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OPERATOR SPLITTING METHODS FOR THE NAVIER-STOKES EQUATIONS WITH NONLINEAR SLIP BOUNDARY CONDITIONS

YUAN LI AND KAITAI LI

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Abstract. In this paper, the θ scheme of operator splitting methods is applied to the Navier-Stokes equations with nonlinear slip boundary conditions whose variational formulation is the variational inequality of the second kind with the Navier-Stokes operator. Firstly, we introduce the multiplier such that the variational inequality is equivalent to the variational identity. Subsequently, we give the θ scheme to compute the variational identity and consider the finite element approximation of the θ scheme. The stability and convergence of the θ scheme are showed. Finally, we give the numerical results.

Key Words. Navier-Stokes Equations, Nonlinear Slip Boundary Conditions, Operator Splitting Method, $\theta-$Scheme, Finite Element Approximation.

1. Introduction

Numerical simulation for the incompressible flow is the fundamental and significant problem in computational mathematics and computational fluid mechanics. It is well known that the mathematical model of viscous incompressible fluid with homogeneous boundary conditions is the Navier-Stokes equations

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \\ \mathrm{div} u = 0. \end{cases}$$

It is obvious that (1) is a coupled system with a first-order nonlinear evolution equation and an imposed incompressible constrain so that the numerical simulation for the Navier-Stokes equations is very difficult. The popular technique to overcome this difficulty is to relax the solenoidal condition in an appropriate method and to result in a pesudo-compressible system, such as the penalty method and the artificial compressible method. The operator splitting method is also very useful to overcome this shortage. The main advantage is that it can decouple the difficulties associated to the nonlinear property with those associated to the incompressible condition. For more detail, see [1].

The operator splitting method has been a popular tool for the numerical simulation of the incompressible viscous flow. Based on the main idea of the operator splitting method, there have some different schemes, such as the Peaceman-Rachford

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scheme [2], the Douglas-Rachford scheme [3] and the θ scheme [4-5]. In this paper, we only apply the θ scheme to the Navier-Stokes equations with nonlinear slip boundary conditions. This class of boundary conditions are introduced by Fujita in [6-7], where he investigated some hydrodynamics problems under nonlinear boundary conditions, such as leak and slip boundary conditions involving a subdifferential property. These types of boundary conditions appear in the modeling of blood flow in a vein of an arterial sclerosis patient and in that of avalanche of water and rocks. Moreover, the variational formulation of the Navier-Stokes equations with these nonlinear boundary conditions is the variational inequality of the second kind.

The stability analysis of the θ scheme for the Navier-Stokes equations with the whole homogeneous Dirichlet boundary conditions has been investigated in [8]. The difficulty lies in the treatment of the trilinear term in the right-hand side. However, in this paper, besides the trilinear term, another difficulty is due to that the variational formulation is the variational inequality. To overcome this difficulty, we introduce the multiplier such that the variational inequality is equivalent to the variational identity.

2. The Navier-Stokes Equations

Consider the following Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, & \text{in } Q_T,\\ \mathrm{div} u = 0, & \text{in } Q_T, \end{cases} \tag{2}$$

where $Q_T = \Omega \times [0,T]$ for some $T > 0$, $u(t,x)$ denotes the velocity, $p(t,x)$ denotes the pressure, $f(t,x)$ denotes the external force and $\nu > 0$ is the kinematic viscous coefficient. The domain $\Omega \subset \mathbb{R}^2$ is a bounded domain.

Given the initial value $u(0,x) = u_0(x)$ in Ω, we consider the following nonlinear slip boundary conditions:

$$\begin{cases} u = 0 & \text{on } \Gamma \times (0,T],\\ u_n = 0, \quad -\sigma_\tau(u) \in g\partial|u_\tau| & \text{on } S \times (0,T], \end{cases} \tag{3}$$

where $\Gamma \cap S = \emptyset, \overline{\Gamma \cup S} = \partial\Omega$ with $|\Gamma| \neq 0, |S| \neq 0$. g is a scalar functions; $u_n = u\cdot n$ and $u_\tau = u - u_n n$ are the normal and tangential components of the velocity, where n stands for the unit vector of the external normal to S; $\sigma_\tau(u) = \sigma - \sigma_n n$, independent of p, is the tangential component of the stress vector σ which is defined by $\sigma_i = \sigma_i(u,p) = (\mu e_{ij}(u) - p\delta_{ij})n_j$, where $e_{ij}(u) = \frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}, i,j = 1,2$. The set $\partial\psi(a)$ denotes a subdifferential of the function ψ at the point a:

$$\partial\psi(a) = \{b \in \mathbb{R}^2 : \psi(h) - \psi(a) \geq b\cdot(h-a), \quad \forall\ h \in \mathbb{R}^2\}.$$

Denote

$$V = \{u \in H^1(\Omega)^2;\ u|_\Gamma = 0,\ u\cdot n|_S = 0\}, \quad V_0 = H_0^1(\Omega)^2; \quad V_\sigma = \{u \in V \mid \mathrm{div} u = 0\};$$

$$H = \{u \in L^2(\Omega)^2;\ u\cdot n|_{\partial\Omega} = 0\}, \quad M = L_0^2(\Omega) = \{q \in L^2(\Omega);\ \int_\Omega q\mathrm{d}x = 0\}.$$

Let $||\cdot||_k$ be the norm of the Hilbert space $H^k(\Omega)$ or $H^k(\Omega)^2$. Let $(\cdot,\cdot)$ and $||\cdot||$ be the inner product and the norm in $L^2(\Omega)^2$ or $L^2(\Omega)$. Then we can equip the inner product and the norm in V by $(\nabla\cdot,\nabla\cdot)$ and $||\cdot||_V = ||\nabla\cdot||$, respectively, because $||\nabla\cdot||$ is equivalent to $||\cdot||_1$ according to the Poincare inequality.

If X is a Banach space, $L^p(0,T,X), 1 \leq p < +\infty$ will be the linear space of measurable functions from the interval $(0,T)$ into X such that

$$\int_0^T ||u(t)||_X^p \mathrm{d}t < \infty.$$

If $p = +\infty$, we require that

$$\operatorname{ess}\sup_{t\in[0,T]} ||u(t)||_X < \infty.$$

Introduce the following bilinear forms:

$$\begin{cases} a(u,v) = \nu(\nabla u, \nabla v) & \forall\ u, v \in V, \\ d(u,p) = (p, \operatorname{div} u) & \forall\ u \in V, p \in M, \\ c(\mu, u) = \displaystyle\int_S \mu u_\tau \mathrm{d}s & \forall\ \mu \in L^2(S), u \in V, \end{cases}$$

and the trilinear form

$$\begin{aligned} b(u,v,w) &= ((u\cdot\nabla)v, w) + \frac{1}{2}((\operatorname{div} u)v, w) \\ &= \frac{1}{2}((u\cdot\nabla)v, w) - \frac{1}{2}((u\cdot\nabla)w, v) \qquad \forall\ u, v, w \in V. \end{aligned}$$

It is obvious that $b(u,v,w)$ satisfies the antisymmetric property:

$$b(u,v,w) = -b(u,w,v) \qquad \forall\ u, v, w \in V.$$

Denote $j(\cdot) = \displaystyle\int_S g|\cdot|\mathrm{d}s$. Given $u_0 \in V_\sigma$, $f \in L^2(0,T,H)$ and $g \in L^2(0,T,L^\infty(S))$ with $g(t) \geq 0$, the variational problem of the problem (2)-(3) is: find $u \in L^\infty(0,T,H) \cap L^2(0,T,V)$ with $u' \in L^2(0,T,V')$ and $p \in L^2(0,T,M)$ such that

$$\begin{cases} < u', v-u > + a(u, v-u) + b(u,u,v-u) - d(v-u, p) \\ \qquad\qquad + j(v_\tau) - j(u_\tau) \geq (f, v-u) \qquad \forall\ v \in V, \\ d(u,q) = 0 \qquad \forall\ q \in M, \end{cases} \tag{4}$$

which is the variational inequality of the second kind with the Navier-Stokes operator.

Using a regularized method, in [9] we show the following theorem about the existence and uniqueness of the solution to (4):

Theorem 2.1 Given $u_0 \in V_\sigma$, $f \in L^2(0,T,H)$ and $g \in L^2(0,T,L^\infty(S))$, there exists a unique solution $u \in L^\infty(0,T,H) \cap L^2(0,T,V)$ with $u' \in L^2(0,T,V')$ and $p \in L^2(0,T,M)$ of the variational inequality (4). Moreover, the following energy inequality holds:

$$\sup_{0\leq t\leq T} ||u(t)||^2 + \nu\int_0^T ||u(\xi)||_V^2 \mathrm{d}\xi \leq \frac{4}{\nu}\int_0^T (||f(\xi)||_{V'}^2 + ||g(\xi)||_S^2)\mathrm{d}\xi + 2||u_0||^2.$$

3. Existence of Multiplier

Define the convex set Λ by

$$\Lambda = \{\mu \in L^2(S),\ |\mu(x)| \leq 1 \text{ a.e. on } S\}.$$

The next theorem from the idea in [10] shows the existence of the multiplier such that the variational inequality (4) is equivalent to the following variational formulation (5).

Theorem 3.1 *For almost everywhere $t \in (0,T]$, that $u \in L^\infty(0,T,H) \cap L^2(0,T,V)$ with $u' \in L^2(0,T,V')$ and $p \in L^2(0,T,M)$ is the solution of the variational inequality (4) if and only if there exists a $\lambda(t) \in \Lambda$ such that*

$$\begin{cases} < u', v > + a(u,v) + b(u,u,v) - d(v,p) + \displaystyle\int_S \lambda g v_\tau\, ds = (f, v) & \forall\ v \in V, \\ d(u,q) = 0 & \forall\ q \in M, \\ \lambda u_\tau = |u_\tau| & a.e.\ on\ S. \end{cases} \tag{5}$$

Proof We first prove that (4) implies (5). Setting $v = 0$ and $v = 2u$ in (4) yields

$$< u', u > +a(u, u) + j(u_\tau) - d(u, p) = (f, u). \tag{6}$$

Hence

$$(f, v) - < u', v > -a(u, v) - b(u, u, v) + d(v, p) \le j(v_\tau) \qquad \forall\ v \in V,$$

which implies that

$$\begin{aligned} &|(f, v) - < u', v > -a(u, v) - b(u, u, v) + d(v, p)| \\ &\le j(v_\tau) = \int_S g|v_\tau|\mathrm{d}s \qquad \forall\ v \in V. \end{aligned} \tag{7}$$

Then there exists a continuous linear functional $l(\cdot)$ on $H^{\frac{1}{2}}(S)$ such that

$$(f, v) - < u', v > -a(u, v) - b(u, u, v) + d(v, p) = l(v_\tau|_S) \qquad \forall\ v \in V. \tag{8}$$

From (7), we have

$$|l(\mu)| \le c||g||_{L^\infty(S)}||\mu||_{L^2(S)} \qquad \forall\ \mu \in H^{\frac{1}{2}}(S).$$

Since $H^{\frac{1}{2}}(S) \subset L^2(S)$, according to the Hahn-Banach theorem, there exists a $\lambda \in \Lambda$ such that

$$l(\mu) = \int_S \lambda g \mu \mathrm{d}s \qquad \forall\ \mu \in H^{\frac{1}{2}}(S).$$

