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THE LOCAL DISCONTINUOUS GALERKIN METHOD FOR OPTIMAL CONTROL PROBLEM GOVERNED BY CONVECTION DIFFUSION EQUATIONS

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Abstract. In this paper we analyze the Local Discontinuous Galerkin (LDG) method for the constrained optimal control problem governed by the unsteady convection diffusion equations. A priori error estimates are obtained for both the state, the adjoint state and the control. For the discretization of the control we discuss two different approaches which have been used for elliptic optimal control problem.

Key Words. Local Discontinuous Galerkin method, unsteady convection diffusion equations, constrained optimal control problem, a priori error estimate.

1. Introduction

In this paper, we consider the following linear-quadratic optimal control problems for state variable y and the control variable u involving pointwise control constraints:

$$\min_{u\in K\subset X}\left\{\frac{1}{2}\int_0^T\int_\Omega (y(x,t)-y_d(x,t))^2dxdt+\frac{\alpha}{2}\int_0^T\int_{\Omega_U}u(x,t)^2dxdt\right\} \tag{1}$$

subject to

$$\begin{aligned} y_t+\nabla\cdot(\vec{\beta}y-\varepsilon\nabla y)&=f+\mathcal{B}u, \quad x\in\Omega,\quad t\in(0,T],\\ (\vec{\beta}y-\varepsilon\nabla y)\cdot\vec{n}&=\tilde{y} \qquad \text{on} \quad \partial\Omega_I,\\ \varepsilon\nabla y\cdot\vec{n}&=0 \qquad \text{on} \quad \partial\Omega_O,\\ y(x,0)&=y_0(x), \qquad x\in\Omega. \end{aligned} \tag{2}$$

Here Ω and Ω_U are bounded open sets in R^2 with boundaries $\partial\Omega$ and $\partial\Omega_U$; $K\subset X$ is bounded convex set. The details will be specified in the next section.

Although the a priori error estimates for finite element discretization of optimal control problem governed by elliptic equations and parabolic equations have been discussed in many publications, see, e.g., [1], [7], [13], [16], there are very few results on the a priori error estimates of optimal control problem governed by convection diffusion equations. Some related work can be find in, e.g., [2], [3], [5], [18].

In the optimal control problem (1)-(2), the state equation is a convection diffusion equation. It is well known that the standard finite element discretizations applied to the convection diffusion problem (2) lead to strong oscillation when ε is small. There are some effective discretization schemes which are introduced

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to improve the approximation properties of standard Galerkin method and to reduce the oscillatory behavior, see, e.g., [4], [11], [12]. Recently, a new discretization scheme was proposed in [6] for the convection diffusion equation, which is called Local Discontinuous Galerkin method. The analysis of Local Discontinuous Galerkin method has been extended to many equations, such as, elliptic equation, nonlinear convection diffusion equation, oseen equations and stokes equations .

In this paper, we use the Local Discontinuous Galerkin method to approximate the state equation in the optimal control problem (1)-(2). For the control discretization we discussed two different methods. The first is the classic finite element discretization. The control variable is discretized by piecewise constant and piecewise linear finite element spaces, respectively. The second is a variational approach proposed in [10], where no explicit discretization of the control variable is used and the discrete control variable is achieved by projecting the discrete adjoint state variable on the admissible control set. For above LDG scheme, a priori error estimates of the semi-discrete and fully-discrete approximation schemes for the state, the adjoint state and the control are derived. To our best knowledge, the similar results has not yet been reported in the open literature.

This paper is organized as follows: In Section 2, we introduce the model problem for the optimal control problem governed by the unsteady convection diffusion equations and present the LDG approximation scheme of the model problem. In Section 3, we prove a priori error estimate of the semi-discretization scheme for the optimal control problem. In Section 4, a priori error estimate of the full discretization scheme for the optimal control problem is derived. In the last section, we briefly summarize the method used, the results obtained and possible future extensions and challenges.

2. LDG scheme for the optimal control problem

Let us introduce some standard notations. We adopt the notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω, with a norm $\|\cdot\|_{m,q,\Omega}$ and a semi-norm $|\cdot|_{m,q,\Omega}$. For q=2, we denote $H^m(\Omega)=W^{m,2}(\Omega)$ and $\|\cdot\|_m=\|\cdot\|_{m,2}$. Furthermore, we set $W_0^{1,q}(\Omega)=\{v\in W^{1,q}(\Omega):\gamma v\mid_{\partial\Omega}=0\}$, where γv is the trace of v on the boundary $\partial\Omega$. The inner products in $L^2(\Omega_U)$ and $L^2(\Omega)$ are indicated by $(\cdot,\cdot)_U$ and $(\cdot,\cdot)$, respectively. For $p\in[1,\infty)$, the internal $[0,T]\subset R$ and the Banach space A with norm $\|\cdot\|_A$, we denote by $L^p(0,T;A)$ the set of measurable functions $y:[0,T]\to A$ such that $\int_0^T\|y\|_A^p\,dt\leq\infty$. The norm on $L^p(0,T;A)$ is defined by

$$\| y(t)\|_{L^p(0,T;A)}=\begin{cases}(\int_0^T\| y(t)\|_A^p\,dt)^{\frac{1}{p}}, & 1\leq p<\infty,\\ ess\sup\limits_{t\in[0,T]}\| y(t)\|_A, & p=\infty.\end{cases}$$

In addition c and C denote generic constants.

In this section we provide a numerical scheme to approximate the distributed convex optimal control problem governed by evolutionary convection diffusion equations. We shall take the control space $X=L^2(0,T;U)$ with $U=L^2(\Omega_U)$ to fix the idea.

Consider the following constrained optimal control problem governed by evolutionary convection diffusion equations:

$$\min_{u\in K\subset X}\left\{\frac{1}{2}\int_0^T\int_\Omega(y(x,t)-y_d(x,t))^2dxdt+\frac{\alpha}{2}\int_0^T\int_{\Omega_U}u(x,t)^2dxdt\right\}\tag{3}$$

subject to

$$
\begin{aligned}
& y_t + \nabla \cdot (\vec{\beta} y - \varepsilon \nabla y) = f + \mathcal{B}u, && x \in \Omega, t \in (0,T], \\
& (\vec{\beta} y - \varepsilon \nabla y) \cdot \vec{n} = \tilde{y}, && \text{on} \quad \partial\Omega_I, \\
& \nabla y \cdot \vec{n} = 0, && \text{on} \quad \partial\Omega_O, \\
& y(x,0) = y_0(x), && x \in \Omega.
\end{aligned}
\tag{4}
$$

Here the bounded open set $\Omega \subset R^2$ is convex polygon with piecewise smooth boundary $\partial\Omega$, $\Omega_U \subset R^2$ is a bounded domain with Lipschitz boundary $\partial\Omega_U$; $\mathcal{B}$ is a bounded linear operator from X to $L^2(0,T;Y')$; $\alpha > 0$ is positive constant. In this paper, we set

$$K = \{v \in X : \ v \geq 0 \ \text{ a.e. in } \ \Omega_U \times [0,T]\}.$$

For the data of the above equations we assume:

(i) f, $\tilde{y}$ are given functions, and $\varepsilon > 0$ is a constant.

(ii) $\vec{\beta}$ denotes a velocity field. We assume that it belongs to $(W^{1,\infty}(\Omega))^2$ and satisfies the incompressible condition, i.e., $\nabla \cdot \vec{\beta} = 0$.

(iii) For boundary conditions, let $\vec{n}$ denote the unit outward normal to $\partial\Omega$. We write

$$\partial\Omega_I = \{x \in \partial\Omega : \vec{\beta} \cdot \vec{n} < 0\},$$

and

$$\partial\Omega_O = \{x \in \partial\Omega : \vec{\beta} \cdot \vec{n} \geq 0\}.$$

In order to define the Local Discontinuous Galerkin approximation scheme for the optimal control problem (3)-(4), we introduce a new variable vector:

$$\vec{q} = -\varepsilon^{\frac{1}{2}} \nabla y.$$

Then the optimal control problem (3)-(4) can be rewritten to

$$\min_{u \in K \subset X} \left\{ \frac{1}{2} \int_0^T \int_\Omega (y(x,t) - y_d(x,t))^2 dxdt + \frac{\alpha}{2} \int_0^T \int_{\Omega_U} u(x,t)^2 dxdt \right\} \tag{5}$$

subject to

$$
\begin{aligned}
& y_t + \nabla \cdot (\vec{\beta} y + \varepsilon^{\frac{1}{2}} \vec{q}) = f + \mathcal{B}u, && x \in \Omega, \ \ t \in (0,T], \\
& \vec{q} = -\varepsilon^{\frac{1}{2}} \nabla y, && x \in \Omega, \ \ t \in (0,T], \\
& (\vec{\beta} y + \varepsilon^{\frac{1}{2}} \vec{q}) \cdot \vec{n} = \tilde{y}, && \text{on} \quad \partial\Omega_I, \\
& \vec{q} \cdot \vec{n} = 0, && \text{on} \quad \partial\Omega_O, \\
& y(x,0) = y_0(x), && x \in \Omega.
\end{aligned}
\tag{6}
$$

