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ANALYSIS OF AN INTERACTION PROBLEM BETWEEN AN ELECTROMAGNETIC FIELD AND AN ELASTIC BODY

ANTONIO BERNARDO, ANTONIO MÁRQUEZ, AND SALIM MEDDAHI

Abstract. This paper deals with an interaction problem between a solid and an electromagnetic field in the frequency domain. More precisely, we aim to compute both the magnetic component of the scattered wave and the elastic vibrations that take place in the solid elastic body. To this end, we solve a transmission problem holding between the bounded domain $\Omega_s \subset \mathbb{R}^3$ representing the obstacle and a sufficiently large annular region surrounding it. We point out here that (following Voigt's model, cf. [12]) we only allow the electromagnetic field to interact with the elastic body through the boundary of Ω_s. We apply the abstract framework developed in the work [3] by A. Buffa to prove that our coupled variational formulation is well posed. We define the corresponding discrete scheme by using the edge element in the electromagnetic domain and standard Lagrange finite elements in the solid domain. Then we show that the resulting Galerkin scheme is uniquely solvable, convergent and we derive optimal error estimates. Finally, we illustrate our analysis with some results from computational experiments.

Key Words. edge finite elements, Maxwell equations and elastodynamics equations.

1. Introduction

In this paper we develop a finite element method for a time-harmonic problem that models the interaction between an elastic body and an electromagnetic field. We consider a solid occupying a bounded region $\Omega_s \subset \mathbb{R}^3$ and assume that it is subject to a given incident electromagnetic wave. Actually, we suppose here that the electromagnetic field occupies an annular region Ω_m whose exterior boundary Γ is located far from the obstacle (the solid body) and impose on this artificial closed surface a boundary condition compatible with the behavior of the scattered field at infinity. Moreover, we assume that the penetration of the electromagnetic field inside the body is not large enough to consider. The interaction between the electromagnetic field and the elastic body is only governed by the equilibrium of tangential forces on the interface $\Sigma := \partial\Omega_s$. This model problem is a simplification of the one presented by Cakoni and Hsiao in [7]. To the best of our knowledge, the numerical study of this interaction problem has not been treated in the literature. Our aim is to provide a finite element Galerkin scheme that permits one to compute both the scattered electromagnetic wave and the elastic vibrations of the solid.

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Once the variational formulation is derived, it becomes clear that the term coupling the elastodynamics equation in Ω_s and Maxwell equations in Ω_m is a compact perturbation (see (29) below). This means that we are almost left with an separate study of both equations in each domain. The primal formulation of the elasticity problem in Ω_s is standard. The operator arising from the corresponding primal formulation of the elastodynamics equation fails to be elliptic due to the "wrong" sign of the zero order term. Nevertheless, the compactness of the embedding $H^1(\Omega_s) \hookrightarrow L^2(\Omega_s)$ allows one to use successfully a Fredholm alternative to analyze its solvability.

The Maxwell problem is more intricate since it does not fit in any classical theory for proving well-posedness. Actually, since the canonical embedding $\mathbf{H}(\mathbf{curl}\,, \Omega_m) \hookrightarrow [L^2(\Omega_m)]^3$ is not compact, it is not possible to employ a Fredholm alternative, at least for the original form of the resulting variational formulation. The difficulty is then related to the noncoerciveness of the sesquilinear form arising in the study of Maxwell equations (written here in terms of the magnetic field). A Helmholtz-type decomposition of the magnetic field is usually proposed in order to reveal hidden compactness properties that permits to deal with the study of this problem through a classical analysis, see [3, 13] and the references cited therein. Actually, Buffa [3] succeeded in setting up this technique in a general abstract framework. We follow here this technique, our analysis is based on a suitable decomposition of $\mathbf{H}_\Gamma(\mathbf{curl}\,, \Omega_m)$ (see (18) below) that renders possible the application of a Fredholm alternative to the whole coupled problem. The corresponding discrete scheme is defined with the first order Nédélec element (also known as the edge element) in the electromagnetic domain and traditional first order Lagrange finite elements in the solid. The stability and convergence of this Galerkin method also relies on a stable decomposition of the finite element space used to approximate the magnetic field.

The remainder of the paper is organized as follows. In the next section we recall some essential tools related with tangential trace operators in the space $\mathbf{H}(\mathbf{curl}\,, \Omega)$. In sections 3 and 4 we give a brief description of the model problem and derive its coupled variational formulation. In section 5, we use a Fredholm alternative to show that, under some regularity conditions on the coefficients, the problem is well-posed. The corresponding Galerkin scheme is analyzed in section 6. Finally, in section 7 we provide results from numerical experiments that confirm our theoretical assertions.

We end this section with some notations to be used below. Since in the sequel we deal with complex valued functions, we let $\mathbb{C}$ be the set of complex numbers, use the symbol $\imath$ for $\sqrt{-1}$, and denote by $\overline{z}$ and $|z|$ the conjugate and modulus, respectively, of each $z \in \mathbb{C}$. In addition, given any Hilbert space U, $[U]^3$ denotes the space of vectors of order 3 with entries in U. Given $\boldsymbol{\sigma} := (\sigma_{ij})$, $\boldsymbol{\tau} := (\tau_{ij}) \in \mathbb{C}^{3\times 3}$, we define as usual the transpose tensor $\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})$, the trace $\operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{3} \tau_{ii}$ and the tensor product $\boldsymbol{\sigma} : \boldsymbol{\tau} := \sum_{i,j=1}^{3} \sigma_{ij}\, \tau_{ij}$. Finally, in what follows we utilize the standard terminology for Sobolev spaces and norms, employ $\mathbf{0}$ to denote a generic null vector, and use C , with or without subscripts, to denote generic constants independent of the discretization parameters, which may take different values at different places.

2. Preliminaries

We denote by $\Omega \subset \mathbb{R}^3$ a generic bounded polyhedral domain and let $\boldsymbol{n}$ be the outward normal vector on its boundary Σ. We recall that

$$\mathbf{H}(\mathbf{curl}\,, \Omega) := \left\{ \boldsymbol{w} \in [L^2(\Omega)]^3; \quad \mathbf{curl}\, \boldsymbol{w} \in [L^2(\Omega)]^3 \right\}$$

endowed with the norm $\|\boldsymbol{w}\|^2_{\mathbf{H}(\mathbf{curl}\,,\Omega)} := \|\boldsymbol{w}\|^2_{[L^2(\Omega)]^3} + \|\mathbf{curl}\,\boldsymbol{w}\|^2_{[L^2(\Omega)]^3}$ is a Hilbert space and that $[\mathcal{C}^\infty(\overline{\Omega})]^3$ (the space of indefinitely differentiable vector field functions in $\overline{\Omega}$) is dense in $\mathbf{H}(\mathbf{curl}\,,\Omega)$.

We give here a brief summary of some fundamental tools related with tangential trace operators in $\mathbf{H}(\mathbf{curl}\,,\Omega)$, see [4]. To this end, we begin by introducing the space

$$\mathbf{L}^2_{\mathbf{t}}(\Sigma) = \left\{\boldsymbol{\mu} \in [L^2(\Sigma)]^3; \quad \boldsymbol{\mu}\cdot\boldsymbol{n} = 0 \quad \text{on } \Sigma\right\}$$

and the tangential trace mapping

$$\begin{array}{rcl} \boldsymbol{\gamma}_{\mathbf{t}} : \ [\mathcal{C}^\infty(\overline{\Omega})]^3 & \to & \mathbf{L}^2_{\mathbf{t}}(\Sigma) \\ \boldsymbol{v} & \mapsto & \boldsymbol{\gamma}_{\mathbf{t}}\boldsymbol{v} := \boldsymbol{v}|_\Sigma \times \boldsymbol{n} \end{array}$$

together with the tangential projection operator

$$\begin{array}{rcl} \boldsymbol{\pi}_{\mathbf{t}} : \ [\mathcal{C}^\infty(\overline{\Omega})]^3 & \to & \mathbf{L}^2_{\mathbf{t}}(\Sigma) \\ \boldsymbol{v} & \mapsto & \boldsymbol{\pi}_{\mathbf{t}}\boldsymbol{v} := \boldsymbol{n} \times (\boldsymbol{v}|_\Sigma \times \boldsymbol{n}). \end{array}$$

Let us now consider

$$\mathbf{H}^{1/2}_{\perp}(\Sigma) := \boldsymbol{\gamma}_{\mathbf{t}}([H^1(\Omega)]^3) \quad \text{and} \quad \mathbf{H}^{1/2}_{\parallel}(\Sigma) := \boldsymbol{\pi}_{\mathbf{t}}([H^1(\Omega)]^3).$$

These two spaces are endowed with the natural Hilbert space structure that makes $\boldsymbol{\gamma}_{\mathbf{t}} : \ [H^1(\Omega)]^3 \to \mathbf{H}^{1/2}_{\perp}(\Sigma)$ and $\boldsymbol{\pi}_{\mathbf{t}} : \ [H^1(\Omega)]^3 \to \mathbf{H}^{1/2}_{\parallel}(\Sigma)$ bounded and surjective. We refer to [4] for an explicit definition of these spaces in the case of Lipschitz boundaries with piecewise smooth components.

We introduce the dual $\mathbf{H}^{-1/2}_{\perp}(\Sigma)$ of $\mathbf{H}^{1/2}_{\perp}(\Sigma)$ and the dual $\mathbf{H}^{-1/2}_{\parallel}(\Sigma)$ of $\mathbf{H}^{1/2}_{\parallel}(\Sigma)$ with respect to the pivot space $\mathbf{L}^2_{\mathbf{t}}(\Sigma)$. In the following, we will also write $\boldsymbol{\gamma}_{\mathbf{t}}\boldsymbol{\varphi}$ (or $\boldsymbol{\pi}_{\mathbf{t}}\boldsymbol{\varphi}$) for $\boldsymbol{\varphi} \in [H^{1/2}(\Sigma)]^3$, this should be understood as $\boldsymbol{\gamma}_{\mathbf{t}} \circ \gamma^{-1}\boldsymbol{\varphi}$ (or $\boldsymbol{\pi}_{\mathbf{t}} \circ \gamma^{-1}\boldsymbol{\varphi}$) where $\gamma^{-1} : [H^{1/2}(\Sigma)]^3 \to [H^1(\Omega)]^3$ is any bounded right-inverse of the usual trace operator γ.

It is easy to deduce from the Green formula

$$\int_\Omega \boldsymbol{u}\cdot\mathbf{curl}\,\boldsymbol{v} - \mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{v} = \int_\Sigma \boldsymbol{\gamma}_{\mathbf{t}}\boldsymbol{u}\cdot\boldsymbol{\pi}_{\mathbf{t}}\boldsymbol{v} \quad \forall \boldsymbol{u},\boldsymbol{v} \in [\mathcal{C}^\infty(\overline{\Omega})]^3 \tag{1}$$

that $\boldsymbol{\gamma}_{\mathbf{t}}$ and $\boldsymbol{\pi}_{\mathbf{t}}$ can be extended to define bounded tangential mappings from $\mathbf{H}(\mathbf{curl},\Omega)$ onto $\mathbf{H}^{-1/2}_{\parallel}(\Sigma)$ and from $\mathbf{H}(\mathbf{curl},\Omega)$ onto $\mathbf{H}^{-1/2}_{\perp}(\Sigma)$, respectively. A more precise result is given by the following theorem, see [5]. (We refer to [4, 5] for the definition of the differential operators div_Σ and curl_Σ on piecewise smooth Lipschitz boundaries.)