Therefore, according to (8), we have

$$(f, v) - < u', v > -a(u, v) - b(u, u, v) + d(v, p) = \int_S \lambda g v_\tau \mathrm{d}s \qquad \forall\ v \in V,$$

which shows the first formulation of (5). Taking $v = u$ in the above equation, we obtain

$$< u', u > +a(u, u) - d(u, p) + \int_S \lambda g u_\tau \mathrm{d}s = (f, u).$$

In view of (6), one has

$$\int_S (\lambda g u_\tau - g|u_\tau|)\mathrm{d}s = 0. \tag{9}$$

Since $|\lambda| \le 1$,

$$g|u_\tau| - \lambda g u_\tau \ge 0,$$

which, together with (9), implies that

$$\lambda u_\tau = |u_\tau| \quad \text{a.e. on } S.$$

This completes the proof of (5). Next, we will show that (5) implies (4). For almost everywhere $t \in (0, T]$, let $(u'(t), u(t), p(t), \lambda(t)) \in V' \times V \times M \times \Lambda$ be the solution of (5), then

$$< u', v-u > +a(u, v-u) + b(u, u, v-u) - d(v-u, p) + \int_S \lambda g(v_\tau - u_\tau)\mathrm{d}s = (f, v-u).$$

From the third identity in (5), we obtain

$$< u', v-u > +a(u, v-u) + b(u, u, v-u) - d(v-u, p) + \int_S \lambda g v_\tau \mathrm{d}s - \int_S g|u_\tau|\mathrm{d}s = (f, v-u).$$

Since $\lambda v_\tau \le |v_\tau|$ a.e. on S,

$$< u', v-u > +a(u, v-u) + b(u, u, v-u) - d(v-u, p) + \int_S g|v_\tau|\mathrm{d}s - \int_S g|u_\tau|\mathrm{d}s \ge (f, v-u).$$

$\square$

4. The θ Scheme

In this section, we will give the θ scheme to solve the variational problem (5). Let $0 < \theta < \dfrac{\sqrt{24}-3}{10}$, we divide the time interval $[t_n, t_{n+1}]$ of length Δt into three parts $[t_n, t_{n+\theta}), [t_{n+\theta}, t_{n+1-\theta})$ and $[t_{n+1-\theta}, t_{n+1}]$, where the lengths are $\theta\Delta t, (1-2\theta)\Delta t$ and $\theta\Delta t$, respectively. Denote $u^n = u(t_n, x), p^n = p(t_n, x), \lambda^n = \lambda(t_n, x), f^n = f(t_n, x), g^n = g(t_n, x)$. Let $\alpha > 0$ and $\beta > 0$ with $\alpha + \beta = 1$. We give the following θ scheme:

Step I:

$$u^0 = u_0 \in V_\sigma, \qquad \lambda^0 \in \Lambda \qquad \text{is given.} \tag{10}$$

For every $n \geq 0$, u^n, λ^n, we can compute $u^{n+\theta}, u^{n+1-\theta}$ and u^{n+1} as follows

Setp II:

$$\begin{cases} \text{Find } (u^{n+\theta}, p^{n+\theta}) \in V \times M \text{ such that} \\ \dfrac{1}{\theta\Delta t}(u^{n+\theta}, v) + \alpha a(u^{n+\theta}, v) - d(v, p^{n+\theta}) = \dfrac{1}{\theta\Delta t}(u^n, v) - \beta a(u^n, v) \\ \quad -b(u^n, u^n, v) - c(\lambda^n, g^n v) + (f^n, v) \qquad \forall\, v \in V, \\ d(u^{n+\theta}, q) = 0 \qquad \forall\, q \in M. \end{cases} \tag{11}$$

Setp III:

$$\lambda^{n+1-\theta} = P_\Lambda(\lambda^n + \rho g^{n+\theta} u_\tau^{n+\theta}) \qquad \rho > 0, \tag{12}$$

where

$$P_\Lambda(\mu) = \sup(-1, \inf(1, \mu)) \qquad \forall\, \mu \in L^2(S).$$

Step IV:

$$\begin{cases} \text{Find } u^{n+1-\theta} \in V \text{ such that} \\ \dfrac{1}{(1-2\theta)\Delta t}(u^{n+1-\theta}, v) + \beta a(u^{n+1-\theta}, v) + b(u^{n+1-\theta}, u^{n+1-\theta}, v) \\ \qquad = \dfrac{1}{(1-2\theta)\Delta t}(u^{n+\theta}, v) - \alpha a(u^{n+\theta}, v) + d(v, p^{n+\theta}) \\ \qquad -c(\lambda^{n+1-\theta}, g^{n+1-\theta} v) + (f^n, v) \quad \forall\, v \in V. \end{cases} \tag{13}$$

Step V:

$$\begin{cases} \text{Find } (u^{n+1}, p^{n+1}) \in V \times M \text{ such that} \\ \dfrac{1}{\theta\Delta t}(u^{n+1}, v) + \alpha a(u^{n+1}, v) - d(v, p^{n+1}) = \dfrac{1}{\theta\Delta t}(u^{n+1-\theta}, v) \\ \qquad -\beta a(u^{n+1-\theta}, v) - b(u^{n+1-\theta}, u^{n+1-\theta}, v) \\ \qquad -c(\lambda^{n+1-\theta}, g^{n+1-\theta} v) + (f^{n+1}, v) \qquad \forall\, v \in V \\ d(u^{n+1}, q) = 0 \qquad \forall\, q \in M. \end{cases} \tag{14}$$

Step VI:

$$\lambda^{n+1} = P_\Lambda(\lambda^{n+1-\theta} + \rho g^{n+1} u_\tau^{n+1}) \qquad \rho > 0. \tag{15}$$

5. Finite Element Approximation

Let T_h be a family of regular triangular partitions of Ω into triangles of diameter not greater than $0 < h < 1$ [11]. Let $V_h \subset V$ and $M_h \subset M$ be the conforming finite element subspaces, which satisfy the discrete inf-sup condition, i.e., there exists a positive constant $\beta > 0$ independent of h such that

$$\beta ||p_h|| \leq \sup_{v_h \in V_h} \frac{d(v_h, p_h)}{||v_h||_V}.$$

Denote $V_{\sigma h}$ the discretized solenoidal subspace of V_h. For $u_h \in V_h$, we have the following inverse inequality:

$$||u_h||_V \le c_1 h^{-1}||u_h||,$$

where $c_1 > 0$ is independent of h. According to the definition of $b(\cdot,\cdot,\cdot)$, we have

$$b(u_h, v_h, v_h) \equiv 0 \qquad \forall\ u_h, v_h \in V_h.$$

Moreover, it satisfies

$$b(u_h, v_h, w_h) \le c_2 h^{-1}||u_h|| \cdot ||v_h||_V \cdot ||w_h|| \qquad \forall\ u_h, v_h, w_h \in V_h,$$

where $c_2 > 0$ is independent of h.

Denote $\Lambda_h = \{w_h : \ |w_h(x_S)| \le 1, \quad \forall\ x_S \in N_S\}$, where N_S is the set of all nodes on S. For every $\lambda_h \in \Lambda_h$ and $v_h \in V_h$, we have

$$c(\lambda_h, g v_h) \le c_4 ||g||_{L^\infty(S)} ||v_h||_V.$$

For simplicity, we assume $c_i = 1, i = 1, 2, 3$.

For initial value $u_0 \in V_\sigma$, the discretized initial value $u_{0h} \in V_{\sigma h}$ is defined as follows

$$a(u_{0h}, v_h) = a(u_0, v_h) \qquad \forall\ v_h \in V_{\sigma h}.$$

The finite element approximation of the θ scheme (10)-(15) is

Step I:

$$u_h^0 = u_{0h} \in V_{\sigma h}, \qquad \lambda_h^0 \in \Lambda_h \qquad \text{is given.} \tag{16}$$

For every $n \ge 0$, u_h^n, λ_h^n, we can compute $u_h^{n+\theta}, u_h^{n+1-\theta}$ and u_h^{n+1} as follows:

Step II:

$$\begin{cases} \text{Find } (u_h^{n+\theta}, p_h^{n+\theta}) \in V_h \times M_h \text{ such that} \\ \dfrac{1}{\theta\Delta t}(u_h^{n+\theta}, v_h) + \alpha a(u_h^{n+\theta}, v_h) - d(v_h, p_h^{n+\theta}) = \dfrac{1}{\theta\Delta t}(u_h^n, v_h) \\ \qquad -\beta a(u_h^n, v_h) - b(u_h^n, u_h^n, v_h) - c(\lambda_h^n, g^n v_h) + (f^n, v_h) \qquad \forall\ v_h \in V_h, \\ d(u_h^{n+\theta}, q_h) = 0 \qquad \forall\ q_h \in M_h. \end{cases} \tag{17}$$

Step III:

$$\lambda_h^{n+1-\theta}(x_S) = P_{\Lambda_h}(\lambda_h^n(x_S) + \rho g^{n+\theta}(x_S) u_{h\tau}^{n+\theta}(x_S)) \quad \forall\ x_S \in N_S,\ \rho > 0, \tag{18}$$

where P_{Λ_h} is the projection operator from $\mathbb{R}$ to $[-1, 1]$.

Step IV:

$$\begin{cases} \text{Find } u_h^{n+1-\theta} \in V_h \text{ such that} \\ \dfrac{1}{(1-2\theta)\Delta t}(u_h^{n+1-\theta}, v_h) + \beta a(u_h^{n+1-\theta}, v_h) + b(u_h^{n+1-\theta}, u_h^{n+1-\theta}, v_h) \\ = \dfrac{1}{(1-2\theta)\Delta t}(u_h^{n+\theta}, v_h) - \alpha a(u_h^{n+\theta}, v_h) + d(v_h, p_h^{n+\theta}) \\ \quad -c(\lambda_h^{n+1-\theta}, g^{n+1-\theta} v_h) + (f^n, v_h) \quad \forall\ v_h \in V_h. \end{cases} \tag{19}$$

Step V:

$$\begin{cases} \text{Find } (u_h^{n+1}, p_h^{n+1}) \in V_h \times M_h \text{ such that} \\ \dfrac{1}{\theta\Delta t}(u_h^{n+1}, v_h) + \alpha a(u_h^{n+1}, v_h) - d(v_h, p_h^{n+1}) = \dfrac{1}{\theta\Delta t}(u_h^{n+1-\theta}, v_h) \\ \qquad -\beta a(u_h^{n+1-\theta}, v_h) - b(u_h^{n+1-\theta}, u_h^{n+1-\theta}, v_h) \\ \qquad -c(\lambda_h^{n+1-\theta}, g^{n+1-\theta} v_h) + (f^{n+1}, v_h) \quad \forall\ v_h \in V_h \\ d(u_h^{n+1}, q_h) = 0 \qquad \forall\ q_h \in M_h. \end{cases} \tag{20}$$

Step VI:

$$\lambda_h^{n+1}(x_S) = P_{\Lambda_h}(\lambda_h^{n+1-\theta}(x_S) + \rho g^{n+1}(x_S) u_{h\tau}^{n+1}(x_S)) \quad \forall\ x_S \in N_S,\ \rho > 0. \tag{21}$$

6. Stability Analysis

In this section, we will show the stability property of the finite element discretized θ scheme (16)-(21).

