INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 7, Number 4, Pages 766–784

AN ANISOTROPIC NONCONFORMING ELEMENT FOR FOURTH ORDER ELLIPTIC SINGULAR PERTURBATION PROBLEM

SHAOCHUN CHEN, MINGFANG LIU, AND ZHONGHUA QIAO

(Communicated by Xuecheng Tai)

Abstract. A new nonconforming element constructed by the Double Set Parameter method, is applied to the fourth order elliptic singular perturbation problem. The convergence uniformly in the perturbation parameter ε, is proved under the anisotropic meshes and optimal convergence rate $O(h)$ is obtained. Numerical results are given to demonstrate validity of our theoretical analysis.

Key Words. Nonconforming finite element, Double set parameter method, Anisotropic, Fourth order elliptic singular perturbation problem, Uniform convergence.

1. Introduction

In the discretization of the non-stationary Navier-Stokes systems and the non-stationary oscillation model, the following singular perturbation problem is often considered [11, 15]:

$$(1)\qquad \begin{cases} \varepsilon^2 \triangle^2 u - \Delta u = f, & \text{in} \quad \Omega, \\ u = \dfrac{\partial^2 u}{\partial n^2} = 0, & \text{on} \quad \partial\Omega, \end{cases}$$

where $f \in L^2(\Omega)$, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator, $\Delta^2 = (\partial^2/\partial x^2 + \partial^2/\partial y^2)^2$, $\Omega \subset R^2$ is a bounded rectangle domain, $\partial\Omega$ is the boundary of Ω, and $\partial/\partial n$, $\partial/\partial s$ denote the outer normal derivative and tangential derivative on $\partial\Omega$, respectively. Because Ω is a rectangle, we have

$$\Delta u|_{\partial\Omega} = \left(\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial n^2}\right)\Big|_{\partial\Omega} = 0.$$

In (1) ε is a real parameter such that $0 < \varepsilon \le 1$. In particular, we are interested in the regime when ε is close to zero. Obviously, if ε tends to zero the differential equation (1) formally degenerates to the Poisson equation. Hence, a plate model may degenerate towards an elastic membrane problem.

The problem (1) but with boundary condition $u = \dfrac{\partial u}{\partial n} = 0$ on $\partial\Omega$ has been studied in [7, 12, 17]. [12] presented a nine parameter C^0 triangular element, [17] presented a modified triangular Morley's element and a modified rectangular Morley's element by changing the discrete variational problem, and [7] presented two non-C^0 nonconforming elements.

Received by the editors January 1, 2009 and, in revised form, March 22, 2010.
2000 *Mathematics Subject Classification.* 65N12, 65N30.
This research was supported by NSFC(No.10771198, No.10590353).

The classical finite element approximation relies on the regular or non-degenerate condition, i.e., there exists a constant C such that

$$\frac{h_K}{\rho_K} \leq C, \quad \text{for each element } K, \tag{2}$$

where h_K is the diameter of K and ρ_K is the diameter of the biggest ball contained in K, see e.g. [3, 8] for details. But the solution of some problems may have the anisotropic behavior in parts of the domain, which means the solution varies significantly in certain direction. For such problems using regular element meshes will make the computation expensive. It is inclined to use anisotropic meshes with a small mesh size in the direction of the rapid variation of the solution and a large mesh size in the perpendicular direction. Recently much attention is paid to anisotropic finite elements, see e.g. [1, 2, 6, 9]. The main point is to get the error estimate independent of the above regular or nondegenerate condition. The fourth order elliptic singular perturbation problem (1) is such a problem which may have the boundary layers. The anisotropic behavior will happen near some boundaries. However, all the analysis results in [7, 12, 17] were got based on the regular and quasi-uniform assumption of the mesh. The convergence order $O(h^{\frac{1}{2}})$ were obtained. In [19], we constructed a nonconforming finite element by the Double Set Parameter Method for solving the plate bending problem. The goal of this paper is to use this element for solving the singular perturbation problem (1), which is uniformly convergent for ε under anisotropic meshes with the optimal convergence order $O(h)$.

Double Set Parameter method is one of the useful nonstandard methods for constructing nonconforming finite elements, which is firstly proposed by the first author and his coworker in [5]. The key step to construct a finite element is to choose suitable and matched shape function space and degrees of freedom. The degrees of freedom determine the global continuity of the whole finite element space, so they should be chosen carefully to satisfy the convergence demand. On the other hand, the degrees of freedom represent the unknowns of discrete finite element equations, therefore they should be chosen to be simple and convenient so that the size of discrete system is small. These two demands for degrees of freedom are sometimes difficult to meet each other. To overcome this difficulty the double set parameter method separates the two demands for degrees of freedom. The essential point is to choose two sets of parameters, which can be chosen independently with each other. The first set of parameters are discretized into the second one according to suitable numerical rules, which will make the degrees of freedom having small perturbations. In principle, the first set of parameters, which determine the smoothness of the shape function across elements, are selected to meet convergence requirements, while the second set of parameters, which are real degrees of freedom, are chosen to be simple to make the total number of unknowns in the resulting discrete system small. Recently, we have found the new application of double set parameter method in the anisotropic elements. In some cases this method can improve the behavior of the element to make the element anisotropically convergent, while the corresponding single set parameter form of the element is not anisotropically convergent. The element in this paper is one example and the double set parameter rotated-Q_1 element in [18] is another example.

The rest of this paper is organized as follows. In the following section, we construct a rectangular element using Double Set Parameter method, and prove it's anisotropy. Next, we prove the new element is uniformly convergent in ε for the

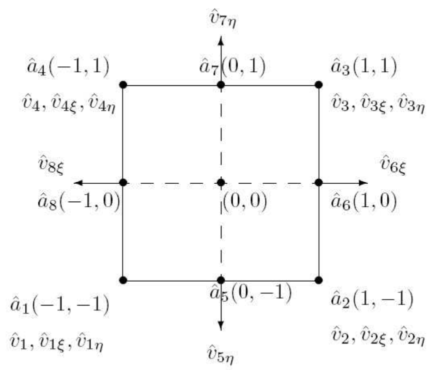

FIGURE 1

singular perturbation problem, and give the error bound. Finally, two numerical experiments are given to demonstrate validity of our theoretical analysis.

2. A Nonconforming Element and Anisotropic Interpolation

In this section, to make the completeness of the paper, we present the nonconforming finite element introduced in [19].

Let reference element $\widehat{K}$ be a square on (ξ,η) plane with vertices $\widehat{a}_1(-1,-1)$, $\widehat{a}_2(1,-1)$, $\widehat{a}_3(1,1)$, $\widehat{a}_4(-1,1)$, sides $\widehat{l}_1 = \overline{\widehat{a}_1\widehat{a}_2}$, $\widehat{l}_2 = \overline{\widehat{a}_2\widehat{a}_3}$, $\widehat{l}_3 = \overline{\widehat{a}_3\widehat{a}_4}$, $\widehat{l}_4 = \overline{\widehat{a}_4\widehat{a}_1}$ and middle points of sides $\widehat{a}_5(0,-1)$, $\widehat{a}_6(1,0)$, $\widehat{a}_7(0,1)$, $\widehat{a}_8(-1,0)$, respectively. See Figure 1.

Its shape function space is taken as

$$\widehat{P} = P_2(\widehat{K})\bigcup\{\xi^3,\eta^3\} = \mathrm{span}\{\widehat{p_i}(\xi,\eta),\ i=1,\cdots,8\},$$

where $P_i(\widehat{K})$ is the space of polynomials of degree less than or equal to i on $\widehat{K}$ and

$$\widehat{p_1}(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta), \quad \widehat{p_2}(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta),$$

$$\widehat{p_3}(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta), \quad \widehat{p_4}(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta),$$

$$\widehat{p_5}(\xi,\eta) = (1-\xi^2), \quad \widehat{p_6}(\xi,\eta) = (1-\eta^2), \quad \widehat{p_7}(\xi,\eta) = \xi(1-\xi^2), \quad \widehat{p_8}(\xi,\eta) = \eta(1-\eta^2).$$

We use the Double Set Parameter method introduced in [5] to construct the element. The first set of parameters are

$$D(\hat{v}) = \{\hat{v}_1,\hat{v}_2,\hat{v}_3,\hat{v}_4,\hat{v}_{5\eta},\hat{v}_{6\xi},\hat{v}_{7\eta},\hat{v}_{8\xi}\}, \tag{3}$$

where $\hat{v}_i = \hat{v}(\widehat{a}_i), i=1,\cdots,4$, $\hat{v}_{i\eta} = \partial\hat{v}/\partial\eta(\widehat{a}_i), i=5,7$, $\hat{v}_{i\xi} = \partial\hat{v}/\partial\xi(\widehat{a}_i), i=6,8$. Then $\forall\hat{v}\in\hat{P}$,

$$\hat{v} = \sum_{i=1}^{8}\beta_i\hat{p}_i(\xi,\eta), \tag{4}$$

where

$$
\begin{cases}
\beta_i = \hat{v}_i, 1 \le i \le 4, \\
\beta_5 = \dfrac{1}{4}(\hat{v}_{8\xi} - \hat{v}_{6\xi}), \\
\beta_6 = \dfrac{1}{4}(\hat{v}_{5\eta} - \hat{v}_{7\eta}), \\
\beta_7 = \dfrac{1}{8}(-\hat{v}_1 + \hat{v}_2 + \hat{v}_3 - \hat{v}_4 - 2\hat{v}_{8\xi} - 2\hat{v}_{6\xi}), \\
\beta_8 = \dfrac{1}{8}(-\hat{v}_1 - \hat{v}_2 + \hat{v}_3 + \hat{v}_4 - 2\hat{v}_{5\eta} - 2\hat{v}_{7\eta}).
\end{cases}
\tag{5}
$$

In remark 2.1 (ii) we will prove the single set parameter element defined by (3)-(5) is not anisotropically convergent element, and its double set parameter form is considered in the following.