Theorem 2.1. *Let*

$$\mathbf{H}^{-1/2}(\mathrm{div}_\Sigma,\Sigma) := \left\{\boldsymbol{\mu} \in \mathbf{H}^{-1/2}_{\parallel}(\Sigma); \quad \mathrm{div}_\Sigma\boldsymbol{\mu} \in H^{-1/2}(\Sigma)\right\}$$

and

$$\mathbf{H}^{-1/2}(\mathrm{curl}_\Sigma,\Sigma) := \left\{\boldsymbol{\mu} \in \mathbf{H}^{-1/2}_{\perp}(\Sigma); \quad \mathrm{curl}_\Sigma\boldsymbol{\mu} \in H^{-1/2}(\Sigma)\right\}.$$

Then

$$\boldsymbol{\gamma}_{\mathbf{t}} : \ \mathbf{H}(\mathbf{curl},\Omega) \to \mathbf{H}^{-1/2}(\mathrm{div}_\Sigma,\Sigma) \quad \textit{and} \quad \boldsymbol{\pi}_{\mathbf{t}} : \ \mathbf{H}(\mathbf{curl},\Omega) \to \mathbf{H}^{-1/2}(\mathrm{curl}_\Sigma,\Sigma)$$

are bounded, surjective and possess a continuous right inverse. Moreover, the $\mathbf{L}^2_{\mathbf{t}}(\Sigma)$*-inner product can be extended to define a duality product* $\langle\,\cdot,\cdot\,\rangle_{\mathbf{t},\Sigma}$ *between the spaces* $\mathbf{H}^{-1/2}(\mathrm{div}_\Sigma,\Sigma)$ *and* $\mathbf{H}^{-1/2}(\mathrm{curl}_\Sigma,\Sigma)$.

As a consequence of this theorem, Green's formula (1) can be extended to functions $\boldsymbol{u}$, $\boldsymbol{v}$ in $\mathbf{H}(\mathbf{curl}, \Omega)$ if the boundary integral of the right hand side is interpreted as $\langle\, \boldsymbol{\gamma}_{\mathbf{t}} \boldsymbol{u}, \boldsymbol{\pi}_{\mathbf{t}} \boldsymbol{v}\, \rangle_{\mathbf{t},\Sigma}$.

Finally, we recall that the closure $\mathbf{H}_0(\mathbf{curl}\,, \Omega)$ of $[\mathcal{C}_0^\infty(\Omega)]^3$ (the space of indefinitely differentiable functions with compact support in Ω) in $\mathbf{H}(\mathbf{curl}\,, \Omega)$ may also be characterised as the kernel of $\boldsymbol{\gamma}_{\mathbf{t}}$, i.e.,

$$\mathbf{H}_0(\mathbf{curl}\,, \Omega) := \{\boldsymbol{w} \in \mathbf{H}_0(\mathbf{curl}\,, \Omega); \quad \boldsymbol{\gamma}_{\mathbf{t}} \boldsymbol{w} = 0 \quad \text{on } \Sigma\}\,.$$

3. The model problem

We consider a bounded, connected and simply connected polyhedra $\Omega_s \subset \mathbb{R}^3$ representing an homogeneous elastic body immersed in an electromagnetic medium filling the whole space. We assume that the boundary $\Sigma := \partial\Omega_s$ is connected and that the system consisting in the electromagnetic field and the elastic body only interacts through this interface Σ.

Let ϵ, μ, and σ be respectively, the electric permittivity, the magnetic permeability and the conductivity of the medium. These coefficients are piecewise regular real valued scalar functions satisfying in $\mathbb{R}^3 \setminus \Omega_s$,

$$\mu_0 \le \mu(\boldsymbol{x}) \le \bar{\mu}, \qquad \epsilon_0 \le \epsilon(\boldsymbol{x}) \le \bar{\epsilon} \quad \text{and} \quad 0 \le \sigma(\boldsymbol{x}) \le \bar{\sigma}\,,$$

where the constants ϵ_0 and μ_0 denote respectively, the electric permittivity and magnetic permeability of free space. Moreover, we assume that we have vacuum conditions sufficiently far from the obstacle, i.e., there exists $R > 0$ such that

$$\mu(\boldsymbol{x}) = \mu_0, \qquad \epsilon(\boldsymbol{x}) = \epsilon_0 \quad \text{and} \quad \sigma(\boldsymbol{x}) = 0\,,$$

for all $|\boldsymbol{x}| \ge R$.

The incident electromagnetic field $(\boldsymbol{E}^i, \boldsymbol{H}^i)$ is supposed to exhibit a time-harmonic behavior with frequency ω. Hence, the scattered electric and magnetic fields have also a time harmonic behavior with frequency ω. Namely,

$$\boldsymbol{E}(\boldsymbol{x}, t) = \mathrm{Re}[exp(-\imath\omega t)\epsilon_0^{-1/2}\boldsymbol{e}(\boldsymbol{x})]$$
$$\boldsymbol{H}(\boldsymbol{x}, t) = \mathrm{Re}[exp(-\imath\omega t)\mu_0^{-1/2}\boldsymbol{h}(\boldsymbol{x})]$$

and the complex amplitudes $\boldsymbol{e}$ and $\boldsymbol{h}$ satisfy

$$\begin{aligned} \mathbf{curl}\ \boldsymbol{e} - \imath k\, b\, \boldsymbol{h} &= \mathbf{0} \qquad \text{in} \quad \mathbb{R}^3\backslash\Omega_s\,, \\ \mathbf{curl}\ \boldsymbol{h} + \imath k\, a\, \boldsymbol{e} &= \mathbf{0} \qquad \text{in} \quad \mathbb{R}^3\backslash\Omega_s\,, \end{aligned} \tag{2}$$

where $k := \omega\sqrt{\epsilon_0\mu_0}$ is the wave number, $a(\boldsymbol{x}) := \dfrac{\epsilon(\boldsymbol{x})}{\epsilon_0} + \imath\dfrac{\sigma(\boldsymbol{x})}{\epsilon_0\omega}$ and $b(\boldsymbol{x}) := \dfrac{\mu(\boldsymbol{x})}{\mu_0}$ $\forall\, \boldsymbol{x} \in \mathbb{R}^3$.

The solid is supposed to be isotropic and linearly elastic with mass density ρ_s and Lamé constants μ^* and λ, which means, in particular, that the corresponding constitutive equation is given by

$$\boldsymbol{\sigma} = \mathcal{C}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) \qquad \text{in} \quad \Omega_s\,. \tag{3}$$

Here, $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}\,(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathtt{t}})$ is the infinitesimal strain tensor, ∇ is the gradient tensor, and $\mathcal{C}$ is the elasticity operator given by Hooke's law,

$$\mathcal{C}\,\boldsymbol{\tau} := \lambda\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbf{I} + 2\,\mu^*\,\boldsymbol{\tau} \qquad \forall\,\boldsymbol{\tau} \in [L^2(\Omega_s)]^{3\times 3}\,, \tag{4}$$

where $\mathbf{I}$ stands for the identity matrix in $\mathbb{C}^{3\times 3}$. As the elastic displacement is also a time-harmonic field with the same frequency ω the unknown $\boldsymbol{u}$ satisfies the following equilibrium equation:

$$\mathbf{div}\,(\mathcal{C}\,\boldsymbol{\varepsilon}(\boldsymbol{u})) + \kappa_s^2\,\boldsymbol{u} \quad = \quad \mathbf{0} \qquad \text{in} \quad \Omega_s\,, \tag{5}$$

where $\kappa_s := \sqrt{\rho_s}\,\omega$ is the wave number in the obstacle and $\mathbf{div}$ stands for the usual divergence operator acting on each row of the tensor $\boldsymbol{\sigma}$.

Let us denote by $\boldsymbol{n}$ the unit normal on Σ oriented towards the exterior of Ω_s. According to Voigt's model (see [7, 12]), the transmission conditions coupling (2) and (5) on Σ are given by

$$\begin{aligned} (\boldsymbol{e}+\boldsymbol{e}^i)\times\boldsymbol{n} \quad &= \quad \boldsymbol{u}\times\boldsymbol{n} \qquad \text{on} \quad \Sigma\,,\\ \tfrac{\imath}{k}\,(\boldsymbol{h}+\boldsymbol{h}^i)\times\boldsymbol{n} \quad &= \quad -\,\boldsymbol{\sigma}\boldsymbol{n} \qquad \text{on} \quad \Sigma\,. \end{aligned} \tag{6}$$

Finally, the electromagnetic field exhibits the Silver-Muller asymptotic behaviour

$$\boldsymbol{e}\times\frac{\boldsymbol{x}}{|\boldsymbol{x}|} + \boldsymbol{h} = o(\frac{1}{|\boldsymbol{x}|})\,, \tag{7}$$

as $|\boldsymbol{x}| \to +\infty$, uniformly for all directions $\dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}$.

We notice that, the asymptotic behaviour (7) implies that the outgoing waves are absorbed by the far field. Motivated by this fact, and aiming to obtain a suitable simplification of our model problem, we now introduce a sufficiently large sphere Γ centered at the origin, define Ω_m as the domain bounded by Σ and Γ, and consider the boundary condition:

$$\boldsymbol{h}\times\boldsymbol{n} = \mathbf{0} \quad \text{on} \quad \Gamma\,. \tag{8}$$

Actually, in order to avoid introducing later a nonconforming Galerkin scheme, we may simply think of Γ as the polyhedral surface resulting from a sufficiently accurate approximation of the given sphere. We will also use $\boldsymbol{n}$ to denote the unit outward normal on Γ.

Equations (2), (5), (6), (8) and the expression $\boldsymbol{e} = -(\imath k a)^{-1}\mathbf{curl}\,\boldsymbol{h}$, of the electric field in terms of $\boldsymbol{h}$ lead us to the following formulation of the problem: find $\boldsymbol{e} : \Omega_m \to \mathbb{C}^3$ and $\boldsymbol{u} : \Omega_s \to \mathbb{C}^3$ such that

$$\begin{aligned} \mathbf{curl}\,(a^{-1}\mathbf{curl}\,\boldsymbol{h}) - k^2\,b\,\boldsymbol{h} \quad &= \quad \mathbf{0} && \text{in} \quad \Omega_m\,,\\ \boldsymbol{\sigma} \quad &= \quad \mathcal{C}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) && \text{in} \quad \Omega_s\,,\\ \mathbf{div}\,\boldsymbol{\sigma} + \kappa_s^2\,\boldsymbol{u} \quad &= \quad \mathbf{0} && \text{in} \quad \Omega_s\,,\\ a^{-1}\mathbf{curl}\,\boldsymbol{h}\times\boldsymbol{n} + \imath k\boldsymbol{u}\times\boldsymbol{n} \quad &= \quad \imath k\boldsymbol{e}^i\times\boldsymbol{n} && \text{on} \quad \Sigma\,,\\ k^2\boldsymbol{\sigma}\boldsymbol{n} + \imath k\boldsymbol{h}\times\boldsymbol{n} \quad &= \quad -\imath k\boldsymbol{h}^i\times\boldsymbol{n} && \text{on} \quad \Sigma\,,\\ \boldsymbol{h}\times\boldsymbol{n} \quad &= \quad \mathbf{0} && \text{on} \quad \Gamma \end{aligned} \tag{9}$$

the datum $(\boldsymbol{e}^i,\boldsymbol{h}^i)$ is the complex amplitudes of an incident electromagnetic wave satisfying (2) with $a \equiv b \equiv 1$.

