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SUBGRID MODEL FOR THE STATIONARY INCOMPRESSIBLE NAVIER-STOKES EQUATIONS BASED ON THE HIGH ORDER POLYNOMIAL INTERPOLATION

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Abstract. In this paper, we propose a subgrid finite element method for the two-dimensional (2D) stationary incompressible Naver-Stokes equation (NSE) based on high order finite element polynomial interpolations. This method yields a subgrid eddy viscosity which does not act on the large scale flow structures. The proposed eddy viscous term consists of the fluid flow fluctuation stress. The fluctuation stress can be calculated by means of simple reduced-order polynomial projections. Assuming some regular results of NSE, we give a complete error analysis. Finally, in the part of numerical tests, the numerical computations show that the numerical results agree with some benchmark solutions and theoretical analysis very well.

Key Words. Navier-Stokes equation, subgrid method, eddy viscosity, error analysis and numerical tests.

1. Introduction

In this paper, we focus on formulating a subgrid eddy viscosity method for the stationary incompressible Navier-Stokes equation. For the subgrid method, we must admit that there exists a scale separation between large and small scales. This model can be viewed as a viscous correction for large scale fluid flows. For the laminar fluid flows, the added subgrid viscosity term should not affect the large scale structures of fluid flow fields and should tend to vanish. These kinds of subgrid methods are flexible and effective for high Reynolds number fluid flows.

It is well-known that for most problems of fluid flows, the numerical algorithms capturing all scales of fluid flows are impossible. In complex fluid flows, there often exist several scales that span from the large scales to the small Kolmogorov scales which hardly be resolved by state-of-the-art computers for most engineering problems. Especially, for the convection-dominated fluid flows, we often need to consider the dispersive effects of unresolved scales on resolved scales. The eddy viscosity models are often utilized to model and solve this kind of problems by engineers, which have been achieved many successes in engineering practice [1]. These kinds of models are firstly proposed by Boussinesq [2], developed by Taylor and Prandlt [3], and introduced a dissipative mechanism by Smagorinsky [4]. At present, these models have been further improved by various numerical methods [5, 6, 7]. In existing mathematical models, these eddy viscosity models are established by introducing the scale separation based on L^2 and elliptic projection. Recently,

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Hughes *et al* has proposed a variational multi-scale method (VMM) in which the diffusion acts only on the finest resolved scales. This VMM is very effective to model this complex multi-scale phenomena. The key problem focuses on introducing a reasonable scale separation (coarse and fine scales). Generally, there exist many different ways to define coarse and fine scales according to the VMM framework [8]. According to the idea of VMM, the subgrid methods in this paper are variational multiscale methods.

In this paper, we will implement a subgrid method to remove the dispersive effects from small scales by virtue of low-order polynomial projections. The added subgrid term does not need special treatments for implementing calculations. The added subgrid term is calculated by simple treatments of basis functions, which will be given in the section of numerical tests. And you can find an analogous treatment in [9]. But, the method in [9] is based on a projection from a fine finite element space to a coarse finite element space.

The adopted finite element pair is the P_2/P_1 pair to approach velocity-pressure fields. For low Reynolds number fluid flows, the results indicate that this method has a convergence rate of the same order as the standard Garlerkin method. By the numerical tests, it is shown that the proposed subgrid correction model can simulate the fluid flows correctly and does not act on the large scale flow structures.

The outline of the paper is organized as follows. In the next section we introduce the Navier-Stokes equations (NSE) and the corresponding function settings. In section 2, we give the NSE and the corresponding functional settings. In section 3, the subgrid viscous term is introduced into the NSE and the standard Galerkin discretization of the Navier-Stokes equations is given. In section 4, we show the results of the error estimates. Some numerical results are presented in section 5, which show the correctness and efficiency of the methods. Finally, we give some conclusions.

2. Navier-Stokes equations and functional settings

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz continuous boundary $\Gamma = \partial\Omega$. We consider the stationary Navier-Stokes equation

$$\left\{ \begin{aligned} -\nu\Delta u + \nabla p + (u\cdot\nabla)u &= f, \quad \text{in } \Omega \\ \text{div } u &= 0, \quad \text{in } \Omega \\ u &= 0, \quad \text{on } \Gamma \end{aligned} \right. \tag{1}$$

where $u = (u_1, u_2)$ represents the velocity vector, p denotes the pressure, f is the body force and $\nu > 0$ is the viscosity.

We introduce the following functional settings

$$X := H_0^1(\Omega)^2, V := \{v \in X, \text{div} v = 0\}, Y := (L^2(\Omega))^2$$
$$Q := L_0^2(\Omega) = \{q \in L^2(\Omega) : \textstyle\int_\Omega q dx = 0\}.$$

We denote by $(\cdot,\cdot)$ and $\|\cdot\|_0$ are the inner product and norm in $L^2(\Omega)$ or $L^2(\Omega)^2$. The space $H^k(\Omega)$ or $H^k(\Omega)^2$ denotes the standard Sobolev spaces with norm $\|\cdot\|_k$ and semi-norm$|\cdot|_k$. The space $H_0^1(\Omega)$ or $H_0^1(\Omega)^2$ is equipped with the following scalar product and norm

$$((u,v)) = (\nabla u, \nabla v), |u|_1 = ((u,u))^{1/2}.$$

The space $H^{-1}(\Omega)^2$ is the dual space of $H_0^1(\Omega)^2$ equipped with the norm

$$\|z\|_{-1} = \sup_{v\in H_0^1(\Omega)^2} \frac{|(z,v)|}{|v|_1}$$

For convenience, we introduce the following bilinear form $a(\cdot,\cdot)$ on $X \times X$ which is coercive in X: There exists a constant $0 < C_0 \leq 1$ such that

