1 \le m \le M-1. \tag{42}$$

Taking the inner product of (29) with $-\Delta\tilde{e}^{n+\frac{1}{2}}$, thanks to (7), (40) and Lemma 4.1, we get

$$\|\Delta\tilde{e}^{n+\frac{1}{2}}\|_0^2 \le N + c(Q^n, -\Delta\tilde{e}^{n+\frac{1}{2}}).$$

On the other hand, using (5) and (42), we derive from (32) that

$$(Q^n, -\Delta\tilde{e}^{n+\frac{1}{2}}) \le \frac{1}{2}\|\Delta\tilde{e}^{n+\frac{1}{2}}\|_0^2 + N, \quad \forall\ 1 \le n \le M-1.$$

Therefore, $\|\Delta\tilde{e}^{n+\frac{1}{2}}\|_0 \le N$, $\forall\ 1 \le n \le M-1$.

Main result in this paper is the following theorem.

Theorem 4.4. Suppose that hypothesis (6) and (8) hold. Then for $t_0 \in (0,T)$, $\beta > \frac{1}{2}$, let (u^n, p^n) be the solution of (12)-(13) for $n = 2, \cdots, m$, and let $(u(t), p(t))$ be the strong solution of problem (1) up to $t_m = T$. There exists a positive constant N depending on the data and t_0, such that

$$k\sum_{n=1}^{m}\|u(t_n)-u^n\|_0^2 + k^2\|\nabla(u(t_m)-u^m)\|_0^2 + k^2\|p(t_m)-p^m\|_0^2 \le Nk^4, \quad \forall\ 1 \le m \le M, \tag{43}$$

where (u^1, p^1) is the solution of (10)-(11). Theorem 4.4 is proved in the remainder of this section, it will be carried out through a sequence of error estimates presented below.

Firstly, we consider the following auxiliary linear problem and denote the solutions of this problem by $(\tilde{v}^{n+1}, v^{n+1}, r^{n+1})$.

$$\begin{cases} \frac{1}{k}\left(\tilde{v}^{n+1} - v^n\right) - \nu\Delta\tilde{v}^{n+\frac{1}{2}} + \nabla r^n = f(t_{n+\frac{1}{2}}) - \tilde{B}(\tilde{u}(t_{n+\frac{1}{2}}), \tilde{u}(t_{n+\frac{1}{2}})), \\ (\tilde{v}^{n+1} + v^n)|_{\partial\Omega} = 0, \end{cases} \tag{44}$$

and

$$\begin{cases} v^{n+1} - \tilde{v}^{n+1} + \beta k\nabla(r^{n+1} - r^n) = 0, \\ \operatorname{div} v^{n+1} = 0, \\ v^{n+1}\cdot\vec{n}|_{\partial\Omega} = 0, \end{cases} \tag{45}$$

with $(v^0, r^0) = (u^0, p^0)$.

It is obvious that the results in Lemma 4.3 for (12)-(13) are also valid for this auxiliary linear system.

$$\|\nabla v^{m+1}\|_0^2 + \|\nabla\tilde{v}^{m+1}\|_0^2 + \|\Delta(\tilde{v}^{m+1} + v^m)\|_0^2 + \|\nabla r^{m+1}\|_0^2 \le N, \quad \forall\ 1 \le m \le M-1. \tag{46}$$

Similar with the literature [24], we have the following result.

Lemma 4.5. Under the assumptions of Theorem 4.4, we obtain

$$k\sum_{n=1}^{m}\|u(t_n) - \tilde{v}^n\|_0^2 + k^2\|\nabla(u(t_m) - \tilde{v}^m)\|_0^2 + k^2\|p(t_m) - r^m\|_0^2 \le Nk^4, \quad \forall\ 1 \le m \le M. \tag{47}$$

$$k\sum_{n=1}^{m}\|u(t_n) - v^n\|_0^2 + k^2\|\nabla(u(t_m) - v^m)\|_0^2 \le Nk^4,\ \forall\ 1 \le m \le M. \tag{48}$$

Denoting $\xi^n = u(t_n) - v^n,\ \tilde{\xi}^n = u(t_n) - \tilde{v}^n$ and $\phi^n = p(t_n) - r^n$.