Lemma 6.1 *If* $(u_h^{n+\theta}, p_h^{p+\theta}) \in V_h \times M_h$ *is the solution of the problem (17), then there exists some positive constant* $C_0 > 0$ *such that*

$$
\begin{aligned}
&||u_h^{n+\theta}||^2 + \alpha\theta\Delta t\nu||u_h^{n+\theta}||_V^2 + \beta\theta\Delta t\nu||u_h^n||_V^2 + (\frac{1}{2} - \frac{\beta\theta\Delta t\nu}{h^2})||u_h^{n+\theta} - u_h^n||^2 \\
&\le ||u_h^n||^2 + C_0(\Delta t h^{-1})^2||u_h^n||^2||u_h^n||_V^2 + \frac{C_0\Delta t}{\nu}(||f^n||^2 + ||g^n||^2_{L^\infty(S)}).
\end{aligned} \tag{22}
$$

Proof Taking $v_h = u_h^{n+\theta}$ in (17) gives

$$
\begin{aligned}
&\frac{1}{\theta\Delta t}||u_h^{n+\theta}||^2 + \alpha\nu||u_h^{n+\theta}||_V^2 = \frac{1}{\theta\Delta t}(u_h^n, u_h^{n+\theta}) - \beta a(u_h^n, u_h^{n+\theta}) \\
&\quad -b(u_h^n, u_h^n, u_h^{n+\theta}) + (f^n, u_h^{n+\theta}) - c(\lambda_h^n, g^n u_h^{n+\theta}).
\end{aligned}
$$

In terms of $(u,v) = \frac{1}{2}|u|^2 + \frac{1}{2}|v|^2 - \frac{1}{2}|u-v|^2$, the right hand of the above identity is equivalent to

$$
\begin{aligned}
&-\frac{1}{2\theta\Delta t}||u_h^{n+\theta} - u_h^n||^2 + \frac{1}{2\theta\Delta t}||u_h^{n+\theta}||^2 + \frac{1}{2\theta\Delta t}||u_h^n||^2 + \frac{\beta\nu}{2}||u_h^{n+\theta} - u_h^n||_V^2 \\
&\quad -\frac{\beta\nu}{2}||u_h^n||_V^2 - \frac{\beta\nu}{2}||u_h^{n+\theta}||_V^2 - b(u_h^n, u_h^n, u_h^{n+\theta}) + (f^n, u_h^{n+\theta}) - c(\lambda_h^n, g^n u_h^{n+\theta}).
\end{aligned}
$$

According to Young's inequality, we have

$$
\begin{aligned}
&||u_h^{n+\theta}||^2 + (1+\alpha)\nu\theta\Delta t||u_h^{n+\theta}||_V^2 + \nu\beta\theta\Delta t||u_h^n||_V^2 + (1 - \frac{\beta\nu\theta\Delta t}{h^2})||u_h^{n+\theta} - u_h^n||^2 \\
&\le ||u_h^n||^2 - 2\theta\Delta t b(u_h^n, u_h^n, u_h^{n+\theta}) + 2\theta\Delta t(f^n, u_h^{n+\theta}) - 2\theta\Delta t c(\lambda_h^n, g^n u_h^{n+\theta}) \\
&\le ||u_h^n||^2 - 2\theta\Delta t b(u_h^n, u_h^n, u_h^{n+\theta} - u_h^n) + 2\theta\Delta t(f^n, u_h^{n+\theta}) - 2\theta\Delta t c(\lambda_h^n, g^n u_h^{n+\theta}) \\
&\le ||u_h^n||^2 + 2\theta\Delta t h^{-1}||u_h^n||||u_h^n||_V||u_h^{n+\theta} - u_h^n|| \\
&\quad +2\theta\Delta t||f^n||||u_h^{n+\theta}||_V + 2\theta\Delta t||g^n||_{L^\infty(S)}||u_h^{n+\theta}||_V \\
&\le ||u_h^n||^2 + \nu\theta\Delta t||u_h^{n+\theta}||_V^2 + \frac{1}{2}||u_h^{n+\theta} - u_h^n||^2 \\
&\quad +C_0(\Delta t h^{-1})^2||u_h^n||^2||u_h^n||_V^2 + \frac{C_0\Delta t}{\nu}(||f^n||^2 + ||g^n||^2_{L^\infty(S)}).
\end{aligned}
$$

This completes the proof of (22). □

Lemma 6.2 *If* $u_h^{n+1-\theta} \in V_h$ *is the solution of the problem (19), then there exists the positive constant* $C_0 > 0$ *such that*

$$
\begin{aligned}
&||u_h^{n+1-\theta}||^2 + (1-2\theta)\beta\nu\Delta t||u_h^{n+1-\theta}||_V^2 + \frac{1}{2}||u_h^{n+1-\theta} - u_h^{n+\theta}||^2 \\
&\quad +(1-2\theta)\alpha\nu\Delta t||u_h^{n+\theta}||_V^2 \\
&\le ||u_h^{n+\theta}||^2 + C_0(\nu\Delta t h^{-1})^2||u_h^n||_V^2 + C_0(\Delta t h^{-1})^2||u_h^n||^2||u_h^n||_V^2 \\
&\quad +\frac{C_0\Delta t}{\nu}(||f^n||^2 + ||g^n||^2_{L^\infty(S)} + ||g^{n+1-\theta}||^2_{L^\infty(S)}) \\
&\quad +C_0(\Delta t h^{-1})^2||g^n||^2_{L^\infty(S)}.
\end{aligned} \tag{23}
$$

Proof Taking $v_h = u_h^{n+1-\theta}$ in (19) gives

$$
\begin{aligned}
&\frac{1}{(1-2\theta)\Delta t}||u_h^{n+1-\theta}||^2 + \beta\nu||u_h^{n+1-\theta}||_V^2 \\
&= \frac{1}{(1-2\theta)\Delta t}(u_h^{n+\theta}, u_h^{n+1-\theta}) - \alpha a(u_h^{n+\theta}, u_h^{n+1-\theta}) + d(u_h^{n+1-\theta}, p_h^{n+\theta}) \\
&\quad +(f^n, u_h^{n+1-\theta}) - c(\lambda_h^{n+1-\theta}, g^{n+1-\theta}u_h^{n+1-\theta}).
\end{aligned}
$$

According to (17), we have

$$\begin{aligned}
&d(u_h^{n+1-\theta}, p_h^{n+\theta}) = d(u_h^{n+1-\theta} - u_h^{n+\theta}, p_h^{n+\theta})\\
=& \frac{1}{\theta\Delta t}(u_h^{n+\theta} - u_h^n, u_h^{n+1-\theta} - u_h^{n+\theta}) + \alpha a(u_h^{n+\theta}, u_h^{n+1-\theta} - u_h^{n+\theta})\\
&+\beta a(u_h^n, u_h^{n+1-\theta} - u_h^{n+\theta}) + b(u_h^n, u_h^n, u_h^{n+1-\theta} - u_h^{n+\theta})\\
&-(f^n, u_h^{n+1-\theta} - u_h^{n+\theta}) + c(\lambda_h^n, g^n(u_h^{n+1-\theta} - u_h^{n+\theta})).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{1}{(1-2\theta)\Delta t}||u_h^{n+1-\theta}||^2 + \beta\nu||u_h^{n+1-\theta}||_V^2\\
=& \frac{1}{(1-2\theta)\Delta t}(u_h^{n+\theta}, u_h^{n+1-\theta}) - \alpha a(u_h^{n+\theta}, u_h^{n+1-\theta}) + (f^n, u_h^{n+1-\theta})\\
&-c(\lambda_h^{n+1-\theta}, g^{n+1-\theta}u_h^{n+1-\theta}) + \frac{1}{\theta\Delta t}(u_h^{n+\theta} - u_h^n, u_h^{n+1-\theta} - u_h^{n+\theta})\\
&+\alpha a(u_h^{n+\theta}, u_h^{n+1-\theta} - u_h^{n+\theta}) + \beta a(u_h^n, u_h^{n+1-\theta} - u_h^{n+\theta})\\
&+b(u_h^n, u_h^n, u_h^{n+1-\theta} - u_h^{n+\theta}) - (f^n, u_h^{n+1-\theta} - u_h^{n+\theta}) + c(\lambda_h^n, g^n(u_h^{n+1-\theta} - u_h^{n+\theta}))\\
=& \frac{1}{(1-2\theta)\Delta t}(u_h^{n+\theta}, u_h^{n+1-\theta}) - \alpha\nu||u_h^{n+\theta}||_V^2 + \frac{1}{\theta\Delta t}(u_h^{n+\theta} - u_h^n, u_h^{n+1-\theta} - u_h^{n+\theta})\\
&+\beta a(u_h^n, u_h^{n+1-\theta} - u_h^{n+\theta}) + b(u_h^n, u_h^n, u_h^{n+1-\theta} - u_h^{n+\theta})\\
&+(f^n, u_h^{n+\theta}) - c(\lambda_h^{n+1-\theta}, g^{n+1-\theta}u_h^{n+1-\theta}) + c(\lambda_h^n, g^n(u_h^{n+1-\theta} - u_h^{n+\theta}))\\
=& -\frac{1}{2(1-2\theta)\Delta t}||u_h^{n+1-\theta} - u_h^{n+\theta}||^2 + \frac{1}{2(1-2\theta)\Delta t}||u_h^{n+1-\theta}||^2\\
&+\frac{1}{2(1-2\theta)\Delta t}||u_h^{n+\theta}||^2 - \alpha\nu||u_h^{n+\theta}||_V^2 + \frac{1}{\theta\Delta t}(u_h^{n+\theta} - u_h^n, u_h^{n+1-\theta} - u_h^{n+\theta})\\
&+\beta a(u_h^n, u_h^{n+1-\theta} - u_h^{n+\theta}) + b(u_h^n, u_h^n, u_h^{n+1-\theta} - u_h^{n+\theta}) + (f^n, u_h^{n+\theta})\\
&-c(\lambda_h^{n+1-\theta}, g^{n+1-\theta}u_h^{n+1-\theta}) + c(\lambda_h^n, g^n(u_h^{n+1-\theta} - u_h^{n+\theta})).
\end{aligned}$$