To obtain the weak formulation for the state equation, we simply multiply the above equations by smooth test functions $w, \vec{v}$ and integrate on Ω. Then we have

$$
\begin{aligned}
& (y_t, w) - (\vec{\beta} y + \varepsilon^{\frac{1}{2}} \vec{q}, \nabla w) + \langle y\vec{n} \cdot \vec{\beta}, w \rangle_{\partial\Omega_O} = (f + \mathcal{B}u, w) - \langle \tilde{y}, w \rangle_{\partial\Omega_I}, \\
& (\vec{q}, \vec{v}) - (y, \nabla \cdot (\varepsilon^{\frac{1}{2}} \vec{v})) + \langle y, \varepsilon^{\frac{1}{2}} \vec{v} \cdot \vec{n} \rangle_{\partial\Omega} = 0,
\end{aligned}
$$

where

$$< w, v >_L = \int_L wvds$$

describes the integral on part of the boundary or edge of the element. Thus the weak formulation of the optimal control problem (5)-(6) can be expressed as follows

$$\min_{u \in K \subset X} \left\{ \frac{1}{2} \int_0^T \int_\Omega (y(x,t) - y_d(x,t))^2 dxdt + \frac{\alpha}{2} \int_0^T \int_{\Omega_U} u(x,t)^2 \right\} dxdt \tag{7}$$

subject to: $\forall\ (w,\vec{v})\in H^1(\Omega)\times(H^1(\Omega))^2$,

$$(y_t,w)-(\vec{\beta}y+\varepsilon^{\frac12}\vec{q},\nabla w)+\langle y\vec{n}\cdot\vec{\beta},w\rangle_{\partial\Omega_O}=(f+\mathcal{B}u,w)-\langle\tilde{y},w\rangle_{\partial\Omega_I},\tag{8}$$

$$(\vec{q},\vec{v})-(y,\nabla\cdot(\varepsilon^{\frac12}\vec{v}))+\langle y,\varepsilon^{\frac12}\vec{v}\cdot\vec{n}\rangle_{\partial\Omega}=0,\tag{9}$$

$$y(x,0)=y_0(x),\quad x\in\Omega.\tag{10}$$

It can be derived by the standard technique (see, e.g., [9] and [15]) that the control problem (7)-(10) has a unique solution $(y,\vec{q},u)$, and that a triple $(y,\vec{q},u)$ is the solution of (7)-(10) if and only if there is adjoint state $(z,\vec{p})$, such that $(y,\vec{q},z,\vec{p},u)$ satisfies the following optimality conditions: for $\forall\ (w,\vec{v})\in H^1(\Omega)\times(H^1(\Omega))^2$, $\forall\ (\phi,\vec{\psi})\in H^1(\Omega)\times(H^1(\Omega))^2$ and $\forall\ \tilde{v}\in K\subset U$,

$$(y_t,w)-(\vec{\beta}y+\varepsilon^{\frac12}\vec{q},\nabla w)+\langle y\vec{n}\cdot\vec{\beta},w\rangle_{\partial\Omega_O} = (f+\mathcal{B}u,w)-\langle\tilde{y},w\rangle_{\partial\Omega_I},\tag{11}$$

$$(\vec{q},\vec{v})-(y,\nabla\cdot(\varepsilon^{\frac12}\vec{v}))+\langle y,\varepsilon^{\frac12}\vec{v}\cdot\vec{n}\rangle_{\partial\Omega} = 0,\tag{12}$$

$$-(z_t,\phi)+(\vec{\beta}z+\varepsilon^{\frac12}\vec{p},\nabla\phi)-\langle z\vec{n}\cdot\vec{\beta},\phi\rangle_{\partial\Omega_I} = (y-y_d,\phi),\tag{13}$$

$$(\vec{p},\vec{\psi})+(z,\nabla\cdot(\varepsilon^{\frac12}\vec{\psi}))-\langle z,\varepsilon^{\frac12}\vec{\psi}\cdot\vec{n}\rangle_{\partial\Omega} = 0,\tag{14}$$

$$\int_0^T(\alpha u+\mathcal{B}^*z,\tilde{v}-u)_U dt \geq 0,\tag{15}$$

$$y(x,0)=y_0(x),\quad z(x,T)=0,\qquad x\in\Omega.\tag{16}$$

Here $\mathcal{B}^*$ is the adjoint operator of $\mathcal{B}$.

To describe the Local Discontinuous Galerkin procedure, we need introduce the finite element mesh partition on the domain Ω. Let T^h be the regular triangulation of Ω, so that $\bar{\Omega}=\cup_{e\in T^h}\bar{e}$. Let $h=\max\limits_{e\in T^h}h_e$, where h_e denotes the diameter of the element e. Moreover, let E_h^i and E_h^∂ denote the sets of internal and external edges, respectively.

For any function $w\in H^1(e)$, $e\in T^h$, let l denote an edge in the mesh, and $\vec{n}_l$ a unit vector normal to the edge l, with $\vec{n}_l=\vec{n}$ on $\partial\Omega$. Set

$$w^+(x)=\lim_{t\to0^+}w(x+t\vec{n}_l),$$
$$w^-(x)=\lim_{t\to0^-}w(x+t\vec{n}_l).$$

Then we define

$$[w] = w^- - w^+,$$
$$\{w\} = (w^+ + w^-)/2.$$

Therefore for any function $w\in H^1(e)$, $\vec{v}\in(H^1(e))^2$, we obtain the following formulations by multiplying the equations (6) by test functions $w,\vec{v}$ and integrate on every element e:

$$\begin{aligned}(y_t,w)_e &- (\vec{\beta}y+\varepsilon^{\frac12}\vec{q},\nabla w)_e+\langle(\vec{\beta}y+\varepsilon^{\frac12}\vec{q})\cdot\vec{n}_e,w\rangle_{\partial e\setminus\partial\Omega}\\ &+ \langle y\vec{\beta}\cdot\vec{n}_e,w\rangle_{\partial e\cap\partial\Omega_O}=(f+\mathcal{B}u,w)_e-\langle\tilde{y},w\rangle_{\partial e\cap\partial\Omega_I},\end{aligned}\tag{17}$$

$$(\vec{q},\vec{v})_e - (y,\nabla\cdot(\varepsilon^{\frac12}\vec{v}))_e+\langle y,\varepsilon^{\frac12}\vec{v}\cdot\vec{n}_e\rangle_{\partial e}=0.\tag{18}$$

Let $W_{h,e}\subset H^1(e)$ denote the set of all polynomials of degree at most r on e, and $V^h=\{v\in L^2(\Omega),v|_e\in W_{h,e}\}$. The Local Discontinuous Galerkin approximation scheme for the state equation can be obtained by simply discretizing the above systems by discontinuous Galerkin method. We approximate y by $y_h\in V^h$, and $\vec{q}$ by $\vec{q}_h\in(V^h)^2$. Then we have terms involving y and $\vec{q}$ on ∂e. Since y_h and $\vec{q}_h$ are discontinuous across these edges, we must provide the definition for approximating these terms. According to [8], we approximate the value of y_h in (17) by the upwind value defined as follows:

$$\hat{y}_h = \begin{cases} y_h^-, & \vec{n}_e \cdot \vec{\beta} > 0, \\ y_h^+, & \vec{n}_e \cdot \vec{\beta} \le 0. \end{cases}$$

The value of $\vec{q}_h$ on $\partial e \backslash \partial\Omega$ is approximated by $\{\vec{q}_h\}$. The approximation of the value of y_h on $\partial e \setminus \partial\Omega$ in (18) is chosen as $\{y_h\}$. Finally, the value of y_h on $l \cap \partial\Omega_O \subset \partial e$ in (17) and on $l \cap \partial\Omega \subset \partial e$ in (18) is simply approximated by y_h^-. Incorporating these edge approximations and summing (17)-(18) over all elements, we can derive that

$$\begin{aligned} (y_{ht}, w_h) \quad &- \quad \sum_e (\vec{\beta} y_h + \varepsilon^{\frac{1}{2}} \vec{q}_h, \nabla w_h)_e + \sum_{l \in E_h^i} \langle (\vec{\beta} \hat{y}_h + \varepsilon^{\frac{1}{2}} \{\vec{q}_h\}) \cdot \vec{n}_l, [w_h] \rangle_l \\ &+ \quad \sum_{l \in E_h^\partial} \langle \vec{\beta} \cdot \vec{n}_l y_h^-, w_h \rangle_{l \cap \partial\Omega_O} = (f + \mathcal{B} u_h, w_h) - \langle \tilde{y}, w_h \rangle_{\partial\Omega_I}, \\ (\vec{q}_h, \vec{v}_h) \quad &- \quad \sum_e (y_h, \nabla \cdot (\varepsilon^{\frac{1}{2}} \vec{v}_h))_e + \sum_{l \in E_h^i} \langle \{y_h\}, \varepsilon^{\frac{1}{2}} [\vec{v}_h] \cdot \vec{n}_l \rangle_l \\ &+ \quad \sum_{l \in E_h^\partial} \langle y_h^-, \varepsilon^{\frac{1}{2}} \vec{v}_h \cdot \vec{n}_l \rangle_{l \cap \partial\Omega} = 0. \end{aligned}$$

Next, let us consider the discretization of the control variable. Let T_U^h be another regular triangulation of Ω_U, so that $\bar{\Omega}_U = \cup_{e_U \in T_U^h} \bar{e}_U$. Let $h_U = \max_{e_U \in T_U^h} h_{e_U}$, where h_{e_U} denotes the diameter of the element e_U. In this paper, we consider the piecewise constant finite element space:

$$U^h = \{u_h \in U,\ u_h \mid_{e_U} = \text{constant},\ \forall e_U \in T_U^h\},$$

or the piecewise linear finite element space:

$$U^h = \{u_h \in U,\ u_h \mid_{e_U} \in P_1(e_U),\ \forall e_U \in T_U^h\}.$$

Set $K^h = U^h \cap K$. It is easy to see that $K^h \subset K$.

Then the semidiscrete Local Discontinuous Galerkin approximation scheme for optimal control problem (5)-(6) can be written as follows: for $\forall\ (w_h, \vec{v}_h) \in V^h \times (V^h)^2$,

$$\min_{u_h \in K^h \subset X} \left\{ \frac{1}{2} \int_0^T \int_\Omega (y_h(x,t) - y_d(x,t))^2 dxdt + \frac{\alpha}{2} \int_0^T \int_{\Omega_U} u_h(x,t)^2 dxdt \right\} \tag{19}$$

subject to

$$\begin{aligned} (y_{ht}, w_h) \quad &- \quad \sum_e (\vec{\beta} y_h + \varepsilon^{\frac{1}{2}} \vec{q}_h, \nabla w_h)_e + \sum_{l \in E_h^i} \langle (\vec{\beta} \hat{y}_h + \varepsilon^{\frac{1}{2}} \{\vec{q}_h\}) \cdot \vec{n}_l, [w_h] \rangle_l \\ &+ \quad \sum_{l \in E_h^\partial} \langle \vec{\beta} \cdot \vec{n}_l y_h^-, w_h \rangle_{l \cap \partial\Omega_O} = (f + \mathcal{B} u_h, w_h) - \langle \tilde{y}, w_h \rangle_{\partial\Omega_I}, \qquad (20) \\ (\vec{q}_h, \vec{v}_h) \quad &- \quad \sum_e (y_h, \nabla \cdot (\varepsilon^{\frac{1}{2}} \vec{v}_h))_e + \sum_{l \in E_h^i} \langle \{y_h\}, \varepsilon^{\frac{1}{2}} [\vec{v}_h] \cdot \vec{n}_l \rangle_l \\ &+ \quad \sum_{l \in E_h^\partial} \langle y_h^-, \varepsilon^{\frac{1}{2}} \vec{v}_h \cdot \vec{n}_l \rangle_{l \cap \partial\Omega} = 0, \qquad (21) \\ y_h(x,0) \quad &= \quad y_0^h(x), \qquad (22) \end{aligned}$$

where $y_0^h \in V^h$ is the approximation of y_0.