Subtracting (44)-(45) from (20), we obtain the error equations:

$$\begin{cases} \frac{1}{k}\left(\tilde{\xi}^{n+1} - \xi^n\right) - \nu\Delta\tilde{\xi}^{n+\frac{1}{2}} + \nabla(\frac{p(t_{n+1}) - p(t_n)}{2} + \phi^n) = R^n, \\ (\tilde{\xi}^{n+1} + \xi^n)|_{\partial\Omega} = 0. \end{cases}$$

Using the Lagrange mean value theorem, we have $p(t_{n+1}) - p(t_n) = kp_t(\xi_n)$, which parameter ξ_n is from t_n to t_{n+1}. We derive from the above inequality that

$$\|\frac{\tilde{\xi}^{n+1} - \xi^n}{k}\|_{-1} \le N(\|\tilde{\xi}^{n+\frac{1}{2}}\|_1 + \|\phi^n\|_0 + \|R^n\|_0 + k\|p_t(\xi_n)\|_0) \le Nk, \tag{49}$$

for all $1 \le n \le M-1$. Thanks to (45), we obtain

$$\frac{\tilde{\xi}^{n+1} - \xi^{n+1}}{\beta k} + \nabla(r^{n+1} - r^n) = 0.$$

We derive from the above inequality that

$$\|\frac{\tilde{\xi}^M - \xi^M}{\beta k}\|_{-1} \le c\|r^M - r^{M-1}\|_0 \le Nk.$$

In the following, we give error estimates for the nonlinear problem.

Denoting $\eta^n = v^n - u^n,\ \tilde{\eta}^n = \tilde{v}^n - \tilde{u}^n$ and $\psi^n = r^n - p^n$. Since $\eta^0 = 0$ and $\psi^0 = 0$, we can easily prove that

$$\|\eta^1\|_0^2 + \frac{\beta k^2}{2}\|\nabla\psi^1\|_0^2 \le Nk^4.$$

When $n \geq 1$, subtracting (12)-(13) from (44)-(45), we obtain

$$\begin{cases} \frac{1}{k}\left(\tilde{\eta}^{n+1}-\eta^{n}\right)-\nu\Delta\tilde{\eta}^{n+\frac{1}{2}}+\nabla\psi^{n}=Q^{n}, \\ \tilde{\eta}^{n+\frac{1}{2}}|_{\partial\Omega}=0, \end{cases} \tag{50}$$

and

$$\begin{cases} \eta^{n+1}-\tilde{\eta}^{n+1}+\beta k\nabla(\psi^{n+1}-\psi^{n})=0, \\ \operatorname{div}\eta^{n+1}=0, \\ \eta^{n+1}\cdot\vec{n}|_{\partial\Omega}=0. \end{cases} \tag{51}$$

Lemma 4.6. Under the assumptions of Theorem 4.4, we obtain

$$\|\eta^{m+1}\|_0^2+\frac{\beta k^2}{2}\|\nabla\psi^{m+1}\|_0^2+\sum_{n=1}^{m}\big(\frac{2\beta-1}{2\beta}\|\eta^{n+1}-\tilde{\eta}^{n+1}\|_0^2+k\nu\|\nabla\tilde{\eta}^{n+\frac{1}{2}}\|_0^2\big)\leq Nk^4,$$
$$\forall\ 1\leq m\leq M-1.$$

Proof. Taking the inner product of (50) with $2k\tilde{\eta}^{n+\frac{1}{2}}$, we obtain

$$\|\tilde{\eta}^{n+1}\|_0^2-\|\eta^{n}\|_0^2+2k\nu\|\nabla\tilde{\eta}^{n+\frac{1}{2}}\|_0^2+2k(\nabla\psi^{n},\tilde{\eta}^{n+\frac{1}{2}})=2k(Q^{n},\tilde{\eta}^{n+\frac{1}{2}}). \tag{52}$$

Using the similar method in Lemma 4.3 for (51), we have

$$\|\eta^{n+1}\|_0^2-\|\tilde{\eta}^{n+1}\|_0^2+\frac{2\beta-1}{2\beta}\|\eta^{n+1}-\tilde{\eta}^{n+1}\|_0^2=-\frac{k}{2}(\nabla(\psi^{n+1}-\psi^{n}),\tilde{\eta}^{n+1}). \tag{53}$$