The second set of parameters, which are real degrees of freedom, are taken as

$$
Q(\hat{v}) = \{\hat{v}_i, \hat{v}_{i\xi}, \hat{v}_{i\eta}, i = 1, \cdots, 4\},
\tag{6}
$$

where $\hat{v}_{i\xi} = \partial\hat{v}/\partial\xi(\widehat{a}_i)$, $\hat{v}_{i\eta} = \partial\hat{v}/\partial\eta(\widehat{a}_i)$, $1 \le i \le 4$.

Since $\dfrac{\partial\hat{v}}{\partial\eta}|_{\eta=\text{const}} \in P_1$, and $\dfrac{\partial\hat{v}}{\partial\xi}|_{\xi=\text{const}} \in P_1$, we have

$$
\hat{v}_{5\eta} = \frac{1}{2}(\hat{v}_{1\eta} + \hat{v}_{2\eta}),\ \hat{v}_{6\xi} = \frac{1}{2}(\hat{v}_{2\xi} + \hat{v}_{3\xi}),\ \hat{v}_{7\eta} = \frac{1}{2}(\hat{v}_{3\eta} + \hat{v}_{4\eta}),\ \hat{v}_{8\xi} = \frac{1}{2}(\hat{v}_{4\xi} + \hat{v}_{1\xi}).
$$

Substituting the above formulas into β_i, we get

$$
\begin{cases}
\beta_i = \hat{v}_i, 1 \le i \le 4, \\
\beta_5 = \dfrac{1}{8}(\hat{v}_{1\xi} - \hat{v}_{2\xi} - \hat{v}_{3\xi} + \hat{v}_{4\xi}), \\
\beta_6 = \dfrac{1}{8}(\hat{v}_{1\eta} + \hat{v}_{2\eta} - \hat{v}_{3\eta} - \hat{v}_{4\eta}), \\
\beta_7 = \dfrac{1}{8}(-\hat{v}_1 + \hat{v}_2 + \hat{v}_3 - \hat{v}_4 - \hat{v}_{1\xi} - \hat{v}_{2\xi} - \hat{v}_{3\xi} - \hat{v}_{4\xi}), \\
\beta_8 = \dfrac{1}{8}(-\hat{v}_1 - \hat{v}_2 + \hat{v}_3 + \hat{v}_4 - \hat{v}_{1\eta} - \hat{v}_{2\eta} - \hat{v}_{3\eta} - \hat{v}_{4\eta}).
\end{cases}
\tag{7}
$$

Let $H^m(\hat{K})$ be the usual Sobolev space, $\|\cdot\|_{m,\hat{K}}$ and $|\cdot|_{m,\hat{K}}$ be the norm and semi-norm of $H^m(\hat{K})$, respectively. The interpolation operator of this element is defined by

$$
\hat{I} : H^3(\hat{K}) \to \hat{P}, \quad \hat{I}\hat{v} = \sum_{i=1}^{8} \beta_i \hat{p}_i(\xi, \eta),
\tag{8}
$$

where $\beta_i,\ 1 \le i \le 8$ are defined as (7). By the simple computations, we have

$$\begin{cases} (\hat{I}\hat{v})_i = \hat{v}_i, 1 \le i \le 4, \\ (\hat{I}\hat{v})_{5\eta} = \dfrac{1}{2}(\hat{v}_{1\eta} + \hat{v}_{2\eta}), \qquad (\hat{I}\hat{v})_{6\xi} = \dfrac{1}{2}(\hat{v}_{2\xi} + \hat{v}_{3\xi}), \\ (\hat{I}\hat{v})_{7\eta} = \dfrac{1}{2}(\hat{v}_{3\eta} + \hat{v}_{4\eta}), \qquad (\hat{I}\hat{v})_{8\xi} = \dfrac{1}{2}(\hat{v}_{1\xi} + \hat{v}_{4\xi}), \end{cases}$$

$$\begin{cases} (\hat{I}\hat{v})_{1\xi} + (\hat{I}\hat{v})_{4\xi} = \hat{v}_{1\xi} + \hat{v}_{4\xi}, \quad (\hat{I}\hat{v})_{2\xi} + (\hat{I}\hat{v})_{3\xi} = \hat{v}_{2\xi} + \hat{v}_{3\xi}, \\ (\hat{I}\hat{v})_{1\eta} + (\hat{I}\hat{v})_{2\eta} = \hat{v}_{1\eta} + \hat{v}_{2\eta}, \quad (\hat{I}\hat{v})_{3\eta} + (\hat{I}\hat{v})_{4\eta} = \hat{v}_{3\eta} + \hat{v}_{4\eta}. \end{cases}$$

But $(\hat{I}\hat{v})_{i\xi} \ne \hat{v}_{i\xi}$, $(\hat{I}\hat{v})_{i\eta} \ne \hat{v}_{i\eta}$, $1 \le i \le 4$. We have

$$(9) \qquad |(\hat{I}\hat{v})_{i\xi} - \hat{v}_{i\xi}| \le \Big\| \frac{\partial^3 \hat{v}}{\partial\xi^2\partial\eta} \Big\|_{0,\hat{K}}, \quad |(\hat{I}\hat{v})_{i\eta} - \hat{v}_{i\eta}| \le \Big\| \frac{\partial^3 \hat{v}}{\partial\xi\partial\eta^2} \Big\|_{0,\hat{K}}, 1 \le i \le 4.$$

In fact,

$$\begin{aligned} |(\hat{I}\hat{v})_{1\xi} - \hat{v}_{1\xi}| &= \Big| \frac{1}{4}(-\hat{v}_1 + \hat{v}_2 - \hat{v}_3 + \hat{v}_4 - 2\hat{v}_{1\xi} + 2\hat{v}_{4\xi}) \Big| \\ &= \Big| \frac{1}{4} \int_{-1}^{1} \int_{-1}^{\xi} \int_{-1}^{1} \frac{\partial^3 \hat{v}}{\partial\xi^2\partial\eta}(\xi,\eta) \mathrm{d}\eta \mathrm{d}\xi \mathrm{d}\xi \Big| \\ &\le \Big\| \frac{\partial^3 \hat{v}}{\partial\xi^2\partial\eta} \Big\|_{0,\hat{K}} . \end{aligned}$$

It is similar for the other cases.

Obviously, the interpolation $\hat{I}$ will preserve the quadratical polynomials, i.e.,

$$(10) \qquad \forall \hat{v} \in P_2(\hat{K}), \qquad \hat{I}\hat{v} = \hat{v}.$$

Lemma 2.1. Let $\hat{P}$ be a polynomial space on $\hat{K}$, $\hat{\Pi}$ be an interpolation operator on $\hat{P}$, α be a multi-index. Suppose $P_l(\hat{K}) \subset \hat{D}^\alpha \hat{P}$ and

$$\hat{D}^\alpha \hat{\Pi} \hat{v} = \sum_{j=1}^{r} \hat{\beta}_j(\hat{v}) \hat{q}_j,$$

where $r = \dim(\hat{D}^\alpha \hat{P})$ and $\{\hat{q}_i\}_i^r$ is a set of base functions of $\hat{D}^\alpha \hat{P}$. If $\hat{\beta}_j(v)$ can be expressed by

$$\hat{\beta}_j(v) = R_j(\hat{D}^\alpha \hat{v}), \qquad 1 \le j \le r,$$

and

$$R_j \in (H^{l+1}(\hat{K}))', \qquad 1 \le j \le r,$$

then there is a constant $C > 0$ such that

$$\|\hat{D}^\alpha(\hat{v} - \hat{\Pi}\hat{v})\|_{0,\hat{K}} \le C |\hat{D}^\alpha \hat{v}|_{l+1,\hat{K}}.$$

Proof. See Theorem 2.2 of [4] or Theorem 2.3 of [6]. □

Lemma 2.2. The interpolation operator $\hat{I}$ have the following behavior: there is a constant $C > 0$, such that $\forall\ \alpha, |\alpha| = 1, 2$,

$$(11) \qquad \|\hat{D}^\alpha(\hat{v} - \hat{I}\hat{v})\|_{0,\ \hat{K}} \le C |\hat{D}^\alpha \hat{v}|_{3-|\alpha|,\ \hat{K}}, \qquad \forall\ \hat{v} \in H^3(\hat{K}).$$

Proof. By Lemma 2.1, we only need to check that $\hat{D}^\alpha \hat{I}\hat{v}$ can be expressed as

$$\hat{D}^\alpha \hat{I}\hat{v} = \sum_{i=1}^{r} R_i(\hat{D}^\alpha \hat{v}) q_i,$$

where $\{q_i\}_{i=1}^r$ are basis of $\hat{D}^\alpha \hat{P}$ and

$$|R_i(\hat{w})| \leq \hat{C}\|\hat{w}\|_{3-|\alpha|}.$$

(I) $\alpha = (2,0)$,

$$\hat{D}^\alpha \hat{I}\hat{v} = \sum_{i=1}^{8} \beta_i \frac{\partial^2 \hat{p}_i}{\partial \xi^2} = -2\hat{\beta}_5 - 6\xi\hat{\beta}_7.$$