Again, it can be shown that the control problem (19)-(22) has a unique solution $(y_h, \vec{q}_h, u_h)$, and that a triple $(y_h, \vec{q}_h, u_h)$ is the solution of (19)-(22) if and only if there is adjoint state $(z_h, \vec{p}_h)$, such that $(y_h, \vec{q}_h, z_h, \vec{p}_h, u_h)$ satisfies the following optimality conditions: for $\forall\ (w_h, \vec{v}_h) \in V^h \times (V^h)^2$, $\forall\ (\phi_h, \vec{\psi}_h) \in V^h \times (V^h)^2$ and $\forall\ \tilde{v}_h \in K^h$,

$$\begin{aligned}(y_{ht}, w_h) \quad &- \quad \sum_e (\vec{\beta} y_h + \varepsilon^{\frac{1}{2}} \vec{q}_h, \nabla w_h)_e + \sum_{l\in E_h^i} \langle (\vec{\beta} \hat{y}_h + \varepsilon^{\frac{1}{2}} \{\vec{q_h}\}) \cdot \vec{n}_l, [w_h] \rangle_l \\ &+ \quad \sum_{l\in E_h^\partial} \langle \vec{\beta} \cdot \vec{n}_l y_h^-, w_h \rangle_{l\cap\partial\Omega_O} = (f + \mathcal{B} u_h, w_h) - \langle \tilde{y}, w_h \rangle_{\partial\Omega_I},\end{aligned} \tag{23}$$

$$\begin{aligned}(\vec{q}_h, \vec{v}_h) \quad &- \quad \sum_e (y_h, \nabla \cdot (\varepsilon^{\frac{1}{2}} \vec{v}_h))_e + \sum_{l\in E_h^i} \langle \{y_h\}, \varepsilon^{\frac{1}{2}} [\vec{v}_h] \cdot \vec{n}_l \rangle_l \\ &+ \quad \sum_{l\in E_h^\partial} \langle y_h^-, \varepsilon^{\frac{1}{2}} \vec{v}_h \cdot \vec{n}_l \rangle_{l\cap\partial\Omega} = 0,\end{aligned} \tag{24}$$

$$\begin{aligned}-(z_{ht}, \phi_h) \quad &+ \quad \sum_e (\vec{\beta} z_h + \varepsilon^{\frac{1}{2}} \vec{p}_h, \nabla \phi_h)_e - \sum_{l\in E_h^i} \langle (\vec{\beta} \breve{z}_h + \varepsilon^{\frac{1}{2}} \{\vec{p_h}\}) \cdot \vec{n}_l, [\phi_h] \rangle_l \\ &- \quad \sum_{l\in E_h^\partial} \langle \vec{\beta} \cdot \vec{n}_l z_h^-, \phi_h \rangle_{l\cap\partial\Omega_I} = (y_h - y_d, \phi_h),\end{aligned} \tag{25}$$

$$\begin{aligned}(\vec{p}_h, \vec{\psi}_h) \quad &+ \quad \sum_e (z_h, \nabla \cdot (\varepsilon^{\frac{1}{2}} \vec{\psi}_h))_e - \sum_{l\in E_h^i} \langle \{z_h\}, \varepsilon^{\frac{1}{2}} [\vec{\psi}_h] \cdot \vec{n}_l \rangle_l \\ &- \quad \sum_{l\in E_h^\partial} \langle z_h^-, \varepsilon^{\frac{1}{2}} \vec{\psi}_h \cdot \vec{n}_l \rangle_{l\cap\partial\Omega} = 0,\end{aligned} \tag{26}$$

$$\int_0^T (\alpha u_h + \mathcal{B}^* z_h, \tilde{v}_h - u_h)_U dt \geq 0, \tag{27}$$

$$y_h(x, 0) = y_0^h(x), \quad z_h(x, T) = 0, \tag{28}$$

where $y_0^h \in V^h$ is the approximation of y_0, and

$$\breve{z}_h = \begin{cases} z_h^+, & \vec{n}_e \cdot \vec{\beta} > 0, \\ z_h^-, & \vec{n}_e \cdot \vec{\beta} \leq 0. \end{cases}$$

Next, let us consider the full discretization scheme of the Local Discontinuous Galerkin approximation for above optimal control problem by using the backward Euler scheme in time. Let $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$, $k_i = t_i - t_{i-1}, i = 1, 2, \cdots N$, $k = \max_{i\in[1,N]} k_i$. For $i = 1, 2, \cdots, N$, constructing finite element spaces V_i^h with the mesh T_i^h. Similarly, we construct the finite element spaces U_i^h with the mesh $(T_U^h)_i$. Let $K_i^h \subset U_i^h \cap K$. Then the full discretization approximation scheme for the optima control problem (5)-(6) is to find $(y_h^i, u_h^i) \in V_i^h \times K_i^h$ such that for $\forall\ (w_h, \vec{v}_h) \in V_i^h \times (V_i^h)^2$,

$$\min_{u_h^i \in K_i^h} \{\frac{1}{2} \sum_{i=1}^N k_i (\| y_h^i - y_d^i \|_{0,\Omega}^2 + \alpha \| u_h^i \|_{0,\Omega_U}^2)\} \tag{29}$$

subject to

$$(\frac{y_h^i - y_h^{i-1}}{k_i}, w_h) - \sum_e (\vec{\beta} y_h^i + \varepsilon^{\frac{1}{2}} \vec{q}_h^i, \nabla w_h)_e + \sum_{l\in E_h^i} \langle (\vec{\beta} \hat{y}_h^i + \varepsilon^{\frac{1}{2}} \{\vec{q}_h^i\}) \cdot \vec{n}_l, [w_h] \rangle_l$$

$$+\sum_{l\in E_h^\partial}\langle\vec{\beta}\cdot\vec{n}_l {y_h^i}^-, w_h\rangle_{l\cap\partial\Omega_O} = (f^i+\mathcal{B}u_h^i, w_h) - \langle\tilde{y}^i, w_h\rangle_{\partial\Omega_I},\ i=1,2,...,N, \tag{30}$$

$$(\vec{q}_h^i,\vec{v}_h) - \sum_e (y_h^i, \nabla\cdot(\varepsilon^{\frac{1}{2}}\vec{v}_h))_e + \sum_{l\in E_h^i}\langle\{y_h^i\},\varepsilon^{\frac{1}{2}}[\vec{\psi}_h]\cdot\vec{n}_l\rangle_l$$

$$+\sum_{l\in E_h^\partial}\langle {y_h^i}^-,\varepsilon^{\frac{1}{2}}\vec{v}_h\cdot\vec{n}_l\rangle_{l\cap\partial\Omega}=0,\quad i=1,2,...,N, \tag{31}$$

$$y_h^0(x)=y_0^h(x). \tag{32}$$

Similar to the semi-discrete case, we can derive the following optimality conditions:

$$(\frac{y_h^i-y_h^{i-1}}{k_i}, w_h) - \sum_e(\vec{\beta}y_h^i+\varepsilon^{\frac{1}{2}}\vec{q}_h^i,\nabla w_h)_e + \sum_{l\in E_h^i}\langle(\vec{\beta}\hat{y}_h^i+\varepsilon^{\frac{1}{2}}\{\vec{q}_h^i\})\cdot\vec{n}_l,[w_h]\rangle_l$$

$$+\sum_{l\in E_h^\partial}\langle\vec{\beta}\cdot\vec{n}_l {y_h^i}^-, w_h\rangle_{l\cap\partial\Omega_O} = (f^i+\mathcal{B}u_h^i, w_h) - \langle\tilde{y}^i, w_h\rangle_{l\cap\partial\Omega_I}, \forall w_h\in V_i^h, \tag{33}$$

$$(\vec{q}_h^i,\vec{v}_h) - \sum_e (y_h^i, \nabla\cdot(\varepsilon^{\frac{1}{2}}\vec{v}_h))_e + \sum_{l\in E_h^i}\langle\{y_h^i\},\varepsilon^{\frac{1}{2}}[\vec{v}_h]\cdot\vec{n}_l\rangle_l$$

$$+\sum_{l\in E_h^\partial}\langle {y_h^i}^-,\varepsilon^{\frac{1}{2}}\vec{v}_h\cdot\vec{n}_l\rangle_{l\cap\partial\Omega}=0, \forall\vec{v}_h\in(V_i^h)^2, i=1,2,...,N, \tag{34}$$

$$(\frac{z_h^{i-1}-z_h^i}{k_i},\phi_h) + \sum_e(\vec{\beta}z_h^{i-1}+\varepsilon^{\frac{1}{2}}\vec{p}_h^{i-1},\nabla\phi_h)_e - \sum_{l\in E_h^\partial}\langle {z_h^{i-1}}^-\vec{\beta}\cdot\vec{n}_l,\phi_h\rangle_{l\cap\partial\Omega_I}$$

$$-\sum_{l\in E_h^i}\langle(\vec{\beta}\check{z}_h^{i-1}+\varepsilon^{\frac{1}{2}}\{\vec{p}_h^{i-1}\})\cdot\vec{n}_l,[\phi_h]\rangle_l = (y_h^i-y_d^i,\phi_h), \forall\phi_h\in V_i^h, \tag{35}$$

$$(\vec{p}_h^{i-1},\vec{\psi}_h) + \sum_e(z_h^{i-1},\nabla\cdot(\varepsilon^{\frac{1}{2}}\vec{\psi}_h))_e - \sum_{l\in E_h^i}\langle\{z_h^{i-1}\},\varepsilon^{\frac{1}{2}}[\vec{\psi}_h]\cdot\vec{n}_l\rangle_l$$

$$-\sum_{l\in E_h^\partial}\langle {z_h^{i-1}}^-,\varepsilon^{\frac{1}{2}}\vec{\psi}_h\cdot\vec{n}_l\rangle_{l\cap\partial\Omega}=0, \forall\vec{\psi}_h\in(V_i^h)^2,\quad i=N,...,2,1, \tag{36}$$

$$(\alpha u_h^i+\mathcal{B}^*z_h^{i-1},\tilde{v}_h^i-u_h^i)_U\geq 0,\quad\forall\tilde{v}_h^i\in K_i^h,\quad i=1,2,...,N, \tag{37}$$

$$y_h^0(x)=y_0^h(x),\quad z_h^N(x)=0,\ x\in\Omega. \tag{38}$$

3. A priori error estimates for semi-discrete scheme

In this section we will derive a priori error estimates for the semi-discrete scheme. In order to do it, we make the following definitions.

