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ON THE OPTIMAL CONTROL PROBLEM OF LASER SURFACE HARDENING

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Abstract. We discuss an optimal control problem of laser surface hardening of steel which is governed by a dynamical system consisting of a semilinear parabolic equation and an ordinary differential equation with a non differentiable right hand side function f_+. To avoid the numerical and analytic difficulties posed by f_+, it is regularized using a monotone Heaviside function and the regularized problem has been studied in literature. In this article, we establish the convergence of solution of the regularized problem to that of the original problem. The estimates, in terms of the regularized parameter, justify the existence of solution of the original problem. Finally, a numerical experiment is presented to illustrate the effect of regularization parameter on the state and control errors.

Key Words. Laser surface hardening of steel, semilinear parabolic equation, ODE with non-differentiable forcing function, regularized Heaviside function, regularised problem, convergence with respect to regularization parameter, numerical experiments.

1. Introduction

In this paper, we discuss an optimal control problem described by the laser surface hardening of steel. The purpose of surface hardening is to increase the hardness of the boundary layer of a workpiece by rapid heating and subsequent quenching (see Figure 1). The desired hardening effect is achieved as the heat treatment leads to a change in micro structure. A few applications include cutting tools, wheels, driving axles, gears, etc. Let $\Omega \subset \mathbb{R}^2$, denoting the workpiece, be a convex, bounded domain with piecewise Lipschitz continuous boundary $\partial\Omega$, $Q = \Omega \times I$ and $\Sigma = \partial\Omega \times I$, where $I = (0, T)$, $T < \infty$. Following Leblond and Devaux[7], the evolution of volume fraction of the austenite $a(t)$ for a given temperature evolution $\theta(t)$ is described by the initial value problem:

$$\partial_t a = f_+(\theta, a) = \frac{1}{\tau(\theta)}[a_{eq}(\theta) - a]_+ \text{ in } Q, \tag{1}$$

$$a(0) = 0 \text{ in } \Omega, \tag{2}$$

where $a_{eq}(\theta(t))$, denoted as $a_{eq}(\theta)$ for notational convenience, is the equilibrium volume fraction of austenite and τ is a time constant. The term $[a_{eq}(\theta) - a]_+ =$

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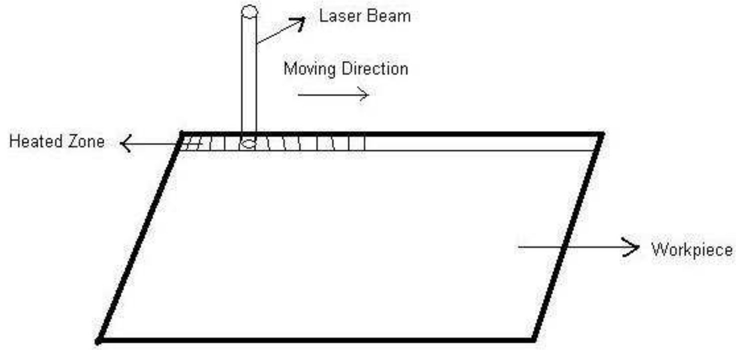

FIGURE 1. Laser Hardening Process

$(a_{eq}(\theta) - a)\mathcal{H}(a_{eq}(\theta) - a)$, where $\mathcal{H}$ is the Heaviside function

$$\mathcal{H}(s) = \begin{cases} 1 & s > 1 \\ 0 & s \leq 0, \end{cases}$$

denotes the non-negative part of $a_{eq}(\theta) - a$, that is,

$$[a_{eq}(\theta) - a]_+ = \frac{(a_{eq}(\theta) - a) + |a_{eq}(\theta) - a|}{2}.$$

Neglecting the mechanical effects and using the Fourier law of heat conduction, the temperature evolution can be obtained by solving the following energy balance equation:

$$\rho c_p \partial_t \theta - K \triangle \theta = -\rho L \partial_t a + \alpha u \quad \text{in } Q, \tag{3}$$

$$\theta(0) = \theta_0 \quad \text{in } \Omega, \tag{4}$$

$$\frac{\partial \theta}{\partial n} = 0 \quad \text{on } \Sigma, \tag{5}$$

where the density ρ, the heat capacity c_p, the thermal conductivity K and the latent heat L are assumed to be positive constants. Further, θ_0 denotes the initial temperature. The term $u(t)\alpha(x,t)$ describes the volumetric heat source due to laser radiation and the laser energy $u(t)$ is a time dependent control variable. Since the main cooling effect is a self-cooling of the workpiece, a homogeneous Neumann condition is assumed on the boundary.