Define $R_1(\hat{w}) = \frac{1}{8}\left(-\int_{\hat{l_1}} \hat{w} d\hat{s} + \int_{\hat{l_3}} \hat{w} d\hat{s}\right)$, then from (7) we have

$$\hat{\beta}_5(\hat{v}) = \frac{1}{8}\left(-\int_{\hat{l}_1} \frac{\partial^2 \hat{v}}{\partial \xi^2} \mathrm{d}\hat{s} + \int_{\hat{l}_3} \frac{\partial^2 \hat{v}}{\partial \xi^2} \mathrm{d}\hat{s}\right) = R_1(\hat{D}^\alpha \hat{v}).$$

Define $R_2(\hat{w}) = \frac{1}{16}\left[\int_{-1}^{1} d\xi \int_{-1}^{\xi} (\hat{w}(t,-1) + \hat{w}(t,1)) dt - \int_{-1}^{1} d\xi \int_{\xi}^{1} (\hat{w}(t,-1) + \hat{w}(t,1)) dt\right]$, then from (7) we have

$$\begin{aligned} \hat{\beta}_7(\hat{v}) &= \frac{1}{8}\left[\int_{-1}^{1} \frac{\partial \hat{v}}{\partial \xi}(\xi,-1)\mathrm{d}\xi - \frac{1}{2}\int_{-1}^{1}\left(\frac{\partial \hat{v}}{\partial \xi}(-1,-1) + \frac{\partial \hat{v}}{\partial \xi}(1,-1)\right)\mathrm{d}\xi\right] \\ &\quad + \frac{1}{8}\left[\int_{-1}^{1} \frac{\partial \hat{v}}{\partial \xi}(\xi,1)\mathrm{d}\xi - \frac{1}{2}\int_{-1}^{1}\left(\frac{\partial \hat{v}}{\partial \xi}(1,1) + \frac{\partial \hat{v}}{\partial \xi}(-1,1)\right)\mathrm{d}\xi\right] \\ &= R_2(\hat{D}^\alpha \hat{v}). \end{aligned}$$

From Hölder's inequality and the trace theorem [8], we have

$$|R_1(\hat{w})| \leq \hat{c}|\hat{w}|_{0,\partial\hat{K}} \leq \hat{c}\|\hat{w}\|_{1,\hat{K}},$$

$$|R_2(\hat{w})| \leq \frac{1}{4}\left[\int_{-1}^{1} |\hat{w}(\xi,-1)|\mathrm{d}\xi + \int_{-1}^{1} |\hat{w}(\xi,1)|\mathrm{d}\xi\right] \leq \hat{c}|\hat{w}|_{0,\partial\hat{K}} \leq \hat{c}\|\hat{w}\|_{1,\hat{K}}.$$

(II) $\alpha = (0,2)$,

$$\hat{D}^\alpha \hat{I}\hat{v} = \sum_{i=1}^{8} \beta_i \frac{\partial^2 \hat{p}_i}{\partial \eta^2} = -2\beta_6 - 6\eta\beta_8.$$

Similarly, we have

$$\beta_6(\hat{v}) = \widetilde{R}_1\left(\frac{\partial^2 \hat{v}}{\partial \eta^2}\right), \qquad \beta_8(\hat{v}) = \widetilde{R}_2\left(\frac{\partial^2 \hat{v}}{\partial \eta^2}\right), \qquad |\widetilde{R}_i(\hat{w})| \leq \hat{c}\|\hat{w}\|_{1,\hat{K}},\ i = 1,2.$$

(III) $\alpha = (1,1)$, define $R(\hat{w}) = \frac{1}{4}\int_{\hat{K}} \hat{w} d\hat{x}$, then

$$\hat{D}^\alpha \hat{I}\hat{v} = \sum_{i=1}^{8} \beta_i \frac{\partial^2 \hat{p}_i}{\partial \xi \partial \eta} = \frac{1}{4}(\hat{v}_1 - \hat{v}_2 + \hat{v}_3 - \hat{v}_4) = \frac{1}{4}\int_{\hat{K}} \frac{\partial^2 \hat{v}}{\partial \xi \partial \eta} \mathrm{d}\xi \mathrm{d}\eta = R(\hat{D}^\alpha \hat{v}),$$

$$|R(\hat{w})| = \frac{1}{4}\left|\int_{\hat{K}} \hat{w} d\xi d\eta\right| \leq \hat{c}\|\hat{w}\|_{1,\hat{K}}.$$

(IV) $\alpha = (1,0)$, $\hat{D}^\alpha \hat{P} = \mathrm{span}\{1, \xi, \eta, \xi^2\}$,

$$\hat{D}^\alpha \hat{I}\hat{v} = \frac{1}{4}\hat{\phi}_1(\hat{v}) + \frac{\eta}{4}\hat{\phi}_2(\hat{v}) - \frac{\xi}{4}\hat{\phi}_3(\hat{v}) + \frac{(1-3\xi^2)}{8}\hat{\phi}_4(\hat{v}),$$

with

$$\begin{cases} \hat{\phi}_1(\hat{v}) = -\widehat{v}_1 + \widehat{v}_2 + \widehat{v}_3 - \widehat{v}_4, \\ \hat{\phi}_2(\hat{v}) = \widehat{v}_1 - \widehat{v}_2 + \widehat{v}_3 - \widehat{v}_4, \\ \hat{\phi}_3(\hat{v}) = \widehat{v}_{1\xi} - \widehat{v}_{2\xi} - \widehat{v}_{3\xi} + \widehat{v}_{4\xi}, \\ \hat{\phi}_4(\hat{v}) = -\widehat{v}_1 + \widehat{v}_2 + \widehat{v}_3 - \widehat{v}_4 - \widehat{v}_{1\xi} - \widehat{v}_{2\xi} - \widehat{v}_{3\xi} - \widehat{v}_{4\xi}. \end{cases}$$

By $H^2(\hat{K}) \hookrightarrow C^0(\hat{K})$, Hölder inequality and the trace theorem,

$$\begin{cases} \hat{\phi}_1(\hat{v}) = \int_{\widehat{l}_1} \hat{D}^\alpha \hat{v} \mathrm{d}\hat{s} + \int_{\widehat{l}_3} \hat{D}^\alpha \hat{v} \mathrm{d}\hat{s} \le C\|\hat{D}^\alpha \hat{v}\|_{2,\hat{K}}, \\ \hat{\phi}_2(\hat{v}) = -\int_{\widehat{l}_1} \hat{D}^\alpha \hat{v} \mathrm{d}\hat{s} + \int_{\widehat{l}_3} \hat{D}^\alpha \hat{v} \mathrm{d}\hat{s} \le C\|\hat{D}^\alpha \hat{v}\|_{2,\hat{K}}, \\ \hat{\phi}_3(\hat{v}) = \widehat{v}_{1\xi} - \widehat{v}_{2\xi} - \widehat{v}_{3\xi} + \widehat{v}_{4\xi} \le C\|\hat{D}^\alpha \hat{v}\|_{2,\hat{K}}, \\ \hat{\phi}_4(\hat{v}) = \int_{\widehat{l}_1} \hat{D}^\alpha \hat{v} \mathrm{d}\hat{s} + \int_{\widehat{l}_3} \hat{D}^\alpha \hat{v} \mathrm{d}\hat{s} - \widehat{v}_{1\xi} - \widehat{v}_{2\xi} - \widehat{v}_{3\xi} - \widehat{v}_{4\xi} \le C\|\hat{D}^\alpha \hat{v}\|_{2,\hat{K}}. \end{cases}$$

(V) $\alpha = (0, 1)$, we have the similar corresponding results with case (IV). □

Remark 2.1. (i) Following the literatures [1, 9], if the element satisfies the inequality like (11), we call the element anisotropic, because from (11) we can get the interpolation error independent of the shape regular assumption (2). (ii) The single set parameter element by (3)-(5) is not anisotropic since (11) does not hold in this case. In fact, let $\hat{v} = \xi\eta^2$, then $\frac{\partial^2 \hat{v}}{\partial \xi^2} = 0$, but $\frac{\partial^2 \hat{I}\hat{v}}{\partial \xi^2} = -3\xi$.

3. Error Estimate

For the elliptic singular perturbation problem (1), its weak formulation is as in [12]: To find $u \in V = \{v \in H^2(\Omega);\ v = \frac{\partial^2 v}{\partial n^2} = 0 \text{ on } \partial\Omega\}$, such that

$$\varepsilon^2 a(u,v) + b(u,v) = (f,v), \quad \forall v \in V, \tag{12}$$

where

$$a(u,v) = \int_\Omega D^2 u : D^2 v \mathrm{d}x\mathrm{d}y, \quad b(u,v) = \int_\Omega \nabla u \cdot \nabla v \mathrm{d}x\mathrm{d}y.$$

$\forall w \in V$, $D^2 w$ is the Hessian matrix of w, and $D^2 u : D^2 v$ is the summation of each corresponding component product. The energy norm is defined by $|||v|||_\varepsilon^2 = \varepsilon^2 a(v,v) + b(v,v) = \varepsilon^2 |v|_2^2 + |v|_1^2$.