Firstly, we define the element integral averaging operator $\pi_h: U\to U_h$, such that for all $\tilde{u}\in U$,

$$\pi_h\tilde{u}|_{\tau_U}=\frac{\int_{\tau_U}\tilde{u}}{\int_{\tau_U}1}.$$

Then we have the following approximation property (see, e.g., [14]):

$$\|\tilde{u}-\pi_h\tilde{u}\|_{s,\Omega_U}\leq Ch_U^{1-s}|\tilde{u}|_{1,\Omega_U},\quad s=0,1,\ \ \tilde{u}\in H^1(\Omega_U). \tag{39}$$

Moreover, noting that

$$K=\{v\in X:\ v\geq 0\ \text{ a.e. in }\ \Omega_U\times[0,T]\},$$

we divide the domain Ω_U into three parts:

$$\Omega_U^+=\{\cup\tau_U:\tau_U\subset\Omega_U, u|_{\tau_U}>0\},$$

$$\Omega_U^0 = \{\cup \tau_U : \tau_U \subset \Omega_U, u|_{\tau_U} = 0\},$$
$$\Omega_U^b = \Omega_U \backslash (\Omega_U^+ \cup \Omega_U^0).$$

In this paper we assume that u and T_U^h are regular such that $\mathrm{meas}(\Omega_U^b) \leq Ch_U$. Furthermore, set

$$\Omega^+ = \{x \in \Omega_U : u(x,t) > 0\}.$$

Then it is easy to see that $\Omega_U^+ \subset \Omega^+$.

For simplicity, we define:

$$\begin{aligned}
a_y(y_h, \vec{q}_h; w_h) &= \sum_e (\vec{\beta} y_h + \varepsilon^{\frac{1}{2}} \vec{q}_h, \nabla w_h)_e - \sum_{l\in E_h^i} \langle (\vec{\beta}\hat{y}_h + \varepsilon^{\frac{1}{2}}\{\vec{q_h}\}) \cdot \vec{n}_l, [w_h]\rangle_l,\\
a_z(z_h, \vec{p}_h; \phi_h) &= \sum_e (\vec{\beta} z_h + \varepsilon^{\frac{1}{2}} \vec{p}_h, \nabla \phi_h)_e - \sum_{l\in E_h^i} \langle (\vec{\beta}\check{z}_h + \varepsilon^{\frac{1}{2}}\{\vec{p_h}\}) \cdot \vec{n}_l, [\phi_h]\rangle_l,\\
b(y_h, \vec{v}_h) &= \sum_e (y_h, \nabla\cdot(\varepsilon^{\frac{1}{2}}\vec{v}_h))_e - \sum_{l\in E_h^i} \langle \{y_h\}, \varepsilon^{\frac{1}{2}}[\vec{v}_h]\cdot\vec{n}_l\rangle_l\\
&\quad - \sum_{l\in E_h^\partial} \langle y_h^-, \varepsilon^{\frac{1}{2}}\vec{v}_h\cdot\vec{n}_l\rangle_{l\cap\partial\Omega},\\
E_y(y_h, w_h) &= \sum_{l\in E_h^\partial} \langle \vec{\beta}\cdot\vec{n}_l y_h^-, w_h\rangle_{l\cap\partial\Omega_O}, E_z(z_h,\phi_h) = \sum_{l\in E_h^\partial} \langle \vec{\beta}\cdot\vec{n}_l z_h^-, \phi_h\rangle_{l\cap\partial\Omega_I},\\
F(u_h, w_h) &= (f + \mathcal{B}u_h, w_h) - \langle \tilde{y}, w_h\rangle_{\partial\Omega_I}, G(y_h,\phi_h) = (y_h - y_d, \phi_h).
\end{aligned}$$

Then the optimality condition (23)-(27) can be rewritten as:

$$\begin{aligned}
(y_{ht}, w_h) - a_y(y_h, \vec{q}_h; w_h) + E_y(y_h, w_h) = F(u_h, w_h), &\quad \forall w_h \in V^h,\\
(\vec{q}_h, \vec{v}_h) - b(y_h, \vec{v}_h) = 0, &\quad \forall \vec{v}_h \in (V^h)^2,\\
-(z_{ht}, \phi_h) + a_z(z_h, \vec{p}_h; \phi_h) - E_z(z_h, \phi_h) = G(y_h, \phi_h), &\quad \forall \phi_h \in V^h,\\
(\vec{p}_h, \vec{\psi}_h) + b(z_h, \vec{\psi}_h) = 0, &\quad \forall \psi_h \in (V^h)^2,\\
\int_0^T (\alpha u_h + \mathcal{B}^* z_h, \tilde{v}_h - u_h)_U \geq 0, &\quad \forall \tilde{v}_h \in K^h,\\
y_h(x,0) = y_0^h(x), \quad z_h(x,T) = 0, \quad x \in \Omega.&
\end{aligned}$$

In order to do the error analysis for the optimal control problems, we derive the following error estimates for the auxiliary problems using the technique as in [8].

Lemma 3.1. *Let $(y, \vec{q})$ be the solution of the equation (11)-(12). Let $(y_h(u), \vec{q}_h(u))$ be the solution of the following system:*

$$(y_{ht}(u), w_h) - a_y(y_h(u), \vec{q}_h(u); w_h) + E_y(y_h(u), w_h) = F(u, w_h), \tag{40}$$

$$(\vec{q}_h(u), \vec{v}_h) - b(y_h(u), \vec{v}_h) = 0, \tag{41}$$

$$y_h(u)(x,0) = y_0^h(x). \tag{42}$$

Assume that $z \in H^{r+1}(\Omega)$ and $y \in H^{r+1}(\Omega)$. Then we have the following estimate

$$|||\ (y - y_h(u), \vec{q} - \vec{q}_h(u))\ |||_* \leq Ch^r, \tag{43}$$

where r is the order of the finite element space, and

$$\begin{aligned}
|||\ (y_h, \vec{q}_h)\ |||_*^2 &= \max_{0\leq t\leq T} \|\ y_h(t)\ \|^2 + \int_0^T \|\ \vec{q}_h\ \|^2\, dt\\
&\quad + \frac{1}{2}\int_0^T [\langle |\vec{n}\cdot\vec{\beta}|, (y_h^-)^2\rangle_{\partial\Omega} + \sum_{l\in E_h^i} \langle |\vec{n}\cdot\vec{\beta}|, [y_h]^2\rangle_l] dt.
\end{aligned}$$

Proof. It is easy to see that $(y_h(u), \vec{q}_h(u))$ is the LDG approximation of $(y, \vec{q})$. Thus, according to [8], the estimate (43) holds. □

Corollary 3.2. *Let $(y, \vec{q}, z, \vec{p}, u)$ and $(y_h, \vec{q}_h, z_h, \vec{p}_h, u_h)$ be the solutions of the equations (11)-(16) and (23)-(28), respectively. Assume that the conditions of Lemma 3.1 hold. Then*

$$|||\ (y - y_h, \vec{q} - \vec{q}_h)\ |||_* \leq Ch^r + C\|u - u_h\|_{L^2(0,T;L^2(\Omega_U)}. \tag{44}$$

Proof. Recall that $(y_h, \vec{q}_h)$ is the solution of (23)-(24). Subtracting (40)-(41) from (23)-(24), we have that

$$(y_{ht} - y_{ht}(u), w_h) - a_y(y_h - y_h(u), \vec{q}_h - \vec{q}_h(u); w_h) + E_y(y_h - y_h(u), w_h) = F(u_h - u, w_h),$$
$$(\vec{q}_h - \vec{q}_h(u), \vec{v}_h) - b(y_h - y_h(u), \vec{v}_h) = 0.$$

Then setting $w_h = y_h - y_h(u)$, $\vec{v}_h = \vec{q}_h - \vec{q}_h(u)$ and using the stability property of LDG method (see, e.g., [6], [8]), we can derive that

$$|||\ (\vec{q}_h - \vec{q}_h(u), y_h - y_h(u))\ |||_* \leq C\ \|\ u - u_h\ \|_{L^2(0,T;L^2(\Omega_U))}\ . \tag{45}$$

Combining Lemma 3.1 and (45) yields (44). □

Next we will consider the error estimate of $|||\ (z - z_h, \vec{p} - \vec{p}_h)\ |||_*$. Similar to Lemma 3.1, we can obtain the following estimate:

Lemma 3.3. *Let $(y, \vec{q}, z, \vec{p}, u)$ be the solution of the equations (11)-(16), and let $(z_h(u), \vec{p}_h(u))$ be the solution of following equations:*

$$-(z_{ht}(u), \phi_h) + a_z(z_h(u), \vec{p}_h(u); \phi_h) - E_z(z_h(u), \phi_h) = G(y_h(u), \phi_h), \tag{46}$$

$$(\vec{p}_h(u), \vec{\psi}_h) + b(z_h(u), \vec{\psi}_h) = 0, \tag{47}$$

$$z_h(u)(x, T) = 0, \tag{48}$$

where $y_h(u)$ is the solution of the system (40)-(42). Assume that $z \in H^{r+1}(\Omega)$ and $y \in H^{r+1}(\Omega)$. Then

$$|||\ (z - z_h(u), \vec{p} - \vec{p}_h(u))\ |||_* \leq Ch^r.$$

Proof. Let $(z_h(y), \vec{p}_h(y))$ be the solutions of following equations:

$$-(z_{ht}(y), \phi_h) + a_z(z_h(y), \vec{p}_h(y); \phi_h) - E_z(z_h(y), \phi_h) = G(y, \phi_h), \tag{49}$$

$$(\vec{p}_h(y), \vec{\psi}_h) + b(z_h(y), \vec{\psi}_h) = 0, \tag{50}$$

$$z_h(y)(x, T) = 0. \tag{51}$$

Comparing (49)-(51) to (13)-(14), it is easy to see that $(z_h(y), \vec{p}_h(y))$ is the LDG approximation solution of $(z, \vec{q})$, then by the result of LDG method (see, e.g., [6], [8]) we have that

$$|||\ (z - z_h(y), \vec{p} - \vec{p}_h(y))\ |||_* \leq Ch^r. \tag{52}$$

Recall that $(z_h(u), \vec{p}_h(u))$ is the solution of (46)-(48). By the stability estimates of LDG method we obtain that

$$|||\ (z_h(u) - z_h(y), \vec{p}_h(u) - \vec{p}_h(y))\ |||_* \leq C\ \|\ y - y_h(u)\ \|_{L^2(0,T;L^2(\Omega))}\ . \tag{53}$$