To maintain the quality of the workpiece surface, it is important to avoid the melting of the surface. In the case of laser hardening, it is a quite delicate problem to obtain parameters that avoid melting but nevertheless lead to the right amount of hardening. Mathematically, this corresponds to an optimal control problem in which we minimize the cost functional defined by:

$$J(\theta, a, u) = \frac{\beta_1}{2} \int_\Omega |a(T) - a_d|^2 dx + \frac{\beta_2}{2} \int_0^T \int_\Omega [\theta - \theta_m]_+^2 dx ds + \frac{\beta_3}{2} \int_0^T |u|^2 ds \tag{6}$$

$$\text{subject to } (1) - (5) \text{ in the set of admissible controls } U_{ad}, \tag{7}$$

where $U_{ad} = \{v \in U : \|v\|_U \leq M, \quad \text{for fixed positive } M\}$ with $U = L^2(I)$, β_1, β_2 and β_3 are positive constants and a_d is the given desired fraction of the austenite. The second term in (6) is a penalizing term that penalizes the temperature below the melting temperature θ_m.

The mathematical model for the laser surface hardening of steel has been studied in [4] and [7]. For an extensive survey on mathematical models for laser material treatments, we refer to [9]. In this article, we follow the Leblond-Devaux model [7]. In [1], [4], the mathematical model for the laser hardening problem is discussed and results on existence, regularity and stability are derived. In [3], the authors have investigated two different methods of surface hardening: laser and induction hardening and then for numerical approximation, they have applied finite volume method for space discretization and finite difference for temporal discetization of the regularised problem. In [5], the optimal control problem is analyzed and related error estimates for the regularised state system are derived using proper orthogonal decomposition (POD) Galerkin method. In [12], a nonlinear conjugate gradient method has been used to solve the optimal control problem and a finite element method has been used for the purpose of space discretization. Recently in [10], the authors have derived *a priori* error estimates for the regularized laser surface hardening problem.

The presence of the term $[a_{eq}-a]_+$ in the right hand side of (1) creates a problem in developing analytical results and finding numerical solution. In order to overcome this difficulty, the function $f_+ = \frac{1}{\tau(\theta)}[a_{eq} - a]_+$ is regularized using a regularized Heaviside function in literature (see [3]-[6], [12]). Although the numerical schemes in [3]-[6] and [12] are discretizations of the regularized problem, there are hardly any convergence results available which establish the fact that the solution of the regularized problem converges to that of the original problem as the regularization parameter ϵ tends to zero. In this paper, it is shown that the error between the solution of regularized problem and that of the original problem is of order $O(\epsilon)$ and a convergence analysis for the regularized laser surface hardening of steel problem is discussed.

The outline of this paper is as follows. In Section 2, we describe the regularized optimal control problem of laser surface hardening of steel and its weak formulation with results of existence and uniqueness of solution, which are already available in the literature. A stability result for the temperature is also established. In Section 3, the existence of a unique solution for (1)-(5) is proved for a fixed control u and then the convergence of the solution of the regularized problem to that of the original problem is proved. Finally, Section 4 gives numerical results, which justifies the theoretical results obtained in Section 3.

2. The Regularized Problem

In this section, we first present a regularized problem and recall some related results on existence, uniqueness and regularity.

With $\epsilon > 0$ as regularization parameter, we replace the Heaviside function by a regularized function $\mathcal{H}_\epsilon \in C^{1,1}(\mathbb{R})$, where $\mathcal{H}_\epsilon$ is a monotone approximation of the Heaviside function satisfying $\mathcal{H}_\epsilon(x) = 0$ for $x \leq 0$. Thus, we arrive at the following regularized problem:

$$\min_{u_\epsilon \in U_{ad}} J(\theta_\epsilon, a_\epsilon, u_\epsilon) \text{ subject to} \tag{1}$$

$$\partial_t a_\epsilon = f_\epsilon(\theta_\epsilon, a_\epsilon) \ \text{ in } Q, \tag{2}$$