That is

$$\begin{aligned}
&\frac{1}{2}||u_h^{n+1-\theta}||^2 + \beta\nu(1-2\theta)\Delta t||u_h^{n+1-\theta}||_V^2 + \frac{1}{2}||u_h^{n+1-\theta} - u_h^{n+\theta}||^2\\
\le& \frac{1}{2}||u_h^{n+\theta}||^2 - (1-2\theta)\Delta t\alpha\nu||u_h^{n+\theta}||_V^2 + \frac{1-2\theta}{\theta}||u_h^{n+\theta} - u_h^n|| \cdot ||u_h^{n+1-\theta} - u_h^{n+\theta}||\\
&+\beta(1-2\theta)\Delta t a(u_h^n, u_h^{n+1-\theta} - u_h^{n+\theta}) + (1-2\theta)\Delta t b(u_h^n, u_h^n, u_h^{n+1-\theta} - u_h^{n+\theta})\\
&+(1-2\theta)\Delta t(f^n, u_h^{n+\theta}) - (1-2\theta)\Delta t c(\lambda_h^{n+1-\theta}, g^{n+1-\theta}u_h^{n+1-\theta})\\
&+(1-2\theta)\Delta t c(\lambda_h^n, g^n(u_h^{n+1-\theta} - u_h^{n+\theta}))\\
\le& \frac{1}{2}||u_h^{n+\theta}||^2 - (1-2\theta)\Delta t\alpha\nu||u_h^{n+\theta}||_V^2 + \frac{1}{4}||u_h^{n+1-\theta} - u_h^{n+\theta}||^2\\
&+\frac{C_1}{2}(\frac{1-2\theta}{\theta})^2||u_h^{n+\theta} - u_h^n||^2 + \frac{C_1}{2}(\nu\Delta t h^{-1})^2||u_h^n||_V^2 + \frac{C_1}{2}(\Delta t h^{-1})^2||u_h^n||^2||u_h^n||_V^2\\
&+\frac{(1-2\theta)\Delta t\alpha\nu}{2}||u_h^{n+\theta}||_V^2 + \frac{\beta\nu(1-2\theta)\Delta t}{2}||u_h^{n+1-\theta}||_V^2 + \frac{C_1}{2}(\Delta t h^{-1})^2||g^n||_{L^\infty(S)}^2\\
&+\frac{C_1\Delta t}{2\nu}(||f^n||^2 + ||g^{n+1-\theta}||_{L^\infty(S)}^2),
\end{aligned}$$

where $C_1 > 0$ is some positive constant. Thus we obtain

$$\begin{aligned}
&||u_h^{n+1-\theta}||^2 + \beta\nu(1-2\theta)\Delta t||u_h^{n+1-\theta}||_V^2 + (1-2\theta)\Delta t\alpha\nu||u_h^{n+\theta}||_V^2\\
&\quad+\frac{1}{2}||u_h^{n+1-\theta} - u_h^{n+\theta}||^2\\
\le& ||u_h^{n+\theta}||^2 + C_1(\frac{1-2\theta}{\theta})^2||u_h^{n+\theta} - u_h^n||^2 + C_1(\nu\Delta t h^{-1})^2||u_h^n||_V^2\\
&+C_1(\Delta t h^{-1})^2||u_h^n||^2||u_h^n||_V^2 + C_1(\Delta t h^{-1})^2||g^n||_{L^\infty(S)}^2\\
&+\frac{C_1\Delta t}{\nu}(||f^n||^2 + ||g^{n+1-\theta}||_{L^\infty(S)}^2).
\end{aligned} \tag{24}$$

Taking $v_h = u_h^{n+\theta} - u_h^n \in V_{\sigma h}$ in (17) gives

$$\begin{aligned}
0 &= d(u_h^{n+\theta} - u_h^n, p^{n+\theta}) \\
&= \frac{1}{\theta\Delta t}||u_h^{n+\theta} - u_h^n||^2 + \alpha a(u_h^{n+\theta}, u_h^{n+\theta} - u_h^n) + \beta a(u_h^n, u_h^{n+\theta} - u_h^n) \\
&\quad + b(u_h^n, u_h^n, u_h^{n+\theta} - u_h^n) + c(\lambda_h^n, g^n(u_h^{n+\theta} - u_h^n)) - (f^n, u_h^{n+\theta} - u_h^n).
\end{aligned}$$

Then

$$\begin{aligned}
&||u_h^{n+\theta} - u_h^n||^2 + \theta\Delta t\alpha\nu||u_h^{n+\theta} - u_h^n||_V^2 \\
\leq\ & \theta\Delta t\nu||u_h^n||_V \cdot ||u_h^{n+\theta} - u_h^n||_V + \theta\Delta t h^{-1}||u_h^n|| \cdot ||u_h^n||_V \cdot ||u_h^{n+\theta} - u_h^n|| \\
&+ \theta\Delta t||g^n||_{L^\infty(S)}||u_h^{n+\theta} - u_h^n||_V + \theta\Delta t||f^n||||u_h^{n+\theta} - u_h^n||_V \\
\leq\ & \frac{1}{2}||u_h^{n+\theta} - u_h^n||^2 + \frac{\theta\Delta t\alpha\nu}{2}||u_h^{n+\theta} - u_h^n||_V^2 + \frac{C_2}{2}(\Delta t\nu h^{-1})^2||u_h^n||_V^2 \\
&+ \frac{C_2}{2}(\Delta t h^{-1})^2||u_h^n||^2||u_h^n||_V^2 + \frac{C_2\Delta t}{2\nu}(||f^n||^2 + ||g^n||^2_{L^\infty(S)}),
\end{aligned}$$

where $C_2 > 0$ is some positive constant. That is

$$\begin{aligned}
&||u_h^{n+\theta} - u_h^n||^2 + \theta\Delta t\alpha\nu||u_h^{n+\theta} - u_h^n||_V^2 \\
\leq\ & C_2(\Delta t\nu h^{-1})^2||u_h^n||_V^2 + C_2(\Delta t h^{-1})^2||u_h^n||^2||u_h^n||_V^2 \\
&+ C_2\frac{\Delta t}{\nu}(||f^n||^2 + ||g^n||^2_{L^\infty(S)}).
\end{aligned} \tag{25}$$

Substituting (25) into (24), we shows (23). □

Lemma 6.3 *If $(u_h^{n+1}, p_h^{p+1}) \in V_h \times M_h$ is the solution of the problem (20), if Δt and h satisfy $h^2 > 8\beta\theta\Delta t\nu$, then there exists the constant $C_0 > 0$ such that*

$$\begin{aligned}
&||u_h^{n+1}||^2 + \alpha\theta\Delta t\nu||u_h^{n+1}||_V^2 + (\frac{1}{4} - \frac{2\beta\nu\theta\Delta t}{h^2})||u_h^{n+1} - u_h^{n+1-\theta}||^2 \\
&\quad + \frac{1}{2}\theta\Delta t\beta\nu||u_h^{n+1-\theta}||_V^2 \\
\leq\ & ||u_h^{n+1-\theta}||^2 + [(\frac{2\theta}{1-2\theta})^2 + \frac{1}{8}]||u_h^{n+1-\theta} - u_h^{n+\theta}||^2 \\
&+ C_0(\Delta t h^{-1})^2||u_h^n||^2||u_h^n||_V^2 + \frac{33}{8}||u_h^{n+\theta} - u_h^n||^2 \\
&+ \frac{C_0\Delta t}{\nu}(||f^{n+1}||^2 + ||g^n||^2_{L^\infty(S)} + ||g^{n+1-\theta}||^2_{L^\infty(S)}).
\end{aligned} \tag{26}$$

Proof Similar to the proof of Lemma 6.1, we have

$$\begin{aligned}
&||u_h^{n+1}||^2 + (1+\alpha)\nu\theta\Delta t||u_h^{n+1}||_V^2 + \nu\beta\theta\Delta t||u_h^{n+1-\theta}||_V^2 \\
&\quad + (1 - \frac{\beta\nu\theta\Delta t}{h^2})||u_h^{n+1} - u_h^{n+1-\theta}||^2 \\
\leq\ & ||u_h^{n+1-\theta}||^2 - 2\theta\Delta t b(u_h^{n+1-\theta}, u_h^{n+1-\theta}, u_h^{n+1}) \\
&+ 2\theta\Delta t(f^{n+1}, u_h^{n+1}) - 2\theta\Delta t c(\lambda_h^{n+1-\theta}, g^{n+1-\theta}u_h^{n+1}).
\end{aligned} \tag{27}$$

Taking $v_h = u_h^{n+1} - u_h^{n+1-\theta}$ in (19) and observing that $b(u_h^{n+1-\theta}, u_h^{n+1-\theta}, u_h^{n+1-\theta}) = 0$, then we obtain

$$\begin{aligned}
&-b(u_h^{n+1-\theta}, u_h^{n+1-\theta}, u_h^{n+1}) = -b(u_h^{n+1-\theta}, u_h^{n+1-\theta}, u_h^{n+1} - u_h^{n+1-\theta}) \\
=\ & \frac{1}{(1-2\theta)\Delta t}(u_h^{n+1-\theta} - u_h^{n+\theta}, u_h^{n+1} - u_h^{n+1-\theta}) \\
&+ \beta a(u_h^{n+1-\theta}, u_h^{n+1} - u_h^{n+1-\theta}) + \alpha a(u_h^{n+\theta}, u_h^{n+1} - u_h^{n+1-\theta}) \\
&+ c(\lambda_h^{n+1-\theta}, g^{n+1-\theta}(u_h^{n+1} - u_h^{n+1-\theta})) \\
&- d(u_h^{n+1} - u_h^{n+1-\theta}, p^{n+\theta}) - (f^n, u_h^{n+1} - u_h^{n+1-\theta}).
\end{aligned} \tag{28}$$

Taking $v_h = u_h^{n+1} - u_h^{n+1-\theta}$ in (17) gives

$$\begin{aligned}
&-d(u_h^{n+1} - u_h^{n+1-\theta}, p_h^{n+\theta}) \\
=&\ \frac{1}{\theta\Delta t}(u_h^n - u_h^{n+\theta}, u_h^{n+1} - u_h^{n+1-\theta}) - \alpha a(u_h^{n+\theta}, u_h^{n+1} - u_h^{n+1-\theta}) \\
&-\beta a(u_h^n, u_h^{n+1} - u_h^{n+1-\theta}) - b(u_h^n, u_h^n, u_h^{n+1} - u_h^{n+1-\theta}) \\
&-c(\lambda_h^n, g^n(u_h^{n+1} - u_h^{n+1-\theta})) + (f^n, u_h^{n+1} - u_h^{n+1-\theta}).
\end{aligned}$$

Substituting the above identity into (28), we have

$$\begin{aligned}
&-2\theta\Delta t b(u_h^{n+1-\theta}, u_h^{n+1-\theta}, u_h^{n+1}) = \frac{2\theta}{1-2\theta}(u_h^{n+1-\theta} - u_h^{n+\theta}, u_h^{n+1} - u_h^{n+1-\theta}) \\
&+2\theta\Delta t\beta a(u_h^{n+1-\theta}, u_h^{n+1} - u_h^{n+1-\theta}) + 2\theta\Delta t c(\lambda_h^{n+1-\theta}, g^{n+1-\theta}(u_h^{n+1} - u_h^{n+1-\theta})) \\
&+2(u_h^n - u_h^{n+\theta}, u_h^{n+1} - u_h^{n+1-\theta}) - 2\theta\Delta t\beta a(u_h^n, u_h^{n+1} - u_h^{n+1-\theta}) \\
&-2\theta\Delta t b(u_h^n, u_h^n, u_h^{n+1} - u_h^{n+1-\theta}) - 2\theta\Delta t c(\lambda_h^n, g^n(u_h^{n+1} - u_h^{n+1-\theta})).
\end{aligned}$$