Using the result of Lemma 3.1 and combining (52)-(53) leads to the theorem result. □

Corollary 3.4. *Let $(y, \vec{q}, z, \vec{p}, u)$ and $(y_h, \vec{q}_h, z_h, \vec{p}_h, u_h)$ be the solutions of the equations (11)-(16) and (23)-(28), respectively. Assume that the conditions of Lemma 3.3 hold. Then the following error estimate holds*

$$|||\ (z - z_h, \vec{p} - \vec{p}_h)\ |||_* \ \leq\ Ch^r + C\|u - u_h\|_{L^2(0,T;L^2(\Omega_U))}.$$

Proof. Subtracting (46)-(48) from (25)-(26), it is deduced that

$$(z_{ht}(u)-z_{ht},\phi_h)+a_z(z_h-z_h(u),\vec{p}_h-\vec{p}_h(u);\phi_h)-E_z(z_h-z_h(u),\phi_h) = G(y_h-y_h(u),\phi_h),$$
$$(\vec{p}_h-\vec{p}_h(u),\vec{\psi}_h)+b(z_h-z_h(u),\vec{\psi}_h)=0,$$

Let $\phi_h=z_h-z_h(u)$, $\vec{\psi}_h=\vec{p}_h-\vec{p}_h(u)$, then by the stability estimate of LDG method and (45) we can obtain that

$$\begin{aligned}|||\,(z_h-z_h(u),\vec{p}_h-\vec{p}_h(u))\,|||_* &\leq C\parallel y_h-y_h(u)\parallel_{L^2(0,T;L^2(\Omega))}\\ &\leq C\parallel u-u_h\parallel_{L^2(0,T;L^2(\Omega_U))}.\end{aligned} \tag{54}$$

Summing up, it follows from (54) and Lemma 3.3 that

$$|||\,(z-z_h,\vec{p}-\vec{p}_h)\,|||_* \leq Ch^r+C\|u-u_h\|_{L^2(0,T;L^2(\Omega_U))}.$$

□

3.1. Finite element discretization for the control u.

Theorem 3.5. *Let $(y,\vec{q},z,\vec{p},u)$ and $(y_h,\vec{q}_h,z_h,\vec{p}_h,u_h)$ be the solutions of the equations (11)-(16) and (23)-(28), respectively. Assume that $u\in W^{1,\infty}(\Omega_U)$, $u|_{\Omega^+}\in H^2(\Omega^+)$, $z\in W^{1,\infty}(\Omega)\cap H^{r+1}(\Omega)$, and $y\in H^{r+1}(\Omega)$. Then we have*

$$\parallel u-u_h\parallel_{L^2(0,T;L^2(\Omega_U))}+|||\,(y-y_h,\vec{q}-\vec{q}_h)\,|||_*+|||\,(z-z_h,\vec{p}-\vec{p}_h)\,|||_*\leq C(h_U^{1+m/2}+h^r),$$

where h and h_U are the sizes of the meshes T^h and T_U^h, respectively, $m=0$ or 1 is the order of the finite element space for control variable, and r is the order of the finite element space for the state and the adjoint state.

Proof. Let

$$(J_h'(u),v-u)_U=(\alpha u+\mathcal{B}^*z_h(u),v-u)_U,$$

where $z_h(u)$ is the solution of (46)-(48). Note that

$$(J_h'(v),v-u)_U-(J_h'(u),v-u)_U=(\alpha(v-u),v-u)_U+(\mathcal{B}^*z_h(v)-\mathcal{B}^*z_h(u),v-u)_U.$$

Moreover, it follows from (40)-(42) and (46)-(48) that

$$(y_{ht}(v)-y_{ht}(u),w_h)-a_y(y_h(v)-y_h(u),\vec{q}_h(v)-\vec{q}_h(u);w_h) +E_y(y_h(v)-y_h(u),w_h)=(\mathcal{B}(v-u),w_h),$$
$$(\vec{q}_h(v)-\vec{q}_h(u),\vec{v}_h)-b(y_h(v)-y_h(u),\vec{\psi}_h)=0,$$

and

$$(z_{ht}(u)-z_{ht}(v),\phi_h)+a_z(z_h(v)-z_h(u),\vec{p}_h(v)-\vec{p}_h(u);\phi_h) -E_z(z_h(v)-z_h(u),\phi_h)=G(y_h(v)-y_h(u),\phi_h),$$
$$(\vec{p}_h(v)-\vec{p}_h(u),\vec{\psi}_h)+b(z_h(v)-z_h(u),\vec{\psi}_h)=0,$$

Taking $w_h=z_h(v)-z_h(u)$, $\vec{v}_h=\vec{p}_h(v)-\vec{p}_h(u)$ and $\phi_h=y_h(v)-y_h(u)$, $\vec{\psi}_h=\vec{q}_h(v)-\vec{q}_h(u)$ in above equalities, we have that

$$(\mathcal{B}^*z_h(v)-\mathcal{B}^*z_h(u),v-u)_U=(y_h(v)-y_h(u),y_h(v)-y_h(u))\geq 0. \tag{55}$$

Then (55) imply that

$$(J_h'(v),v-u)_U-(J_h'(u),v-u)_U\geq\alpha\parallel v-u\parallel_{0,\Omega_U}^2. \tag{56}$$

Let $Q_hu\in K^h$ be an approximation of u, then it follows from (15), (27) and (56) that

$$\alpha\parallel u-u_h\parallel^2_{L^2(0,T;L^2(\Omega_U))}$$

$$
\begin{aligned}
&\leq \int_0^T (J_h'(u) - J_h'(u_h), u - u_h)_U dt \\
&= \int_0^T (\alpha u + \mathcal{B}^* z, u - u_h)_U dt + \int_0^T (\mathcal{B}^* z_h(u) - \mathcal{B}^* z, u - u_h)_U dt \\
&\quad + \int_0^T (\alpha u_h + \mathcal{B}^* z_h, u_h - Q_h u)_U dt + \int_0^T (\alpha u_h + \mathcal{B}^* z_h, Q_h u - u)_U dt \\
&\leq \int_0^T (\mathcal{B}^* z_h(u) - \mathcal{B}^* z, u - u_h)_U dt + \int_0^T (\alpha u_h + \mathcal{B}^* z_h, Q_h u - u)_U dt.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\int_0^T (\alpha u_h + \mathcal{B}^* z_h, Q_h u - u)_U dt \\
&= \int_0^T (\alpha u + \mathcal{B}^* z, Q_h u - u)_U dt + \int_0^T (\alpha u_h - \alpha u, Q_h u - u) dt \\
&\quad + \int_0^T (\mathcal{B}^*(z_h - z_h(u)), Q_h u - u) dt + \int_0^T (\mathcal{B}^*(z_h(u) - z), Q_h u - u) dt \\
&\leq \int_0^T (\alpha u + \mathcal{B}^* z, Q_h u - u)_U dt + C(\delta) \parallel Q_h u - u \parallel^2_{L^2(0,T;L^2(\Omega_U))} \\
&\quad + C\delta \parallel \alpha u - \alpha u_h \parallel^2_{L^2(0,T;L^2(\Omega_U))} + C\delta \parallel \mathcal{B}^*(z_h - z_h(u)) \parallel^2_{L^2(0,T;L^2(\Omega_U))} \\
&\quad + C\delta \parallel \mathcal{B}^*(z - z_h(u)) \parallel^2_{L^2(0,T;L^2(\Omega_U))},
\end{aligned}
$$

where δ is an arbitrarily small positive number. Therefore,

$$
\begin{aligned}
&\parallel u - u_h \parallel^2_{L^2(0,T;L^2(\Omega_U))} \\
&\leq C \int_0^T (\alpha u + \mathcal{B}^* z, Q_h u - u)_U dt + C(\delta) \parallel Q_h u - u \parallel^2_{L^2(0,T;L^2(\Omega_U))} \\
&\quad + C\delta \parallel u - u_h \parallel^2_{L^2(0,T;L^2(\Omega_U))} + C\delta \parallel \mathcal{B}^*(z_h - z_h(u)) \parallel^2_{L^2(0,T;L^2(\Omega_U))} \\
&\quad + C\delta \parallel \mathcal{B}^*(z - z_h(u)) \parallel^2_{L^2(0,T;L^2(\Omega_U))}.
\end{aligned}
$$

Then using Lemma 3.3 and (54) we get that

$$
\begin{aligned}
&\parallel u - u_h \parallel^2_{L^2(0,T;L^2(\Omega_U))} \\
&\leq C \int_0^T (\alpha u + \mathcal{B}^* z, Q_h u - u)_U dt + C \parallel Q_h u - u \parallel^2_{L^2(0,T;L^2(\Omega_U))} \\
&\quad + C \parallel \mathcal{B}^*(z - z_h(u)) \parallel^2_{L^2(0,T;L^2(\Omega_U))} \\
&\leq C \int_0^T (\alpha u + \mathcal{B}^* z, Q_h u - u)_U dt + C \parallel Q_h u - u \parallel^2_{L^2(0,T;L^2(\Omega_U))} + Ch^{2r}.
\end{aligned} \tag{57}
$$

In the following argument we shall consider the error estimates for the control variable under different finite element spaces. Firstly, let us consider the case that U^h is the piecewise constant finite element space. Let $Q_h u \in U^h$ be the element integral average of u. Using the property of the operator Q_h, we can derive that

$$
\begin{aligned}
(\alpha u + \mathcal{B}^* z, Q_h u - u)_U &= (\alpha u + \mathcal{B}^* z - Q_h(\alpha u + \mathcal{B}^* z), Q_h u - u)_U \\
&\leq Ch_U^2 (\parallel u \parallel^2_{1,\Omega_U} + \parallel z \parallel^2_{1,\Omega}).
\end{aligned} \tag{58}
$$

Therefore, it follows from (57)-(58) that

$$
\parallel u - u_h \parallel_{L^2(0,T;L^2(\Omega_U))} \leq Ch_U + Ch^r. \tag{59}
$$