Summing up (52) and (53), we obtain

$$\begin{aligned} &\|\eta^{n+1}\|_0^2-\|\eta^{n}\|_0^2+\frac{2\beta-1}{2\beta}\|\eta^{n+1}-\tilde{\eta}^{n+1}\|_0^2+2k\nu\|\nabla\tilde{\eta}^{n+\frac{1}{2}}\|_0^2 \\ &\quad=2k(Q^{n},\tilde{\eta}^{n+\frac{1}{2}})-\frac{k}{2}(\nabla(\psi^{n+1}+\psi^{n}),\tilde{\eta}^{n+1}) \\ &\quad=2k(Q^{n},\tilde{\eta}^{n+\frac{1}{2}})-\frac{\beta k^2}{2}(\|\nabla\psi^{n+1}\|_0^2-\|\nabla\psi^{n}\|_0^2). \end{aligned} \tag{54}$$

We note that $\tilde{e}^{n+\frac{1}{2}}=\tilde{\xi}^{n+\frac{1}{2}}+\tilde{\eta}^{n+\frac{1}{2}}$ and $e^{n}=\xi^{n}+\eta^{n}$. Therefore,

$$\begin{aligned} Q^{n}=&-\tilde{B}(\phi(u(t_{n+1}))-\phi(u^{n+1}),\tilde{u}(t_{n+\frac{1}{2}}))+\tilde{B}(\phi(u(t_{n+1}))-\phi(u^{n+1}),\tilde{e}^{n+\frac{1}{2}}) \\ &-\tilde{B}(\phi(u(t_{n+1})),\tilde{e}^{n+\frac{1}{2}})-\tilde{B}(\tilde{u}(t_{n+\frac{1}{2}})-\phi(u(t_{n+1})),\tilde{u}(t_{n+\frac{1}{2}})) \\ =&-\tilde{B}(\phi(\xi^{n+1}+\eta^{n+1}),\tilde{u}(t_{n+\frac{1}{2}}))+\tilde{B}(\phi(\xi^{n+1}+\eta^{n+1}),\tilde{\xi}^{n+\frac{1}{2}}+\tilde{\eta}^{n+\frac{1}{2}}) \\ &-\tilde{B}(\phi(u(t_{n+1})),\tilde{\xi}^{n+\frac{1}{2}}+\tilde{\eta}^{n+\frac{1}{2}})-\tilde{B}(\tilde{u}(t_{n+\frac{1}{2}})-\phi(u(t_{n+1})),\tilde{u}(t_{n+\frac{1}{2}})). \end{aligned}$$

By using (5), (7) and (46), we have

$$\begin{aligned} 2k\tilde{b}(\phi(\xi^{n+1}+\eta^{n+1}),\tilde{u}(t_{n+\frac{1}{2}}),\tilde{\eta}^{n+\frac{1}{2}}) &\leq ck\|\phi(\xi^{n+1}+\eta^{n+1})\|_0\|\Delta\tilde{u}(t_{n+\frac{1}{2}})\|_0\|\nabla\tilde{\eta}^{n+\frac{1}{2}}\|_0 \\ &\leq \frac{\nu k}{4}\|\nabla\tilde{\eta}^{n+\frac{1}{2}}\|_0^2+Nk(\|\phi(\xi^{n+1})\|_0^2+\|\phi(\eta^{n+1})\|_0^2), \end{aligned}$$

$$\begin{aligned} 2k\tilde{b}(\phi(\xi^{n+1}+\eta^{n+1}),\tilde{\xi}^{n+\frac{1}{2}},\tilde{\eta}^{n+\frac{1}{2}}) &\leq ck\|\phi(\xi^{n+1}+\eta^{n+1})\|_0\|\Delta\tilde{\xi}^{n+\frac{1}{2}}\|_0\|\nabla\tilde{\eta}^{n+\frac{1}{2}}\|_0 \\ &\leq \frac{\nu k}{4}\|\nabla\tilde{\eta}^{n+\frac{1}{2}}\|_0^2+Nk(\|\phi(\xi^{n+1})\|_0^2+\|\phi(\eta^{n+1})\|_0^2), \end{aligned}$$

$$\begin{aligned} 2k\tilde{b}(\phi(u(t_{n+1})),\tilde{\xi}^{n+\frac{1}{2}},\tilde{\eta}^{n+\frac{1}{2}}) &\leq ck\|\Delta\phi(u(t_{n+1}))\|_0\|\tilde{\xi}^{n+\frac{1}{2}}\|_0\|\nabla\tilde{\eta}^{n+\frac{1}{2}}\|_0 \\ &\leq \frac{\nu k}{4}\|\nabla\tilde{\eta}^{n+\frac{1}{2}}\|_0^2+Nk(\|\tilde{\xi}^{n+1}\|_0^2+\|\xi^{n}\|_0^2), \end{aligned}$$