$$a(u,u) = ((u,u)) \geq C_0 |u|_1^2, \forall u \in X, \tag{2}$$

and $d(\cdot,\cdot)$ on $X \times Q$ defined by

$$d(v,q) = (q, \text{div } v), \forall v \in X, q \in Q,$$

A trilinear term is defined by

$$\begin{aligned} b(u;v,w) &= ((u\cdot\nabla)v,w) + \tfrac{1}{2}((\text{div } u)v,w) \\ &= \tfrac{1}{2}((u\cdot\nabla)v,w) - \tfrac{1}{2}((u\cdot\nabla)w,v), \forall u,v,w \in X, \end{aligned}$$

which is the skew-symmetric form of the convective term. It is easy to gain

$$b(u;v,w) = -b(u;w,v). \tag{3}$$

Whilst we give the following estimates of the trilinear term [10]

$$|b(u,v,w)| \leq N\|\nabla u\|_0 \|\nabla v\|_0 \|\nabla w\|_0, \forall u,v,w \in X, \tag{4}$$

where N is a positive constant depending only on the domain Ω. For a given $f \in Y$, the weak formulation of Eq.(1) reads: Find $(u,p) \in (X,Q)$ such that

$$\begin{aligned} \nu a(u,v) + b(u;u,v) - d(v,p) = (f,v), &\quad \forall v \in X. \\ d(u,q) = 0, &\quad \forall q \in Q. \end{aligned} \tag{5}$$

The inf-sup condition [11]

$$\sup_{v\in X} \frac{d(v,q)}{|v|_1} \geq \beta_1 \|q\|_0, \tag{6}$$

where β is positive constant, guarantees that there is a unique solution of (5).

In order to implement finite element analysis, the following regularity assumption is given:

Theorem 2.1. [10] *Assuming that (u,p) is a nonsingular solution of the Navier-Stokes equation (1), the solution (u,p) satisfies the following regularities*

$$u \in X \cap H^3(\Omega)^2, \ p \in Q \cap H^2(\Omega). \tag{7}$$

For low Reynolds number fluid flow, f and ν satisfy the following uniqueness condition :

$$N\|f\|_{-1} \leq C_0 \nu^2, \tag{8}$$

where N is the constant of (4) and C_0 is the constant of (2).

If $f \in H^1(\Omega)^2$ and (8) hold, then the solution (u,p) satisfies that

$$\nu\|u\|_3 + \|p\|_2 \leq c\|f\|_1, \tag{9}$$

where c is a positive constant depending on the domain Ω, which stands for different values at different occurrences.

If we restrict the domain Ω to be a convex-polygonal domain in a 2D plane, Theorem 2.1 is invalid. For current polygonal domain, we make the following regularity assumption of the solution (u,p) of Eq. (1):

Assumption 2.1. *When the domain Ω is a convex, polygonal domain in a 2-dimensional plane, we assume that (u,p) satisfies*

$$u \in X \cap H^3(\Omega)^2, \ p \in Q \cap H^2(\Omega). \tag{10}$$

If $f \in H^1(\Omega)^2$ and (8) hold, then the solution (u,p) satisfies that

$$\nu\|u\|_3 + \|p\|_2 \leq c\|f\|_1, \tag{11}$$

where c is a positive constant depending on the domain Ω*, which stands for different values at different occurrences.*

If we implement the finite element discretization for Navier-Stokes equation (1), the computational domain will become a polygonal domain. Under this case, we will use regularity results of Assumption 2.1 to carry out the finite element analysis.

3. Discretization of the Navier-Stokes equations and Subgrid model

We give a family τ_h, which is a partition of Ω into triangles or quadrilaterals, assumed to be regular in the usual sense [12]. The diameter of the cell K is denoted by h_K. The mesh parameter h describes the maximum diameter of the cells $K \in \tau_h$.

We introduce the finite-dimensional subspace X_h and Q_h,

$$X_h := \{v_h \in (C^0(\Omega))^2 \cap X : v_h|_K \in P_2(K)^2, \forall K \in \tau_h\},\quad Q_h := \{q_h \in Q : q_h|_K \in P_1(K), \forall K \in \tau_h\}, \tag{12}$$

and define the discrete analogue of the space V denoted by V_h:

$$V_h := \{v_h \in X_h : d(v_h, q_h) = 0, \forall q_h \in Q_h\} \tag{13}$$

Under Assumption 2.1, we assume that for the finite element space (X_h, Q_h), the following approximation properties hold:

$$\inf_{v_h \in X_h}\{\|u - v_h\|_0 + h\|\nabla(u - v_h)\|_0\} \le ch^3|u|_3,\quad \inf_{q_h \in M_h}\|p - q_h\|_0 \le ch^2|p|_2, \tag{14}$$

Meanwhile, the velocity-pressure pare in (X_h, Q_h) satisfies the following well-known discrete *inf*-*sup* condition

$$\inf_{q_h \in Q_h}\sup_{v_h \in X_h}\frac{d(v_h, q_h)}{\|q_h\|_0|v_h|_1} \ge \beta > 0. \tag{15}$$

Remark 3.1. [12, 13, 14] *Let* $\Pi : Q \to R_0$ *be the standard* L^2*-projection with the following properties*

$$(q, q_h) = (\Pi q, q_h), \forall q \in Q, q_h \in R_0, \tag{16}$$

$$\|\Pi q\|_0 \le c\|q\|_0, \forall q \in Q, \tag{17}$$

$$\|q - \Pi q\|_0 \le ch\|q\|_1, \forall q \in H^1(\Omega) \cap Q, \tag{18}$$

where $R_0 = \{q_h \in Q : q_h|_K \text{ is constant}, \forall K \in K_h\}$.