Let a general element K be a rectangle on (x,y) plane with vertices $a_1(x_K - h_{K_1}, y_K - h_{K_2})$, $a_2(x_K + h_{K_1}, y_K - h_{K_2})$, $a_3(x_K + h_{K_1}, y_K + h_{K_2})$, $a_4(x_K - h_{K_1}, y_K + h_{K_2})$, sides $l_1 = \overline{a_1 a_2}$, $l_2 = \overline{a_2 a_3}$, $l_3 = \overline{a_3 a_4}$, $l_4 = \overline{a_4 a_1}$ and middle points of sides $a_5(x_K, y_K - h_{K_2})$, $a_6(x_K + h_{K_1}, y_K)$, $a_7(x_K, y_K + h_{K_2})$, $a_8(x_K - h_{K_1}, y_K)$, respectively. The affine mapping F_K from $\widehat{K}$ to K is :

$$\begin{cases} x = h_{K_1}\xi + x_K, \\ y = h_{K_2}\eta + y_K. \end{cases}$$

Let J be the Jacobian matrix of F_K, then

$$|J| = h_{K_1} h_{K_2}, \quad |J^{-1}| = h_{K_1}^{-1} h_{K_2}^{-1}, \quad D^\alpha u = h_K^{-\alpha} \widehat{D}^\alpha \widehat{u}, \quad \widehat{D}^\alpha \widehat{u} = h_K^\alpha D^\alpha u,$$

where $h_K^\alpha = h_{K_1}^{\alpha_1} h_{K_2}^{\alpha_2}$. Put $v = \hat{v} \circ F_K^{-1}$. Let $\mathcal{T}_h$ be a rectangular partition of Ω, $\Omega = \bigcup_{K \in \mathcal{T}_h} K$, K is the rectangular element. Let the shape function space on K be $P_K = P_2(K) \cup \{x^3, y^3\}$ and the degrees of freedom be $\{v_i, v_{ix}, v_{iy}, 1 \le i \le 4\}$, where $v_i = v(a_i)$, $v_{ix} = \frac{\partial v}{\partial x}(a_i)$, $v_{iy} = \frac{\partial v}{\partial y}(a_i), 1 \le i \le 4$. Obviously $P_K = \hat{P} \circ F_K^{-1}$.

The interpolation operator $I_K : H^3(K) \to P_K$ is defined by

$$I_K v = \hat{I}\hat{v} \circ F_K^{-1}.$$

It is easy to see that I_K is affine equivalent and

$$I_K v = \sum_{i=1}^{8} \beta_i p_i(x, y), \tag{13}$$

where $p_i = \hat{p}_i \circ F_K^{-1}$ and

$$\begin{cases} \beta_i = v_i, \quad 1 \le i \le 4, \\ \beta_5 = \dfrac{h_{K1}}{8}(v_{1x} - v_{2x} - v_{3x} + v_{4x}), \\ \beta_6 = \dfrac{h_{K2}}{8}(v_{1y} + v_{2y} - v_{3y} - v_{4y}), \\ \beta_7 = \dfrac{1}{8}(-v_1 + v_2 + v_3 - v_4 - h_{K1}v_{1x} - h_{K1}v_{2x} - h_{K1}v_{3x} - h_{K1}v_{4x}), \\ \beta_8 = \dfrac{1}{8}(-v_1 - v_2 + v_3 + v_4 - h_{K2}v_{1y} - h_{K2}v_{2y} - h_{K2}v_{3y} - h_{K2}v_{4y}). \end{cases} \tag{14}$$

It is easy to get

$$\begin{cases} (I_K v)_i = v_i, 1 \le i \le 4 \\ (I_K v)_{5y} = \dfrac{1}{2}(v_{1y} + v_{2y}), \qquad (I_K v)_{6x} = \dfrac{1}{2}(v_{2x} + v_{3x}), \\ (I_K v)_{7y} = \dfrac{1}{2}(v_{3y} + v_{4y}), \qquad (I_K v)_{8x} = \dfrac{1}{2}(v_{1x} + v_{4x}). \end{cases} \tag{15}$$

By (9) we have

$$|(I_K v)_{ix} - v_{ix}| \le \frac{1}{2}(h_{K_1} h_{K_2})^{\frac{1}{2}} || \frac{\partial^3 v}{\partial^2 x \partial y} ||_{0,K},$$

$$|(I_K v)_{iy} - v_{iy}| \le \frac{1}{2}(h_{K_1} h_{K_2})^{\frac{1}{2}} || \frac{\partial^3 v}{\partial x \partial^2 y} ||_{0,K}.$$

The corresponding finite element space V_h is defined by

$$\begin{aligned} V_h = \{ \quad & v_h : \ v_h|_K \in P_K, v_h|_K = I_K v \quad \forall K \in \mathcal{T}_h; \\ & v_h(a) = v_{h_x}(a) = v_{h_y}(a) = 0, \ \text{for all nodes } a \text{ on } \partial\Omega\}. \end{aligned} \tag{16}$$

The interpolation operator I_h on $\overline{\Omega}$ is defined as

$$I_h|_K = I_K = \hat{I} \circ F_K^{-1}, \qquad \forall \ K \in \mathcal{T}_h.$$

The discrete problem of (12) is: To find $u_h \in V_h$, such that

$$\varepsilon^2 a_h(u_h, v_h) + b_h(u_h, v_h) = (f, v_h), \qquad \forall \ v_h \in V_h, \tag{17}$$

where

$$a_h(u_h,v_h)=\sum_{K\in\mathcal{T}_h}\int_K D^2u_h:D^2v_h\mathrm{d}x\mathrm{d}y,\quad b_h(u_h,v_h)=\sum_{K\in\mathcal{T}_h}\int_K \nabla u_h\cdot\nabla v_h\mathrm{d}x\mathrm{d}y.$$

We define the discrete energy norm as

$$(18)\qquad |||v_h|||^2_{\varepsilon,h}=\varepsilon^2 a_h(v_h,v_h)+b_h(v_h,v_h)=\varepsilon^2|v_h|^2_{2,h}+|v_h|^2_{1,h},\ \forall\ v_h\in V_h,$$

where $|\cdot|^2_{i,h}=\sum\limits_K|\cdot|^2_{i,K}, i=1,2$. It is easy to see that $|||\cdot|||_{\varepsilon,h}$ is a norm on V_h for $\varepsilon\in[0,1]$, so (17) has a unique solution by Lax-Milgram Lemma [8].

Let $[v]$ be the jump of v between elements. Then $[v]=v$ on $\partial\Omega$, which means that suppose $l=K_1\cap K_2, K_1,K_2\in\mathcal{T}_h$, define $[v]|_l=(v|_{K_1}-v|_{K_2})|_l$, we have

Lemma 3.1. $\forall w_h\in V_h$, w_h is continuous at all nodes of $\mathcal{T}_h$ and is zero at the nodes on $\partial\Omega$, and

$$(19)\qquad \begin{cases}\displaystyle\int_l\left[\frac{\partial w_h}{\partial n}\right]\mathrm{d}s=0,\quad \text{for all interior sides } l \text{ of } \mathcal{T}_h,\\[2ex] \displaystyle\int_l\frac{\partial w_h}{\partial n}ds=0,\quad \text{for all sides } l \text{ on } \partial\Omega.\end{cases}$$

Proof. Suppose the degrees of freedom of w_h are $\{v_i,v_{ix},v_{iy},a_i\in\mathcal{T}_h\}$ which of course are continuous between elements and are zero on $\partial\Omega$. From (15)(now $w_h=I_hv$), $w_h(a_i)=v_i$, so w_h is continuous at all nodes of $\mathcal{T}_h$ and is zero at the nodes on $\partial\Omega$. Let $l=a_ia_j$ be a side of an element parallel to $y-$axis, a_{ij} be the middle point of l, then from (15), $\frac{\partial w_h}{\partial n}(a_{ij})=\frac{\partial w_h}{\partial x}(a_{ij})=\frac{1}{2}(v_{ix}+v_{jx})$ is continuous across l and is zero as $l\subset\partial\Omega$. The same is true for l parallel to $x-$axis. Since $\frac{\partial w_h}{\partial n}|_l\in P_1(l)$, (19) holds. □

Let w_h^I be the piecewise bilinear interpolation of w_h, $w_h^I\in C_0^0(\overline{\Omega})=\{v\in C^0(\overline{\Omega});v|_{\partial\Omega}=0\}$, and then we have

Lemma 3.2. $\forall w_h\in V_h$,

$$(20)\qquad \int_K\nabla(w_h-w_h^I)\mathrm{d}x\mathrm{d}y=0.$$

Proof.

$$\begin{aligned}\int_K\frac{\partial(w_h-w_h^I)}{\partial x}\mathrm{d}x\mathrm{d}y &= \int_{\hat K}\frac{\partial(\hat w_h-\hat w_h^I)}{\partial\xi}h_1^{-1}(h_1h_2)\mathrm{d}\xi\mathrm{d}\eta\\ &= h_2\int_{-1}^1\int_{-1}^1(-2\xi\beta_5+(1-3\xi^2)\beta_7)\mathrm{d}\xi\mathrm{d}\eta=0.\end{aligned}$$

Similarly,

$$\int_K\frac{\partial(w_h-w_h^I)}{\partial y}\mathrm{d}x\mathrm{d}y=0.$$

Then (20) holds. □

The main theorem of this paper is as follows:

Theorem 3.1. Suppose u, u_h are the solution of (12) and (17), respectively. We have

$$|||u-u_h|||_{\varepsilon,h} \leq C\Big\{ \sum_{K\in\mathcal{T}_h} [\varepsilon^2 h_K^2 |\Delta u|_{1,K}^2 + \varepsilon^2 \sum_{|\alpha|=1} h_K^{2\alpha} |D^\alpha u|_{2,K}^2 + \sum_{|\alpha|=1} h_K^{2\alpha} |D^\alpha u|_{1,K}^2 + h_K^2 \|f\|_{0,K}^2] \Big\}^{1/2},$$

where C is independent of shape regular condition (2).