$$\begin{aligned} 2k\tilde{b}(\tilde{u}(t_{n+\frac{1}{2}})-\phi(u(t_{n+1})),\tilde{u}(t_{n+\frac{1}{2}}),\tilde{\eta}^{n+\frac{1}{2}}) &\leq ck\|E_u^n\|_0\|\Delta\tilde{u}(t_{n+\frac{1}{2}})\|_0\|\nabla\tilde{\eta}^{n+\frac{1}{2}}\|_0 \\ &\leq \frac{\nu k}{4}\|\nabla\tilde{\eta}^{n+\frac{1}{2}}\|_0^2+Nk\|E_u^n\|_0^2. \end{aligned}$$

Since $\eta^0=0$, $\psi^0=0$ and

$$\|\eta^1\|_0^2+\frac{\beta k^2}{2}\|\nabla\psi^1\|_0^2\leq Nk^4.$$

Collecting the above inequalities into (54), we have

$$\begin{aligned} &\|\eta^{m+1}\|_0^2+\frac{\beta k^2}{2}\|\nabla\psi^{m+1}\|_0^2+\sum_{n=1}^{m}(\frac{2\beta-1}{2\beta}\|\eta^{n+1}-\tilde{\eta}^{n+1}\|_0^2+k\nu\|\nabla\tilde{\eta}^{n+\frac{1}{2}}\|_0^2) \\ = \;& \|\eta^1\|_0^2+\frac{\beta k^2}{2}\|\nabla\psi^1\|_0^2+k\sum_{n=1}^{m}\|\eta^n\|_0^2+Nk\sum_{n=1}^{m}(\|\xi^n\|_0^2+\|\tilde{\xi}^{n+1}\|_0^2+\|E_u^n\|_0^2) \\ (55) \leq \;& Nk^4+k\sum_{n=1}^{m}\|\eta^n\|_0^2. \end{aligned}$$

We conclude the results by applying Lemma 2.1 to (55).

Lemma 4.7. Under the assumptions of Theorem 4.4, we obtain

$$\nu(\|\nabla\tilde{\eta}^{n+1}\|_0^2+\|\nabla\eta^{n+1}\|_0^2)\leq Nk^2, \quad \forall\ 1\leq n\leq M-1.$$

Proof. Taking the inner product of (50) with $\tilde{\eta}^{n+1}-\eta^n$, we get

$$\frac{1}{k}\|\tilde{\eta}^{n+1}-\eta^n\|_0^2+\frac{\nu}{2}(\|\nabla\tilde{\eta}^{n+1}\|_0^2-\|\nabla\eta^n\|_0^2)+(\nabla\psi^n,\tilde{\eta}^{n+1}-\eta^n)=(Q^n,\tilde{\eta}^{n+1}-\eta^n).$$

Thanks to the inequality (2), we derive from the last inequality that

$$\frac{1}{k}\|\tilde{\eta}^{n+1}-\eta^n\|_0^2+\frac{\nu}{2}(\|\nabla\eta^{n+1}\|_0^2-\|\nabla\eta^n\|_0^2)\leq(Q^n-\nabla\psi^n,\tilde{\eta}^{n+1}-\eta^n), \tag{56}$$

$$-(\nabla\psi^n,\tilde{\eta}^{n+1}-\eta^n)\leq\frac{1}{2k}\|\tilde{\eta}^{n+1}-\eta^n\|_0^2+ck\|\nabla\psi^n\|_0^2.$$