We know that, for high Reynolds number fluid flows, when the fluid convection dominates fluid flow fields, under the finite resolution of meshes, the flow become very instable. When the mesh scales can not resolve the smallest scale in fluid flows, we must add some term into the Navier-Stokes equations to smear out the effect from the unresolve scales. Here, we chose the following subgrid stabilization term to control the effect from the unresolve scales

$$M(u_h, v_h) = \alpha((I - \Pi)\nabla u_h, (I - \Pi)\nabla v_h), \tag{19}$$

where α is the user-selected stabilization parameter and typically, $\alpha = h^s$ (s is a real number). The analogous stabilization is used to circumvent the pressure stabilization LBB condition for Stokes problems [15]. S. Kaya and B. Rivière [9] had proposed a analogous subgrid model, but their mode was based on two-level finite element spaces (dependent on two different kinds of mesh partition). Our proposed subgrid model is only dependent on one kind of mesh partition (mixed finite element spaces are adopted, which guarantee the LBB stability condition).

Using the above stabilization term, we give the following stabilization finite element discretization form of the variational form Eq. (5): find $(u_h, q_h) \in (X_h, Q_h)$ satisfying

$$\begin{aligned} \nu a(u_h, v_h) + b(u_h; u_h, v_h) - d(v_h, p_h) + M(u_h, v_h) = (f, v_h), \quad \forall v_h \in X_h, \\ d(u_h, q_h) = 0, \quad \forall q_h \in Q_h, \end{aligned} \tag{20}$$

Under the inf-sup condition (15), the formulation (20) is equivalent to the following problem [16]: find $u_h \in V_h$ such that

$$\nu a(u_h, v_h) + b(u_h, u_h, v_h) + M(u_h, v_h) = (f, v_h), \forall v_h \in V_h. \tag{21}$$

Theorem 3.1. [16] *Suppose the uniqueness condition (8) holds. Under the condition (15), then the variational form (20) has a unique solution $(u_h, p_h) \in (X_h, Q_h)$.*

4. Error Analysis

The basic principle of subgrid method focuses on enhancing the numerical stability of solving the discrete Navier-Stokes equations. For this purpose, we will analysis the numerical scheme (20) by H^1 and L^2 estimates of velocity and L^2 estimate of pressure. The theoretical results of error analysis are classical[9]. To derive error estimates for the finite elements solution (u_h, p_h), we first give the following lemma:

Lemma 4.1. *The finite element approximation of velocity for the variational form(20) is stable*

$$\nu |u_h|_1^2 + 2\alpha \|(I - \Pi)\nabla u_h\|_0^2 \leq \frac{1}{\nu\sqrt{C_0}} \|f\|_{-1}^2 \tag{22}$$

where C_0 is the constant of (2).

Proof. The result is easily obtained by setting $v_h = u_h$, $p_h = q_h$ in the variational form (20) and Cauchy Schwarz, Young's inequalities.

Remark 4.1. *Lemma 4.1 directly implies that*

$$|u_h|_1 \leq \frac{1}{\nu\sqrt{C_0}} \|f\|_{-1}. \tag{23}$$

Theorem 4.1. *Suppose the uniqueness condition (8) holds, then we have*

$$\begin{aligned} &\nu |u - u_h|_1^2 + 2\alpha \|(I - \Pi)\nabla(u - u_h)\|_0^2 \\ &\leq C \inf_{w_h \in V_h} \{\nu |u|_1^2 + \tfrac{N^2}{\nu}(|u|_1 + |u_h|_1)^2 |u - w_h|_1^2 + \alpha \|(I - \Pi)\nabla u\|_0^2 \\ &+ \alpha \|(I - \Pi)\nabla(u - w_h)\|_0^2\} + C \inf_{q_h \in M_h} \tfrac{1}{\nu} \|p - q_h\|_0^2. \end{aligned} \tag{24}$$

where C is independent of parameters ν, α and h.

Proof. First, we know that the true solution satisfies the following equation

$$\begin{aligned} \nu a(u, v_h) + b(u, u, v_h) - d(v_h, p) - d(u, q_h) + M(u, v_h) \\ = (f, v_h) + M(u, v_h), \ \forall v_h \in X_h, q_h \in Q_h. \end{aligned} \tag{25}$$

By subtracting Eq. (20) from Eq. (25):

$$\begin{aligned} &\nu a(u - u_h, v_h) + b(u, u, v_h) - b(u_h, u_h, v_h) - d(v_h, p - p_h) \\ &- d(u - u_h, q_h) + M(u - u_h, v_h) = M(u, v_h), \forall v_h \in X_h, q_h \in Q_h, \end{aligned} \tag{26}$$

Now setting $u - u_h = \eta - \phi_h$, with $\eta = u - w_h$ and $\phi_h = u_h - w_h$, where w_h is any function in V_h. Taking $v_h = \phi_h \in V_h$ in the Eq.(26), we obtain:

$$\begin{aligned} \nu a(\phi_h, \phi_h) + M(\phi_h, \phi_h) = \nu a(\eta, \phi_h) + b(u, u, \phi_h) - b(u_h, u_h, \phi_h) \\ + M(\eta, \phi_h) - d(\phi_h, p - q_h) - M(u, \phi_h), \forall q_h \in Q_h \end{aligned} \tag{27}$$

To bound the nonlinear terms, we rewrite two trilinear terms as follows:

$$b(u,u,\phi_h) - b(u_h,u_h,\phi_h) = b(u,\eta,\phi_h) + b(\eta,u_h,\phi_h) - b(\phi_h,u_h,\phi_h). \tag{28}$$