Substituting above identity into (27) and according to $\frac{\theta\beta\Delta t\nu}{h^2} \le \frac{1}{8}$, we have

$$\begin{aligned}
&||u_h^{n+1}||^2 + (1+\alpha)\nu\theta\Delta t||u_h^{n+1}||_V^2 \\
&\quad +\nu\beta\theta\Delta t||u_h^{n+1-\theta}||_V^2 + (1 - \frac{\beta\nu\theta\Delta t}{h^2})||u_h^{n+1} - u_h^{n+1-\theta}||^2 \\
\le&\ ||u_h^{n+1-\theta}||^2 + \frac{2\theta}{1-2\theta}(u_h^{n+1-\theta} - u_h^{n+\theta}, u_h^{n+1} - u_h^{n+1-\theta}) \\
&+2(u_h^n - u_h^{n+\theta}, u_h^{n+1} - u_h^{n+1-\theta}) + 2\theta\Delta t\beta a(u_h^{n+1-\theta} - u_h^n, u_h^{n+1} - u_h^{n+1-\theta}) \\
&+2\theta\Delta t c(\lambda_h^{n+1-\theta}, g^{n+1-\theta}(u_h^{n+1} - u_h^{n+1-\theta})) - 2\theta\Delta t b(u_h^n, u_h^n, u_h^{n+1} - u_h^{n+1-\theta}) \\
&-2\theta\Delta t c(\lambda_h^n, g^n(u_h^{n+1} - u_h^{n+1-\theta})) + 2\theta\Delta t(f^{n+1}, u_h^{n+1}) \\
&-2\theta\Delta t c(\lambda_h^{n+1-\theta}, g^{n+1-\theta}u_h^{n+1}) \\
\le&\ ||u_h^{n+1-\theta}||^2 + \theta\Delta t\nu||u_h^{n+1}||_V^2 + \frac{3}{4}||u_h^{n+1} - u_h^{n+1-\theta}||^2 + 4||u_h^{n+\theta} - u_h^n||^2 \\
&+(\frac{2\theta}{1-2\theta})^2||u_h^{n+1-\theta} - u_h^{n+\theta}||^2 + \frac{\theta\Delta t\beta\nu}{h^2}||u_h^{n+1} - u_h^{n+1-\theta}||^2 + \\
&\theta\Delta t\beta\nu||u_h^{n+1-\theta} - u_h^n||_V^2 + C_0(\Delta t h^{-1})^2||u_h^n||^2||u_h^n||_V^2 \\
&+C_0\frac{\Delta t}{\nu}(||g^n||_{L^\infty(S)}^2 + ||g^{n+1-\theta}||_{L^\infty(S)}^2 + ||f^{n+1}||^2) \\
\le&\ ||u_h^{n+1-\theta}||^2 + \theta\Delta t\nu||u_h^{n+1}||_V^2 + \frac{3}{4}||u_h^{n+1} - u_h^{n+1-\theta}||^2 + 4||u_h^{n+\theta} - u_h^n||^2 \\
&+(\frac{2\theta}{1-2\theta})^2||u_h^{n+1-\theta} - u_h^{n+\theta}||^2 + \frac{\theta\Delta t\beta\nu}{h^2}||u_h^{n+1} - u_h^{n+1-\theta}||^2 \\
&+\frac{\theta\beta\Delta t\nu}{h^2}||u_h^{n+1-\theta} - u_h^{n+\theta}||^2 + \frac{\theta\beta\Delta t\nu}{h^2}||u_h^{n+\theta} - u_h^n||^2 + C_0(\Delta t h^{-1})^2||u_h^n||^2||u_h^n||_V^2 \\
&+C_0\frac{\Delta t}{\nu}(||g^n||_{L^\infty(S)}^2 + ||g^{n+1-\theta}||_{L^\infty(S)}^2 + ||f^{n+1}||^2) \\
\le&\ ||u_h^{n+1-\theta}||^2 + \theta\Delta t\nu||u_h^{n+1}||_V^2 + \frac{3}{4}||u_h^{n+1} - u_h^{n+1-\theta}||^2 + \frac{33}{8}||u_h^{n+\theta} - u_h^n||^2 \\
&+[(\frac{2\theta}{1-2\theta})^2 + \frac{1}{8}]||u_h^{n+1-\theta} - u_h^{n+\theta}||^2 + \frac{\theta\Delta t\beta\nu}{h^2}||u_h^{n+1} - u_h^{n+1-\theta}||^2 \\
&+C_0(\Delta t h^{-1})^2||u_h^n||^2||u_h^n||_V^2 + C_0\frac{\Delta t}{\nu}(||g^n||_{L^\infty(S)}^2 + ||g^{n+1-\theta}||_{L^\infty(S)}^2 + ||f^{n+1}||^2).
\end{aligned}$$

Simplifying the above inequality gives (26). □

Since $0<\theta\le\dfrac{\sqrt{24}-3}{10}$, then $(\dfrac{2\theta}{1-2\theta})^2+\dfrac{1}{8}\le\dfrac{1}{2}$. Thus, substituting (22) and (23) into (26) gives

$$
\begin{aligned}
&||u_h^{n+1}||^2+\alpha\theta\Delta t\nu||u_h^{n+1}||_V^2\\
&\quad+(\frac{1}{4}-\frac{2\beta\nu\theta\Delta t}{h^2})||u_h^{n+1}-u_h^{n+1-\theta}||^2+\frac{1}{2}\theta\Delta t\beta\nu||u_h^{n+1-\theta}||_V^2\\
\le\ &||u_h^{n+1-\theta}||^2+\frac{1}{2}||u_h^{n+1-\theta}-u_h^{n+\theta}||^2+C_0(\Delta th^{-1})^2||u_h^n||^2||u_h^n||_V^2\\
&\quad+4||u_h^{n+\theta}-u_h^n||^2+\frac{C_0\Delta t}{\nu}(||f^{n+1}||^2+||g^n||^2_{L^\infty(S)}+||g^{n+1-\theta}||^2_{L^\infty(S)})\\
\le\ &||u_h^{n+\theta}||^2+4||u_h^{n+\theta}-u_h^n||^2+C_0(\nu\Delta th^{-1})^2||u_h^n||_V^2\\
&\quad+2C_0(\Delta th^{-1})^2||u_h^n||^2||u_h^n||_V^2+C_0(\Delta th^{-1})^2||g^n||^2_{L^\infty(S)}\\
&\quad+\frac{C_0\Delta t}{\nu}(||f^n||^2+||f^{n+1}||^2+2||g^n||^2_{L^\infty(S)}+2||g^{n+1-\theta}||^2_{L^\infty(S)})\\
\le\ &||u_h^n||^2+C_0(\nu\Delta th^{-1})^2||u_h^n||_V^2\\
&\quad+3C_0(\Delta th^{-1})^2||u_h^n||^2||u_h^n||_V^2+C_0(\Delta th^{-1})^2||g^n||^2_{L^\infty(S)}\\
&\quad+\frac{2C_0\Delta t}{\nu}(||f^n||^2+||f^{n+1}||^2+||g^n||^2_{L^\infty(S)}+||g^{n+1-\theta}||^2_{L^\infty(S)}),
\end{aligned}
$$

where we require that $\dfrac{1}{2}-\dfrac{\beta\theta\Delta t\nu}{h^2}>0$, that is $h^2>2\beta\theta\Delta t\nu$. Then for $n=0,1,\cdots,r,r\in Z^+$, we have

$$
\begin{aligned}
&||u_h^{r+1}||^2+\alpha\theta\Delta t\nu\sum_{n=0}^{r}||u_h^{n+1}||_V^2+(\frac{1}{4}-\frac{2\beta\nu\theta\Delta t}{h^2})\sum_{n=0}^{r}||u_h^{n+1}-u_h^{n+1-\theta}||^2\\
&\quad+\frac{1}{2}\theta\Delta t\beta\nu\sum_{n=0}^{r}||u_h^{n+1-\theta}||_V^2\\
\le\ &||u_{0h}||^2+C_0(\nu\Delta th^{-1})^2\sum_{n=0}^{r}||u_h^n||_V^2+3C_0(\Delta th^{-1})^2\sum_{n=0}^{r}||u_h^n||^2||u_h^n||_V^2\\
&\quad+\frac{2C_0\Delta t}{\nu}\sum_{n=0}^{r}(||f^n||^2+||f^{n+1}||^2+||g^n||^2_{L^\infty(S)}+||g^{n+1-\theta}||^2_{L^\infty(S)})\\
&\quad+C_0(\Delta th^{-1})^2\sum_{n=0}^{r}||g^n||^2_{L^\infty(S)}\\
\le\ &||u_{0h}||^2+C_0(\nu\Delta th^{-1})^2\sum_{n=0}^{r}||u_h^n||_V^2+3C_0(\Delta th^{-1})^2\sum_{n=0}^{r}||u_h^n||^2||u_h^n||_V^2\\
&\quad+\frac{8C_0\Delta t}{\nu}\sum_{n=0}^{r}(||f^n||^2+||g^n||^2_{L^\infty(S)}),
\end{aligned}
\tag{29}
$$

where we use $(\Delta th^{-1})^2\le\dfrac{\Delta t}{2\beta\theta\nu}$. Denote

$$
\begin{aligned}
\Lambda_T\ &=||u_{0h}||^2+C_0(\nu\Delta th^{-1})^2||u_{0h}||_V^2+3C_0(\Delta th^{-1})^2||u_{0h}||^2||u_{0h}||_V^2\\
&\quad+\frac{8C_0}{\nu}(||f||^2_{L^2(0,T,H)}+||g||^2_{L^2(0,T,L^\infty(S))})\\
&\le||u_{0h}||^2+\frac{C_0\nu\Delta t}{2\beta\theta}||u_{0h}||_V^2+\frac{3C_0\Delta t}{2\beta\theta\nu}||u_{0h}||^2||u_{0h}||_V^2\\
&\quad+\frac{8C_0}{\nu}(||f||^2_{L^2(0,T,H)}+||g||^2_{L^2(0,T,L^\infty(S))}).
\end{aligned}
$$

Lemma 6.4 *For every $0<\delta<1$ and $r\in Z^+$, if*

$$C_0(\nu\Delta th^{-1})^2+3C_0(\Delta th^{-1})^2\Lambda_T\le(1-\delta)\alpha\theta\Delta t\nu,\tag{30}$$

then we have

(31)

$$||u_h^{r+1}||^2+\delta\alpha\theta\Delta t\nu\sum_{n=0}^{r}||u_h^{n+1}||_V^2+(\frac{1}{4}-\frac{2\beta\nu\theta\Delta t}{h^2})\sum_{n=0}^{r}||u_h^{n+1}-u_h^{n+1-\theta}||^2$$
$$+\frac{1}{2}\theta\Delta t\beta\nu\sum_{n=0}^{r}||u_h^{n+1-\theta}||_V^2\leq\Lambda_r,$$

where

$$\begin{aligned}\Lambda_r\ &=||u_{0h}||^2+C_0(\nu\Delta th^{-1})^2||u_{0h}||_V^2+3C_0(\Delta th^{-1})^2||u_{0h}||^2||u_{0h}||_V^2\\&\quad+\frac{8C_0\Delta t}{\nu}\sum_{n=0}^{r}(||f^n||^2+||g^n||^2_{L^\infty(S)}).\end{aligned}$$

Proof We complete the proof of this lemma by the method of a mathematical induction. When $r=0$, (31) holds following (29). Assume that when $r=k\in Z^+$, (31) holds, that is

$$\begin{aligned}&||u_h^{k+1}||^2+\delta\alpha\theta\Delta t\nu\sum_{n=0}^{k}||u_h^{n+1}||_V^2+(\frac{1}{4}-\frac{2\beta\nu\theta\Delta t}{h^2})\sum_{n=0}^{k}||u_h^{n+1}-u_h^{n+1-\theta}||^2\\&+\frac{1}{2}\theta\Delta t\beta\nu\sum_{n=0}^{k}||u_h^{n+1-\theta}||_V^2\leq\Lambda_k.\end{aligned}\tag{32}$$