We derive from (5), (7) and $\|\Delta\tilde{e}^{n+\frac{1}{2}}\|_0\leq N$, that

$$\begin{aligned} (Q^n,\tilde{\eta}^{n+1}-\eta^n)=&-\tilde{B}(\phi(u(t_{n+1}))-\phi(u^{n+1}),\tilde{u}(t_{n+\frac{1}{2}}),\tilde{\eta}^{n+1}-\eta^n) \\ &+\tilde{B}(\phi(u(t_{n+1}))-\phi(u^{n+1}),\tilde{e}^{n+\frac{1}{2}},\tilde{\eta}^{n+1}-\eta^n) \\ &-\tilde{B}(\phi(u(t_{n+1})),\tilde{e}^{n+\frac{1}{2}},\tilde{\eta}^{n+1}-\eta^n) \\ &-\tilde{B}(\tilde{u}(t_{n+\frac{1}{2}})-\phi(u(t_{n+1})),\tilde{u}(t_{n+\frac{1}{2}}),\tilde{\eta}^{n+1}-\eta^n) \\ \leq&\frac{1}{2k}\|\tilde{\eta}^{n+1}-\eta^n\|_0^2+Nk(\|\nabla\phi(\xi^{n+1})\|_0^2+\|\nabla\phi(\eta^{n+1})\|_0^2) \\ &+Nk(\|\nabla\tilde{e}^{n+\frac{1}{2}}\|_0^2+\|\nabla E_u^n\|_0^2). \end{aligned}$$

Now, taking the sum of (56) for n from 1 to m, we arrive to

$$\begin{aligned} \nu\|\nabla\eta^{m+1}\|_0^2&\leq\nu\|\nabla\eta^1\|_0^2+Nk\sum_{n=1}^{m}(\|\nabla\eta^n\|_0^2+\|\nabla\xi^n\|_0^2+\|\nabla\tilde{e}^{n+\frac{1}{2}}\|_0^2+\|\nabla E_u^n\|_0^2+\|\nabla\psi^n\|_0^2) \\ &\leq Nk^2+Nk\sum_{n=1}^{m}\|\nabla\eta^n\|_0^2. \end{aligned}$$

By applying the discrete Gronwall Lemma to the last inequality, we have

$$\nu\|\nabla\eta^{m+1}\|_0^2\leq Nk^2.$$

Since $\|\nabla\tilde{\eta}^{n+1}\|_0^2 \le \|\nabla\eta^n\|_0^2 + \|\nabla\tilde{\eta}^{n+\frac{1}{2}}\|_0^2$, we also have $\nu\|\nabla\tilde{\eta}^{n+1}\|_0^2 \le Nk^2$.

We derive from (50) that

$$\begin{aligned}\|\frac{\tilde{\eta}^{n+1}-\eta^n}{k}\|_{-1} \le N(\|\nabla\tilde{\eta}^{n+\frac{1}{2}}\|_0 + \|\nabla\psi^n\|_0 + \|Q^n\|_0) \le Nk,\\ \forall\ 1 \le n \le M-1.\end{aligned} \tag{57}$$

Thanks to (49) and (57), we obtain

$$\|\frac{\tilde{e}^{n+1}-e^n}{k}\|_{-1} \le \|\frac{\tilde{\xi}^{n+1}-\xi^n}{k}\|_{-1} + \|\frac{\tilde{\eta}^{n+1}-\eta^n}{k}\|_{-1} \le Nk,\ \forall\ 1 \le n \le M-1.$$

Finally, we derive from (29), Lemma 4.1, Lemma 4.5, Lemma 4.6 and above inequalities that

$$\begin{aligned}\|q^n\|_0 \le N\|\frac{\tilde{e}^{n+1}-e^n}{k}\|_{-1} + \|R^n\|_0 + \|Q^n\|_0 + \nu\|\nabla\tilde{e}^{n+\frac{1}{2}}\|_0 + k\|p_t(\xi_n)\|_0 \le Nk,\\ \forall\ 1 \le n \le M-1.\end{aligned}$$

When $n = M$, in view of (30), we have

$$\begin{aligned}\|q^M\|_0 &\le k\|p_t(\xi_M)\|_0 + \|q^{M-1}\|_0 + \|\frac{\tilde{e}^M - e^M}{\beta k}\|_{-1}\\ &\le k\|p_t(\xi_M)\|_0 + \|q^{M-1}\|_0 + \|\frac{\tilde{\xi}^M - \xi^M}{\beta k}\|_{-1} + \|\frac{\tilde{\eta}^M - \eta^M}{\beta k}\|_0 \le Nk.\end{aligned}$$

The proof of Theorem 4.4 is completed.

References

[1] F. Brezzi and J. Pitkaranta, On the stabilization of finite element approximation of the Stokes problem, in efficient solutions of elliptic systems, W. Hackbusch. ed. Vieweg, Braunschweig, Germany, 1984.

[2] D.L. Brown, R. Cortez and M.L. Minion, Accurate projection methods for the incompressible Navier-Stokes equations, J. Comput. Phys. 168(2)(2001)464-499.