4. The variational formulation

Taking into account that the natural space for the magnetic field is the closed subspace

$$\mathbf{H}_\Gamma(\mathbf{curl}\,,\Omega_m) := \{\boldsymbol{w} \in \mathbf{H}(\mathbf{curl}\,,\Omega_m); \quad \boldsymbol{\gamma}_{\mathbf{t}}\boldsymbol{w} = 0 \text{ on } \Gamma\}$$

of $\mathbf{H}(\mathbf{curl}\,,\,\Omega_m)$, we test the first equation of (9) with $\boldsymbol{w} \in \mathbf{H}_\Gamma(\mathbf{curl}\,,\,\Omega_m)$ and use (1) to obtain

$$\int_{\Omega_m} (a^{-1}\mathbf{curl}\,\boldsymbol{h}\cdot\mathbf{curl}\,\boldsymbol{w} - k^2 b\,\boldsymbol{h}\cdot\boldsymbol{w}) + \langle\, \boldsymbol{\gamma}_{\mathbf{t}}(a^{-1}\mathbf{curl}\,\boldsymbol{h}), \boldsymbol{\pi}_{\mathbf{t}}\boldsymbol{w}\,\rangle_{\mathbf{t},\Sigma} = 0. \tag{10}$$

Therefore, using the first transmission condition on Σ yields

$$E_m(\boldsymbol{h},\boldsymbol{w}) + \imath k\,\langle\, \boldsymbol{\gamma}_{\mathbf{t}}\boldsymbol{w}, \boldsymbol{\pi}_{\mathbf{t}}\boldsymbol{u}\,\rangle_{\mathbf{t},\Sigma} = L_m(\boldsymbol{w}) \qquad \forall \boldsymbol{w} \in \mathbf{H}_\Gamma(\mathbf{curl}\,,\Omega_m) \tag{11}$$

where

$$E_m(\boldsymbol{h},\boldsymbol{w}) := \int_{\Omega_m} (a^{-1}\mathbf{curl}\,\boldsymbol{h}\cdot\mathbf{curl}\,\boldsymbol{w} - k^2 b\,\boldsymbol{h}\cdot\boldsymbol{w})$$

and

$$L_m(\boldsymbol{w}) := \imath k\,\langle\, \boldsymbol{\gamma}_{\mathbf{t}}\boldsymbol{e}^i, \boldsymbol{\pi}_{\mathbf{t}}\boldsymbol{w}\,\rangle_{\mathbf{t},\Sigma}.$$

In the obstacle Ω_s, we test (5) with $\boldsymbol{v} \in [H^1(\Omega_s)]^3$ and apply a Green's formula to obtain

$$\int_{\Omega_s} -\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) + \kappa_s^2 \int_{\Omega_s} \boldsymbol{u}\cdot\boldsymbol{v} + \int_\Sigma \boldsymbol{\sigma}\boldsymbol{n}\cdot\boldsymbol{v} = 0 \quad \forall \boldsymbol{v} \in [H^1(\Omega_s)]^3.$$

Using the remaining tramsmission formula on Σ leads to the following variational formulation in Ω_s

$$E_s(\boldsymbol{u},\boldsymbol{v}) + \imath k\langle\, \boldsymbol{\gamma}_{\mathbf{t}}\boldsymbol{h}, \boldsymbol{\pi}_{\mathbf{t}}\boldsymbol{v}\,\rangle_{\mathbf{t},\Sigma} = L_s(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in [H^1(\Omega_s)]^3, \tag{12}$$

where

$$E_s(\boldsymbol{u},\boldsymbol{v}) := k^2\left(\int_{\Omega_s} \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) - \kappa_s^2 \int_{\Omega_s} \boldsymbol{u}\cdot\boldsymbol{v}\right)$$

and

$$L_s(\boldsymbol{v}) := -\imath k\langle\, \boldsymbol{\gamma}_{\mathbf{t}}\boldsymbol{h}^i, \boldsymbol{\pi}_{\mathbf{t}}\boldsymbol{v}\,\rangle_{\mathbf{t},\Sigma}.$$

We deduce from (11) and (12) that the global variational formulation of problem (9) reads as follows:

$$\begin{aligned} &\text{find } \boldsymbol{h} \in \mathbf{H}_\Gamma(\mathbf{curl}\,,\,\Omega_m) \text{ and } \boldsymbol{u} \in [H^1(\Omega_s)]^3 \text{ such that} \\ &E_m(\boldsymbol{h},\boldsymbol{w}) + \imath k\,\langle\, \boldsymbol{\gamma}_{\mathbf{t}}\boldsymbol{w},\ \boldsymbol{\pi}_{\mathbf{t}}\boldsymbol{u}\,\rangle_{\mathbf{t},\Sigma} = L_m(\boldsymbol{w}) \quad \forall \boldsymbol{w} \in \mathbf{H}_\Gamma(\mathbf{curl}\,,\,\Omega_m) \\ &E_s(\boldsymbol{u},\boldsymbol{v}) + \imath k\langle\, \boldsymbol{\gamma}_{\mathbf{t}}\boldsymbol{h}, \boldsymbol{\pi}_{\mathbf{t}}\boldsymbol{v}\,\rangle_{\mathbf{t},\Sigma} = L_s(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in [H^1(\Omega_s)]^3 \end{aligned} \tag{13}$$

Let us now introduce the space

$$\mathbb{X} := \mathbf{H}_\Gamma(\mathbf{curl}\,,\,\Omega_m) \times [H^1(\Omega_s)]^3$$

endowed with the Hilbertian norm $\|(\boldsymbol{w},\boldsymbol{v})\|_{\mathbb{X}}^2 := \|\boldsymbol{w}\|^2_{\mathbf{H}(\mathbf{curl}\,,\Omega_m)} + \|\boldsymbol{v}\|^2_{[H^1(\Omega_s)]^3}$. We also consider the sesquilinear form

$$\boldsymbol{A}((\boldsymbol{h},\boldsymbol{u}),(\boldsymbol{w},\boldsymbol{v})) := E_m(\boldsymbol{h},\boldsymbol{w}) + E_s(\boldsymbol{u},\boldsymbol{v}) + \imath k\langle\, \boldsymbol{\gamma}_{\mathbf{t}}\boldsymbol{w},\ \boldsymbol{\pi}_{\mathbf{t}}\boldsymbol{u}\,\rangle_{\mathbf{t},\Sigma} + \imath k\langle\, \boldsymbol{\gamma}_{\mathbf{t}}\boldsymbol{h}, \boldsymbol{\pi}_{\mathbf{t}}\boldsymbol{v}\,\rangle_{\mathbf{t},\Sigma}.$$

With this notation, problem (13) may be equivalently written:

$$\begin{aligned} &\text{find } (\boldsymbol{h},\boldsymbol{u}) \in \mathbb{X} \text{ such that} \\ &\boldsymbol{A}((\boldsymbol{h},\boldsymbol{u}),(\boldsymbol{w},\boldsymbol{v})) = \mathcal{F}((\boldsymbol{w},\boldsymbol{v})) \quad \forall (\boldsymbol{w},\boldsymbol{v}) \in \mathbb{X}, \end{aligned} \tag{14}$$

where $\mathcal{F} :\ \mathbb{X} \to \mathbb{C}$ is the linear form given by $\mathcal{F}((\boldsymbol{w},\boldsymbol{v})) := L_m(\boldsymbol{w}) + L_s(\boldsymbol{v})$.

5. Analysis of the continuous problem

5.1. A splitting of $\mathbf{H}_\Gamma(\mathbf{curl}\,,\,\Omega_m)$. Let us consider the open and simply connected set Ω given by the interior of $\overline{\Omega}_s \cup \overline{\Omega}_m$. We introduce the spaces

$$\mathbf{V}(\Omega) := \{\boldsymbol{w} \in \mathbf{H}_0(\mathbf{curl}\,,\Omega); \quad \operatorname{div}\boldsymbol{w} = 0 \quad \text{in } \Omega_s\},$$

$$\mathbf{V}_0(\Omega_s) := \{\boldsymbol{w} \in \mathbf{H}_0(\mathbf{curl}\,,\Omega_s); \quad \operatorname{div}\boldsymbol{w} = 0 \quad \text{in } \Omega_s\}$$

and recall the following useful result.

Lemma 5.1. *With our hypotheses on Ω_s, the seminorm $\boldsymbol{w} \mapsto \|\mathbf{curl}\,\boldsymbol{w}\|_{[L^2(\Omega_s)]^3}$ is a norm on $\mathbf{V}_0(\Omega_s)$ equivalent to the usual norm in $\mathbf{H}(\mathbf{curl}\,,\Omega_s)$.*

Proof. See for instance [2, Corollary 3.19]. □

Lemma 5.2. *The linear extension mapping*

$$\begin{array}{rcl} \mathcal{E}:\ \mathbf{H}_\Gamma(\mathbf{curl}\,,\,\Omega_m) & \to & \mathbf{V}(\Omega) \\ \boldsymbol{w} & \mapsto & \mathcal{E}\boldsymbol{w} \end{array}$$

characterized by $\mathcal{E}\boldsymbol{w}|_{\Omega_m} = \boldsymbol{w}$ and

$$\int_{\Omega_s} \mathbf{curl}\,\mathcal{E}\boldsymbol{w} \cdot \mathbf{curl}\,\boldsymbol{q} = 0 \qquad \forall \boldsymbol{q} \in \mathbf{V}_0(\Omega_s) \tag{15}$$

is bounded.