$$a_\epsilon(0) = 0 \ \text{ in } \Omega, \tag{3}$$

$$\rho c_p \partial_t \theta_\epsilon - K \triangle \theta_\epsilon = -\rho L \partial_t a_\epsilon + \alpha u_\epsilon \ \text{ in } Q, \tag{4}$$

$$\theta_\epsilon(0) = \theta_0 \ \text{ in } \Omega, \tag{5}$$

$$\frac{\partial \theta_\epsilon}{\partial n} = 0 \ \text{ on } \Sigma, \tag{6}$$

where

$$J(\theta_\epsilon, a_\epsilon, u_\epsilon) = \frac{\beta_1}{2}\int_\Omega |a_\epsilon(T) - a_d|^2 dx + \frac{\beta_2}{2}\int_0^T\int_\Omega [\theta_\epsilon - \theta_m]_+^2 dx ds + \frac{\beta_3}{2}\int_0^T |u_\epsilon|^2\, ds,$$

and

$$f_\epsilon(\theta_\epsilon, a_\epsilon) = \frac{1}{\tau(\theta_\epsilon)}(a_{eq}(\theta_\epsilon) - a_\epsilon)\mathcal{H}_\epsilon(a_{eq}(\theta_\epsilon) - a_\epsilon).$$

We now make the following assumptions [5]:

(A1) $a_{eq}(s) \in (0,1)$ for all $s \in \mathbb{R}$ and $\|a_{eq}\|_{C^1(\mathbb{R})} \leq c_a$;
(A2) $0 < \underline{\tau} \leq \tau(s) \leq \bar{\tau}$ for all $s \in \mathbb{R}$ and $\|\tau\|_{C^1(\mathbb{R})} \leq c_\tau$;
(A3) $\theta_0 \in H^1(\Omega)$, $\theta_0 \leq \theta_m$ a.e. in Ω, where the constant $\theta_m > 0$ denotes the melting temperature of the steel;
(A4) $\alpha \in L^\infty(I, L^\infty(\Omega))$;
(A5) $u \in L^2(I)$;
(A6) $a_d \in L^\infty(\Omega)$ with $0 \leq a_d \leq 1$ a.e. in Ω.

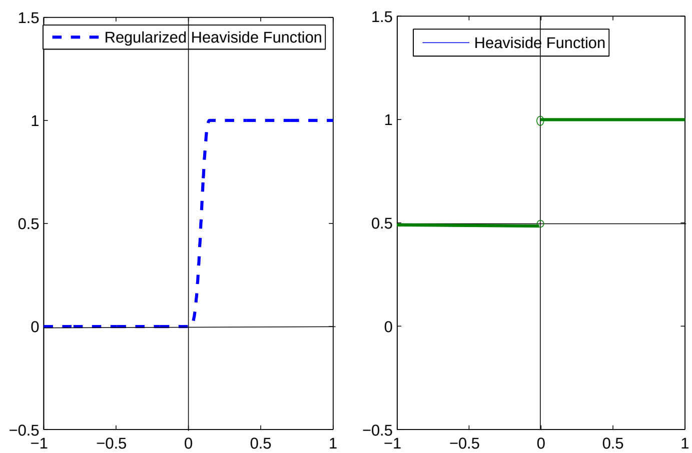

FIGURE 2. Regularized Heaviside($\mathcal{H}_\epsilon$) and Heaviside($\mathcal{H}$) Functions

Below, we discuss the weak formulation corresponding to the regularized problem (1)-(6). Let $X = \{v \in L^2(I; H^1(\Omega)) : v_t \in L^2(I; H^{-1}(\Omega))\}$. The Hilbert space $H^1(\Omega)$ and its dual $H^{-1}(\Omega)$ build a Gelfand triple $H^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$. The duality pairing between $H^1(\Omega)$ and $H^{-1}(\Omega)$ is denoted by $< \cdot,\cdot >= < \cdot,\cdot >_{H^{-1}(\Omega)\times H^1(\Omega)}$. Let the inner product and norm in $L^2(I)$ be denoted by $(\cdot,\cdot)_{L^2(I)}$ and $\|\cdot\|_{L^2(I)}$, respectively. Now the weak formulation corresponding to the regularized problem (1)-(6) is given by