[3] E. Burman, Pressure projection stabilizations for Galerkin approximations of Stokes' and Darcy's problem, Numer. Meth. Part. Differ. Equ. 24(2008)127-143.

[4] A.J. Chorin, Numerical solution of the Navier-Stokes equations, Math. Comput. 22(1968)745-762.

[5] W. E and J.G. Liu, Projection method I: Convergence and numerical boundary layers, SIAM J. Numer. Anal. 32(1995)1017-1057.

[6] C. Févrière, J. Laminie, P. Poullet and P. Poullet, On the penalty-projection method for the Navier-Stokes equations with the MAC mesh, J. Comput. Appl. Math. 226(2009)228-245.

[7] V. Girault and P.A. Raviart, Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms, Springer-Verlag, Berlin, Heidelberg, 1987.

[8] K. Goda, A multistep technique with implicit difference schemes for calculating two or three dimensional cavity flows, J. Comput. Phys. 30(1979)76-95.

[9] J.L. Guermond, Un résultat de convergence d'ordre deux en temps pour l'approximation des équations de Navier-Stokes par une technique de projection incrémentale, M2AN Math. Model. Numer. Anal. 33(1)(1999)169-189.

[10] J.L. Guermond, P. Minev and J. Shen, An overview of projection methods for incompressible flows, Comput. Meth. Appl. Mech. Eng. 195(2006)6011-6045.

[11] J.L. Guermond and J. Shen, On the error estimates for the rotational pressure-correction projection methods, Math. Comput. 73(2004)1719-1737.

[12] Y.N. He, Y.P. Lin and W.W. Sun, Stabilized finite element methods for the nonstationary Navier-Stokes problem, Discret. Contin. Dyn. Syst. B 6(1)(2006)41-68.

[13] Y.N. He and W.W. Sun, Stability and convergence of the Crank-Nicolson/Adams-Bashforth scheme for the time-dependent Navier-Stokes equations, SIAM J. Numer. Anal. 45 (2007)837-869.

[14] J.G. Heywood and R. Rannacher, Finite element approximation of the nonstationary Navier-Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization, SIAM J. Numer. Anal. 19(1982)275-311.
[15] J.G. Heywood and R. Rannacher, Finite element approximation of the nonstationary Navier-Stokes problem. IV. Error analysis for second-order time discretization, SIAM J. Numer. Anal. 27(1990)353-384.
[16] M. Jobelin, C. Lapuerta, J.-C. Latché, Ph. Angot and B. Piar, A finite element penalty-projection method for incompressible flows, J. Comput. Phys. 217(2006)502-518.
[17] J. Li, Penalty finite element approximations for the Stokes equations by L^2 projection, Math. Meth. Appl. Sci. 32(2009)470-479.
[18] C.H. Min and F. Gibou, A second order accurate projection method for the incompressible Navier-Stokes equations on non-graded adaptive grids, J. Comput. Phys. 219(2006)912-929.
[19] R. Rannacher, Numerical analysis of the Navier-Stokes equations, Applications of Mathematics, 38(1993)361-380.
[20] J. Shen, On error estimates of some higher order projection and penalty-projection methods for Navier-Stokes equations, Numer. Math. 62(1992)49-73.
[21] J. Shen, A remark on the projection-3 method, Int. J. Numer. Methods Fluids, 16(1993)249-253.
[22] J. Shen, On error estimates of the penalty method for unsteady Navier-Stokes equations, SIAM J. Numer. Anal. 32(1995)386-403.
[23] J. Shen, On a new pseudo-compressibility method for the incompressible Navier-Stokes equations, Appl. Numer. Math. 21(1996)71-90.
[24] J. Shen, On error estimates of projection methods for the Navier-Stokes equations: second-order schemes, Math. Comput. 65(1996)1039-1065.
[25] A. Sokolov, M. A. Olshanskii and S. Turek, A discrete projection method for incompressible viscous flow with coriolis force, Comput. Meth. Appl. Mech. Eng. 197(2008)4512-4520.
[26] R. Temam, Sur méthode d'approximation de la solution des équations de Navier-Stokes par la méthode des pas fractionnaires II, Arch. Rat. Mech. Anal. 33(1969)377-385.
[27] R. Temam, Navier-Stokes Equations: Theory and Numerical Analysis, North-Holland, Amsterdam, 1984.
[28] J. van Kan, A second-order accurate pressure-correction scheme for viscous incompressible flow, SIAM J. Sci. Stat. Comp. 7(1986)870-891.

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