Proof. Let us denote by $\gamma_{\mathbf{t}}^+$ and $\gamma_{\mathbf{t}}^-$ the tangential traces on Σ taken from Ω_m and Ω_s, respectively.We know from Theorem 2.1 that there exists a continuous right inverse $(\gamma_{\mathbf{t}}^-)^{-1}$ of $\gamma_{\mathbf{t}}^-$. It follows that the linear operator

$$\begin{array}{rcl} \mathcal{L}:\ \mathbf{H}_\Gamma(\mathbf{curl}\,,\,\Omega_m) & \to & \mathbf{H}(\mathbf{curl}\,,\Omega_s) \\ \boldsymbol{w} & \mapsto & \mathcal{L}\boldsymbol{w} := (\gamma_{\mathbf{t}}^-)^{-1}(\gamma_{\mathbf{t}}^+\boldsymbol{w}) \end{array}$$

is bounded, namely, there exists a constant $C_0 > 0$ such that

$$\|\mathcal{L}\boldsymbol{w}\|_{\mathbf{H}(\mathbf{curl}\,,\Omega_s)} \leq C_0 \|\boldsymbol{w}\|_{\mathbf{H}(\mathbf{curl}\,,\Omega_m)} \quad \forall \boldsymbol{w} \in \mathbf{H}_\Gamma(\mathbf{curl}\,,\,\Omega_m). \tag{16}$$

Now, given $\boldsymbol{w} \in \mathbf{H}_\Gamma(\mathbf{curl}\,,\,\Omega_m)$, consider the problem of finding $\boldsymbol{z}_w \in \mathcal{L}\boldsymbol{w} + \mathbf{H}_0(\mathbf{curl}\,,\Omega_s)$ and $\chi \in H_0^1(\Omega_s)$ satisfying

$$\begin{array}{rcl} \displaystyle\int_{\Omega_s} \mathbf{curl}\,\boldsymbol{z}_w \cdot \mathbf{curl}\,\boldsymbol{q} + \int_{\Omega_s} \boldsymbol{q} \cdot \nabla\chi & = & 0 \quad \forall \boldsymbol{q} \in \mathbf{H}_0(\mathbf{curl}\,,\Omega_s) \\ \displaystyle\int_{\Omega_s} \boldsymbol{z}_w \cdot \nabla\theta & = & 0 \quad \forall \theta \in H_0^1(\Omega_s). \end{array}$$

The well-posedness of this problem is guaranteed by the Babuška-Brezzi theory. Indeed, on the one hand, the Poincaré inequality and the fact that $\nabla(H_0^1(\Omega_s)) \subset \mathbf{H}_0(\mathbf{curl}\,,\Omega_s)$ yield the inf-sup condition

$$\sup_{\boldsymbol{q} \in \mathbf{H}_0(\mathbf{curl}\,,\Omega_s)} \frac{\displaystyle\int_{\Omega_s} \boldsymbol{q} \cdot \nabla\theta}{\|\boldsymbol{q}\|_{\mathbf{H}(\mathbf{curl}\,,\Omega_s)}} \geq \frac{\displaystyle\int_{\Omega_s} |\nabla\theta|^2}{\|\nabla\theta\|_{\mathbf{H}(\mathbf{curl}\,,\Omega_s)}} = \|\nabla\theta\|_{[L^2(\Omega_s)]^3} \geq \beta\, \|\theta\|_{H^1(\Omega_s)},$$

for all $\theta \in H_0^1(\Omega_s)$.

On the other hand, Lemma 5.1 ensures the ellipticity on the kernel

$$\mathbf{V}_0(\Omega_s) = \left\{ \boldsymbol{q} \in \mathbf{H}_0(\mathbf{curl}\,,\Omega_s); \quad \int_{\Omega_s} \boldsymbol{q} \cdot \nabla\chi = 0 \quad \forall \chi \in H_0^1(\Omega_s) \right\}.$$

In other words, there exists $C_1 > 0$ such that

$$\int_{\Omega_s} |\mathbf{curl}\,\boldsymbol{q}|^2 \geq C_1 \|\boldsymbol{q}\|^2_{\mathbf{H}(\mathbf{curl}\,,\Omega_s)} \quad \forall \boldsymbol{q} \in \mathbf{V}_0(\Omega_s).$$

It is clear now that $\mathcal{E}\boldsymbol{w} := \begin{cases} \boldsymbol{w} & \text{in } \Omega_m \\ \boldsymbol{z}_w & \text{in } \Omega_s \end{cases}$ satisfies (15). Moreover, by virtue of the stability results provided by the Babuška-Brezzi theory, there exists a constant $C_2 > 0$ such that

$$\|\mathcal{E}\boldsymbol{w}\|_{\mathbf{H}(\mathbf{curl}\,,\Omega_s)} \leq C_2 \|\mathcal{L}\boldsymbol{w}\|_{\mathbf{H}(\mathbf{curl}\,,\Omega_s)}.$$

Finally, (16) yields the estimate

$$\|\mathcal{E}\boldsymbol{w}\|_{\mathbf{H}(\mathbf{curl}\,,\Omega)} \leq \sqrt{1 + (C_0 C_2)^2}\; \|\boldsymbol{w}\|_{\mathbf{H}(\mathbf{curl}\,,\Omega_m)} \quad \forall \boldsymbol{w} \in \mathbf{H}_\Gamma(\mathbf{curl}\,,\,\Omega_m).$$

$\square$

Lemma 5.3. *There exists a linear and bounded operator*

$$\mathcal{R} :\ \mathbf{H}_0(\mathbf{curl}\,,\Omega) \to \mathbf{H}_0(\mathbf{curl}\,,\Omega)$$

satisfying $\mathbf{curl}\,(\mathcal{R}\boldsymbol{w}) = \mathbf{curl}\,\boldsymbol{w}$ *and* $\operatorname{div}(\mathcal{R}\boldsymbol{w}) = 0$.

Proof. See Section 3.5 of [2]. $\square$

With the aid of these tools, we are able to introduce the linear and bounded operator

$$\begin{aligned} \mathcal{P} :\ \mathbf{H}_\Gamma(\mathbf{curl}\,,\,\Omega_m) \quad &\to \quad \mathbf{H}_\Gamma(\mathbf{curl}\,,\,\Omega_m) \\ \boldsymbol{w} \quad &\mapsto \quad \mathcal{P}\boldsymbol{w} := (\mathcal{R}\mathcal{E}\boldsymbol{w})|_{\Omega_m}. \end{aligned}$$

Lemma 5.4. *It holds that* $\mathcal{P} \circ \mathcal{P} = \mathcal{P}$ *and*

$$\mathbf{curl}\,(\mathcal{P}\boldsymbol{w}) = \mathbf{curl}\,\boldsymbol{w}, \quad \forall \boldsymbol{w} \in \mathbf{H}_\Gamma(\mathbf{curl}\,,\,\Omega_m). \tag{17}$$

Proof. The property (17) follows immediately from Lemma 5.3, indeed,

$$\mathbf{curl}\,\mathcal{P}\boldsymbol{w} = (\mathbf{curl}\,\mathcal{R}(\mathcal{E}\boldsymbol{w}))|_{\Omega_m} = (\mathbf{curl}\,(\mathcal{E}\boldsymbol{w}))|_{\Omega_m} = \mathbf{curl}\,\boldsymbol{w} \quad \forall \boldsymbol{w} \in \mathbf{H}_\Gamma(\mathbf{curl}\,,\,\Omega_m).$$

To prove that $\mathcal{P}$ is a projector we first notice that, for any $\boldsymbol{w} \in \mathbf{H}_0(\mathbf{curl}\,,\Omega)$, $\mathcal{R}(\boldsymbol{w} - \mathcal{R}\boldsymbol{w}) \in \mathbf{H}_0(\mathbf{curl}\,,\Omega)$, $\operatorname{div}\mathcal{R}(\boldsymbol{w} - \mathcal{R}\boldsymbol{w}) = 0$ and $\mathbf{curl}\,\mathcal{R}(\boldsymbol{w} - \mathcal{R}\boldsymbol{w}) = \mathbf{0}$. Hence, Lemma 5.1 (which is also valid when Ω_s is replaced by Ω) proves that $\mathcal{R}(\boldsymbol{w} - \mathcal{R}\boldsymbol{w})$ vanishes identically in Ω, which is to say that $\mathcal{R} \circ \mathcal{R} = \mathcal{R}$.

Now, notice that the field $\boldsymbol{z} := \mathcal{E}((\mathcal{R}\mathcal{E}\boldsymbol{w})|_{\Omega_m}) - \mathcal{R}\mathcal{E}\boldsymbol{w}$ vanishes identically in Ω_m. Moreover, it is straightforward that $\boldsymbol{z}|_{\Omega_s} \in \mathbf{V}_0(\Omega_s)$. Hence, by virtue of (15),

$$\begin{aligned} \int_{\Omega_s} \mathbf{curl}\,\boldsymbol{z} \cdot \mathbf{curl}\,\boldsymbol{z} \quad &= \quad \int_{\Omega_s} \mathbf{curl}\,\mathcal{E}((\mathcal{R}\mathcal{E}\boldsymbol{w})|_{\Omega_m}) \cdot \mathbf{curl}\,\boldsymbol{z} - \int_{\Omega_s} \mathbf{curl}\,\mathcal{R}\mathcal{E}\boldsymbol{w} \cdot \mathbf{curl}\,\boldsymbol{z} \\ &= \quad -\int_{\Omega_s} \mathbf{curl}\,\mathcal{E}\boldsymbol{w} \cdot \mathbf{curl}\,\boldsymbol{z} = 0, \end{aligned}$$

which proves that $\mathbf{curl}\,\boldsymbol{z} = \mathbf{0}$ in Ω_s. Consequently, thanks again to Lemma 5.1, $\boldsymbol{z}$ also vanishes identically in Ω_s. This means that

$$\mathcal{E}((\mathcal{R}\mathcal{E}\boldsymbol{w})|_{\Omega_m}) = \mathcal{R}\mathcal{E}\boldsymbol{w} \quad \text{in } \Omega.$$

Using the last identity together with the fact that $\mathcal{R}$ is idempotent yield

$$\begin{aligned} \mathcal{P}(\mathcal{P}\boldsymbol{w}) = (\mathcal{R}\mathcal{E}(\mathcal{P}\boldsymbol{w}))|_{\Omega_m} = \Big(\mathcal{R}\mathcal{E}\Big(\mathcal{R}\mathcal{E}\boldsymbol{w}\Big)|_{\Omega_m}\Big)|_{\Omega_m} = \Big(\mathcal{R}\mathcal{R}\mathcal{E}\boldsymbol{w}\Big)|_{\Omega_m} \\ = (\mathcal{R}\mathcal{E}\boldsymbol{w})|_{\Omega_m} = \mathcal{P}\boldsymbol{w} \end{aligned}$$

and the result follows. $\square$

We deduce from the last results that $\mathcal{P}$ provides the stable and direct Helmholtz-type decomposition

$$\mathbf{H}_\Gamma(\mathbf{curl}\,,\,\Omega_m) = \mathcal{P}(\mathbf{H}_\Gamma(\mathbf{curl}\,,\,\Omega_m)) \;\oplus\; (\mathcal{I}-\mathcal{P})(\mathbf{H}_\Gamma(\mathbf{curl}\,,\,\Omega_m)), \tag{18}$$

where $\mathcal{I}$ represents here the identity operator. This means that any element $\boldsymbol{h} \in \mathbf{H}_\Gamma(\mathbf{curl}\,,\,\Omega_m)$ admits the unique splitting

$$\boldsymbol{h} = \boldsymbol{h}^0 + \boldsymbol{h}^\perp$$

with $\boldsymbol{h}^0 := \mathcal{P}\boldsymbol{w}$ and $\boldsymbol{h}^\perp := \boldsymbol{w} - \mathcal{P}\boldsymbol{w}$ and the norm

$$\boldsymbol{w} \to |||\boldsymbol{w}|||_{\mathbf{H}(\mathbf{curl}\,,\,\Omega_m)} := \left(\|\boldsymbol{w}^0\|^2_{\mathbf{H}(\mathbf{curl}\,,\,\Omega_m)} + \|\boldsymbol{w}^\perp\|^2_{\mathbf{H}(\mathbf{curl}\,,\,\Omega_m)}\right)^{1/2}$$

is equivalent to $\boldsymbol{w} \to \|\boldsymbol{w}\|_{\mathbf{H}(\mathbf{curl}\,,\,\Omega_m)}$ on $\mathbf{H}_\Gamma(\mathbf{curl}\,,\,\Omega_m)$. Namely, as $\|\mathcal{P}\| = \|\mathcal{I}-\mathcal{P}\|$ (see Lemma 5 of [15]),

$$\frac{1}{\sqrt{2}\|\mathcal{P}\|}|||\boldsymbol{w}|||_{\mathbf{H}(\mathbf{curl}\,,\,\Omega_m)} \le \|\boldsymbol{w}\|_{\mathbf{H}(\mathbf{curl}\,,\,\Omega_m)} \le \sqrt{2}\,|||\boldsymbol{w}|||_{\mathbf{H}(\mathbf{curl}\,,\,\Omega_m)}, \tag{19}$$

for all $\boldsymbol{w} \in \mathbf{H}_\Gamma(\mathbf{curl}\,,\,\Omega_m)$.