$$\min_{u_\epsilon \in U_{ad}} J(\theta_\epsilon, a_\epsilon, u_\epsilon) \text{ subject to} \tag{7}$$

$$(\partial_t a_\epsilon, w) = (f_\epsilon(\theta_\epsilon, a_\epsilon), w), \tag{8}$$

$$a_\epsilon(0) = 0, \tag{9}$$

$$\rho c_p(\partial_t \theta_\epsilon, v) + K(\nabla\theta_\epsilon, \nabla v) = -\rho L(\partial_t a_\epsilon, v) + (\alpha u_\epsilon, v), \tag{10}$$

$$\theta_\epsilon(0) = \theta_0, \tag{11}$$

for all $(w, v) \in L^2(\Omega) \times H^1(\Omega)$, a.e. in I, where $f_\epsilon(\theta, a) = \frac{1}{\tau(\theta)}(a_{eq}(\theta) - a)\mathcal{H}_\epsilon(a_{eq}(\theta) - a)$. The following theorem ([12], Theorem 2.1) ensures the existence and uniqueness of solution of the regularized problem (8)-(11).

Theorem 2.1. *Suppose that assumptions* (A1)-(A6) *are satisfied. Then, for a given $u_\epsilon \in U_{ad}$ the system* (8)-(11) *has a unique solution*

$$(\theta_\epsilon, a_\epsilon) \in H^{1,1}(Q) \times W^{1,\infty}(I; L^\infty(\Omega)),$$

where $H^{1,1} = L^2(I; H^1(\Omega)) \cap H^1(I; L^2(\Omega))$. Moreover, a_ϵ satisfies

$$0 \le a_\epsilon < 1 \ a.e. \ in \ Q.$$

Remark 2.1. *Due to* (A1)-(A2) *and nature of the regularized Heaviside function, there exists a constant $c_f > 0$ independent of θ_ϵ and a_ϵ such that*

$$\max(\|f_\epsilon(\theta_\epsilon, a_\epsilon)\|_{L^\infty(Q)}, \|\frac{\partial f_\epsilon}{\partial a}(\theta_\epsilon, a_\epsilon)\|_{L^\infty(Q)}, \|\frac{\partial f_\epsilon}{\partial \theta}(\theta_\epsilon, a_\epsilon)\|_{L^\infty(Q)}) \le c_f$$

for all $(\theta_\epsilon, a_\epsilon) \in L^2(Q) \times L^\infty(Q)$.

The existence of a global solution to the optimal control problem (1)-(6) is guaranteed by the following theorem ([12], Theorem 2.3).

Theorem 2.2. *Suppose that the assumptions* (A1)-(A6) *are satisfied. Then the optimal control problem* (1)-(6) *has at least one(global) solution.*

The next lemma shows the stability result for the temperature θ_ϵ when $a_\epsilon \in W^{1,\infty}(I, L^\infty(\Omega))$.

Lemma 2.1. *Suppose that the assumptions* (A1)-(A6) *are satisfied. Then, for a fixed $u_\epsilon \in U_{ad}$, the first component of the solution $(\theta_\epsilon, a_\epsilon) \in H^{1,1} \times W^{1,\infty}(I, L^\infty(\Omega))$ of* (4)-(6), *satisfies*

$$\|\theta_\epsilon\|_{L^\infty(I, H^1(\Omega))} \le C,$$

where $C > 0$ is a finite constant.

Proof. Set $v = \theta_\epsilon$ in (10) to obtain

$$\frac{\rho c_p}{2}\frac{d}{dt}\|\theta_\epsilon\|^2 + \|\nabla\theta_\epsilon\|^2 = -\rho L(\partial_t a_\epsilon, \theta_\epsilon) + (\alpha u_\epsilon, \theta_\epsilon)$$

Integrating from 0 to t, using Cauchy-Schwarz and Young's inequality, we find that

$$\|\theta_\epsilon(t)\|^2 + \int_0^t \|\nabla\theta_\epsilon\|^2 dt \le C\Big(\|\theta_0\|^2 + \int_0^t (\|\partial_t a_\epsilon\|^2 + |u_\epsilon|^2)dt + \int_0^t \|\theta_\epsilon\|^2 dt\Big). \tag{12}$$