By using Eq.(4), Young's inequality, Eq.(23) and Eq.(8), we have:

$$\begin{aligned} |b(u,u,\phi_h) - b(u_h,u_h,\phi_h)| &\le N(|u|_1 + |u_h|_1)|\eta|_1|\phi|_1 + N|u_h|_1|\phi_h|_1^2 \\ &\le \tfrac{CN^2}{\nu}(|u|_1 + |u_h|_1)^2|\eta|_1|^2 + \tfrac{1}{4}|\phi_h|_1^2. \end{aligned} \tag{29}$$

To bound the linear terms in the right-hand side of (27), we use the Cauchy Schwarz inequality and Young's inequality and get

$$|\nu a(\eta,\phi_h)| \le \nu|\eta|_1|\phi|_1 \le \nu|\eta|_1^2 + \frac{1}{4}|\phi_h|_1^2, \tag{30}$$

$$|d(\phi_h, p - q_h)| \le \nu\|p - q_h\|_0^2 + \frac{1}{4}|\phi_h|_1^2, \tag{31}$$

$$\begin{aligned} |M(\eta,\phi_h)| &\le \alpha\|(I-\Pi)\nabla\eta\|_0\|(I-\Pi)\nabla\phi_h\|_0 \\ &\le \tfrac{\alpha}{4}\|\|\|(I-\Pi)\nabla\phi_h\|_0^2 + \alpha\|(I-\Pi)\nabla\eta\|_0^2, \end{aligned} \tag{32}$$

$$|M(u,\phi_h)| \le \alpha\|(I-\Pi)\nabla u\|_0^2 + \frac{\alpha}{4}\|(I-\Pi)\nabla\phi_h\|_0^2. \tag{33}$$

Combining all the above bounds gives

$$\begin{aligned} \nu|\phi_h|_1^2 + 2\alpha\|(I-\Pi)\nabla\phi_h\|_0^2 &\le C\{\nu|\eta|_1^2 + \tfrac{N^2}{\nu}(|u|_1 + |u_h|_1)^2|\eta|_1^2 \\ &+\alpha\|(I-\Pi)\nabla\eta\|_0^2 + \alpha\|(I-\Pi)\nabla u\|_0^2 \\ &+\tfrac{1}{\nu}\|p - q_h\|_0^2\} \end{aligned} \tag{34}$$

The final result is easily obtained by using the triangle inequality

$$\begin{aligned} &\nu|u-u_h|_1^2 + 2\alpha\|(I-\Pi)\nabla(u-u_h)\|_0^2 \\ &\le C\{\nu|u-w_h|_1^2 + 2\alpha\|(I-\Pi)\nabla(u-w_h)\|_0^2 \\ &+\nu|\phi_h|_1^2 + 2\alpha\|(I-\Pi)\nabla\phi_h\|_0^2\} \end{aligned} \tag{35}$$

By choosing the proper parameters α and h, and using Eq.(14) and (3.1), we can obtain the following corollary.

Corollary 4.1. *Under the assumption of Theorem 3.1 and the regularity assumption of* $(u,p) \in (H^3(\Omega)^2 \cap X, H^2(\Omega) \cap Q)$ *in Assumption 2.1, there exists a constant* C *independent of* α *and* h *such that:*

$$\begin{aligned} &\nu|u-u_h|_1^2 + 2\alpha\|(I-\Pi)\nabla(u-u_h)\|_0^2 \\ &\le Ch^4|u|_3^2(\nu + \tfrac{1}{\nu}(1+\tfrac{1}{\nu})^2 + \alpha) + \tfrac{C}{\nu}h^4|p|_2^2 + C\alpha h^2|u|_3^2. \end{aligned} \tag{36}$$

In particular,

$$|u-u_h|_1 \le Ch^2,\ if\ \alpha = h^2 \tag{37}$$

Now, we give the estimation of the discrete pressure in the following Theorem:

Theorem 4.2. *Suppose the uniqueness condition (8) holds, then the pressure error satisfies*

$$\begin{aligned} \|p-p_h\|_0 \le\ & C((\nu+1)|u-u_h|_1 + |u-u_h|_1^2 + \alpha\|(I-\Pi)\nabla(u-u_h)\|_0 \\ &+\alpha\|(I-\Pi)\nabla u\|_0) + C\inf_{q_h\in Q_h}\|p-q_h\|_0, \end{aligned} \tag{38}$$

where C *is independent of* ν, α, h.

Proof. Setting the error of the velocity $e = u - u_h$ and introducing an approximation of the pressure $\widetilde{p_h} \in Q_h$ in Eq.(26), we have:

$$d(v_h, p_h - \widetilde{p_h}) = d(v_h, p - \widetilde{p_h}) - \nu a(e, v_h) - (b(u, u, v_h) - b(u_h, u_h, v_h)) - M(e, v_h) + M(u, v_h), \ \forall v_h \in X_h. \tag{39}$$

From Eq.(4), the nonlinear terms are bounded as:

$$\begin{aligned} |b(u, u, v_h) - b(u_h, u_h, v_h)| = &\ |b(u, e, v_h) + b(e, u, v_h) - b(e, e, v_h)| \\ &\le C(|e|_1 + |u|_1)|e|_1|v_h|_1. \end{aligned} \tag{40}$$

To bound the linear terms in the right-hand side of (39), we apply Cauchy Schwarz inequality and (3.1), the term $M(u, v_h)$ is bounded as in (32). Combining all the bounds, we have:

$$\begin{aligned} |d(v_h, p_h - \widetilde{p_h})| \le &\ C\{\|p - \widetilde{p_h}\|_0 + \nu|e|_1 + (|e|_1 + |u|_1)|e|_1 \\ &+ \alpha\|(I - \Pi)\nabla e\|_0 + \alpha\|(I - \Pi)\nabla u\|_0\}|v_h|_1. \end{aligned} \tag{41}$$