Lemma 5.5. *The mapping $\mathcal{P}:\ \mathbf{H}_\Gamma(\mathbf{curl}\,,\,\Omega_m) \to [L^2(\Omega_m)]^3$ is compact.*

Proof. It is known (see e.g., [2]) that there exists $s_\Omega \in (1/2, 1]$ such that

$$\mathcal{R}(\mathbf{H}_0(\mathbf{curl}\,,\Omega)) \subset \{\boldsymbol{w} \in \mathbf{H}_0(\mathbf{curl}\,,\Omega);\quad \operatorname{div}\boldsymbol{w} = 0\} \hookrightarrow [H^{s_\Omega}(\Omega)]^3. \tag{20}$$

The result is then a consequence of the compactness of the canonical injection $H^{s_\Omega}(\Omega_m) \hookrightarrow L^2(\Omega_m)$. □

5.2. A Fredholm alternative. We introduce the sesquilinear form

$$\boldsymbol{A}_0((\boldsymbol{h},\boldsymbol{w}),(\boldsymbol{\sigma},\boldsymbol{\tau})) := E^0_m(\boldsymbol{h},\boldsymbol{w}) + E^0_s(\boldsymbol{\sigma},\boldsymbol{\tau})$$

where

$$E^0_s(\boldsymbol{\sigma},\boldsymbol{\tau}) := k^2\left(\int_{\Omega_s}\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}) + \kappa_s^2\int_{\Omega_s}\boldsymbol{u}\cdot\boldsymbol{v}\right)$$

and

$$E^0_m(\boldsymbol{h},\boldsymbol{w}) := -E^+_m(\boldsymbol{h}^0,\boldsymbol{w}^0) + E^+_m(\boldsymbol{h}^\perp,\boldsymbol{w}^\perp)$$

with

$$E^+_m(\boldsymbol{h},\boldsymbol{w}) := \int_{\Omega_m}(a^{-1}\mathbf{curl}\,\boldsymbol{h}\cdot\mathbf{curl}\,\boldsymbol{w} + k^2 b\,\boldsymbol{h}\cdot\boldsymbol{w}).$$

Lemma 5.6. *The sesquilinear form $\boldsymbol{A}_0$ is weakly coercive on $\mathbb{X}$ in the sense that there exists $\alpha > 0$ such that*

$$\sup_{(\boldsymbol{w},\boldsymbol{v})\in\mathbb{X}}\frac{|\boldsymbol{A}_0((\boldsymbol{h},\boldsymbol{u}),(\boldsymbol{w},\boldsymbol{v}))|}{\|(\boldsymbol{w},\boldsymbol{v})\|_{\mathbb{X}}} \;\ge\; \alpha\,\|(\boldsymbol{h},\boldsymbol{u})\|_{\mathbb{X}} \qquad \forall\,(\boldsymbol{h},\boldsymbol{u}) \in \mathbb{X}\,. \tag{21}$$

In addition, there holds

$$\sup_{(\boldsymbol{h},\boldsymbol{u})\in\mathbb{X}}|\,\boldsymbol{A}_0((\boldsymbol{h},\boldsymbol{u}),(\boldsymbol{w},\boldsymbol{v}))\,| \;>\; 0 \qquad \forall\,(\boldsymbol{w},\boldsymbol{v}) \in \mathbb{X}\,,\ (\boldsymbol{w},\boldsymbol{v}) \neq \mathbf{0}\,. \tag{22}$$

Proof. We deduce from (19) that the linear operator

$$\begin{aligned}\Xi:\ \mathbb{X} &\ \to\ \mathbb{X}\\ (\boldsymbol{w},\boldsymbol{v}) &\ \mapsto\ \Xi((\boldsymbol{w},\boldsymbol{v})) := ((\boldsymbol{w}^\perp - \boldsymbol{w}^0),\boldsymbol{v})\end{aligned}$$

is bounded; i.e., there exists a constant $C_1 > 0$ such that

$$\|\Xi((\boldsymbol{w},\boldsymbol{v}))\|_{\mathbb{X}} \le C_1\|(\boldsymbol{w},\boldsymbol{v})\|_{\mathbb{X}} \quad \forall(\boldsymbol{w},\boldsymbol{v}) \in \mathbb{X}. \tag{23}$$

Notice that

$$\begin{aligned}\boldsymbol{A}_0((\boldsymbol{w},\boldsymbol{v}),\Xi((\overline{\boldsymbol{w}},\overline{\boldsymbol{v}})) &= E_s^0(\boldsymbol{v},\overline{\boldsymbol{v}}) + E_m^0(\boldsymbol{w}^0+\boldsymbol{w}^\perp,(\overline{\boldsymbol{w}}^\perp-\overline{\boldsymbol{w}}^0))\\ &= E_s^0(\boldsymbol{v},\overline{\boldsymbol{v}}) + E_m^+(\boldsymbol{w}^0,\overline{\boldsymbol{w}}^0) + E_m^+(\boldsymbol{w}^\perp,\overline{\boldsymbol{w}}^\perp).\end{aligned}$$

Now, on the one hand, Korn's inequality (see e.g. [9, Theorem 3.78]) shows that there exists $C_2>0$ such that

$$E_s^0(\boldsymbol{v},\overline{\boldsymbol{v}}) \geq C_2\|\boldsymbol{v}\|^2_{[H^1(\Omega_s)]^3} \quad \forall \boldsymbol{v}\in[H^1(\Omega_s)]^3. \tag{24}$$

On the other hand,

$$\operatorname{Re}[E_m^+(\boldsymbol{w},\overline{\boldsymbol{w}})] = \int_{\Omega_m} \frac{\epsilon}{\epsilon_0|a|}\,|\mathbf{curl}\,\boldsymbol{w}|^2 + k^2 b\,|\boldsymbol{w}|^2.$$

We deduce from our hypotheses on the coefficients that $b(\boldsymbol{x})\geq 1$ and $\frac{\epsilon(\boldsymbol{x})}{\epsilon_0|a(\boldsymbol{x})|}\geq \frac{\epsilon_0}{\sqrt{(\bar\epsilon)^2+(\bar\sigma/\omega)^2}}$ for a.e. $\boldsymbol{x}\in\Omega_m$. Hence,

$$\operatorname{Re}[E_m^+(\boldsymbol{w},\overline{\boldsymbol{w}})] \geq C_3\,\|\boldsymbol{w}\|^2_{\mathbf{H}(\mathbf{curl}\,,\,\Omega_m)} \quad \forall \boldsymbol{w}\in \mathbf{H}_\Gamma(\mathbf{curl}\,,\,\Omega_m) \tag{25}$$

with $C_3=\min(k^2,\frac{\epsilon_0}{\sqrt{(\bar\epsilon)^2+(\bar\sigma/\omega)^2}})$. Summing up, (24), (25) and (23) yield

$$\begin{aligned}\operatorname{Re}\Big\{\boldsymbol{A}_0((\boldsymbol{w},\boldsymbol{v}),\Xi(\overline{\boldsymbol{w}},\overline{\boldsymbol{v}}))\Big\} &\geq \min(C_2,C_3)\Big(|||\boldsymbol{w}|||^2_{\mathbf{H}(\mathbf{curl}\,,\,\Omega_m)}+\|\boldsymbol{v}\|^2_{[H^1(\Omega_s)]^3}\Big)\\ &\geq \frac{\min(C_2,C_3)}{\sqrt{2}}\|(\boldsymbol{w},\boldsymbol{v})\|^2_{\mathbb{X}} \geq \frac{\min(C_2,C_3)}{\sqrt{2}C_1}\|(\boldsymbol{w},\boldsymbol{v})\|_{\mathbb{X}}\|\Xi((\boldsymbol{w},\boldsymbol{v}))\|_{\mathbb{X}}\end{aligned} \tag{26}$$

for all $(\boldsymbol{w},\boldsymbol{v})\in\mathbb{X}$. This proves the inf-sup condition (21). Finally, the symmetry of $\boldsymbol{A}_0$ and the same estimates yield the inf-sup condition (22). $\square$

Let us consider now the linear mapping

$$\begin{aligned}\mathcal{G}^0:\ \mathbb{X}' &\to \mathbb{X}\\ \mathcal{W} &\mapsto \mathcal{G}^0(\mathcal{W}) := (\boldsymbol{h}_0,\boldsymbol{u}_0)\end{aligned} \tag{27}$$

where $(\boldsymbol{h}_0,\boldsymbol{u}_0)\in\mathbb{X}$ solves the problem

$$\boldsymbol{A}_0((\boldsymbol{h}_0,\boldsymbol{u}_0),(\boldsymbol{w},\boldsymbol{v})) \;=\; \mathcal{W}((\boldsymbol{w},\boldsymbol{v})) \quad \forall(\boldsymbol{w},\boldsymbol{v})\in\mathbb{X} \tag{28}$$

Theorem 5.1. *The operator* $\mathcal{G}_0:\ (\mathbb{X}\times\mathbb{M})'\to(\mathbb{X}\times\mathbb{M})$ *is an isomorphism.*

Proof. The result is a direct consequence of Lemma 5.6 and the well-known Nečas theorem, see [9, Theorem 3.2.3]. $\square$

Lemma 5.7. *The sesquilinear form* $\boldsymbol{A}-\boldsymbol{A}_0$ *is compact.*

Proof. Let us first notice that, by virtue of Lemma 5.5,

$$\begin{aligned}E_m(\boldsymbol{h},\boldsymbol{w})-E_m^0(\boldsymbol{h},\boldsymbol{w}) &= E_m(\boldsymbol{h}^0,\boldsymbol{w}^\perp)+E_m(\boldsymbol{h}^\perp,\boldsymbol{w}^0)-2k^2\int_{\Omega_m}\boldsymbol{h}^\perp\cdot\boldsymbol{w}^\perp\\ &= \int_{\Omega_m}\boldsymbol{h}^\perp\cdot\boldsymbol{w}^0+\int_{\Omega_m}\boldsymbol{h}^0\cdot\boldsymbol{w}^\perp-2k^2\int_{\Omega_m}\boldsymbol{h}^\perp\cdot\boldsymbol{w}^\perp\end{aligned}$$

is compact. Moreover, as the embedding $H^1(\Omega_s)\hookrightarrow L^2(\Omega_s)$ is compact, we deduce that

$$E_s(\boldsymbol{\sigma},\boldsymbol{v})-E_s^0(\boldsymbol{\sigma},\boldsymbol{v}) \;:=\; -2k^2\kappa_s^2\int_{\Omega_s}\boldsymbol{u}\cdot\boldsymbol{v}$$

is also compact.