Proof. From the second Strang's Lemma [3, 8], we get

$$|||u-u_h|||_{\varepsilon,h} \leq C \left(\inf_{v_h\in V_h} |||u-v_h|||_{\varepsilon,h} + \sup_{w_h\in V_h\setminus\{0\}} \frac{E_{\varepsilon,h}(u,w_h)}{|||w_h|||_{\varepsilon,h}} \right), \tag{21}$$

where

$$E_{\varepsilon,h}(u,w_h) = \varepsilon^2 a_h(u,w_h) + b_h(u,w_h) - (f,w_h). \tag{22}$$

The first term of (21) is called approximation error and the second one is called consistency error.

Firstly, we will consider the approximation error. From the above Lemma 2.2, we have

$$\begin{aligned}
&\inf_{v_h\in V_h} |||u-v_h|||_{\varepsilon,h} \leq |||u-I_h u|||_{\varepsilon,h} \\
&= \left(\sum_{K\in\mathcal{T}_h} \left(\varepsilon^2 |u-I_K u|_{2,K}^2 + |u-I_K u|_{1,K}^2 \right) \right)^{\frac{1}{2}} \\
&= \left(\sum_{K\in\mathcal{T}_h} \left(\sum_{|\alpha|=2} \varepsilon^2 \|D^\alpha(u-I_K u)\|_{0,K}^2 + \sum_{|\alpha|=1} \|D^\alpha(u-I_K u)\|_{0,K}^2 \right) \right)^{\frac{1}{2}} \\
&\leq C \left(\sum_{K\in\mathcal{T}_h} h_{K1}h_{K2} \left(\sum_{|\alpha|=2} h_K^{-2\alpha} \varepsilon^2 |\widehat{D}^\alpha \widehat{u}|_{1,\widehat{K}}^2 + \sum_{|\alpha|=1} h_K^{-2\alpha} |\widehat{D}^\alpha \widehat{u}|_{1,\widehat{K}}^2 \right) \right)^{\frac{1}{2}} \qquad (23) \\
&\leq C \left(\sum_{K\in\mathcal{T}_h} \left(\sum_{\substack{|\alpha|=2\\|\beta|=1}} h_K^{2\beta} \varepsilon^2 \|D^{\alpha+\beta} u\|_{0,K}^2 + \sum_{\substack{|\alpha|=1\\|\beta|=1}} h_K^{2\beta} \|D^{\alpha+\beta} u\|_{0,K}^2 \right) \right)^{\frac{1}{2}} \\
&\leq C \left(\sum_{K\in\mathcal{T}_h} \left(\varepsilon^2 \sum_{|\alpha|=1} h_K^{2\alpha} |D^\alpha u|_{2,K}^2 + \sum_{|\alpha|=1} h_K^{2\alpha} |D^\alpha u|_{1,K}^2 \right) \right)^{\frac{1}{2}}.
\end{aligned}$$

Then we will analyze the consistent error. Since

$$\begin{aligned}
D^2 u : D^2 w_h &= u_{xx} w_{hxx} + u_{yy} w_{hyy} + 2u_{xy} w_{hxy} \\
&= \triangle u \triangle w_h + (2u_{xy} w_{hxy} - u_{xx} w_{hyy} - u_{yy} w_{hxx}),
\end{aligned}$$

from Green's formula, we have

$$\int_K \triangle u \triangle w_h \mathrm{d}x\mathrm{d}y = \int_{\partial K} \triangle u \frac{\partial w_h}{\partial n} \mathrm{d}S - \int_K \nabla\triangle u \cdot \nabla w_h \mathrm{d}x\mathrm{d}y,$$

$$\int_K (2u_{xy} w_{hxy} - u_{xx} w_{hyy} - u_{yy} w_{hxx}) \mathrm{d}x\mathrm{d}y = \int_{\partial K} \left(\frac{\partial^2 u}{\partial n \partial s} \frac{\partial w_h}{\partial s} - \frac{\partial^2 u}{\partial s^2} \frac{\partial w_h}{\partial n} \right) \mathrm{d}S.$$

Hence,

$$
\begin{aligned}
a_h(u, w_h) &= \sum_{K\in\mathcal{T}_h} \int_K D^2u : D^2w_h \mathrm{d}x\mathrm{d}y \\
&= \sum_{K\in\mathcal{T}_h} \left\{ \int_{\partial K} \left[(\triangle u - \frac{\partial^2 u}{\partial s^2}) \frac{\partial w_h}{\partial n} + \frac{\partial^2 u}{\partial n \partial s} \frac{\partial w_h}{\partial s} \right] \mathrm{d}S \right. \\
&\qquad \left. - \int_K \nabla \triangle u \cdot \nabla w_h \mathrm{d}x\mathrm{d}y \right\}.
\end{aligned} \tag{24}
$$

Because $w_h^I \in H_0^1(\Omega)$, we have

$$
\begin{aligned}
(f, w_h) &= (f, w_h^I) + (f, w_h - w_h^I) = (\varepsilon^2 \Delta^2 u - \Delta u, w_h^I) + (f, w_h - w_h^I) \\
&= \sum_{K\in\mathcal{T}_h} \int_K (-\varepsilon^2 \nabla \Delta u \cdot \nabla w_h^I + \nabla u \cdot \nabla w_h^I) \mathrm{d}x\mathrm{d}y + (f, w_h - w_h^I).
\end{aligned} \tag{25}
$$

Substituting (24), (25) into (22), it yields

$$
\begin{aligned}
E_{\varepsilon,h}(u, w_h) &= \sum_{K\in J_h} \{ \varepsilon^2 \int_{\partial K} [(\Delta u - \frac{\partial^2 u}{\partial s^2}) \frac{\partial \omega_h}{\partial n} + \frac{\partial^2 u}{\partial s \partial n} \frac{\partial \omega_h}{\partial s}] \mathrm{d}S \\
&\qquad - \varepsilon^2 \int_K \nabla \triangle u \cdot \nabla (w_h - w_h^I) \mathrm{d}x\mathrm{d}y \\
&\qquad + \int_K \nabla u \cdot \nabla (w_h - w_h^I) \mathrm{d}x\mathrm{d}y \} - (f, w_h - w_h^I) \\
&\triangleq J_1 + J_2 + J_3 + J_4.
\end{aligned} \tag{26}
$$

Let $\Pi_i v = \frac{1}{|l_i|} \int_{l_i} v \mathrm{d}S$, $1 \le i \le 4$, because $\int_{l_i} \frac{\partial \omega_h}{\partial n}, \int_{l_i} \frac{\partial \omega_h}{\partial s}$ are continuous between the elements by Lemma 3.1, and $(\Delta u - \frac{\partial^2 u}{\partial s^2})|_{\partial\Omega} = 0, \Pi_i(\frac{\partial w_h}{\partial s}) = 0$ on $\partial\Omega$, we have

$$
\begin{aligned}
J_1 &= \sum_{K\in\mathcal{T}_h} \varepsilon^2 \sum_{i=1}^4 \int_{l_i} [(\Delta u - \frac{\partial^2 u}{\partial s^2})(\frac{\partial w_h}{\partial n} - \Pi_i(\frac{\partial w_h}{\partial n})) \\
&\qquad + \frac{\partial^2 u}{\partial s \partial n}(\frac{\partial w_h}{\partial s} - \Pi_i(\frac{\partial w_h}{\partial s}))] \mathrm{d}S.
\end{aligned} \tag{27}
$$

In order to get the error bounds independent of K, we will use a new method treating J_1, which is different from the traditional method in [10]. For the sake of simplicity, we denote

$$
I_i \triangleq \int_{l_i} [(\Delta u - \frac{\partial^2 u}{\partial s^2})(\frac{\partial w_h}{\partial n} - \Pi_i(\frac{\partial w_h}{\partial n}))] \mathrm{d}S,
$$

then

$$I_1 + I_3 = \int_{x_k - h_{K_1}}^{x_k + h_{K_1}} [(\Delta u - \frac{\partial^2 u}{\partial x^2})(\frac{\partial w_h}{\partial y} - \Pi_3(\frac{\partial w_h}{\partial y}))(x, y_K + h_{K_2}) \tag{28}$$
$$-(\Delta u - \frac{\partial^2 u}{\partial x^2})(\frac{\partial w_h}{\partial y} - \Pi_1(\frac{\partial w_h}{\partial y}))(x, y_K - h_{K_2})]\mathrm{d}x\mathrm{d}y,$$

where

$$(\frac{\partial w_h}{\partial y} - \Pi_3(\frac{\partial w_h}{\partial y}))(x, y_K + h_{K_2})$$
$$= \frac{\partial w_h}{\partial y}(x, y_K + h_{K_2}) - \frac{1}{2h_{K_1}} \int_{x_k - h_{K_1}}^{x_k + h_{K_1}} \frac{\partial w_h}{\partial y}(t, y_K + h_{K_2})\mathrm{d}t \tag{29}$$
$$= \frac{1}{2h_{K_1}} \int_{x_k - h_{K_1}}^{x_k + h_{K_1}} \int_t^x \frac{\partial^2 w_h}{\partial r \partial y}(r, y_K + h_{K_2})\mathrm{d}r\mathrm{d}t.$$