Next, let us consider the case that U^h is the piecewise linear finite element space (which can be continuous or discontinuous). Set $Q_hu \in U^h$ be the standard Lagrange interpolation of u such that $Q_hu(x) = u(x)$ for all vertices x. Then it is easy to see that $Q_hu \in K^h$. Note that $u \in W^{1,\infty}(\Omega_U)$ and $u|_{\Omega^+} \in H^2(\Omega^+)$. We get

$$\| u - Q_hu \|_{0,\Omega_U^+} \le Ch_U^2 \| u \|_{2,\Omega_U^+}, \quad \| u - Q_hu \|_{0,\infty,\Omega_U^b} \le Ch_U \| u \|_{1,\infty,\Omega_U^b},$$

and hence,

$$\begin{aligned} \| u - Q_hu \|_{0,\Omega_U}^2 &= \int_{\Omega_U^+}(u - Q_hu)^2 + \int_{\Omega_U^0}(u - Q_hu)^2 + \int_{\Omega_U^b}(u - Q_hu)^2 \\ &\le Ch_U^4 \| u \|_{2,\Omega_U^+}^2 + 0 + Ch_U^2 \| u \|_{1,\infty,\Omega_U^b}^2 \, meas(\Omega_U^b) \\ &\le Ch_U^4 \| u \|_{2,\Omega_U^+}^2 + Ch_U^3 \| u \|_{1,\infty,\Omega_U^b}^2 \\ &\le Ch_U^3(\| u \|_{2,\Omega^+}^2 + \| u \|_{1,\infty,\Omega_U}^2) \le Ch_U^3. \end{aligned} \tag{60}$$

Moreover, it follows from (15) that $\alpha u + \mathcal{B}^*z = 0$ on Ω_U^+. It is easy to see that $Q_hu - u = 0$ on Ω_U^0. Note that for all element $\tau_U^b \subset \Omega_U^b$, there is $x_0 \in \tau_U^b$ such that $u(x_0) > 0$, and hence $(\alpha u + \mathcal{B}^*z)(x_0) = 0$. Then

$$\|\alpha u + \mathcal{B}^*z\|_{0,\infty,\tau_U^b} = \|\alpha u + \mathcal{B}^*z - (\alpha u + \mathcal{B}^*z)(x_0)\|_{0,\infty,\tau_U^b} \le Ch_U\|\alpha u + \mathcal{B}^*z\|_{1,\infty,\tau_U^b}.$$

Thus,

$$\begin{aligned} (\alpha u + \mathcal{B}^*z, Q_hu - u)_U &= \int_{\Omega_U^+}(\alpha u + \mathcal{B}^*z)(Q_hu - u) + \int_{\Omega_U^0}(\alpha u + \mathcal{B}^*z)(Q_hu - u) \\ &\quad + \int_{\Omega_U^b}(\alpha u + \mathcal{B}^*z)(Q_hu - u) \\ &= 0 + 0 + \int_{\Omega_U^b}(\alpha u + \mathcal{B}^*z)(Q_hu - u) \\ &\le \| \alpha u + \mathcal{B}^*z \|_{0,\infty,\Omega_U^b} \| u - Q_hu \|_{0,\infty,\Omega_U^b} \, meas(\Omega_U^b) \\ &\le Ch_U^3. \end{aligned} \tag{61}$$

Combining (57) and (60)-(61) leads to

$$\| u - u_h \|_{L^2(0,T;L^2(\Omega_U))} \le Ch_U^{\frac{3}{2}} + Ch^r. \tag{62}$$

Therefore, the theorem result follows from (59), (62) and Corollary 3.2 and 3.4.

□

3.2. Variational discretization for the control u**.** In this section, we will introduce a variational discrete concept for control u and a priori error estimates will be derived.

Using a pointwise projection on the admissible set K,

$$P_K : U \longrightarrow K, \quad P_Kv = \max(0, v), \tag{63}$$

the optimal condition (16) can be expressed as follows:

$$u = P_K(-\frac{1}{\alpha}(\mathcal{B}^*z)).$$

Similarly, employing the projection (63) the optimal condition (27) can be rewritten as follows:

$$u_h = P_K(-\frac{1}{\alpha}(\mathcal{B}^*z_h)).$$

Here it should be pointed that $u_h \in K$ and we make minimization on the infinite dimensional space K instead of the finite element space. In general, u_h is not a finite element function corresponding to the mesh T_U^h, especially on the element crossing the discrete free boundary. This fact requires more care for the construction of the algorithms for computing u_h, see [10] for details.

Theorem 3.6. *Let $(y, \vec{q}, z, \vec{p}, u)$ and $(y_h, \vec{q}_h, z_h, \vec{p}_h, u_h)$ be the solutions of the equations (11)-(16) and (23)-(28), respectively, with K^h displaced by K. Assume that $z \in H^{r+1}(\Omega)$ and $y \in H^{r+1}(\Omega)$. Then we have that*

$$\| u - u_h \|_{L^2(0,T;L^2(\Omega))} + ||| (y - y_h, \vec{q} - \vec{q}_h) |||_* + ||| (\vec{p} - \vec{p}_h, z - z_h) |||_* \leq Ch^r,$$

where r is the order of the finite element space for the state and the adjoint state.

Proof. Let $(J_h'(u), v-u)_U = (\alpha u + \mathcal{B}^* z_h(u), v-u)_U$, it has been proved in Theorem 3.5 that

$$(J_h'(v), v-u)_U - (J_h'(u), v-u)_U \geq \alpha \| v-u \|_{0,\Omega_U}^2 . \tag{64}$$

Then it follows from (64), (15) and (27) that

$$\begin{aligned} & \alpha \| u - u_h \|^2_{L^2(0,T;L^2(\Omega_U))} \\ \leq & \int_0^T (J_h'(u) - J_h'(u_h), u - u_h)_U dt \\ = & \int_0^T (\alpha u + \mathcal{B}^* z, u - u_h)_U dt + \int_0^T (\mathcal{B}^* z_h(u) - \mathcal{B}^* z, u - u_h)_U dt \\ & + \int_0^T (\alpha u_h + \mathcal{B}^* z_h, u_h - u)_U dt \\ \leq & \ 0 + \int_0^T (\mathcal{B}^* z_h(u) - \mathcal{B}^* z, u - u_h)_U dt + 0 \\ \leq & \ \| z_h(u) - z \|_{L^2(0,T;L^2(\Omega))} \| u - u_h \|_{L^2(0,T;L^2(\Omega_U))} . \end{aligned}$$

Therefore

$$\| u - u_h \|_{L^2(0,T;L^2(\Omega_U))} \leq C \| z_h(u) - z \|_{L^2(0,T;L^2(\Omega))} .$$

Using the result of Lemma 3.3, we can derive that

$$\| u - u_h \|_{L^2(0,T;L^2(\Omega_U))} \leq Ch^r. \tag{65}$$

Combining Corollary 3.2, Corollary 3.4 and (65) we can obtain the theorem result. □

4. A priori error estimates for full discretization scheme

In this section, we will consider the error analysis of the fully discrete LDG scheme for the optimal control problem. Similar to Section 3, we define:

$$\begin{aligned} a_y^h(y_h^i, \vec{q}_h^i; w_h) &= \sum_e (\vec{\beta} y_h^i + \varepsilon^{\frac{1}{2}} \vec{q}_h^i, \nabla w_h)_e - \sum_{l \in E_h^i} \langle (\vec{\beta} \hat{y}_h^i + \varepsilon^{\frac{1}{2}} \{\vec{q}_h^{\,i}\}) \cdot \vec{n}_l, [w_h] \rangle_l, \\ a_z^h(z_h^i, \vec{p}_h^i; \phi_h) &= \sum_e (\vec{\beta} z_h^i + \varepsilon^{\frac{1}{2}} \vec{p}_h^i, \nabla \phi_h)_e - \sum_{l \in E_h^i} \langle (\vec{\beta} \check{z}_h^i + \varepsilon^{\frac{1}{2}} \{\vec{p}_h^{\,i}\}) \cdot \vec{n}_l, [\phi_h] \rangle_l, \\ b^h(y_h^i, \vec{v}_h) &= \sum_e (y_h^i, \nabla \cdot (\varepsilon^{\frac{1}{2}} \vec{v}_h))_e - \sum_{l \in E_h^i} \langle \{y_h^i\}, \varepsilon^{\frac{1}{2}} [\vec{v}_h] \cdot \vec{n}_l \rangle_l \\ & \quad - \sum_{l \in E_h^\partial} \langle y_h^{i\,-}, \varepsilon^{\frac{1}{2}} \vec{v}_h \cdot \vec{n}_l \rangle_{l \cap \partial\Omega}, \end{aligned}$$

$$E_y^h(y_h^i, w_h) = \sum_{l\in E_h^\partial} \langle \vec{\beta}\cdot\vec{n}_l {y_h^i}^-, w_h\rangle_{l\cap\partial\Omega_O}, E_z^h(z_h^i,\phi_h) = \sum_{l\in E_h^\partial}\langle\vec{\beta}\cdot\vec{n}_l {z_h^i}^-,\phi_h\rangle_{l\cap\partial\Omega_I},$$
$$F^h(u_h^i, w_h) = (f^i + \mathcal{B}u_h^i, w_h) - \langle\tilde{y}^i, w_h\rangle_{\partial\Omega_I}, G^h(y_h^i,\phi_h) = (y_h^i - y_d^i, \phi_h).$$

Then the optimality condition (33)-(38) can be rewritten as follows:

$$(\frac{y_h^i - y_h^{i-1}}{k_i}, w_h) - a_y^h(y_h^i, \vec{q}_h^i; w_h) + E_y^h(y_h^i, w_h) = F^h(u_h^i, w_h), \tag{66}$$
$$(\vec{q}_h^i, \vec{v}_h) - b^h(y_h^i, \vec{v}_h) = 0, \tag{67}$$
$$(\frac{z_h^{i-1} - z_h^i}{k_i}, \phi_h) + a_z^h(z_h^{i-1}, \vec{p}_h^{i-1}; \phi_h) - E_z^h(z_h^{i-1}, \phi_h) = G^h(y_h^i, \phi_h), \tag{68}$$
$$(\vec{p}_h^{i-1}, \vec{\psi}_h) + b^h(z_h^{i-1}, \vec{\psi}_h) = 0, \tag{69}$$
$$(\alpha u_h^i + \mathcal{B}^* z_h^{i-1}, \tilde{v}_h - u_h^i)_U \geq 0, \tag{70}$$
$$y_h^0(x) = y_0^h(x), \quad z_h^N(x) = 0, \quad x\in\Omega. \tag{71}$$

We define the discrete time-dependent norms:

$$|||\, F \,|||_{L^p(0,T;H^r(\Omega))}^p = \sum_{i=1}^N k_i \parallel F^i \parallel_{r,\Omega}^p,$$
$$|||\, (w,\vec{v}) \,|||^2 = \max_{1\leq i\leq N} \parallel w^i\parallel^2 + \sum_{i=1}^N k_i \parallel \vec{v}^i \parallel^2 + \frac{1}{2}\sum_{i=1}^N k_i(\langle|\vec{n}\cdot\vec{\beta}|, (w^{i-})^2\rangle_{\partial\Omega} + \sum_{l\in E_h^i}\langle|\vec{n}\cdot\vec{\beta}|, [w^i]^2\rangle_l).$$

Using the techniques used in the proof of Lemmas 3.1 and 3.3 and Corollaries 3.2 and 3.4, it can be proved that for the full discretization scheme we have the following estimates for the state and the adjoint state.