Meanwhile, the inf-sup condition (15) implies that there exists a velocity $v_h \in X_h$ such that

$$d(v_h, p_h - \widetilde{p_h}) \ge \beta|v_h|_1\|p_h - \widetilde{p_h}\|_0. \tag{42}$$

In view of (42), we get

$$\|p - p_h\|_0 \le \|p - \widetilde{p_h}\|_0 + \beta^{-1}\frac{|d(v_h, p_h - \widetilde{p_h})|}{|v_h|_1}. \tag{43}$$

By (41) and (43), we obtain the conclusion

$$\begin{aligned} \|p - p_h\|_0 \le &\ C\|p - \widetilde{p_h}\|_0 + C(\nu|e|_1 + |e|_1^2 + |e|_1|u|_1 \\ &+ \alpha\|(I - \Pi)\nabla e\|_0 + \alpha\|(I - \Pi)\nabla u\|_0). \end{aligned} \tag{44}$$

Namely

$$\begin{aligned} \|p - p_h\|_0 \le &\ C((\nu + 1)|u - u_h|_1 + |u - u_h|_1^2 + \alpha\|(I - \Pi)\nabla(u - u_h)\|_0 \\ &+ \alpha\|(I - \Pi)\nabla u\|_0) + C\inf_{q_h \in Q_h}\|p - q_h\|_0. \end{aligned} \tag{45}$$

Corollary 4.2. *From Theorem 4.1, the approximation results (14), and corollary 4.1, we have*

$$\|p - p_h\|_0 \le C(h^2 + \alpha^{\frac{1}{2}}h). \tag{46}$$

In particular,

$$\|p - p_h\|_0 \le Ch^2, \ if \ \alpha = h^2, \tag{47}$$

where C is independent of α and h.

In order to derive the estimate in L^2 space for the velocity, we consider the linearized dual problem of Navier-Stoles equations [9, 13]: Given $\xi \in L^2(\Omega)$, find (ϕ, φ) such that

$$\nu a(\phi, v) + b(u, v, \phi) + b(v, u, \phi) + M(\phi, v) - d(v, \varphi) + d(\phi, q) = (\xi, v), \quad \forall (v, q) \in (X, Q). \tag{48}$$

We assume that, for any $\xi \in L^2(\Omega)$, there exists a unique pair $(\phi, \varphi) \in (H^2(\Omega)^2 \cap X, H^1(\Omega) \cap Q)$ satisfying

$$\|\phi\|_2 + \|\varphi\|_1 \le C\|\xi\|_0. \tag{49}$$

Now we give the L^2 error estimate by the following theorem.

Theorem 4.3. *Suppose that the assumptions of Theorem 4.1 and Theorem 4.2 hold, the dual problem (48) satisfies Eq.(49). We have*

$$\|u-u_h\|_0 \le Ch(1+\alpha)|e|_1 + C|e|_1^2 + C\alpha h^2 + Ch\|p-p_h\|_0 \tag{50}$$

where C is independent of α, h.

Proof. Setting $e = u - u_h$, and subtracting (25) from (20), we can get:

$$\begin{aligned} &\nu a(e,v_h) + b(u,u,v_h) - b(u_h,u_h,v_h) - d(v_h,p-p_h) \\ &-d(e,q_h) + M(e,v_h) - M(u,v_h) = 0, \forall v_h \in X_h, \forall q_h \in Q_h. \end{aligned} \tag{51}$$

Taking $\xi = e$, $v = e$, $q = p_h - p$ in Eq.(48) and subtracting (51), we gain:

$$\begin{aligned} \|e\|_0^2 &\le |\nu a(\phi - v_h, e)| + |b(e,u,\phi) + b(u,e,\phi) - b(u,u,v_h) + b(u_h,u_h,v_h)| \\ &+|d(e,\varphi - q_h)| + |d(\phi - v_h, p - p_h)| + |M(\phi - v_h, e) + M(u,v_h)| \\ &\le C(\nu|e|_1 + \|p-p_h\|_0 + \alpha\|(I-\Pi)\nabla e\|_0)|\phi - v_h|_1 + C\|\varphi - q_h\|_0|e|_1 \\ &+\alpha\|(I-\Pi)\nabla u\|_0\|(I-\Pi)\nabla v_h\|_0 + |b(e,u,\phi) + b(u,e,\phi) - b(u,u,v_h) \\ &+b(u_h,u_h,v_h)|. \end{aligned} \tag{52}$$

Let $\widetilde{\phi}$ and $\widetilde{\varphi}$ be the best approximation of $(\phi,\varphi) \in (X_h, Q_h)$, we have the following approximation properties:

$$\begin{aligned} \|\phi - \widetilde{\phi}\|_1 \le Ch\|\phi\|_2, \\ \|\varphi - \widetilde{\varphi}\|_0 \le Ch\|\varphi\|_1. \end{aligned} \tag{53}$$

Setting $(v_h, q_h) = (\widetilde{\phi}, \widetilde{\varphi})$ and using the Cauchy-Schwarz inequality and the above approximation properties, Eq.(52) becomes:

$$\begin{aligned} \|e\|_0^2 &\le Ch(|e|_1 + \|p-p_h\|_0 + \alpha\|(I-\Pi)\nabla e\|_0)\|\phi\|_2 \\ &+Ch\|\varphi\|_1|e|_1 + \alpha\|(I-\Pi)\nabla u\|_0\|(I-\Pi)\nabla\widetilde{\phi}\|_0 \\ &+|b(u,e,\phi) + b(e,u,\phi) - b(u,u,\widetilde{\phi}) + b(u_h,u_h,\widetilde{\phi})|. \end{aligned} \tag{54}$$