Finally, using that the embedding $\mathbf{H}_{\|}^{1/2}(\Sigma) \hookrightarrow \mathbf{L}^2_{\mathbf{t}}(\Sigma)$ is compact (cf. [11, Lemma 3.2]), we deduce the compactness of

$$\begin{array}{rcl} \mathbf{H}(\mathbf{curl}\,, \Omega_m) \times [H^1(\Omega_s)]^3 & \to & \mathbb{C} \\ (\boldsymbol{w}, \boldsymbol{v}) & \mapsto & \langle\, \boldsymbol{\gamma}_{\mathbf{t}} \boldsymbol{w}, \boldsymbol{\pi}_{\mathbf{t}} \boldsymbol{v}\, \rangle_{\mathbf{t},\Sigma}. \end{array} \tag{29}$$

□

We are now ready to establish the main result of this section.

Theorem 5.2. *Assume that the homogeneous problem associated to* (9) *has only the trivial solution. Then, there exists a unique solution* $(\boldsymbol{h}, \boldsymbol{u}) \in \mathbb{X}$ *to* (14) *or equivalently* (13). *In addition, there exists* $C > 0$ *such that*

$$\|(\boldsymbol{h}, \boldsymbol{u})\|_{\mathbb{X}} \;\leq\; C\, \|\mathcal{L}\|_{\mathbb{X}'}\,. \tag{30}$$

Proof. We introduce the linear operator

$$\begin{array}{rcl} \mathcal{G} : \mathbb{X}' & \to & \mathbb{X} \\ \mathcal{W} & \mapsto & \mathcal{G}(\mathcal{W}) := (\tilde{\boldsymbol{h}}, \tilde{\boldsymbol{u}}) \end{array} \tag{31}$$

where $(\tilde{\boldsymbol{h}}, \tilde{\boldsymbol{u}})$ is the solution of (14) with a right-hand side $\mathcal{W}$ instead of $\mathcal{F}$. Let $\mathcal{I}$ be the identity operator in $\mathbb{X}$. By virtue of Lemma 5.7,

$$\mathcal{G}[\mathcal{G}^0]^{-1} = \mathcal{I} + \mathcal{K}[\mathcal{G}^0]^{-1}$$

with $\mathcal{K} : \mathbb{X}' \to \mathbb{X}$ compact. It follows from the Fredholm Alternative that the well-posedness of problem (14) may be derived from uniqueness, as stated in the Theorem. □

5.3. A uniqueness result. It is important to notice here that there may exist singular frequencies ω for which the homogeneous problem

$$\begin{array}{rcll} \boldsymbol{\sigma} & = & \mathcal{C}\,\boldsymbol{\varepsilon}(\boldsymbol{u}) & \text{in} \quad \Omega_s\,, \\ \mathbf{div}\,\boldsymbol{\sigma} + \omega^2 \rho_s\, \boldsymbol{u} & = & \mathbf{0} & \text{in} \quad \Omega_s\,, \\ \boldsymbol{u} \times \boldsymbol{n} & = & \mathbf{0} & \text{on} \quad \Sigma\,, \\ \boldsymbol{\sigma}\boldsymbol{n} & = & \mathbf{0} & \text{on} \quad \Sigma \end{array} \tag{32}$$

admits a non trivial solution.

We recall that, thanks to our assumptions on Ω_s and Γ, Ω_m is a connected and simply connected Lipschitz polyhedra with boundary $\partial\Omega_m$ consisting of two disjoint connected components Σ and Γ. Furthermore, we assume that Ω_m can be decomposed into J connected polyhedra Ω_m^j such that $\overline{\Omega}_m = \cup_{j=1}^J \overline{\Omega}_m^j$ and $\Omega_m^i \cap \Omega_m^j = \emptyset$ if $i \neq j$.

Theorem 5.3. *Assume that* (32) *only admits the trivial solution. If the magnetic permeability* $\mu(\boldsymbol{x})$ *is constant on each subdomain* Ω_m^j *and the restrictions of* $\epsilon(\boldsymbol{x})$ *and* $\sigma(\boldsymbol{x})$ *to* Ω_m^j *are in* $H^3(\Omega_m^j)$, *for all* $j = 1, \cdots, J$, *then, there is at most one solution to* (13).

Proof. The result is obtained by using the unique continuation principle given in [8, Theorem 9.3] as illustrated in [13, Theorem 4.12]. □

6. The discrete problem

Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of $\overline{\Omega}_s \cup \overline{\Omega}_m$ by tetrahedrons K of diameter h_K with mesh size $h := \max\{\, h_K : \quad K \in \mathcal{T}_h \,\}$. We assume that for all $h > 0$, $\mathcal{T}_h(\Omega_l) := \left\{K \in \mathcal{T}_h; \quad K \subset \overline{\Omega}_l\right\}$ is a triangulation of $\overline{\Omega}_l$ for $l = s, m$.

For any $K \in \mathcal{T}_h(\Omega_m)$, we consider the local representation of the edge finite element of Nédélec $\boldsymbol{\mathcal{N}\mathcal{D}}_1(K) := \left\{\boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x} : \quad \boldsymbol{a}, \boldsymbol{b} \in \mathbb{C}^3\right\}$. We seek the finite element counterpart of $\boldsymbol{h}$ in

$$\mathbf{X}_h^m := \{\boldsymbol{w} \in \mathbf{H}_\Gamma(\mathbf{curl}\,, \Omega_m) : \quad \boldsymbol{w}|_K \in \boldsymbol{\mathcal{N}\mathcal{D}}_1(K), \quad \forall\, K \in \mathcal{T}_h(\Omega_m)\}\;.$$

We will also need the usual space of continuous and piecewise linear functions

$$\mathcal{S}_h^1(\Omega_s) = \left\{v \in \mathcal{C}^0(\overline{\Omega}_s); \quad v|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h(\Omega_s)\right\}.$$

The finite element scheme associated to our coupled problem (14) is given by:

$$\begin{aligned} &\text{Find } (\boldsymbol{h}_h, \boldsymbol{u}_h) \in \mathbb{X}_h := \mathbf{X}_h^m \times [\mathcal{S}_h^1(\Omega_s)]^3 \text{ such that} \\ &\boldsymbol{A}((\boldsymbol{h}_h, \boldsymbol{u}_h), (\boldsymbol{w}, \boldsymbol{v})) \;=\; \mathcal{L}((\boldsymbol{w}, \boldsymbol{v})) \quad \forall (\boldsymbol{w}, \boldsymbol{v}) \in \mathbb{X}_h. \end{aligned} \tag{33}$$

6.1. Technical tools. For any $\delta \geq 0$, we introduce the Sobolev space

$$\mathbf{H}^\delta(\mathbf{curl}\,, \Omega_m) := \left\{\boldsymbol{w} \in [H^\delta(\Omega_m)]^3, \quad \mathbf{curl}\,\boldsymbol{w} \in [H^\delta(\Omega_m)]^3\right\}$$

endowed with its Hilbertian norm

$$\|\boldsymbol{w}\|^2_{\mathbf{H}^\delta(\mathbf{curl}\,, \Omega_m)} = \|\boldsymbol{w}\|^2_{[H^\delta(\Omega_m)]^3} + \|\mathbf{curl}\,\boldsymbol{w}\|^2_{[H^\delta(\Omega_m)]^3}.$$

For any edge E of $\mathcal{T}_h(\Omega_m)$, we denote by $\mathbf{t}_E$ a unit tangential vector along E. It follows from [2, Lemma 4.7] that if $\boldsymbol{w} \in \mathbf{H}^\delta(\mathbf{curl}\,, \Omega_m)$ with $\delta > 1/2$, then the moments $\int_E \boldsymbol{w} \cdot \mathbf{t}_E$ are meaningful. This guarantees that the interpolation operator $\Pi_h^{\mathcal{N}\mathcal{D}} : \mathbf{H}^\delta(\mathbf{curl}\,, \Omega_m) \to \mathbf{X}_h^m$ associated to the edge finite element characterized by

$$\int_E \Pi_h^{\mathcal{N}\mathcal{D}} \boldsymbol{w} \cdot \mathbf{t}_E = \int_E \boldsymbol{w} \cdot \mathbf{t}_E \qquad \text{for all edge } E \text{ of } \mathcal{T}_h(\Omega_m),$$

is uniformly bounded and we have the following interpolation error estimate (see [1, Proposition 5.6]):

$$\|\boldsymbol{w} - \Pi_h^{\mathcal{N}\mathcal{D}} \boldsymbol{w}\|_{\mathbf{H}(\mathbf{curl}\,, \Omega_m)} \leq C\, h^\delta \|\boldsymbol{w}\|_{\mathbf{H}^\delta(\mathbf{curl}\,, \Omega_m)} \quad \forall \boldsymbol{w} \in \mathbf{H}^\delta(\mathbf{curl}\,, \Omega_m), \quad (\delta > 1/2). \tag{34}$$

Another useful property of $\Pi_h^{\mathcal{N}\mathcal{D}}$ is given by the following result.

Lemma 6.1. *If $\boldsymbol{w} \in [H^\delta(\Omega_m)]^3$ with $\delta \in (1/2, 1]$ and $\mathbf{curl}\,\boldsymbol{w} \in \mathbf{curl}\,(\mathbf{X}_h^m)$, then $\Pi_h^{\mathcal{N}\mathcal{D}} \boldsymbol{w}$ is well-defined and there is a constant $C > 0$ independent of h such that*

$$\|\boldsymbol{w} - \Pi_h^{\mathcal{N}\mathcal{D}} \boldsymbol{w}\|_{[L^2(\Omega_m)]^3} \leq C h^\delta \|\boldsymbol{w}\|_{[H^\delta(\Omega_m)]^3}.$$

Proof. See [10, Lemma 4.6] □

In order to establish the global approximation properties of our finite element subspaces, we will also need the following well-known results (see for instante [13]): For each $\epsilon \in (1, 2]$ and for each $\boldsymbol{v} \in [H^{1+\epsilon}(\Omega_s)]^3$, there holds

$$\inf_{\boldsymbol{v}_h \in [\mathcal{S}_h^1(\Omega_s)]^3} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{[H^1(\Omega_s)]^3} \;\leq\; C\, h^\epsilon\, \|\boldsymbol{v}\|_{[H^{1+\epsilon}(\Omega_s)]^3}\;. \tag{35}$$

6.2. Analysis of the discrete problem. Let us denote by $\Pi_h^{\mathcal{RT}}$ the first order Raviart-Thomas interpolation operator associated to the triangulation $\mathcal{T}_h(\Omega_m)$, see [14, 6]. We deduce from the well-known commuting diagram property (cf. [13]) $\mathbf{curl} \circ \Pi_h^{\mathcal{ND}} = \Pi_h^{\mathcal{RT}} \circ \mathbf{curl}$ and from the fact that $\mathbf{curl}\,(\mathbf{X}_h^m)$ is contained in the first order Raviart-Thomas finite element space, that

$$\mathbf{curl}\ \Pi_h^{\mathcal{ND}}(\mathcal{P}\boldsymbol{w}) = \Pi_h^{\mathcal{RT}}(\mathbf{curl}\ \mathcal{P}\boldsymbol{w}) = \Pi_h^{\mathcal{RT}}(\mathbf{curl}\,\boldsymbol{w}) = \mathbf{curl}\,\boldsymbol{w} \in \mathbf{curl}\,(\mathbf{X}_h^m).$$