Lemma 4.1. *Let $(y,\vec{q},z,\vec{p},u)$ and $(y_h^i, \vec{q}_h^i, z_h^{i-1}, \vec{p}_h^{i-1}, u_h^i)$ be the solutions of the equations (11)-(16) and (33)-(38), respectively. Assume that $z, y\in H^1(0,T;H^{r+1}(\Omega))\cap H^2(0,T;L^2(\Omega))$, $y_d\in H^1(0,T;L^2(\Omega))$. Then we have*

$$|||\, (y - Y_h(u), \vec{q} - \vec{Q}_h(u)) \,||| + |||\, (z - Z_h(u), \vec{p} - \vec{P}_h(u)) \,||| \leq C(h^r + k),$$
$$|||\, (y-y_h, \vec{q}-\vec{q}_h) \,||| + |||\, (z - z_h, \vec{p}-\vec{p}_h) \,||| \leq C(h^r + k + |||\, u - u_h \,|||_{L^2(0,T;L^2(\Omega_U))}),$$

where h and r are the element size and the order of the finite element space, k is the time step, and $Y_h^i(u), \vec{Q}_h^i(u), Z_h^{i-1}(u), \vec{P}_h^{i-1}(u)$ are the solutions of the following equations:

$$(\frac{Y_h^i(u) - Y_h^{i-1}(u)}{k_i}, w_h) - a_y^h(Y_h^i(u), \vec{Q}_h^i(u); w_h) + E_y^h(Y_h^i(u), w_h) = F^h(u^i, w_h), \tag{72}$$
$$(\vec{Q}_h^i(u), \vec{v}_h) - b^h(Y_h^i(u), \vec{v}_h) = 0, \tag{73}$$
$$(\frac{Z_h^{i-1}(u) - Z_h^i(u)}{k_i}, \phi_h) + a_z^h(Z_h^{i-1}(u), \vec{P}_h^{i-1}(u); \phi_h) + E_z^h(Z_h^{i-1}(u), \phi_h) \tag{74}$$
$$= G^h(Y_h^i(u), \phi_h), \tag{75}$$
$$(\vec{P}_h^{i-1}(u), \vec{\psi}_h) + b^h(Z_h^{i-1}(u), \vec{\psi}_h) = 0, \tag{76}$$
$$Y_h^0(u) = y_0^h(x), \quad Z_h^N(u) = 0. \tag{77}$$

Next we will discuss the convex property of the full discrete scheme.

Lemma 4.2. *Let* $(y, \vec{q}, z, \vec{p}, u)$ *and* $(y_h^i, \vec{q}_h^i, z_h^{i-1}, \vec{p}_h^{i-1}, u_h^i)$ *be the solutions of the equations (11)-(16) and (33)-(38), respectively. Let*

$$(\hat{J}_h'(u), v-u)_U = \sum_{i=1}^N k_i(\alpha u^i + \mathcal{B}^* Z_h^{i-1}(u), v^i - u^i)_U,$$

where $Z_h^{i-1}(u)$ *is the solution of the equations (72)-(77). Then the following estimate holds:*

$$(\hat{J}_h'(v) - \hat{J}_h'(u), v-u)_U \geq \alpha \, ||| \, v-u \, |||^2_{L^2(0,T;L^2(\Omega_U))} \, .$$

Proof. Note that

$$\begin{aligned} & (\hat{J}_h'(v) - \hat{J}_h'(u), v-u)_U \\ = \; & \sum_{i=1}^N k_i(\alpha v^i - \alpha u^i, v^i - u^i)_U + \sum_{i=1}^N k_i(\mathcal{B}^* Z_h^{i-1}(v) - \mathcal{B}^* Z_h^{i-1}(u), v^i - u^i)_U \\ = \; & \alpha \, ||| \, v-u \, |||^2_{L^2(0,T;L^2(\Omega_U))} + \sum_{i=1}^N k_i(Z_h^{i-1}(v) - Z_h^{i-1}(u), \mathcal{B}(v^i - u^i))_U. \end{aligned}$$

Let $Y^i = Y_h^i(v) - Y_h^i(u)$, $\vec{Q}^i = \vec{Q}_h^i(v) - \vec{Q}_h^i(u)$, $Z^{i-1} = Z_h^{i-1}(v) - Z_h^{i-1}(u)$, and $\vec{P}^{i-1} = \vec{P}_h^{i-1}(v) - \vec{P}_h^{i-1}(u)$, then we have that

$$(\frac{Y^i - Y^{i-1}}{k_i}, w_h) - a_y^h(Y^i, \vec{Q}^i; w_h) + E_y^h(Y^i, w_h) = (\mathcal{B}(v^i - u^i), w_h), \tag{78}$$

$$(\vec{Q}^i, \vec{v}_h) - b^h(Y^i, \vec{v}_h) = 0, \tag{79}$$

$$(\frac{Z^{i-1} - Z^i}{k_i}, \phi_h) + a_z^h(Z^{i-1}, \vec{P}^{i-1}; \phi_h) + E_z^h(Z^{i-1}, \phi_h) = G^h(Y^i, \phi_h), \tag{80}$$

$$(\vec{P}^{i-1}, \vec{\psi}_h) + b^h(Z^{i-1}, \vec{\psi}_h) = 0. \tag{81}$$

Set $w_h = Z^{i-1}$, $\vec{v}_h = \vec{P}^{i-1}$ in (78)-(79) and $\phi_h = Y^i$, $\vec{\psi}_h = \vec{Q}^i$ in (80)-(81), respectively. Similar to the semidiscrete case, it is easy to prove that

$$\sum_{i=1}^N k_i(Z_h^{i-1}(v) - Z_h^{i-1}(u), \mathcal{B}(v^i - u^i))_U \geq 0.$$

Then we can derive the theorem result. □

In the following, we will provide a priori error estimates for two different control discretization approaches (finite element approximation and variational discretization) described in section 3.

4.1. Finite element discretization for the control u.

Theorem 4.3. *Let* $(y, \vec{q}, z, \vec{p}, u)$ *and* $(y_h^i, \vec{q}_h^i, z_h^{i-1}, \vec{p}_h^{i-1}, u_h^i)$ *be the solutions of the equations (11)-(16) and (33)-(38), respectively. Suppose that the conditions of Lemma 4.1 are valid. Moreover, we assume that* $u \in L^2(0,T;W^{1,\infty}(\Omega_U))$, $u|_{\Omega^+} \in L^2(0,T;H^2(\Omega^+))$, $z \in L^2(0,T;W^{1,\infty}(\Omega)) \cap H^1(0,T;L^2(\Omega))$. *Then we have*

$$\begin{aligned} & ||| \, (y-y_h, \vec{q} - \vec{q}_h) \, ||| + ||| \, (z - z_h, \vec{p} - \vec{p}_h) \, ||| + ||| \, u - u_h \, |||_{L^2(0,T;L^2(\Omega_U)} \\ & \quad \leq C(h^r + h_U^{1+m/2} + k). \end{aligned}$$

Proof. Let $\Pi_h u$ be an approximation of u. Following (15), (37) and Lemma 4.2 we obtain that

$$
\begin{aligned}
& \alpha \,|||\, u-u_h \,|||^2_{L^2(0,T;L^2(\Omega_U))} \\
\leq\ & (\hat{J}'_h(u)-\hat{J}'_h(u_h), u-u_h)_U \\
=\ & \sum_{i=1}^N k_i(\alpha u^i+\mathcal{B}^* z^i, u^i-u_h^i)_U + \sum_{i=1}^N k_i(\mathcal{B}^* Z_h^{i-1}(u)-\mathcal{B}^* z^i, u^i-u_h^i)_U \\
& + \sum_{i=1}^N k_i(\alpha u_h^i+\mathcal{B}^* z_h^{i-1}, \Pi_h u^i-u^i)_U + \sum_{i=1}^N k_i(\alpha u_h^i+\mathcal{B}^* z_h^{i-1}, u_h^i-\Pi_h u^i)_U \\
\leq\ & 0+\sum_{i=1}^N k_i(\mathcal{B}^* Z_h^{i-1}(u)-\mathcal{B}^* z^i, u^i-u_h^i)_U + \sum_{i=1}^N k_i(\alpha u_h^i+\mathcal{B}^* z_h^{i-1}, \Pi_h u^i-u^i)_U + 0 \\
=\ & \sum_{i=1}^N k_i(\mathcal{B}^* Z_h^{i-1}(u)-\mathcal{B}^* z^{i-1}, u^i-u_h^i)_U + \sum_{i=1}^N k_i(\mathcal{B}^* z^{i-1}-\mathcal{B}^* z^i, u^i-u_h^i)_U \\
& + \sum_{i=1}^N k_i(\alpha u_h^i+\mathcal{B}^* z_h^{i-1}, \Pi_h u^i-u^i)_U = T_1+T_2+T_3.
\end{aligned}
$$

Now we are in the position to estimate $T_1 \sim T_3$. It follows from $Young's$ inequality that

$$
\begin{aligned}
T_1 \ \leq\ & C(\delta)\sum_{i=1}^N k_i \,\|\, z^{i-1}-Z_h^{i-1}(u) \,\|^2_{0,\Omega} + C\delta \sum_{i=1}^N k_i \,\|\, u^i-u_h^i \,\|^2_{0,\Omega_U} \\
\leq\ & C(\delta)\,|||\, z-Z_h(u) \,|||^2_{L^2(0,T;L^2(\Omega))} + C\delta \,|||\, u-u_h \,|||^2_{L^2(0,T;L^2(\Omega_U))} .
\end{aligned}
$$

Note that

$$
|z^i-z^{i-1}| = |\int_{t_{i-1}}^{t_i} \frac{\partial z}{\partial t} dt| \leq k_i^{\frac{1}{2}} (\int_{t_{i-1}}^{t_i} (\frac{\partial z}{\partial t})^2 dt)^{\frac{1}{2}}.
$$

Then we have

$$
\begin{aligned}
T_2 \ \leq\ & C(\delta)\sum_{i=1}^N k_i \,\|\, z^i-z^{i-1} \,\|^2_{0,\Omega} + C\delta \sum_{i=1}^N k_i \,\|\, u^i-u_h^i \,\|^2_{0,\Omega_U} \\
\leq\ & C(\delta)k^2 \,\|\, \frac{\partial z}{\partial t} \,\|^2_{L^2(0,T;L^2(\Omega))} + C\delta \,|||\, u-u_h \,|||^2_{L^2(0,T;L^2(\Omega_U))} .
\end{aligned}
$$

The estimate of T_3 depends on the choice of the finite element space for the control discretization.