Next we will show that (31) also holds when $r=k+1$. In terms of (29) and (32), we have

$$||u_h^{n+1}||^2\leq\Lambda_k\leq\Lambda_T\qquad n=0,1,\cdots,k.$$

In terms of (29) and (30), we have

$$\begin{aligned}&||u_h^{k+2}||^2+\alpha\theta\Delta t\nu\sum_{n=0}^{k+1}||u_h^{n+1}||_V^2+(\frac{1}{4}-\frac{2\beta\nu\theta\Delta t}{h^2})\sum_{n=0}^{k+1}||u_h^{n+1}-u_h^{n+1-\theta}||^2\\&+\frac{1}{2}\theta\Delta t\beta\nu\sum_{n=0}^{k+1}||u_h^{n+1-\theta}||_V^2\\\leq\ &||u_{0h}||^2+C_0(\nu\Delta th^{-1})^2\sum_{n=0}^{k+1}||u_h^n||_V^2+3C_0(\Delta th^{-1})^2\sum_{n=0}^{k+1}||u_h^n||^2||u_h^n||_V^2\\&+\frac{8C_0\Delta t}{\nu}\sum_{n=0}^{k+1}(||f^n||^2+||g^n||^2_{L^\infty(S)})\\\leq\ &\Lambda_{k+1}+C_0(\nu\Delta th^{-1})^2\sum_{n=1}^{k+1}||u_h^n||_V^2+3C_0(\Delta th^{-1})^2\Lambda_T\sum_{n=1}^{k+1}||u_h^n||_V^2\\\leq\ &\Lambda_{k+1}+\left[C_0(\nu\Delta th^{-1})^2+3C_0(\Delta th^{-1})^2\Lambda_T\right]\sum_{n=1}^{k+1}||u_h^n||_V^2\\\leq\ &\Lambda_{k+1}+(1-\delta)\alpha\theta\Delta t\nu\sum_{n=1}^{k+1}||u_h^n||_V^2.\end{aligned}$$

Simplifying the above inequality gives (31). □

Under these lemmas, we have the following stability theorem.

Theorem 6.1 *If* $0<\theta\leq\dfrac{\sqrt{24}-3}{10}$, $0<\delta<1$, Δt *and* h *satisfy* $h^2>8\beta\theta\Delta t\nu$ *and (30), then for every* $N\in Z^+$, *the solutions* $u_h^{N+\theta},u_h^{N+1-\theta}$ *and* u_h^{N+1} *of the finite element approximation problem (17),(19) and (20) all belong to* $l^2(0,T,V_h)\cap$

$l^{\infty}(0,T,H)$, *where* $l^2(0,T,V_h) = \{v_h \in V_h :\ \Delta t \sum_{n=0}^{N} ||v_h||_V^2 < +\infty\}$, $\quad l^{\infty}(0,T,H) = \{v_h \in V_h :\ ||v_h|| < +\infty\}$.

Proof According to Lemma 6.4, for every $N \in Z^+$, we have

$$||u_h^{N+1}||^2 + \delta\alpha\theta\Delta t\nu \sum_{n=0}^{N} ||u_h^{n+1}||_V^2 + \frac{1}{2}\theta\Delta t\beta\nu \sum_{n=0}^{N} ||u_h^{n+1-\theta}||_V^2 \le \Lambda_T.$$

This shows

$$u_h^{N+1} \in l^2(0,T,V_h) \cap l^{\infty}(0,T,H), \qquad u_h^{N+1-\theta} \in l^2(0,T,V_h).$$

According to Lemma 6.1, we have

$$\begin{aligned} ||u_h^{N+\theta}||^2 &\le ||u_h^N||^2 + C_0(\Delta t h^{-1})^2 \Lambda_T \sum_{n=0}^{N} ||u_h^n||_V^2 \\ &\quad + \frac{C_0\Delta t}{\nu} \sum_{n=0}^{N} (||f^n||^2 + ||g^n||^2_{L^{\infty}(S)}) \\ &\le 2\Lambda_T + (1-\delta)\alpha\theta\Delta t\nu \sum_{n=0}^{N} ||u_h^n||_V^2 \\ &\le 2\Lambda_T + \alpha\theta\Delta t\nu \sum_{n=0}^{N} ||u_h^n||_V^2 \le (2 + \frac{1}{\delta})\Lambda_T. \end{aligned}$$

Hence,

$$u_h^{N+\theta} \in l^{\infty}(0,T,H).$$

According to Lemma 6.2, we have

$$\begin{aligned} &||u_h^{N+1-\theta}||^2 + \frac{1}{2} \sum_{n=0}^{N} ||u_h^{n+1-\theta} - u_h^{n+\theta}||^2 \\ &\le ||u_h^{N+\theta}||^2 + C_0(\nu\Delta t h^{-1})^2 \sum_{n=0}^{N} ||u_h^n||_V^2 \\ &\quad + C_0(\Delta t h^{-1})^2 \sum_{n=0}^{N} ||u_h^n||^2 ||u_h^n||_V^2 + C_0(\Delta t h^{-1})^2 \sum_{n=0}^{N} ||g^n||^2_{L^{\infty}(S)} \\ &\quad + \frac{C_0\Delta t}{\nu} \sum_{n=0}^{N} (||f^n||^2 + ||g^n||^2_{L^{\infty}(S)} + ||g^{n+1-\theta}||^2_{L^{\infty}(S)}) \\ &\le ||u_h^{N+\theta}||^2 + [C_0(\nu\Delta t h^{-1})^2 + C_0(\Delta t h^{-1})^2 \Lambda_T) \sum_{n=0}^{N} ||u_h^n||_V^2 \\ &\quad + \frac{8C_0\Delta t}{\nu} \sum_{n=0}^{N} (||f^n||^2 + ||g^n||^2_{L^{\infty}(S)}) \\ &\le (2 + \frac{1}{\delta})\Lambda_T + \alpha\theta\Delta t\nu \sum_{n=0}^{N} ||u_h^n||_V^2 + 8\Lambda_T \le (10 + \frac{1}{\delta})\Lambda_T. \end{aligned}$$

Hence,

$$u_h^{N+1-\theta} \in l^{\infty}(0,T,H).$$

From (25), we obtain

$$
\begin{aligned}
&\sum_{n=0}^{N}||u_h^{n+\theta}-u_h^n||^2+\theta\Delta t\alpha\nu\sum_{n=0}^{N}||u_h^{n+\theta}-u_h^n||_V^2\\
\le\ & C_2(\Delta t\nu h^{-1})^2\sum_{n=0}^{N}||u_h^n||_V^2+C_2(\Delta t h^{-1})^2\sum_{n=0}^{N}||u_h^n||^2||u_h^n||_V^2\\
&+C_2\frac{\Delta t}{\nu}\sum_{n=0}^{N}(||f^n||^2+||g^n||^2_{L^\infty(S)})\\
\le\ & \frac{C_2}{C_0}[C_0(\Delta t h^{-1})^2\Lambda_T)]\sum_{n=0}^{N}||u_h^n||_V^2+\frac{C_2}{C_0}\Lambda_T\\
\le\ & \frac{C_2}{C_0}\alpha\theta\Delta t\nu\sum_{n=0}^{N}||u_h^n||_V^2+\frac{C_2}{C_0}\Lambda_T\le\frac{C_2}{C_0}(1+\frac{1}{\delta})\Lambda_T.
\end{aligned}
$$

Thus, from the triangle inequality, we have

$$
\begin{aligned}
&\theta\Delta t\alpha\nu\sum_{n=0}^{N}||u_h^{n+\theta}||_V^2\le\theta\Delta t\alpha\nu\sum_{n=0}^{N}||u_h^{n+\theta}-u_h^n||_V^2+\theta\Delta t\alpha\nu\sum_{n=0}^{N}||u_h^n||_V^2\\
\le\ & \frac{C_2}{C_0}(1+\frac{1}{\delta})\Lambda_T+\frac{1}{\delta}\Lambda_T.
\end{aligned}
$$

So,

$$
u_h^{N+\theta}\in l^2(0,T,V_h). \qquad \square
$$

7. Convergence Analysis

Define

$$
\left\{
\begin{array}{l}
u^{(1)}(t)=\left\{\begin{array}{ll} u_h^n & t\in(n\Delta t,(n+\theta)\Delta t],\\ 0 & t\in((n+\theta)\Delta t,(n+1)\Delta t],\end{array}\right.\\
u^{(2)}(t)=\left\{\begin{array}{ll} u_h^{n+\theta} & t\in((n+\theta)\Delta t,(n+1-\theta)\Delta t],\\ 0 & t\in(n\Delta t,(n+\theta)\Delta t]\cup((n+1-\theta)\Delta t,(n+1)\Delta t],\end{array}\right.\\
u^{(3)}(t)=\left\{\begin{array}{ll} u_h^{n+1-\theta} & t\in((n+1-\theta)\Delta t,(n+1)\Delta t],\\ 0 & t\in(n\Delta t,(n+1-\theta)\Delta t],\end{array}\right.
\end{array}
\right.
$$

and

$$
\left\{
\begin{array}{l}
\lambda^{(1)}(t)=\left\{\begin{array}{ll} \lambda_h^n & t\in(n\Delta t,(n+\theta)\Delta t],\\ 0 & t\in((n+\theta)\Delta t,(n+1)\Delta t],\end{array}\right.\\
\lambda^{(3)}(t)=\left\{\begin{array}{ll} \lambda_h^{n+1-\theta} & t\in((n+1-\theta)\Delta t,(n+1)\Delta t],\\ 0 & t\in(n\Delta t,(n+1-\theta)\Delta t],\end{array}\right.
\end{array}
\right.
$$

where $n=0,1,\cdots,N-1, N\cdot\Delta t=T$.

Denote $w_h(t)$ a continuous function from $[0,T]$ to V_h which is a linear function on every interval $[n\Delta t,(n+1)\Delta t]$ and satisfies $w_h(n\Delta t)=u_h^n$. Then $w_h(t)$ can be represented as

$$
w_h(t)=u_h^n+\frac{1}{\Delta t}(t-t_n)(u_h^{n+1}-u_h^n) \qquad t\in[n\Delta t,(n+1)\Delta t], \tag{33}
$$

where $t_n=n\Delta t, n=0,1,\cdots,N-1$. Obviously, we have $\dfrac{dw_h}{dt}=\dfrac{1}{\Delta t}(u_h^{n+1}-u_h^n)$. According to (17), (19) and (20), for every $\phi_h\in V_{\sigma h}$ and $t\in[n\Delta t,(n+1)\Delta t]$, we have

(34)
$$\begin{aligned}&\frac{1}{\theta}\frac{d}{dt}(w_h,\phi_h)+\alpha a(u^{(1)}(t+\Delta t),\phi_h)+\beta a(u^{(1)}(t),\phi_h)+b(u^{(1)}(t),u^{(1)}(t),\phi_h)\\&=\frac{1}{\theta\Delta t}(u^{(3)}(t)-u^{(2)}(t),\phi_h)-\beta a(u^{(3)}(t),\phi_h)-\alpha a(u^{(2)}(t),\phi_h)\\&-c(\lambda^{(3)}(t)+\lambda^{(1)}(t),g(t)\phi_h)-b(u^{(3)}(t),u^{(3)}(t),\phi_h)+(f^n(t)+f^{n+1}(t),\phi_h).\end{aligned}$$