Using Gronwall's Lemma, it follows that

$$\|\theta_\epsilon(t)\|^2 \le C\Big(\|\theta_0\|^2 + \int_0^t (\|\partial_t a_\epsilon\|^2 + |u_\epsilon|^2)dt\Big). \tag{13}$$

Now, multiply (4) with $\partial_t\theta_\epsilon$, integrate over Ω and use Cauchy-Schwarz and Young's inequality to obtain

$$\rho c_p\|\partial_t\theta_\epsilon\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\theta_\epsilon\|^2 \le C\Big(\|\partial_t a_\epsilon\|^2 + |u_\epsilon|^2\Big) + \frac{\rho c_p}{2}\|\partial_t\theta_\epsilon\|^2.$$

Hence, integrating from 0 to t, we arrive at

$$(14)\quad \int_0^t \|\partial_t\theta_\epsilon\|^2 dt + \|\nabla\theta_\epsilon(t)\|^2 \quad\leq\quad C\bigg(\|\nabla\theta_0\|^2 + \int_0^t \Big(\|\partial_t a_\epsilon\|^2 + |u_\epsilon|^2\Big) dt\bigg).$$

Since $a_\epsilon \in W^{1,\infty}(I, L^\infty(\Omega))$ and $u_\epsilon \in U_{ad}$, using (A3), (13) and (14) we obtain the desired result. This completes the proof. □

3. Convergence Analysis

Below, we first present the weak formulation corresponding to the (6)-(7):

$$(1)\qquad \min_{u\in U_{ad}} J(\theta, a, u) \text{ subject to}$$

$$\begin{aligned}
(2)&\qquad & (\partial_t a, w) &= (f_+(\theta,a), w),\\
(3)&\qquad & a(0) &= 0,\\
(4)&\qquad & \rho c_p(\partial_t\theta, v) + K(\nabla\theta, \nabla v) &= -\rho L(\partial_t a, v) + (\alpha u, v),\\
(5)&\qquad & \theta(0) &= \theta_0,
\end{aligned}$$

for all $(w, v) \in L^2(\Omega)\times H^1(\Omega)$ a.e. in I, where $f_+(\theta,a) = \frac{1}{\tau(\theta)}(a_{eq}(\theta)-a)\mathcal{H}(a_{eq}(\theta)-a)$.

In this section, we prove that for a fixed control $u \in U_{ad}$, solution to the problem (8)-(11) converges to the solution of (2)-(5). Then we discuss the existence of solution of the optimal control problem and finally, convergence of the regularized problem as the regularized parameter tends to zero.

Theorem 3.1. *Let the assumptions* (A1)-(A6) *hold true. Then, for a fixed* $u \in U_{ad}$*, there exists a unique solution* (θ, a) *to* (2)-(5) *and for all* $\epsilon \in (0,1)$*,* $t \in I$*, the following estimate holds:*

$$(6)\qquad \|a(t) - a_\epsilon(t)\| + \|\theta(t) - \theta_\epsilon(t)\| \leq C(\Omega, T)\epsilon,$$

where $C(\Omega, T)$ *is a positive constant and* $(\theta_\epsilon, a_\epsilon)$ *is the solution to the problem* (8)-(11) *for the fixed* $u \in U_{ad}$.

Proof. From Theorem 2.1, the sequence $\{(\theta_\epsilon, a_\epsilon)\}$ is uniformly bounded in $H^{1,1}\times W^{1,\infty}(I, L^\infty(\Omega))$, and from Lemma 2.1, the sequence $\{\theta_\epsilon\}$ is uniformly bounded in $L^\infty(I, H^1(\Omega))$. Therefore, using weak and weak* compactness arguments and $H^1(\Omega)$ being compactly imbedded in $L^2(\Omega)$, we obtain

$$\begin{aligned}
(7)&\qquad & \theta_\epsilon &\longrightarrow \theta \text{ strongly in } C(I, L^2(\Omega)),\\
(8)&\qquad & \theta_\epsilon &\longrightarrow \theta \text{ weakly in } H^{1,1},\\
(9)&\qquad & a_\epsilon &\longrightarrow a \text{ weak}^* \text{ in } W^{1,\infty}(I, L^\infty(\Omega)).
\end{aligned}$$