Since $\frac{\partial^2 w_h}{\partial x \partial y}$ is a constant independent of y,

$$w(x) = (\frac{\partial w_h}{\partial y} - \Pi_1(\frac{\partial w_h}{\partial y}))(x, y_K - h_{K_2})$$
$$= (\frac{\partial w_h}{\partial y} - \Pi_3(\frac{\partial w_h}{\partial y}))(x, y_K + h_{K_2}) \tag{30}$$
$$= \frac{1}{4h_{K_1}h_{K_2}} \int_{x_K - h_{K_1}}^{x_K + h_{K_1}} \int_t^x \int_{y_K - h_{K_2}}^{y_K + h_{K_2}} \frac{\partial^2 w_h}{\partial r \partial y}\mathrm{d}y\mathrm{d}r\mathrm{d}t,$$

$$|\omega(x)| \le \frac{1}{2h_{K_2}} \int_{x_k - h_{K_1}}^{x_k + h_{K_1}} \int_{y_k - h_{K_2}}^{y_k + h_{K_2}} |\frac{\partial^2 \omega_h}{\partial x \partial y}(x, y)|\mathrm{d}y\mathrm{d}x \le \sqrt{\frac{h_{K_1}}{h_{K_2}}}|\omega_h|_{2,K}. \tag{31}$$

Substituting (30) and (31) into (28), we get

$$|I_1 + I_3| = |\int_{x_k - h_{K_1}}^{x_k + h_{K_1}} w(x)[(\Delta u - \frac{\partial^2 u}{\partial x^2})(x, y_K + h_{K_2})$$
$$-(\Delta u - \tfrac{\partial^2 u}{\partial x^2})(x, y_K - h_{K_2})]\mathrm{d}x\big|$$
$$= |\int_{x_k - h_{K_1}}^{x_k + h_{K_1}} w(x)[\int_{y_k - h_{K_2}}^{y_k + h_{K_2}} \frac{\partial(\Delta u - \frac{\partial^2 u}{\partial x^2})}{\partial y}(x, y)]\mathrm{d}y\mathrm{d}x\big|$$
$$\le \sqrt{\frac{h_{K_1}}{h_{K_2}}}|w_h|_{2,K} \int_{x_k - h_{K_1}}^{x_k + h_{K_1}} \int_{y_k - h_{K_2}}^{y_k + h_{K_2}} |\frac{\partial(\Delta u - \frac{\partial^2 u}{\partial x^2})}{\partial y}|\mathrm{d}y\mathrm{d}x \tag{32}$$
$$\le \sqrt{\frac{h_{K_1}}{h_{K_2}}}|w_h|_{2,K}\sqrt{4h_{K_1}h_{K_2}}(|\Delta u|_{1,K} + |\frac{\partial u}{\partial x}|_{2,K})$$
$$= 2h_{K_1}(|\Delta u|_{1,K} + |\frac{\partial u}{\partial x}|_{2,K})|w_h|_{2,K}.$$

Similarly, we obtain

$$|I_2 + I_4| \leq 2h_{K_2}(|\Delta u|_{1,K} + \Big|\frac{\partial u}{\partial y}\Big|_{2,K})|w_h|_{2,K}. \tag{33}$$

From (32) and (33), we get

$$\begin{aligned}
&\big|\varepsilon^2 \sum_{K\in\mathcal{T}_h}\{\sum_{i=1}^{4}\int_{F_i}(\Delta u - \frac{\partial^2 u}{\partial s^2})(\frac{\partial w_h}{\partial n} - \Pi_i(\frac{\partial w_h}{\partial n}))\mathrm{d}S\}\big| = \varepsilon^2 \sum_{K\in\mathcal{T}_h}\{\sum_{i=1}^{4} I_i\}\\
&\leq C\sum_{K\in\mathcal{T}_h}\varepsilon^2(h_K|\Delta u|_{1,K} + \sum_{|\alpha|=1} h_K^{\alpha}|D^{\alpha}u|_{2,K})|\omega_h|_{2,K}\\
&\leq C(\sum_{K\in\mathcal{T}_h}\varepsilon^2(h_K^2|\Delta u|_{1,K}^2 + \sum_{|\alpha|=1} h_K^{2\alpha}|D^{\alpha}u|_{2,K}^2))^{1/2}(\sum_{K\in\mathcal{T}_h}\varepsilon^2|w_h|_{2,K}^2)^{1/2}\\
&\leq C\varepsilon(\sum_{K\in\mathcal{T}_h}(h_K^2|\Delta u|_{1,K}^2 + \sum_{|\alpha|=1} h_K^{2\alpha}|D^{\alpha}u|_{2,K}^2))^{1/2}|\|w_h\||_{\varepsilon,h}.
\end{aligned} \tag{34}$$

In the same way, we get

$$\begin{aligned}
&\big|\varepsilon^2 \sum_{K\in\mathcal{T}_h}\sum_{i=1}^{4}\int_{l_i}\frac{\partial^2 u}{\partial s\partial n}(\frac{\partial w_h}{\partial s} - \Pi_i(\frac{\partial w_h}{\partial s}))dS\big|\\
&\qquad\leq 2\varepsilon^2\sum_{K\in\mathcal{T}_h}(\sum_{|\alpha|=1} h_K^{\alpha}|D^{\alpha}u|_{2,K})|w_h|_{2,K}\\
&\qquad\leq C\varepsilon(\sum_{K\in\mathcal{T}_h}\sum_{|\alpha|=1} h_K^{2\alpha}|D^{\alpha}u|_{2,K}^2)^{1/2}|\|w_h\||_{\varepsilon,h}.
\end{aligned} \tag{35}$$

From (34) and (35), we have

$$|J_1| \leq C\varepsilon(\sum_{K\in\mathcal{T}_h}(h_K^2|\Delta u|_{1,K}^2 + \sum_{|\alpha|=1} h_K^{2\alpha}|D^{\alpha}u|_{2,K}^2))^{\frac{1}{2}}|\|w_h\||_{\varepsilon,h}. \tag{36}$$

Next, since the bilinear interpolation is anisotropic [1],

$$\begin{aligned}
&|w_h - w_h^I|_{1,K}^2\\
&= \int_K[(\frac{\partial(w_h - w_h^I)}{\partial x})^2 + (\frac{\partial(w_h - w_h^I)}{\partial y})^2]\mathrm{d}x\mathrm{d}y\\
&= (h_{K_1}h_{K_2})\int_{\widehat{K}}[((\frac{\partial(\hat{w}_h - \hat{w}_h^I)}{\partial \xi})\frac{\partial \xi}{\partial x})^2 + ((\frac{\partial(\hat{w}_h - \hat{w}_h^I)}{\partial \eta})\frac{\partial \eta}{\partial y})^2]\mathrm{d}\xi\mathrm{d}\eta\\
&= (h_{K_1}h_{K_2})(h_{K_1}^{-2}\|\frac{\partial(\hat{w}_h - \hat{w}_h^I)}{\partial \xi}\|_{0,\widehat{K}}^2 + h_{K_2}^{-2}\|\frac{\partial(\hat{w}_h - \hat{w}_h^I)}{\partial \eta}\|_{0,\widehat{K}}^2)\\
&\leq C(h_{K_1}h_{K_2})(h_{K_1}^{-2}|\frac{\partial \hat{w}_h}{\partial \xi}|_{1,\widehat{K}}^2 + h_{K_2}^{-2}|\frac{\partial \hat{w}_h}{\partial \eta}|_{1,\widehat{K}}^2)\\
&\leq \sum_{|\alpha|=1} h_K^{2\alpha}|D^{\alpha}w_h|_{1,K}^2 \leq Ch_K^2|w_h|_{2,K}^2.
\end{aligned} \tag{37}$$

So

$$
\begin{aligned}
|J_2| &= \big|\varepsilon^2 \sum_{K\in\mathcal{T}_h} \int_K \nabla\triangle u\cdot\nabla(w_h-w_h^I)\mathrm{d}x\mathrm{d}y\big| \\
&\le \varepsilon^2 \sum_{K\in\mathcal{T}_h} |\triangle u|_{1,K}|\omega_h - w_h^I|_{1,K} \\
&\le C\varepsilon^2 \sum_{K\in\mathcal{T}_h} h_K|\triangle u|_{1,K}|w_h|_{2,K} \qquad (38)\\
&\le C(\varepsilon^2 \sum_{K\in\mathcal{T}_h} h_K^2|\triangle u|_{1,K}^2)^{1/2}(\varepsilon^2 \sum_{K\in\mathcal{T}_h} |w_h|_{2,K}^2)^{1/2} \\
&\le C\varepsilon(\sum_{K\in\mathcal{T}_h} h_K^2|\triangle u|_{1,K}^2)^{1/2}|||\omega_h|||_{\varepsilon,h}.
\end{aligned}
$$