From the proof of Theorem 6.1, for every $i=1,2,3$, we have

$$\sup_{t\in[0,T]}||u^{(i)}(t)||^2\le(10+\frac{2}{\delta})\Lambda_T. \tag{35}$$

$$\begin{aligned}\int_0^T||u^{(i)}(t)||_V^2\mathrm{d}t&=\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}||u^{(i)}(t)||_V^2\mathrm{d}t=\Delta t\sum_{n=0}^{N-1}||u^{(i)}(t)||_V^2\\&\le[\frac{1}{\delta\alpha\theta\nu}+\frac{2}{\theta\beta\nu}+\frac{C_2}{C_0\alpha\theta\nu}(1+\frac{1}{\delta})+\frac{1}{\alpha\theta\nu\delta}]\Lambda_T.\end{aligned} \tag{36}$$

Lemma 7.1 *If Δt and h satisfy $h^2>16\beta\theta\Delta t\nu$, then for every $i=1,2,3$, as $\Delta t\longrightarrow 0$, we have*

$$||u^{(i)}-w_h||_{L^2(0,T,L^2(\Omega)^2)}\longrightarrow 0. \tag{37}$$

Proof From the definition of w_h, we have

$$\begin{aligned}&||u^{(1)}-w_h||^2_{L^2(0,T,L^2(\Omega)^2)}\\&=(\frac{1}{\Delta t})^2\sum_{n=0}^{N-1}||u_h^{n+1}-u_h^n||^2\int_{t_n}^{t_{n+1}}(t-t_n)^2\mathrm{d}t=\frac{1}{3}\Delta t\sum_{n=0}^{N-1}||u_h^{n+1}-u_h^n||^2\\&\le\frac{1}{3}\Delta t\sum_{n=0}^{N-1}\left(||u_h^{n+1}-u_h^{n+1-\theta}||^2+||u_h^{n+1-\theta}-u_h^{n+\theta}||^2+||u_h^{n+\theta}-u_h^n||^2\right).\end{aligned}$$

In terms of the proof of Theorem 6.1, we have

$$\begin{aligned}&\sum_{n=0}^{N-1}||u_h^{n+\theta}-u_h^n||^2\le\frac{C_2}{C_0}(1+\frac{1}{\delta})\Lambda_T,\\&\sum_{n=0}^{N-1}||u_h^{n+1-\theta}-u_h^{n+\theta}||^2\le(20+\frac{2}{\delta})\Lambda_T.\end{aligned}$$

On the other hand, since $h^2>16\beta\theta\Delta t\nu$, then

$$\frac{1}{4}-\frac{2\beta\nu\theta\Delta t}{h^2}>\frac{1}{8}.$$

From the proof of Lemma 6.4, we obtain

$$\sum_{n=0}^{N-1}||u_h^{n+1}-u_h^{n+1-\theta}||^2\le 8\Lambda_T.$$

Thus,

$$||u^{(1)}-w_h||_{L^2(0,T,L^2(\Omega)^2)}\le[\frac{C_2}{3C_0}(1+\frac{1}{\delta})+(7+\frac{1}{\delta})]\Lambda_T\Delta t.$$

So as $\Delta t\longrightarrow 0$, we show

$$||u^{(1)}-w_h||_{L^2(0,T,L^2(\Omega)^2)}\longrightarrow 0.$$

For $u^{(2)}$, we have

$$\begin{aligned} ||u^{(1)} - u^{(2)}||^2_{L^2(0,T,L^2(\Omega)^2)} &= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} ||u^{(1)}(t) - u^{(2)}(t)||^2 \mathrm{d}t \\ &= \Delta t \sum_{n=0}^{N-1} ||u_h^{n+\theta} - u_h^n||^2 \leq \frac{C_2}{C_0}(1 + \frac{1}{\delta})\Lambda_T \Delta t. \end{aligned}$$

Hence, as $\Delta t \longrightarrow 0$, we have $||u^{(1)} - |u^{(2)}||_{L^2(0,T,L^2(\Omega)^2)} \longrightarrow 0$. From the triangle inequality, we have

$$||u^{(2)} - w_h||_{L^2(0,T,L^2(\Omega)^2)} \longrightarrow 0 \qquad \text{as } \Delta t \longrightarrow 0.$$

Similarly, we can show

$$||u^{(3)} - w_h||_{L^2(0,T,L^2(\Omega)^2)} \longrightarrow 0 \qquad \text{as } \Delta t \longrightarrow 0. \qquad \square$$

Lemma 7.2 *It holds that* $w_h \in L^2(0,T,V)$.

Proof According to the definition of $w_h(t)$, we have

$$\begin{aligned} &\int_0^T ||w_h(t)||_V^2 \mathrm{d}t \\ &\leq \int_0^T ||u^{(1)}(t)||_V^2 \mathrm{d}t + (\frac{1}{\Delta t})^2 \sum_{n=0}^{N-1} ||u_h^{n+1} - u_h^n||_V^2 \int_{t_n}^{t_{n+1}} (t - t_n)^2 \mathrm{d}t \\ &\leq \int_0^T ||u^{(1)}(t)||_V^2 \mathrm{d}t + \frac{1}{3}\Delta t \sum_{n=0}^{N-1} ||u_h^{n+1} - u_h^n||_V^2 \\ &\leq \int_0^T ||u^{(1)}(t)||_V^2 \mathrm{d}t + \frac{2}{3}\Delta t \sum_{n=1}^{N} ||u_h^n||_V^2 \leq \frac{2}{\delta\alpha\theta\nu}\Lambda_T. \qquad \square \end{aligned}$$

From (35), (36) and Lemma 7.2, there exist $u^i \in L^2(0,T,V) \cap L^\infty(0,T,L^2(\Omega))$, $i = 1,2,3$, and $w \in L^2(0,T,V)$ such that as $(h,\Delta t) \longrightarrow 0$, we have

$$\left\{ \begin{array}{l} u^{(i)} \text{ weakly converges to } u^i \text{ in } L^2(0,T,V), \\ u^{(i)} \text{ weakly star converges to } u^i \text{ in } L^\infty(0,T,H), \\ w_h \text{ weakly converges to } w \text{ in } L^2(0,T,V)\ , \\ w_h \text{ strongly converges to } w \text{ in } L^2(0,T,H), \\ w_h \text{ strongly converges to } w \text{ in } L^2(0,T,L^2(S)^2). \end{array} \right. \tag{38}$$

Moreover, from Lemma 7.1, for almost everywhere $t \in (0,T)$, we have

$$u^i(t) = w(t) \qquad \text{a.e. in } \Omega. \tag{39}$$

Next, we will show that w satisfies the variational problem (5). Let $\Pi_h : V \longrightarrow V_h$ be the projection operator and satisfy

$$||v - \Pi_h v||_V \leq ch||v||_2 \qquad \forall\ v \in V \cap H^2(\Omega)^2,$$

where $c > 0$ is independent of h. Let $\psi \in C^1[0,T]$ and satisfy $\psi(T) = 0$. Taking $\phi_h = \Pi_h v$ for $v \in V_\sigma \cap H^2(\Omega)^2$ and integrating t from 0 to T in (34), we have

$$
\begin{aligned}
&-\frac{1}{\theta}\int_0^T (w_h(t), \psi'(t)\Pi_h v)\mathrm{d}t + \alpha\int_0^T a(u^{(1)}(t+\Delta t), \psi(t)\Pi_h v)\mathrm{d}t \\
&+\beta\int_0^T a(u^{(1)}(t), \psi(t)\Pi_h v)\mathrm{d}t + \int_0^T b(u^{(1)}(t), u^{(1)}(t), \psi(t)\Pi_h v)\mathrm{d}t \\
= &\frac{1}{\theta\Delta t}\int_0^T (u^{(3)}(t) - u^{(2)}(t), \psi(t)\Pi_h v)\mathrm{d}t - \beta\int_0^T a(u^{(3)}(t), \psi(t)\Pi_h v)\mathrm{d}t \\
&-\alpha\int_0^T a(u^{(2)}(t), \psi(t)\Pi_h v)\mathrm{d}t - \int_0^T c(\lambda^{(3)}(t) + \lambda^{(1)}(t), g(t)\psi(t)\Pi_h v)\mathrm{d}t \\
&-\int_0^T b(u^{(3)}(t), u^{(3)}(t), \psi(t)\Pi_h v)\mathrm{d}t + \int_0^T (f^n(t) + f^{n+1}(t), \psi(t)\Pi_h v)\mathrm{d}t \\
&+\frac{1}{\theta}(u_{0h}, \Pi_h v)\psi(0),
\end{aligned}
\tag{40}
$$

On the other hand, integrating from 0 to T in (19) yields

$$
\begin{aligned}
&\frac{1}{\theta\Delta t}\int_0^T (u^{(3)}(t) - u^{(2)}(t), \psi(t)\Pi_h v)\mathrm{d}t = -\frac{1-2\theta}{\theta}\beta\int_0^T a(u^{(3)}(t), \psi(t)\Pi_h v)\mathrm{d}t \\
&-\frac{1-2\theta}{\theta}\int_0^T b(u^{(3)}(t), u^{(3)}(t), \psi(t)\Pi_h v)\mathrm{d}t - \frac{1-2\theta}{\theta}\alpha\int_0^T a(u^{(2)}(t), \psi(t)\Pi_h v)\mathrm{d}t \\
&-\frac{1-2\theta}{\theta}\int_0^T c(\lambda^{(3)}(t), g(t)\psi(t)\Pi_h v)\mathrm{d}t + \frac{1-2\theta}{\theta}\int_0^T (f^n(t), \psi(t)\Pi_h v)\mathrm{d}t.
\end{aligned}
\tag{41}
$$

Since $\lambda_h^{(i)} \in [-1,1], i = 1,3$, then as $(h, \Delta t) \longrightarrow 0$, $\lambda_h^{(i)}(t)$ almost everywhere converges to $\lambda^i(t)$. Next, we show $\lambda^1(t) = \lambda^3(t)$. From (18), (21) and (39), one has

$$\lambda^3 = P_\Lambda(\lambda^1 + \rho g w_\tau) \qquad \text{and} \qquad \lambda^1 = P_\Lambda(\lambda^3 + \rho g w_\tau).$$

Subtracting the above two identities and according to the compressibility of the projection operator, we conclude $\lambda^1(t) = \lambda^3(t)$. Here we denote it by $\lambda(t)$. According to (38) and (39), making $(h, \Delta t) \longrightarrow 0$ in (40) and (41) gives

$$
\begin{aligned}
&-\int_0^T (w(t), \psi'(t)v)\mathrm{d}t + \int_0^T a(w(t), \psi(t)v)\mathrm{d}t + \int_0^T b(w(t), w(t), \psi(t)v)\mathrm{d}t \\
&+\int_0^T c(\lambda(t), g(t)\psi(t)v)\mathrm{d}t = (u_0, v)\psi(0) + \int_0^T (f(t), \psi(t)v)\mathrm{d}t \qquad \forall\, v \in V_\sigma,
\end{aligned}
$$

which shows that w satisfies the variational problem (5). Thus, we show the following converges theorem:

Theorem 7.1 *Under the assumptions of Theorem 6.1, if h satisfies $h^2 > 16\beta\theta\Delta t\nu$, then as $(h, \Delta t) \longrightarrow 0$, the solutions $u_h^{N+\theta}, u_h^{N+1-\theta}$ and u_h^{N+1} of the discretized problem (17),(19) and (21) strongly converge to the solution of the problem (5) in $L^2(0,T,H)$.*

If $V_{\sigma h} \subset V_\sigma$, then

Theorem 7.2 *Under the assumptions of Theorem 7.1, as $(h, \Delta t) \longrightarrow 0$, the solutions $u_h^{N+\theta}, u_h^{N+1-\theta}$ and u_h^{N+1} of the discretized problem (17),(19) and (21) strongly converge to the solution of the problem (5) in $L^2(0,T,V)$.*