Using (3.1), the consistency error term in the right-hand side of (54) gives

$$\begin{aligned} \alpha\|(I-\Pi)\nabla u\|_0\|(I-\Pi)\nabla\widetilde{\phi}\|_0 &\le C\alpha h|u|_3 h\|\nabla\widetilde{\phi}\|_1 \\ &\le C\alpha h^2|u|_3\|\widetilde{\phi}\|_2 \\ &\le C\alpha h^2|u|_2(\|\phi - \widetilde{\phi}\|_2 + \|\phi\|_2) \\ &\le C\alpha h^2|u|_3\|\phi\|_2. \end{aligned} \tag{55}$$

To bound the nonlinear terms in (54), we rewrite these terms as follows:

$$\begin{aligned} b(u,e,\phi) + b(e,u,\phi) \quad &-b(u,u,\widetilde{\phi}) + b(u_h,u_h,\widetilde{\phi}) = b(e,e,\phi) + b(u,e,\phi - \widetilde{\phi}) \\ &+b(e,u,\phi - \widetilde{\phi}) + b(e,e,\phi - \widetilde{\phi}). \end{aligned} \tag{56}$$

Using Eq.(4) and Eq.(53), we gain

$$\begin{aligned} &|b(u,e,\phi) + b(e,u,\phi) - b(u,u,\widetilde{\phi}) + b(u_h,u_h,\widetilde{\phi})| \\ &\le C|e|_1^2\|\phi\|_1 + C|u|_1|e|_1\|\phi - \widetilde{\phi}\|_1 + C|e|_1^2\|\phi - \widetilde{\phi}\|_1 \\ &\le C(|e|_1 + h)|e|_1\|\phi\|_2. \end{aligned} \tag{57}$$

Combining all the bounds and using Eq.(49) give the final result

$$\begin{aligned} \|e\|_0 &\le Ch(\nu + \alpha)|e|_1 + Ch\|p-p_h\|_0 + C\alpha h^2|u|_3 + C(|e|_1 + h)|e|_1 \\ &\le Ch(1+\alpha)|e|_1 + C|e|_1^2 + C\alpha h^2 + Ch\|p-p_h\|_0. \end{aligned} \tag{58}$$

Corollary 4.3. *The statement of Theorem 4.3, the result of Corollary 4.1 and Corollary 4.2 imply that*

$$\|u-u_h\|_0 \le C(h^3 + \alpha^{\frac{1}{2}}h^2). \tag{59}$$

In particular,

$$\|u - u_h\|_0 \leq Ch^3, \ if \ \alpha = h^2 \tag{60}$$

where C *is independent of* α, h.

5. Numerical tests

Firstly, we give the algorithm used to deal with the nonlinear term and the subgrid eddy viscosity term. For the nonlinear term the Newtonian iteration method is adopted. Given (u_h^{n-1}, p_h^{n-1}), we find (u_h^n, p_h^n) satisfying

$$\begin{aligned} &\nu a(u_h^n, v_h) - d(v_h, p_h^n) + d(u_h^n, q_h) + M(u_h^n, v_h) + \\ &b(u_h^n, u_h^{n-1}, v_h) + b(u_h^{n-1}, u_h^n, v_h) = (f, v_h) + b(u_h^{n-1}, u_h^{n-1}, v_h). \end{aligned} \tag{61}$$

$$M(u_h^n, v_h) = \alpha((\nabla u_h^n, \nabla v_h) - (\Pi \nabla u_h^{n-1}, \nabla v_h)). \tag{62}$$

In order to calculate the subgrid term $M(u_h, v_h)$, we use a simple treatment. Denoting a basis of R_0 by $\{\phi_{j0}^h\}_{j=1}^N$, and a basis of X_h by $\{\phi_j^h\}_{j=1}^N$. So, we have

$$\Pi \nabla u_h^{n-1} = \sum_{j=1}^N \beta_j^{n-1} \phi_{j0}^h. \tag{63}$$

The coefficients $\beta = (\beta_j^{n-1})_j$ can be calculated by the definition of the projection operator as follows

$$S\beta = (\nabla u_n^{n-1}, \phi_{j0}^h)_{1 \leq j \leq N}, \tag{64}$$

where the matrix S is the mass matrix, which has the form $S_{ij} = (\phi_{i0}^h, \phi_{j0}^h)$. Moreover, u_h^{n-1} can be denoted by

$$u_h^{n-1} = \sum_{j=1}^N \gamma_j^{n-1} \phi_j^h. \tag{65}$$

Then, we have

$$\beta = S^{-1} R^T \gamma, \tag{66}$$

where $\gamma = (\gamma_j^{n-1})_{1 \leq j \leq N}$ and $R_{ij} = (\phi_{j0}^h, \nabla(\phi_j^h))$. Finally, we have

$$(\Pi \nabla u_h^{n-1}, \nabla v_h) = R\beta = RS^{-1} R^T \gamma. \tag{67}$$

Because R_0 consists of piecewise constants, the matrix S is block diagonal and this computation can be implemented on each element. The analogous algorithm can be found in [9].