Let $\Pi_0 v = \frac{1}{|K|}\int_K v\mathrm{d}x\mathrm{d}y = \widehat{\Pi}_0\widehat{v} = \frac{1}{|\widehat{K}|}\int_{\widehat{K}} \widehat{v}\mathrm{d}\xi\mathrm{d}\eta$. By using Lemma 3.2,

$$
\begin{aligned}
\big|\int_K \nabla u\cdot\nabla(w_h-w_h^I)\mathrm{d}x\mathrm{d}y\big| = &\ \big|\int_K (\nabla u - \Pi_0\nabla u)\cdot\nabla(w_h-w_h^I)\mathrm{d}x\mathrm{d}y\big| \qquad (39)\\
&\le \|\nabla u - \Pi_0\nabla u\|_{0,K}|w_h-w_h^I|_{1,K}.
\end{aligned}
$$

Denote $\underline{v} = \nabla u$, then

$$
\begin{aligned}
\|\nabla u - \Pi_0\nabla u\|_{0,K} &= \|\underline{v} - \Pi_0\underline{v}\|_{0,K} = (h_{K_1}h_{K_2})^{1/2}\|\widehat{\underline{v}} - \widehat{\Pi}_0\widehat{\underline{v}}\|_{0,\widehat{K}} \\
&\le (h_{K_1}h_{K_2})^{1/2}|\widehat{\underline{v}}|_{1,\widehat{K}} \\
&\le C(\sum_{|\alpha|=1} h_K^{2\alpha}\|D^\alpha\underline{v}\|_{0,K}^2)^{1/2} \qquad (40)\\
&\le C(\sum_{|\alpha|=1} h_K^{2\alpha}|D^\alpha u|_{1,K}^2)^{1/2}.
\end{aligned}
$$

Substituting (37) and (40) into (39), we get

$$
\begin{aligned}
|J_3| = &\ \big|\sum_{K\in\mathcal{T}_h}\int_K \nabla u\cdot\nabla(w_h-w_h^I)\mathrm{d}x\mathrm{d}y\big| \qquad (41)\\
&\le C(\sum_{K\in\mathcal{T}_h}\sum_{|\alpha|=1} h_K^{2\alpha}|D^\alpha u|_{1,K}^2)^{1/2}|||w_h|||_{\varepsilon,h}.
\end{aligned}
$$

For the last term,

$$
\|w_h - w_h^I\|_{0,K}^2 = (h_{K_1}h_{K_2})\int_{\widehat{K}} |\widehat{w} - \widehat{w}^I|^2 d\xi d\eta \le C(h_{K_1}h_{K_2})|\widehat{w}|_{1,K}^2 \le Ch_K^2|w_h|_{1,K}^2,
$$

$$
\begin{aligned}
|J_4| &= |(f, w_h - w_h^I)| = \big|\sum_{K\in\mathcal{T}_h}\int_K f(w_h-w_h^I)\mathrm{d}x\mathrm{d}y\big| \qquad (42)\\
&\le \sum_{K\in\mathcal{T}_h}\|f\|_{0,K}\|w_h-w_h^I\|_{0,K} \le C\left(\sum_{K\in\mathcal{T}_h} h_K^2\|f\|_{0,K}^2\right)^{1/2}|||w_h|||_{\varepsilon,h}.
\end{aligned}
$$

Substituting (36), (38), (41), (42) into (26), it yields

$$
\begin{aligned}
|E_{\varepsilon,h}(u,w_h)| \quad &\le C\{\sum_{K\in\mathcal{T}_h}[\varepsilon^2 h_K^2|\Delta u|_{1,K}^2+\varepsilon^2\sum_{|\alpha|=1}h_K^{2\alpha}|D^\alpha u|_{2,K}^2 \\
&\quad +\sum_{|\alpha|=1}h_K^{2\alpha}|D^\alpha u|_{1,K}^2+h_K^2\|f\|_{0,K}^2]\}^{\frac{1}{2}}|||w_h|||_{\varepsilon,h}.
\end{aligned}
\tag{43}
$$

Substituting (43) into (21), together (23), the theorem is derived. □

To show that the error estimate is valid uniformly in the singular perturbation parameter ε, we need *a priori* estimate about the exact solution of equation (1). To the best of our knowledge, [13] and [16] both gave a precise asymptotic expansion of the exact solution u for ε in 1D, [12] and [17] gave another *a priori* estimate and both got their uniformly convergence results $O(h^{\frac{1}{2}})$ in 2D. But a better *a priori* regular estimate was presented in [11].

Lemma 3.3. For a given $f\in L^2(\Omega)$, let u be the solution of (1). If Ω is rectangle, then there exists a constant C independent of ε such that

$$
\varepsilon^2\|u\|_{3,\Omega}^2+\|u\|_{2,\Omega}^2\le C\|f\|_{0,\Omega}^2. \tag{44}
$$

Proof. Multiplying (1) by $\triangle u$ and integrating over Ω, we get

$$
\int_\Omega \varepsilon^2\Delta^2 u\Delta u \mathrm{d}x\mathrm{d}y-\int_\Omega \Delta u\Delta u \mathrm{d}x\mathrm{d}y=\int_\Omega f\Delta u \mathrm{d}x\mathrm{d}y\ .
$$

From the boundary condition of (1) and Ω is a rectangle, it is easy to see that

$$
\int_\Omega \varepsilon^2\Delta^2 u\Delta u \mathrm{d}x\mathrm{d}y=-\varepsilon^2|u|_{3,\Omega}^2\ ,
$$

$$
\int_\Omega \Delta u\Delta u \mathrm{d}x\mathrm{d}y=|u|_{2,\Omega}^2\ ,
$$

$$
\Big|\int_\Omega f\Delta u \mathrm{d}x\Big|\le\|f\|_{0,\Omega}\|\Delta u\|_{0,\Omega}\le\frac{1}{2}\|f\|_{0,\Omega}^2+\frac{1}{2}\|\Delta u\|_{0,\Omega}^2=\frac{1}{2}\|f\|_{0,\Omega}^2+\frac{1}{2}|u|_{2,\Omega}^2\ .
$$

Hence

$$
\varepsilon^2|u|_{3,\Omega}^2+|u|_{2,\Omega}^2=-\int_\Omega f\Delta u \mathrm{d}x\le\frac{1}{2}\|f\|_{0,\Omega}^2+\frac{1}{2}|u|_{2,\Omega}^2\ . \qquad \square
$$

If we denote $h=\max_{K\in\mathcal{T}_h}\{h_K\}$, by using the above global *a priori* regular estimate, together with the above Theorem 3.1, we get the following uniformly convergence results in the singular perturbation parameter ε:

Theorem 3.2. Suppose u,u_h are the solution of (12) and (17), respectively, we have the following uniformly anisotropic convergence bounds:

$$
|||u-u_h|||_{\varepsilon,h}\le Ch\|f\|_{0,\Omega}. \tag{45}
$$

Remark 3.1. The uniform error estimate (45) is half order higher than the corresponding results in [12] and [17].

Remark 3.2. All the above analyses are independent of the special meshes. However, it was pointed out by Su and Liu in [15] that the solution $u(x,\ y)$ of (1) has the boundary layers along the four boundaries of Ω and has the representation

$$
\begin{aligned}
u(x,y) \quad &= \quad G(x,y,\varepsilon)+\varepsilon^2\Big(E_1(x,y,\varepsilon)\mathrm{e}^{-x/\varepsilon}+ \\
&\quad E_2(x,y,\varepsilon)\mathrm{e}^{-(1-x)/\varepsilon}+F_1(x,y,\varepsilon)\mathrm{e}^{-y/\varepsilon}+F_2(x,y,\varepsilon)\mathrm{e}^{-(1-y)/\varepsilon}\Big),
\end{aligned}
$$

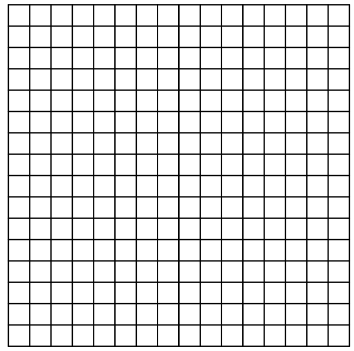

FIGURE 2. Mesh 1 (n=16).

where the functions $G,\ E_1,\ E_2,\ F_1,\ F_2$ have asymptotic power series expansions in ε and they are sufficiently differentiable for $(x,\ y) \in \Omega$. In the following numerical experiments, we will take a special mesh adapted to the boundary layers.

4. Numerical Experiments

Experiment 1. Consider problem (1) on the unit square $\Omega = [0,1]\times[0,1] \subset R^2$, where $f = \varepsilon^2\triangle^2 u - \triangle u$, and $u = (\sin(\pi x)\sin(\pi y))^2$. The domain Ω is subdivided into small rectangles by the following two different meshes:
mesh 1: each edge of Ω is divided into n equal segments. See Figure 2 (case $n = 16$).
mesh 2: each edge of Ω is divided into n segments with $n+1$ points $(1-\cos(\frac{i\pi}{n}))/2, i = 0, 1, \cdots, n$. See Figure 3 (case $n = 16$).