Proof Since $w_h \in V_\sigma$, then taking $v = w - w_h$ in (5) yields

$$
\begin{aligned}
\nu\int_0^T ||w(t)-w_h(t)||_V^2\mathrm{d}t = &-\int_0^T a(w_h(t), w(t)-w_h(t))\mathrm{d}t \\
&-\int_0^T (w'(t), w(t)-w_h(t))\mathrm{d}t - \int_0^T b(w(t), w(t), w(t)-w_h(t))\mathrm{d}t \\
&-\int_0^T c(\lambda(t), g(t)(w(t)-w_h(t)))\mathrm{d}t + \int_0^T (f(t), w(t)-w_h(t))\mathrm{d}t.
\end{aligned} \tag{42}
$$

Because w_h strongly converges to w in $L^2(0,T,H)$ and $L^2(0,T,L^2(S)^2)$, so as $(h,\Delta t) \longrightarrow 0$, we have

$$
\begin{aligned}
&\int_0^T (w'(t), w(t)-w_h(t))\mathrm{d}t + \int_0^T c(\lambda(t), g(t)(w(t)-w_h(t)))\mathrm{d}t \\
&\quad -\int_0^T (f(t), w(t)-w_h(t))\mathrm{d}t \longrightarrow 0.
\end{aligned} \tag{43}
$$

For trilinear form, we have

$$
\begin{aligned}
&\int_0^T b(w(t), w(t), w(t)-w_h(t))\mathrm{d}t \\
\le &\int_0^T ||w(t)||^{\frac{1}{2}}||w(t)||_V^{\frac{3}{2}}||w(t)-w_h(t)||^{\frac{1}{2}}||w(t)-w_h(t)||_V^{\frac{1}{2}}\mathrm{d}t \\
\le &\sup_{t\in[0,T]}||w(t)||^{\frac{1}{2}}\sup_{t\in[0,T]}||w(t)-w_h(t)||^{\frac{1}{2}}\Big(\int_0^T ||w(t)||_V^2\mathrm{d}t\Big)^{\frac{3}{4}}\Big(\int_0^T ||w(t)-w_h(t)||_V^2\mathrm{d}t\Big)^{\frac{1}{4}}.
\end{aligned}
$$

For almost everywhere $t \in [0,T]$, there holds $||w(t)-w_h(t)|| \longrightarrow 0$. Thus,

$$
\int_0^T b(w(t), w(t), w(t)-w_h(t))\mathrm{d}t \longrightarrow 0. \tag{44}
$$

From (38), (43) and (44), making $(h,\Delta t) \longrightarrow 0$ in (42) gives

$$
\int_0^T ||w(t)-w_h(t)||_V^2\mathrm{d}t = 0,
$$

which shows that w_h strongly converges to w in $L^2(0,T,V)$. □

8. Numerical Results

In this section, we give the numerical results to check the theoretical analysis. Assume that the domain Ω is the standard square domain, i.e., $\Omega = [0,1]\times[0,1]$. The exact solutions u and p are

$$
\begin{aligned}
&u(t,x,y) = (u_1(t,x,y), u_2(t,x,y)), \qquad &&p(x,y) = t(2x-1)(2y-1), \\
&u_1(x,y) = tx^2y(x-1)(3y-2), &&u_2(x,y) = -txy^2(y-1)(3x-2).
\end{aligned}
$$

It is easy to verify the exact solution u satisfies $u = 0$ on Γ, $u\cdot\vec{n} = u_1 = 0, u_2 \neq 0$ on S_1 and $u_1 \neq 0, u\cdot\vec{n} = u_2 = 0$ on S_2. Moreover, the tangential vectors τ on S_1 and S_2 are $(0,1)$ and $(-1,0)$, so

$$
\begin{cases}
\sigma_\tau = \sigma_{21} = 4\nu ty^2(y-1) & \text{on } S_1, \\
\sigma_\tau = -\sigma_{12} = 4\nu tx^2(1-x) & \text{on } S_2.
\end{cases}
$$

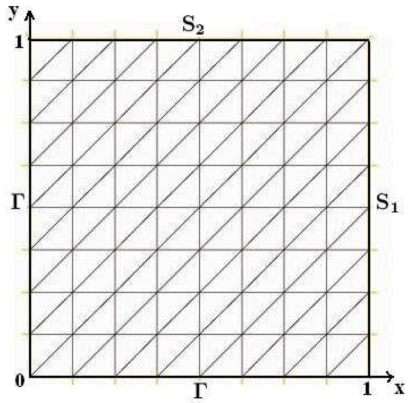

Figure 1 the domain Ω

On the other hand, from the nonlinear boundary conditions (2), we have

$$|\sigma_\tau| \leq g \quad \text{and} \quad \sigma_\tau u_\tau + g|u_\tau| = 0 \qquad \text{on } S = S_1 \cup S_2,$$

so the function g can be chosen such that $g = -\sigma_\tau \geq 0$ on S_1 and $g = \sigma_\tau \geq 0$ on S_2.

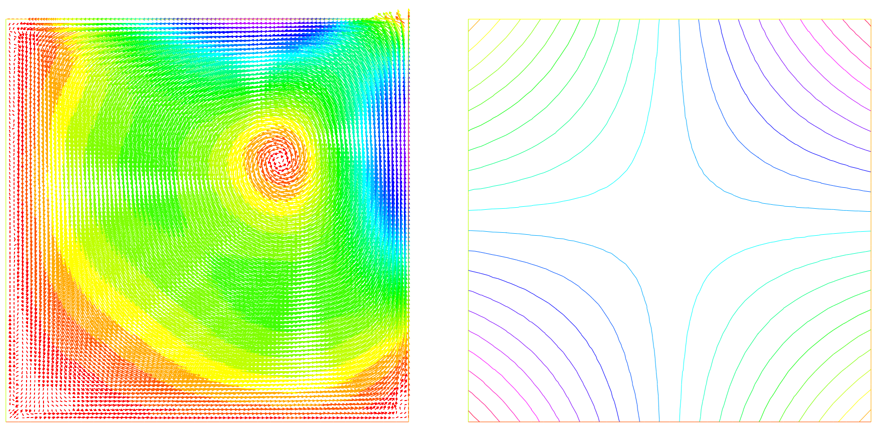

Figure 2 the velocity field and the pressure isovalue

Let $\nu = 0.005$. The external force f can be determined by the first equation of (2). Since the finite element space (V_h, M_h) must satisfy the discretized inf-sup condition, we use the Taylor-Hood element ($P_2 - P_1$ element). Take the initial value $u_0 = 0$, $\lambda^0 = 1$, the time step $\Delta t = 0.01$, the parameter $\rho = 0.1\nu$, $\alpha = 2 - \sqrt{2}, \beta = \sqrt{2} - 1$, $h = 1/32$.

Although $0 < \theta < \dfrac{\sqrt{24} - 3}{10}$ in Theorem 6.1, for the θ scheme, the optimal value of θ is $1 - \sqrt{2}/2$ (e.g.[1]). Here, we give the numerical comparison between $\theta = 0.01, 0.1$ and $1 - \sqrt{2}/2$. Tables 1-3 show the relative errors at different times $T = 0.02, 0.5, 1$ as $\theta = 0.01, 0.1$ and $1 - \sqrt{2}/2$, from which we can see that the errors are the smallest as $\theta = 1 - \sqrt{2}/2$. Figure 2 shows the velocity field and the pressure isovalue at $T = 1$ as $\theta = 1 - \sqrt{2}/2$.

T	$\frac{\|\|u-u_h\|\|_{L^2(0,T,H)}}{\|\|u\|\|_{L^2(0,T,H)}}$	$\frac{\|\|u-u_h\|\|_{L^2(0,T,V)}}{\|\|u\|\|_{L^2(0,T,V)}}$	$\frac{\|\|p-p_h\|\|_{L^2(0,T,L^2(\Omega))}}{\|\|p\|\|_{L^2(0,T,L^2(\Omega))}}$
0.02	0.000584957	0.0150981	1.98118
0.5	0.000737447	0.0159339	1.97464
1	0.000862151	0.0164102	1.97627

Table 1 The relative errors at different time T as $\theta = 0.01$

T	$\frac{\|\|u-u_h\|\|_{L^2(0,T,H)}}{\|\|u\|\|_{L^2(0,T,H)}}$	$\frac{\|\|u-u_h\|\|_{L^2(0,T,V)}}{\|\|u\|\|_{L^2(0,T,V)}}$	$\frac{\|\|p-p_h\|\|_{L^2(0,T,L^2(\Omega))}}{\|\|p\|\|_{L^2(0,T,L^2(\Omega))}}$
0.02	0.000446522	0.0117658	0.160728
0.5	0.000551083	0.0123479	0.160753
1	0.000631107	0.0126645	0.160886

Table 2 The relative errors at different time T as $\theta = 0.1$

T	$\frac{\|\|u-u_h\|\|_{L^2(0,T,H)}}{\|\|u\|\|_{L^2(0,T,H)}}$	$\frac{\|\|u-u_h\|\|_{L^2(0,T,V)}}{\|\|u\|\|_{L^2(0,T,V)}}$	$\frac{\|\|p-p_h\|\|_{L^2(0,T,L^2(\Omega))}}{\|\|p\|\|_{L^2(0,T,L^2(\Omega))}}$
0.02	0.000195296	0.00527557	0.0287233
0.5	0.000228313	0.00540365	0.0284619
1	0.000304089	0.00550197	0.0283958

Table 3 The relative errors at different time T as $\theta = 1 - \sqrt{2}/2$

In the above numerical results, we select the small time step $\Delta t = 0.01$. Then, if $\Delta t > 0.01$, we want to know if the numerical results are acceptant. Let $\theta = 1-\sqrt{2}/2$. Table 4 shows the relative error at $T = 1$ with the different time step Δt. When $\Delta t = 0.02$, the relative error of u is small, but the relative error of p is large. When $\Delta t = 0.05, 0.1$ and 0.2, the errors of u and p both are large. Moreover, the relative errors are larger and larger when Δt becomes large. Hence, it is important to study the numerical methods for solving the time-dependent Navier-Stokes equations with the nonlinear slip boundary conditions using large time steps Δt in the future papers.

Δt	$\frac{\|\|u-u_h\|\|_{L^2(0,T,H)}}{\|\|u\|\|_{L^2(0,T,H)}}$	$\frac{\|\|u-u_h\|\|_{L^2(0,T,V)}}{\|\|u\|\|_{L^2(0,T,V)}}$	$\frac{\|\|p-p_h\|\|_{L^2(0,T,L^2(\Omega))}}{\|\|p\|\|_{L^2(0,T,L^2(\Omega))}}$
0.01	0.00030408	0.005501	0.028395
0.02	0.00109316	0.023815	0.056916
0.05	0.00720578	0.156394	0.143162
0.1	0.03081094	0.593744	0.288979
0.2	0.13937837	2.264180	0.588398

Table 4 The relative errors with the different Δt

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College of Mathematics and Information Science, Wenzhou University, Wenzhou, 325035, P.R. China

E-mail: yuanli1984@yahoo.com.cn

School of Science, Xi'an Jiaotong University, Xi'an, 710049, P.R. China

E-mail: ktli@mail.xjtu.edu.cn