Firstly, let us consider the case that U_i^h is the piecewise constant finite element space. Let $\Pi_h u^i \in U_i^h$ be the the element integral average of u^i. Then

$$
\begin{aligned}
T_3 \ =\ & \sum_{i=1}^N k_i(\alpha u^i+\mathcal{B}^* z^i, \Pi_h u^i-u^i)_U + \sum_{i=1}^N k_i(\alpha(u_h^i-u^i), \Pi_h u^i-u^i)_U \\
& + \sum_{i=1}^N k_i(\mathcal{B}^*(z_h^{i-1}-Z_h^{i-1}(u)), \Pi_h u^i-u^i)_U \\
& + \sum_{i=1}^N k_i(\mathcal{B}^*(Z_h^{i-1}(u)-z^{i-1}), \Pi_h u^i-u^i)_U
\end{aligned}
$$

$$+\sum_{i=1}^{N} k_i(\mathcal{B}^*(z^{i-1}-z^i), \Pi_h u^i - u^i)_U = \sum_{i=1}^{5} I_i. \tag{82}$$

Now let's derive the estimates of $I_1 \sim I_5$, respectively. It follows the property of Π_h and $Young's$-inequality that

$$\begin{aligned}
I_1 &= \sum_{i=1}^{N} k_i(\alpha u^i + \mathcal{B}^* z^i - \Pi_h(\alpha u^i + \mathcal{B}^* z^i), \Pi_h u^i - u^i)_U \\
&\le Ch_U^2(|||\, u\, |||^2_{L^2(0,T;H^1(\Omega_U))} + |||\, z\, |||^2_{L^2(0,T;H^1(\Omega))}). \\
I_2 &\le C\delta\, |||\, u_h - u\, |||^2_{L^2(0,T;L^2(\Omega))} + C(\delta) h_U^2\, |||\, u\, |||^2_{L^2(0,T;H^1(\Omega_U))}\, . \\
I_5 &\le C\sum_{i=1}^{N} k_i^2 \,\| \frac{\partial z}{\partial t} \|^2_{L^2(0,T;L^2(\Omega))} + Ch_U^2\, |||\, u\, |||^2_{L^2(0,T;H^1(\Omega_U))}\, .
\end{aligned}$$

Note that

$$|||z_h - Z_h(u)||| \le C|||u - u_h|||_{L^2(0,T;L^2(\Omega))}.$$

Using the approximation property of Π_h yields

$$I_3 \le C\delta\, |||\, u_h - u\, |||^2_{L^2(0,T;L^2(\Omega))} + C(\delta) h_U^2\, |||\, u\, |||^2_{L^2(0,T;H^1(\Omega_U))}$$

and

$$I_4 \le C\, |||\, Z_h(u) - z\, |||^2_{L^2(0,T;L^2(\Omega))} + Ch_U^2\, |||\, u\, |||^2_{L^2(0,T;H^1(\Omega_U))}\, .$$

Summing up, inserting the estimates of $I_1 \sim I_5$ into (82) results in

$$\begin{aligned}
T_3 &\le Ck^2\, \| \frac{\partial z}{\partial t} \|^2_{L^2(0,T;L^2(\Omega))} + C\, |||\, Z_h(u) - z\, |||^2_{L^2(0,T;L^2(\Omega))} \\
&+ C\delta\, |||\, u_h - u\, |||^2_{L^2(0,T;L^2(\Omega))} + Ch_U^2(|||\, u\, |||^2_{L^2(0,T;H^1(\Omega_U))} + |||\, z\, |||^2_{L^2(0,T;H^1(\Omega))}).
\end{aligned}$$

Combining the estimates of $T_1 \sim T_3$, and setting δ small enough we have the following error estimate:

$$\begin{aligned}
&|||\, u - u_h\, |||^2_{L^2(0,T;L^2(\Omega_U))} \\
\le\; & Ck^2\, \| \frac{\partial z}{\partial t} \|^2_{L^2(0,T;L^2(\Omega))} + C\, |||\, Z_h(u) - z\, |||^2_{L^2(0,T;L^2(\Omega))} \\
& + Ch_U^2(|||\, u\, |||^2_{L^2(0,T;H^1(\Omega_U))} + |||\, z\, |||^2_{L^2(0,T;H^1(\Omega))}).
\end{aligned} \tag{83}$$

Secondly, let us consider the case that U_i^h is the piecewise linear finite element space. Set $\Pi_h u^i \in U_i^h$ be the standard Lagrange interpolation of u such that $\Pi_h u^i(x) = u^i(x)$ for all vertices x. Then it is easy to see that $\Pi_h u^i \in K_i^h$. Similar to Section 3, using the property of Π_h it can be proved that the term T_3 satisfies the following estimate:

$$\begin{aligned}
T_3 &\le Ck^2\, \| \frac{\partial z}{\partial t} \|^2_{L^2(0,T;L^2(\Omega))} + C\, |||\, Z_h(u) - z\, |||^2_{L^2(0,T;L^2(\Omega))} \\
&+ C\delta\, |||\, u_h - u\, |||^2_{L^2(0,T;L^2(\Omega))} + Ch_U^3.
\end{aligned}$$

Thus, combining the estimates of $T_1 \sim T_3$ and setting δ small enough we can derive that

$$\begin{aligned}
|||\, u - u_h\, |||^2_{L^2(0,T;L^2(\Omega_U))} &\le Ck^2\, \| \frac{\partial z}{\partial t} \|^2_{L^2(0,T;L^2(\Omega))} \\
&\quad + C\, |||\, Z_h(u) - z\, |||^2_{L^2(0,T;L^2(\Omega))} + Ch_U^3.
\end{aligned} \tag{84}$$

Summing up, the theorem result can be derived by combining (83), (84) and Lemma 4.1. □

4.2. Variational discretization for the control u. Similar to Section 3.2, we will derive the error estimates of the variational discretization for the control u when the full discretization scheme is applied.

Similarly, employing the projection (63) the optimal condition (27) can be rewritten as

$$u_h^i = P_K(-\frac{1}{\alpha}(\mathcal{B}^* z_h^{i-1})).$$

Then it is easy to see that $u_h^i \in K$.

Theorem 4.4. *Let $(y, \vec{q}, z, \vec{p}, u)$ and $(y_h^i, \vec{q}_h^i, z_h^{i-1}, \vec{p}_h^{i-1}, u_h^i)$ be the solutions of the equations (11)-(16) and (33)-(38), respectively, with K^h replaced by K. Assume that the conditions of Lemma 4.1 are valid. Then we have that*

$$|||\ u - u_h\ |||_{L^2(0,T;L^2(\Omega))} + |||\ (y - y_h, \vec{q} - \vec{q}_h)\ ||| + |||\ (\vec{p} - \vec{p}_h, z - z_h)\ ||| \le C(h^r + k).$$

Proof. It follows from (15), (37) and Lemma 4.2 that

$$\begin{aligned}
& \alpha\ |||\ u - u_h\ |||^2_{L^2(0,T;L^2(\Omega_U))} \\
\le\ & (\hat{J}_h'(u) - \hat{J}_h'(u_h), u - u_h) \\
=\ & \sum_{i=1}^N k_i(\alpha u^i + \mathcal{B}^* z^i, u^i - u_h^i)_U + \sum_{i=1}^N k_i(\mathcal{B}^* Z_h^{i-1}(u) - \mathcal{B}^* z^i, u^i - u_h^i)_U \\
& + \sum_{i=1}^N k_i(\alpha u_h^i + \mathcal{B}^* z_h^{i-1}, u_h^i - u^i)_U \\
\le\ & 0 + \sum_{i=1}^N k_i(\mathcal{B}^* Z_h^{i-1}(u) - \mathcal{B}^* z^i, u^i - u_h^i)_U + 0 \\
=\ & \sum_{i=1}^N k_i(\mathcal{B}^* Z_h^{i-1}(u) - \mathcal{B}^* z^{i-1}, u^i - u_h^i)_U + \sum_{i=1}^N k_i(\mathcal{B}^* z^{i-1} - \mathcal{B}^* z^i, u^i - u_h^i)_U.
\end{aligned}$$

Therefore, by $Young's$-inequality we get

$$|||\ u - u_h\ |||_{L^2(0,T;L^2(\Omega_U))} \le C\ |||\ Z_h(u) - z\ |||_{L^2(0,T;L^2(\Omega))} + Ck\ \|\ \frac{\partial z}{\partial t}\ \|_{L^2(0,T;L^2(\Omega))}.$$

Using the result of Lemma 4.1 yields that

$$|||\ u - u_h\ |||_{L^2(0,T;L^2(\Omega_U))} \le C(h^r + k). \tag{85}$$

Combing Lemma 4.1 and (85) leads to the theorem result. □

5. Discussion

In this paper, we discuss the local discontinuous Galerkin approximation for the constrained optimal control problem governed by unsteady convection dominated diffusion equations, where the control variation is discretized by finite element method and variational discretization, respectively. The a priori error estimates are derived for both semi-discrete and full-discrete schemes. The a posteriori error estimates and the numerical experiments will be addressed in the coming work. In this area there are still many important issues to be addressed, such as optimal control governed by nonlinear problems, the state constrained problems, and more complicated practical problems.

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