This example has also been studied in [12]. We compute the error in the energy norm by taking $\varepsilon = 2^{-2}, 2^{-4}, 2^{-6}$. Table 1-3 denote the convergence trend under two different meshes, respectively.

Remark 4.1. For singular perturbation problems, Shishkin meshes are widely used, see e.g. [16], which are uniformly fine on the boundary layer domains and uniformly spare on the other domains. Recently, [9] presented a new graded meshes, on which it is gradual from fine meshes to spare meshes. Here we use another kind of mesh, i.e., mesh 2, which comes from the Chebyshev polynomial [14].

Experiment 2. Consider problem (1) on the unit square $\Omega = [0,1]\times[0,1] \subset R^2$, where we take f such that $u = (1 - e^{-x(1-x)/\varepsilon})^2(1 - e^{-y(1-y)/\varepsilon})^2$ is the exact solution. The exact solution u has boundary layers, which varies significantly near the boundary of Ω. See Figure 4 (case $\varepsilon = 2^{-6}$). The unit square is subdivided by mesh 1 and mesh 2, respectively. We compute the error in the energy norm by taking $\varepsilon = 2^{-2}, 2^{-4}, 2^{-6}$. Table 4-6 are the comparisons of errors based on two different meshes.

From the numerical experiments it can be seen that these numerical results are consistent with our theoretical analysis. Firstly, this element is convergent on any narrow rectangular meshes for the fourth order elliptic singular perturbation problem. Secondly, for the solution with isotropic property (Experiment 1), the finite element solutions on the uniform mesh 1 have smaller errors than those on the anisotropic mesh 2; and for the solution with anisotropic property (Experiment 2), the anisotropic mesh (mesh 2) can help to get better solutions than uniform mesh 1. Finally, the convergent order is also consistent with the theoretical analysis.

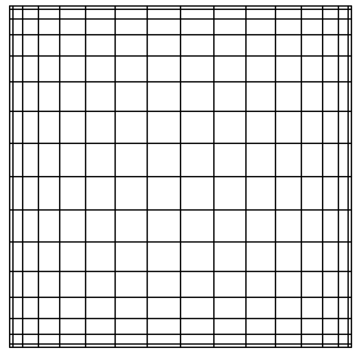

FIGURE 3. Mesh 2 (n=16).

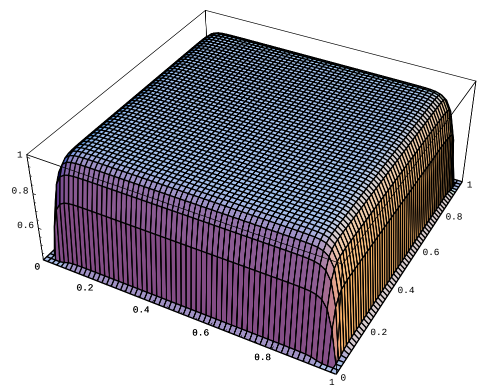

FIGURE 4. The exact solution u when $\varepsilon = 2^{-6}$ in experiment 2.

TABLE 1. Experiment 1, $|||u - u_h|||_{\varepsilon,\ h}$ under two meshes, $\varepsilon = 2^{-2}$.

mesh $\backslash$ $n \times n$	8×8	16×16	32×32	64×64	128×128
mesh 1	0.7057E0	0.3453E0	0.1716E0	0.8569E-1	0.4283E-1
mesh 2	0.9287E0	0.4420E0	0.2179E0	0.1086E0	0.5424E-1

TABLE 2. Experiment 1, $|||u - u_h|||_{\varepsilon,\ h}$ under two meshes, $\varepsilon = 2^{-4}$.

mesh $\backslash$ $n \times n$	8×8	16×16	32×32	64×64	128×128
mesh 1	0.1957E0	0.8867E-1	0.4320E-1	0.2146E-1	0.1071E-1
mesh 2	0.3158E0	0.1227E0	0.5603E-1	0.2734E-1	0.1358E-1

TABLE 3. Experiment 1, $|||u - u_h|||_{\varepsilon,\ h}$ under two meshes, $\varepsilon = 2^{-6}$.

mesh $\backslash$ $n \times n$	8×8	16×16	32×32	64×64	128×128
mesh 1	0.1166E0	0.3235E-1	0.1206E-1	0.5519E-2	0.2697E-2
mesh 2	0.3215E0	0.9480E-1	0.2665E-1	0.8836E-2	0.3670E-2

TABLE 4. Experiment 2, $|||u - u_h|||_{\varepsilon,\ h}$ under two meshes, $\varepsilon = 2^{-2}$.

mesh \ $n \times n$	8×8	16×16	32×32	64×64	128×128
mesh 1	0.1981E0	0.9141E-1	0.4429E-1	0.2195E-1	0.1095E-1
mesh 2	0.1984E0	0.8137E-1	0.3936E-1	0.1955E-1	0.9761E-2

TABLE 5. Experiment 2, $|||u - u_h|||_{\varepsilon,\ h}$ under two meshes, $\varepsilon = 2^{-4}$.

mesh \ $n \times n$	8×8	16×16	32×32	64×64	128×128
mesh 1	0.4818E0	0.2279E0	0.8333E-1	0.3009E-1	0.1272E-1
mesh 2	0.6193E0	0.1096E0	0.3120E-1	0.1368E-1	0.6671E-2

TABLE 6. Experiment 2, $|||u - u_h|||_{\varepsilon,\ h}$ under two meshes, $\varepsilon = 2^{-6}$.

mesh \ $n \times n$	8×8	16×16	32×32	64×64	128×128
mesh 1	0.1015E+1	0.7103E0	0.4397E0	0.1973E0	0.6401E-1
mesh 2	0.3153E+1	0.7385E0	0.1194E0	0.2230E-1	0.7167E-2

Acknowledgments

The authors are grateful to the referees for their valuable comments and careful reading.

References

[1] T. Apel, Anisotropic Finite Elements: Local Estimates And Applications, B.G.Teubner, Stuttgart · Leipzig, 1999.
[2] T. Apel, M. Dobrowolski, Anisotropic interpolation with applications to the finite element method, Computing, 47(1992), 227-293.
[3] S. C. Brenner, L.R.Scott, The Mathematical Theory Of Finite Element Methods, Springer-Verlag, Berlin, 1994.
[4] S. C. Chen, D. Y. Shi, Y. C. Zhao, Anisotropic interpolation and quasi-Wilson element for narrow quadrilateral meshes, IMA J. Numer. Anal., 24(2004), 77-95.
[5] S. C. Chen, Z. C. Shi, Double set parameter method of constructing a stiffness matrix, Chinese J. Numer. Math. Appl., 4(1991), 55-69.
[6] S. C. Chen, Y. C. Zhao, D. Y. Shi, Anisotropic interpolation with application to nonconforming elements, Appl. Numer. Math., 49(2004), 135-152.
[7] S. C. Chen, Y. C. Zhao, D. Y. Shi, Non C^0 nonconforming elements for elliptic fourth order singular perturbation problem, J. Comput. Math., 23(2005), 185-198.
[8] P. G. Ciarlet, The Finite Element Method For Elliptic Problems, North-Holland, Amsterdam, 1978.
[9] R. G. Durán, Error estimates for anisotropic finite elements and applications, Proceedings of the International Congress of Mathematicians, 2006.
[10] P. Lascaux, P. Lesaint, Some noncomforming finite element for the plate bending problem, RAIRO Anal. Numer., R-1(1975), 9-53.
[11] G. Q. Liu, Y. C. Su, A uniformly convergent difference scheme for the singular perturbation problem of a higher order elliptic differential equation, Appl. Math. Mech., 17(1996), 413-421.
[12] T. K. Nilssen, X. C. Tai, R. Winther, A robust nonconforming H^2-element, Math. Comput., 70(2000), 489-505.
[13] R. E. O'Malley, Introdution to singular perturbations, Academic, New York, 1974.
[14] A. Quarteroni, A. Valli, Numerical Approximation Of Partial Differential Equations, Springer-Verlag, Berlin, 1994.
[15] Y. C. Su, G. Q. Liu, Asymptotic solution of singular perturbation problems for the fourth-order elliptic differential equations, Appl. Math. Mech., 11(1990), 597-6109. (In Chinese)

[16] G. F. Sun, M. Stynes, Finite-element methods for singular perturbed high-order elliptic two-point boundary value problems. I: reaction-diffusion-type problems, IMA J. Numer. Anal., 15(1995), 117-139.

[17] M. Wang, J. C. Xu, Y. C. Hu, Modified Morley element method for a fourth order elliptic singular perturbation problem, J. Comput. Math., 24(2006), 113-120.

[18] S. P. Mao, S. C. Chen and H. X. Sun, A quadrilateral, anisotropic, superconvergent, nonconforming double set parameter element, Appl. Numer. Math., 56(2006), 937-961.

[19] S. C. Chen, L. Yin, S. P. Mao. An anisotropic superconvergent nonconforming plate finite element. J. Comput. Appl. Math., 220(2008), 96-110.

Department of Mathematics, Zhengzhou University, Zhengzhou, 450052, China
E-mail: `shchchen@zzu.edu.cn`

College of Mathematics and Information Sciences, Henan University, Kaifeng, 475000, China
E-mail: `liumingfang666@sina.com`

Department of Mathematics, Hong Kong Baptist University, Kowloon Tong, Hong Kong
E-mail: `zqiao@math.hkbu.edu.hk`