5.1. Example of a exact solution. It is essential to investigate the subgrid model (19) for low viscosity fluid flow and validate the flexibility and convergence rates of this model. So, we need to choose a true solution. We consider the equation (1) on the domain $\Omega = [0, 1] \times [0, 1]$, with a body force obtained such that the following true solution is given by $u = (u_1, u_2)$,

$$\left\{ \begin{array}{l} u = 2x^2(x-1)^2 y(y-1)(2y-1), \\ v = -y^2(y-1)^2 2x(x-1)(2x-1), \\ p = y^2 - x^2. \end{array} \right. \tag{68}$$

The viscosity $\nu = 0.01$ and the corresponding Reynolds number $Re = 100$. We choose $\alpha = h^2$. The mesh scales we choose are $h \in \{1/20, 1/30, 1/40, 1/50\}$. The iterative tolerance is 10^{-6}. In Fig. 1, we show the convergence orders by log-log plots. From the figure, it is shown that the convergence orders are up to 2.985 and

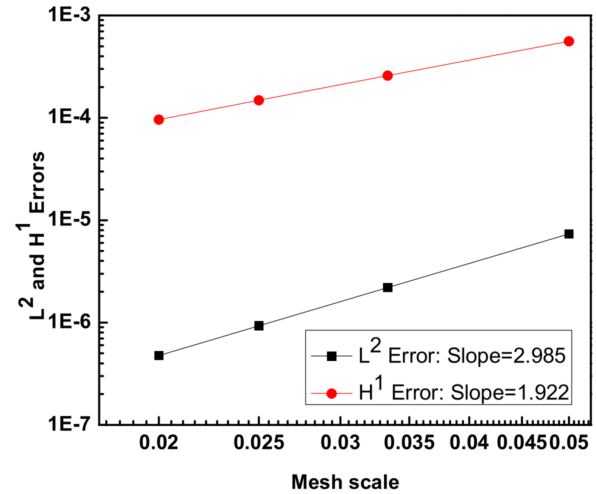

FIGURE 1. L^2 and H^1 convergence order by a log-log plot.

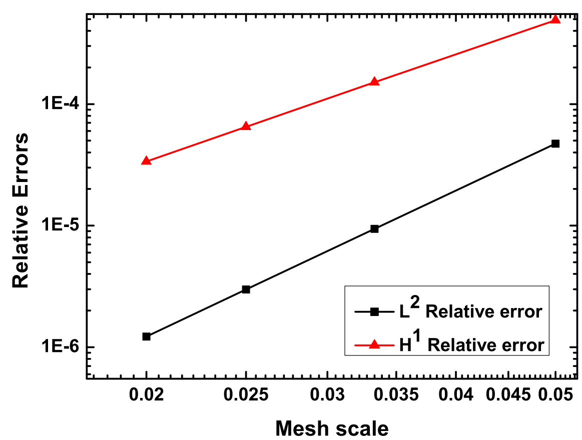

FIGURE 2. L^2 and H^1 relative errors by a log-log plot.

1.922 for L^2 and H^1 norms, respectively. This results coincide with the results of theoretic analysis. In Fig. 2, the relative errors of u and v are given. In Figs. 3-4, the number results and the exact solutions are shown. From the figures, it is obvious that the simulation results by the subgrid method exhibit that the numerical results agree with the exact solution very well. According to all of these, we know that the subgrid term for low Re fluid flows does not act on the large scale structures.

5.2. Lid-driven cavity fluid flows. In this section, we will use the model proposed in this paper to simulate the benchmark lid-driven fluid flows of high Re fluid flows and try out the correctness of the model. The computational domain $\Omega = [0, 1] \times [0, 1]$ and the top boundary velocity $(u, v) = (1, 0)$, and the other three boundaries possess non-slip boundary conditions. The iterative tolerance is 10^{-6}. The Reynolds numbers $Re = LU/\nu = 100$, 400 and 1000. And the corresponding

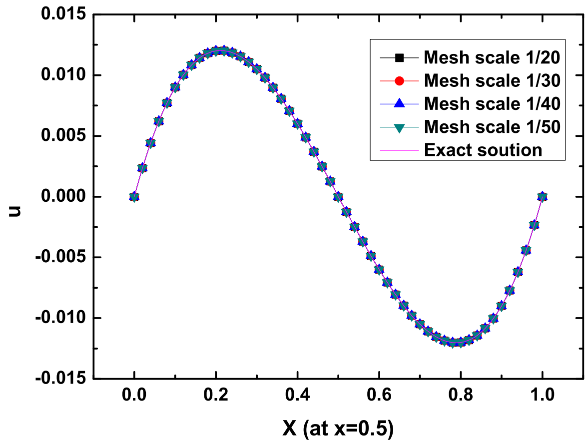

FIGURE 3. The exact solution and the numerical solutions of u.

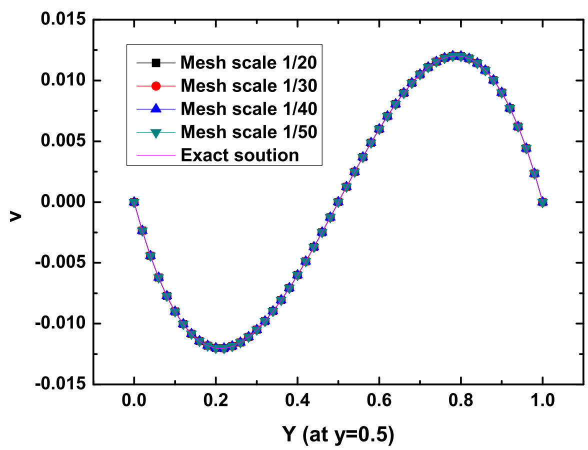

FIGURE 4. The exact solution and the numerical solutions of v.

mesh scale is set as $h = 1/32$ for $Re = 100$ and 400 , and $h = 1/40$ for $Re = 1000$. The comparisons of numerical solutions and Ghia benchmark solutions [17] are shown in Figs. 5-10. According to these comparisons, the numerical results by the current subgrid model coincide with the Ghia's results [17] very well.