Hence, Lemma 6.1 and (20) permit us to define

$$\begin{aligned} \mathcal{P}_h^m : \mathbf{X}_h^m &\to \mathbf{X}_h^m \\ \boldsymbol{w} &\mapsto \mathcal{P}_h\boldsymbol{w} := \Pi_h^{\mathcal{ND}}(\mathcal{P}\boldsymbol{w}). \end{aligned}$$

Lemma 6.2. *There exists a constant $C > 0$ independent of h such that*

$$\|\mathcal{P}\boldsymbol{w} - \mathcal{P}_h\boldsymbol{w}\|_{\mathbf{H}(\mathbf{curl}\,,\,\Omega_m)} \le C h^{s_\Omega} \|\boldsymbol{w}\|_{\mathbf{H}(\mathbf{curl}\,,\,\Omega_m)}.$$

Proof. Notice that as $\mathbf{curl}\,(\mathcal{P}\boldsymbol{w} - \mathcal{P}_h\boldsymbol{w}) = \mathbf{0}$, we have that

$$\|\mathcal{P}\boldsymbol{w} - \mathcal{P}_h\boldsymbol{w}\|_{\mathbf{H}(\mathbf{curl}\,,\,\Omega_m)} = \|\mathcal{P}\boldsymbol{w} - \mathcal{P}_h\boldsymbol{w}\|_{[L^2(\Omega_m)]^3} = \|\mathcal{P}\boldsymbol{w} - \Pi_h^{\mathcal{ND}}(\mathcal{P}\boldsymbol{w})\|_{[L^2(\Omega_m)]^3}.$$

Finally, Lemma 6.1 and (20) ensure that

$$\|\mathcal{P}\boldsymbol{w} - \Pi_h^{\mathcal{ND}}(\mathcal{P}\boldsymbol{w})\|_{[L^2(\Omega_m)]^3} \le C h^{s_\Omega} \|\mathcal{P}\boldsymbol{w}\|_{[H^{s_\Omega}(\Omega_m)]^3} \le C_1 h^{s_\Omega} \|\boldsymbol{w}\|_{\mathbf{H}(\mathbf{curl}\,,\,\Omega_m)}, \quad \forall \boldsymbol{w} \in \mathbf{X}_h^m$$

and the result follows. $\square$

We are now ready to prove the following discrete weak coercivity of $\boldsymbol{A}_0$.

Lemma 6.3. *There exist constants C, $h_0 > 0$, independent of h, such that for each for each $h \le h_0$ there holds*

$$\sup_{(\boldsymbol{w},\boldsymbol{v})\in\mathbb{X}_h} \frac{|\,\boldsymbol{A}_0((\boldsymbol{h},\boldsymbol{u}),(\boldsymbol{w},\boldsymbol{v}))\,|}{\|(\boldsymbol{w},\boldsymbol{v})\|_{\mathbb{X}}} \;\ge\; C\,\|(\boldsymbol{h},\boldsymbol{u})\|_{\mathbb{X}} \qquad \forall\,(\boldsymbol{h},\boldsymbol{u}) \in \mathbb{X}_h\,. \tag{36}$$

Proof. Let us introduce the linear and bounded operator

$$\begin{aligned} \Xi_h : \mathbb{X}_h &\to \mathbb{X}_h \\ (\boldsymbol{w},\boldsymbol{v}) &\mapsto ((\mathcal{I} - 2\,\mathcal{P}_h)\boldsymbol{w},\boldsymbol{v})\,. \end{aligned}$$

It follows from Lemma 6.2 that

$$\|\Xi(\boldsymbol{w},\boldsymbol{v}) \,-\, \Xi_h(\boldsymbol{w},\boldsymbol{v})\|_{\mathbb{X}} \,\le\, C_0\,h^{s_\Omega}\,\|(\boldsymbol{w},\boldsymbol{v})\|_{\mathbb{X}} \qquad \forall\,(\boldsymbol{w},\boldsymbol{v}) \,\in\, \mathbb{X}_h\,.$$

Hence, using the boundedness of $\boldsymbol{A}_0$ and (26), we find that for each $(\boldsymbol{w},\boldsymbol{v}) \in \mathbb{X}_h$ there holds

$$\begin{aligned} &\left|\mathrm{Re}\left\{\,\boldsymbol{A}_0((\boldsymbol{w},\boldsymbol{v}),\Xi_h\overline{(\boldsymbol{w},\boldsymbol{v})})\,\right\}\right| \ge \\ &\qquad \left|\mathrm{Re}\left\{\,\boldsymbol{A}_0((\boldsymbol{w},\boldsymbol{v}),\Xi\overline{(\boldsymbol{w},\boldsymbol{v})}) - \boldsymbol{A}_0((\boldsymbol{w},\boldsymbol{v}),(\Xi-\Xi_h)\overline{(\boldsymbol{w},\boldsymbol{v})})\,\right\}\right| \\ &\qquad \ge \left|\mathrm{Re}\left\{\,\boldsymbol{A}_0((\boldsymbol{w},\boldsymbol{v}),\Xi\overline{(\boldsymbol{w},\boldsymbol{v})})\,\right\}\right| \,-\, C_1\,h^{s_\Omega}\,\|(\boldsymbol{w},\boldsymbol{v})\|_{\mathbb{X}}^2 \\ &\qquad\qquad \ge C_2\,\|(\boldsymbol{w},\boldsymbol{v})\|_{\mathbb{X}}^2 \,-\, C_1\,h^{s_\Omega}\,\|(\boldsymbol{w},\boldsymbol{v})\|_{\mathbb{X}}^2 \,\ge\, \frac{C_2}{2}\,\|(\boldsymbol{w},\boldsymbol{v})\|_{\mathbb{X}}^2, \end{aligned}$$

for all $h \le h_0 := (\frac{C_2}{2C_1})^{1/s_\Omega}$.

On the other hand, Lemma 6.2 shows that Ξ_h is uniformly bounded: there exists $C_3 > 0$ independent of h such that

$$\|\Xi_h(\boldsymbol{w},\boldsymbol{v})\|_{\mathbb{X}} \le C_3\|(\boldsymbol{w},\boldsymbol{v})\|_{\mathbb{X}} \quad \forall(\boldsymbol{w},\boldsymbol{v}) \in \mathbb{X}_h.$$

Hence, (36) follows immediately from

$$\left| \operatorname{Re}\left\{ \boldsymbol{A}_0((\boldsymbol{w},\boldsymbol{v}),\Xi_h(\overline{(\boldsymbol{w},\boldsymbol{v})})) \right\} \right| \geq \frac{C_1}{2C_2}\|(\boldsymbol{w},\boldsymbol{v})\|_{\mathbb{X}}\|\Xi_h(\boldsymbol{w},\boldsymbol{v})\|_{\mathbb{X}} \quad \forall (\boldsymbol{w},\boldsymbol{v})\in\mathbb{X}_h.$$

□

The well-posedness and convergence of the discrete scheme (14) can now be established.

Theorem 6.1. *Assume that the homogeneous problem associated to* (13) *has only the trivial solution. There exists* $h_0 > 0$ *such that, for each for each* $h \leq h_0$*, the Galerkin scheme* (14) *has a unique solution* $(\boldsymbol{h}_h,\boldsymbol{u}_h)\in\mathbb{X}_h$*. In addition, there exist* C_1, $C_2 > 0$*, independent of* h*, such that* $\forall h \leq h_0$

$$\|(\boldsymbol{h}_h,\boldsymbol{u}_h)\|_{\mathbb{X}} \leq C_1\,\|\mathcal{L}\|_{\mathbb{X}'}\,. \tag{37}$$

and

$$\|(\boldsymbol{h},\boldsymbol{u}) - (\boldsymbol{h}_h,\boldsymbol{u}_h)\|_{\mathbb{X}} \leq C_2 \inf_{(\boldsymbol{w}_h,\boldsymbol{v}_h)\in\mathbb{X}_h} \|(\boldsymbol{h},\boldsymbol{u}) - (\boldsymbol{w}_h,\boldsymbol{v}_h)\|_{\mathbb{X}}\,. \tag{38}$$

Furthermore, if there exists $\delta_1\in(1/2,1]$ *and* $\delta_2\in(0,1]$ *such that* $\boldsymbol{h}\in\mathbf{H}^{\delta_1}(\mathbf{curl},\Omega_m)$ *and* $\boldsymbol{u}\in[H^{1+\delta_2}(\Omega_s)]^3$*, then there holds*

$$\|(\boldsymbol{h},\boldsymbol{u}) - (\boldsymbol{h}_h,\boldsymbol{u}_h)\|_{\mathbb{X}} \leq C_3\, h^{\min(\delta_1,\delta_2)} \left\{ \|\boldsymbol{u}\|_{[H^{1+\delta_2}(\Omega_s)]^3} + \|\boldsymbol{h}\|_{\mathbf{H}^{\delta_1}(\mathbf{curl},\Omega_m)} \right\}, \tag{39}$$

with a constant $C_3 > 0$ *independent of* h*.*

Proof. We simplify the notation and denote $\mathcal{H} = (\boldsymbol{h},\boldsymbol{u})\in\mathbb{X}$ and $\mathcal{W} = (\boldsymbol{w},\boldsymbol{v})\in\mathbb{X}$. Thanks to Theorem 5.2, the following global inf-sup condition holds true

$$\exists\boldsymbol{\vartheta} > 0; \qquad \sup_{\mathcal{W}\in\mathbb{X}} \frac{|\boldsymbol{A}(\mathcal{H},\mathcal{W})|}{\|\mathcal{W}\|_{\mathbb{X}}} \geq \boldsymbol{\vartheta}\,\|\mathcal{H}\|_{\mathbb{X}} \quad \forall\mathcal{H}\in\mathbb{X}. \tag{40}$$

We know from Lemma 5.7 that the operator $\boldsymbol{\mathcal{K}}:\ \mathbb{X}\to\mathbb{X}'$ defined by

$$\langle\,\boldsymbol{\mathcal{K}}\mathcal{H},\mathcal{W}\,\rangle = \boldsymbol{A}(\mathcal{H},\mathcal{W}) - \boldsymbol{A}_0(\mathcal{H},\mathcal{W})$$

is compact. Here, $\langle\,\cdot,\cdot\,\rangle$ denotes the $\mathbb{X}'\times\mathbb{X}$-duality bracket.