6. Conclusion

In this paper, we proposed a subgrid model by high-order polynomial interpolations. The theoretical analysis guarantees that the subgrid term does not act on the large scale structures of fluid flows. The stability and error estimates are established in this paper. Meanwhile, the numerical tests are implemented. The proposed subgrid model is simple and easy to be implemented. According to computational

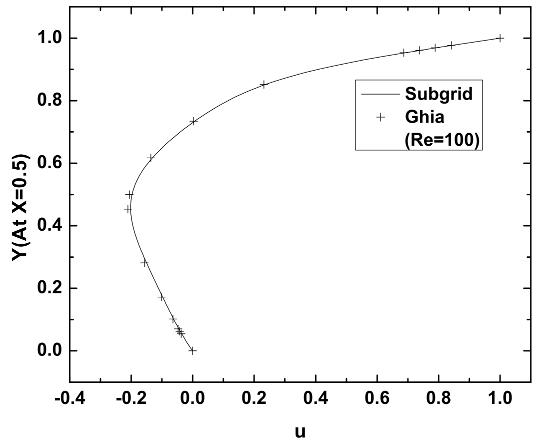

FIGURE 5. The numerical solutions and the benchmark solution of u for $Re = 100$.

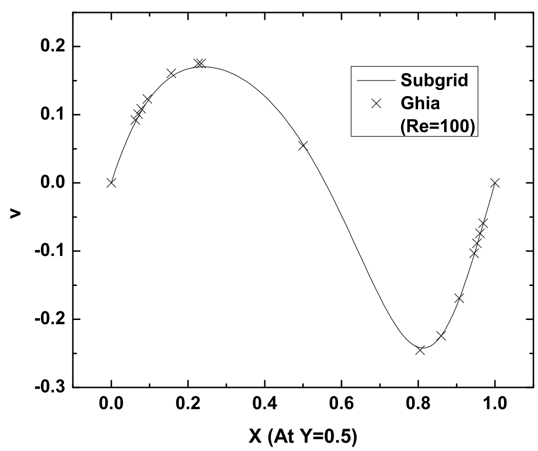

FIGURE 6. The numerical solutions and the benchmark solution of v for $Re = 100$.

results, the subgrid method provides some creditability to engineering applications. The current numerical tests are based on P_2/P_1 polynomial interpolations. We must point out that this method is very easy to extent to higher interpolation finite element spaces. In future, this subgrid method will be attempted to implement some simulations for 2D turbulence fluid flows and 3D high Reynolds fluid flows.

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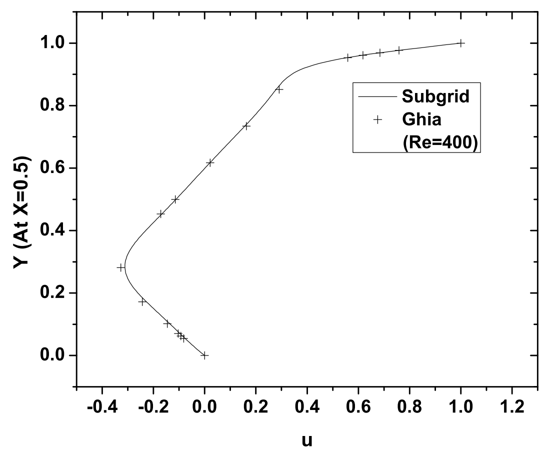

FIGURE 7. The numerical solutions and the benchmark solution of u for $Re = 400$.

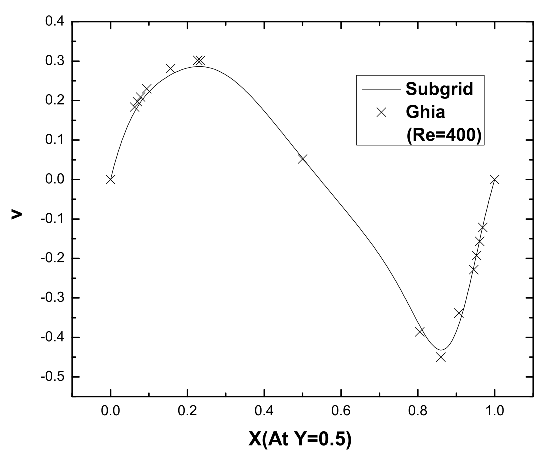

FIGURE 8. The numerical solutions and the benchmark solution of v for $Re = 400$.

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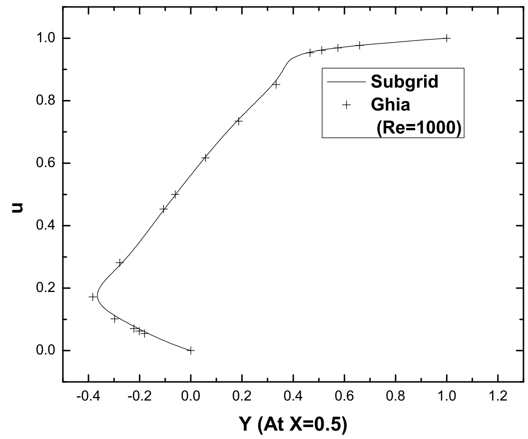

FIGURE 9. The numerical solutions and the benchmark solution of u for $Re = 1000$.

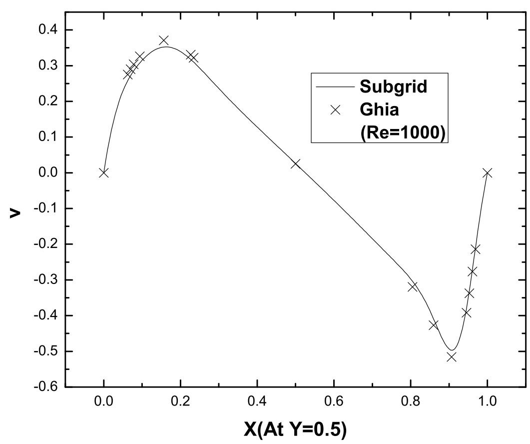

FIGURE 10. The numerical solutions and the benchmark solution of v for $Re = 1000$.

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