Under the hypothesis of the Theorem, the inf-sup condition provided by Lemma 6.3 guarantees that the discrete operator $\boldsymbol{\mathcal{R}}_h:\ \mathbb{X}\to\mathbb{X}_h$ characterized by

$$\boldsymbol{A}_0(\mathcal{W}_h,\boldsymbol{\mathcal{R}}_h\mathcal{H}) = \boldsymbol{A}_0(\mathcal{W}_h,\mathcal{H}) \quad \forall\mathcal{W}_h\in\mathbb{X}_h$$

is well-defined and we have the following Céa estimate

$$\|\mathcal{H} - \boldsymbol{\mathcal{R}}_h\mathcal{H}\|_{\mathbb{X}} \leq C^* \inf_{\mathcal{W}_h\in\mathbb{X}_h}\|\mathcal{H} - \mathcal{W}_h\|_{\mathbb{X}}.$$

Density results and the approximation properties (34) and (35) permit us to conclude that $\boldsymbol{\mathcal{R}}_h$ is pointwise convergent to the identity operator $\boldsymbol{\mathcal{I}}$, i.e.,

$$\lim_{h\to 0}\boldsymbol{\mathcal{R}}_h\mathcal{H} = \mathcal{H} \qquad \forall\mathcal{H}\in\mathbb{X}.$$

Our aim now is to obtain a global discrete inf-sup condition for A. Notice that, for all $\mathcal{W}\in\mathbb{X}$ and for all $\mathcal{H}_h\in\mathbb{X}_h$,

$$\boldsymbol{A}(\mathcal{H}_h,\boldsymbol{\mathcal{R}}_h\mathcal{W}) = \boldsymbol{A}(\mathcal{H}_h,\mathcal{W}) - \boldsymbol{A}(\mathcal{W}_h,\mathcal{H} - \boldsymbol{\mathcal{R}}_h\mathcal{H}) = \boldsymbol{A}(\mathcal{H}_h,\mathcal{W}) - \langle\boldsymbol{\mathcal{K}}\mathcal{H}_h,(\boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{R}}_h)\mathcal{W}\rangle.$$

Hence

$$|\boldsymbol{A}(\mathcal{H}_h,\boldsymbol{\mathcal{R}}_h\mathcal{W})| \geq |\boldsymbol{A}(\mathcal{H}_h,\mathcal{W})| - \|\boldsymbol{\mathcal{K}}'(\boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{R}}_h)\|\|\mathcal{H}_h\|_{\mathbb{X}}\|\mathcal{W}\|_{\mathbb{X}} \tag{41}$$

where $\boldsymbol{\mathcal{K}}'$ is the dual operator of $\boldsymbol{\mathcal{K}}$. Therefore,

$$\sup_{\mathcal{W}_h\in\mathbb{X}_h}\frac{|\boldsymbol{A}(\mathcal{H}_h,\mathcal{W}_h)|}{\|\mathcal{W}_h\|_{\mathbb{X}}}\geq\sup_{\mathcal{W}\in\mathbb{X}}\frac{|\boldsymbol{A}(\mathcal{H}_h,\boldsymbol{\mathcal{R}}_h\mathcal{W})|}{\|\boldsymbol{\mathcal{R}}_h\mathcal{W}\|_{\mathbb{X}}}\geq\frac{1}{\|\boldsymbol{\mathcal{R}}_h\|}\sup_{\mathcal{W}\in\mathbb{X}}\frac{|\boldsymbol{A}(\mathcal{H}_h,\boldsymbol{\mathcal{R}}_h\mathcal{W})|}{\|\mathcal{W}\|_{\mathbb{X}}},$$

and using inequality (41) and the continuous inf-sup condition (40) yield

$$\sup_{\mathcal{W}\in\mathbb{X}}\frac{|\boldsymbol{A}(\mathcal{H}_h,\boldsymbol{\mathcal{R}}_h\mathcal{W})|}{\|\mathcal{W}\|_{\mathbb{X}}}\geq\sup_{\mathcal{W}\in\mathbb{X}}\frac{|\boldsymbol{A}(\mathcal{H}_h,\mathcal{W})|}{\|\mathcal{W}\|_{\mathbb{X}}}-\|\boldsymbol{\mathcal{K}}'(\boldsymbol{\mathcal{I}}-\boldsymbol{\mathcal{R}}_h)\|\|\mathcal{H}_h\|_{\mathbb{X}}\geq$$
$$(\boldsymbol{\vartheta}-\|\boldsymbol{\mathcal{K}}'(\boldsymbol{\mathcal{I}}-\boldsymbol{\mathcal{R}}_h)\|)\|\mathcal{H}_h\|_{\mathbb{X}}.$$

Notice that the pointwise convergence of $\boldsymbol{\mathcal{R}}_h$ to $\boldsymbol{\mathcal{I}}$ and the compactness of $\boldsymbol{\mathcal{K}}'$ provide norm-convergence for $\boldsymbol{\mathcal{K}}'(\boldsymbol{\mathcal{I}}-\boldsymbol{\mathcal{R}}_h)$ to zero. Hence, for sufficiently small h, we can ensure that

$$\sup_{\mathcal{W}_h\in\mathbb{X}_h}\frac{|\boldsymbol{A}(\mathcal{H}_h,\mathcal{W}_h)|}{\|\mathcal{W}_h\|_{\mathbb{X}}}\geq\frac{\boldsymbol{\vartheta}}{2(1+C^*)}\|\mathcal{H}_h\|_{\mathbb{X}}$$

for all $\mathcal{H}_h\in\mathbb{X}_h$.

This discrete inf-sup condition implies the first part of the Theorem whereas the rate of convergence (39) follows directly from the Céa estimate (38) and the approximation properties (34) and (35). □

7. Numerical results

In this section we present two examples illustrating the performance of the finite element scheme (33) on a set of uniform meshes of the domain. We begin by introducing some notations. The variable N stands for the number of degrees of freedom defining the finite element subspaces $\mathbf{X}_h^m$ and $[\mathcal{S}_h^1(\Omega_s)]^3$, and the individual errors are denoted by:

$$e(\boldsymbol{u}):=\|\boldsymbol{u}-\boldsymbol{u}_h\|_{[H^1(\Omega_s)]^3}\quad\text{and}\quad e(\boldsymbol{h}):=\|\boldsymbol{h}-\boldsymbol{h}_h\|_{\mathbf{H}(\mathbf{curl}\,,\Omega_m)}\,.$$

Also, we let $r(\boldsymbol{u})$ and $r(\boldsymbol{h})$ be the experimental rates of convergence given by

$$r(\boldsymbol{u}):=\frac{\log(e(\boldsymbol{u})/e'(\boldsymbol{u}))}{\log(h/h')}\quad\text{and}\quad r(\boldsymbol{h}):=\frac{\log(e(\boldsymbol{h})/e'(\boldsymbol{h}))}{\log(h/h')}\,,$$

where h and h' denote two consecutive meshsizes with corresponding errors e and e'.

We now describe the data of the examples. We consider the domains $\Omega_s:=(0.25,0.75)^3$ and $\Omega_m:=(0,1)^3\setminus[0.25,0.75]^3$, and take the solid parameters $\rho_s=\lambda=\mu^*=1$ and the electromagnetic parameters $\epsilon=\epsilon_0=\mu=\mu_0=1$. We take the frequency $\omega=3$, whence $\kappa_s=k=3$, $a(\boldsymbol{x})=1+\iota\frac{\sigma(\boldsymbol{x})}{3}$ and $b=1$. The function given by :

$$\boldsymbol{u}=\frac{1}{4\pi\mu r}\left[A\begin{pmatrix}1\\0\\0\end{pmatrix}+\frac{B}{r^2}\begin{pmatrix}(x_1-2)^2\\x_2(x_1-2)\\x_3(x_1-2)\end{pmatrix}\right]$$

where

$$A=\left[1+\frac{\iota}{x_T}-\frac{1}{x_T^2}\right]\exp(\iota x_T)-\beta^2\left[\frac{\iota}{x_L}-\frac{1}{x_L^2}\right]\exp(\iota x_L)\,,$$
$$B=\left[\frac{3}{x_T^2}-\frac{3\iota}{x_T}-1\right]\exp(\iota x_T)-\beta^2\left[\frac{3}{x_L^2}-\frac{3\iota}{x_L}-1\right]\exp(\iota x_L)\,,$$
$$r=\sqrt{(x_1-2)^2+x_2^2+x_3^2}\,,\qquad x_L=k_Lr\,,\qquad x_T=k_Tr\,,$$
$$k_L=\frac{\omega}{\sqrt{\frac{\lambda+2\mu}{\rho_s}}}\,,\qquad k_T=\frac{\omega}{\sqrt{\frac{\mu}{\rho_s}}}\qquad\text{and}\qquad\beta^2=\frac{\mu}{\lambda+2\mu}\,,$$

is the fundamental solution, centered at (2,0,0), of the elastodynamics equation in Ω_s. On the other hand, the function:

$$\boldsymbol{h} = \mathbf{curl}\left(\frac{\exp(\iota k\, r_m)}{r_m}, 0, 0\right),$$

with $r_m = \sqrt{(x_1-0.5)^2+(x_2-0.5)^2+(x_3-0.5)^2}$, solves the first equation of (9) in Ω_m. It follows that $(\boldsymbol{u},\boldsymbol{h})$ is solution of (9) with non-homogeneous transmission conditions on Σ and suitable essential boundary conditions on Γ.

We also define $\Omega_m^+ := (0,1)^3\backslash(0.125,0.875)^3$ and $\Omega_m^- := [0.125,0.875]^3\backslash[0.25,0.75]^3$. Thus, in Example 1 we take the conductivity $\sigma = 0$ in Ω_m, while in Example 2 we choose $\sigma(\boldsymbol{x})$ equal to $3(x_1-0.125)(0.875-x_1)(x_2-0.125)(0.875-x_2)(x_3-0.125)(0.875-x_3)$ in Ω_m^- and vanishing identically in Ω_m^+.

N	$e(\boldsymbol{u})$	$r(\boldsymbol{u})$	$e(\boldsymbol{h})$	$r(\boldsymbol{h})$
659	2.520E-01	–	1.160E+01	–
4243	1.224E-01	1.042	6.776E+00	0.777
13251	7.676E-02	1.150	4.735E+00	0.884
30179	5.215E-02	1.344	3.622E+00	0.931
57523	3.742E-02	1.487	2.927E+00	0.955
97779	2.803E-02	1.585	2.453E+00	0.969
153443	2.172E-02	1.653	2.110E+00	0.977
227011	1.731E-02	1.702	1.851E+00	0.982
320979	1.410E-02	1.738	1.648E+00	0.986

TABLE 1. Degrees of freedom, individual errors and rates of convergence (EXAMPLE 1)

N	$e(\boldsymbol{u})$	$r(\boldsymbol{u})$	$e(\boldsymbol{h})$	$r(\boldsymbol{h})$
4243	1.223E-01	–	6.776E+00	–
30179	5.288E-02	1.210	3.622E+00	0.903
97779	2.923E-02	1.462	2.453E+00	0.961
227011	1.887E-02	1.522	1.851E+00	0.979

TABLE 2. Degrees of freedom, individual errors and rates of convergence (EXAMPLE 2)

In Tables 1 and 2 we summarize the convergence history of Examples 1 and 2, respectively, for a sequence of uniform meshes of the computational domain $\overline{\Omega}_s\cup\overline{\Omega}_m$. We observe in each example that the order of convergence provided by Theorem 6.1 when $\delta_1=\delta_2=1$, that is $O(h)$, is attained in both unknowns. On the other hand, we find in both examples that the convergence of $e(\boldsymbol{u})$ is a bit faster than O(h), which could mean either a superconvergence phenomenon of this unknown or a special feature of these particular examples.

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Dpto. de Construcción e Ingeniería de Fabricación, Universidad de Oviedo, Gijón, Spain
E-mail: abernardo@uniovi.es and amarquez@uniovi.es

Facultad de Ciencias, Universidad de Oviedo, Calvo Sotelo s/n, Oviedo, Spain
E-mail: salim@uniovi.es
URL: http://orion.ciencias.uniovi.es/~salim/