For $\theta \in C(I, L^2(\Omega))$ and f_+ being globally Lipschitz continuous, (2)-(3) has a unique solution a (say). Now subtracting (8) from (2), putting $w = a - a_\epsilon$ and using Cauchy-Schwarz's and Young's inequality, we obtain

$$\frac{d}{dt}\|a - a_\epsilon\|^2 \leq \|f_\epsilon(\theta_\epsilon, a_\epsilon) - f_+(\theta, a)\|^2 + \|a - a_\epsilon\|^2.$$

Now integrating from 0 to t, it follows that

$$(10)\; \|(a - a_\epsilon)(t)\|^2 \leq C\bigg(\int_0^t \|f_\epsilon(\theta_\epsilon, a_\epsilon) - f_+(\theta, a)\|^2 dt + \int_0^t \|a - a_\epsilon\|^2 dt\bigg).$$

Note that using triangle inequality, we arrive at

$$(11)\; \|f_\epsilon(\theta_\epsilon, a_\epsilon) - f_+(\theta, a)\|^2 \leq C\bigg(\|f_\epsilon(\theta_\epsilon, a_\epsilon) - f_\epsilon(\theta, a)\|^2 + \|f_\epsilon(\theta, a) - f_+(\theta, a)\|^2\bigg).$$

For the first term on the right hand side of (11), use Remark 2.1 to obtain

$$\|f_\epsilon(\theta_\epsilon, a_\epsilon) - f_\epsilon(\theta, a)\|^2 \leq C\bigg(\|\theta_\epsilon - \theta\|^2 + \|a_\epsilon - a\|^2\bigg). \tag{12}$$

For the second term on the right hand side of (11), using the assumption (A2), we find that

$$\begin{aligned}
&\|f_\epsilon(\theta, a) - f_+(\theta, a)\|^2 \\
&\quad\leq \frac{1}{\mathcal{T}}\|(a_{eq}(\theta) - a)\mathcal{H}(a_{eq}(\theta) - a) - (a_{eq}(\theta) - a)\mathcal{H}_\epsilon(a_{eq}(\theta) - a)\|^2 \\
&\quad= \frac{1}{\mathcal{T}}\int_\Omega (a_{eq}(\theta) - a)^2(\mathcal{H}(a_{eq}(\theta) - a) - \mathcal{H}_\epsilon(a_{eq}(\theta) - a))^2 dx.
\end{aligned}$$

Let $\Omega_1 = \{x \in \Omega : a_{eq} - a \leq 0 \text{ or } a_{eq} - a \geq \epsilon\}$ and $\Omega_2 = \{x \in \Omega : 0 < a_{eq} - a < \epsilon\}$. Since $\Omega = \Omega_1 \bigcup \Omega_2$, we arrive at

$$\begin{aligned}
&\|f_\epsilon(\theta, a) - f_+(\theta, a)\|^2 \\
&\quad\leq \frac{1}{\mathcal{T}}\int_{\Omega_1} (a_{eq}(\theta) - a)^2(\mathcal{H}(a_{eq}(\theta) - a) - \mathcal{H}_\epsilon(a_{eq}(\theta) - a))^2\, dx \\
&\qquad + \frac{1}{\mathcal{T}}\int_{\Omega_2} (a_{eq}(\theta) - a)^2(\mathcal{H}(a_{eq}(\theta) - a) - \mathcal{H}_\epsilon(a_{eq}(\theta) - a))^2\, dx.
\end{aligned}$$

From Figure 2, it follows that

$$\|f_\epsilon(\theta, a) - f_+(\theta, a)\|^2 \leq \frac{1}{\mathcal{T}}\int_{\Omega_2} \epsilon^2 dx \leq C(\Omega)\epsilon^2. \tag{13}$$

Substituting (13) in (11), we obtain

$$\|f_\epsilon(\theta_\epsilon, a_\epsilon) - f_+(\theta, a)\|^2 \leq C\bigg(\|\theta_\epsilon - \theta\|^2 + \|a_\epsilon - a\|^2 + \epsilon^2\bigg). \tag{14}$$

Substituting (14) in (10), we find using Gronwall's lemma that

$$\|a - a_\epsilon\|^2 \leq C(\Omega, T)\bigg(\int_0^t \|\theta - \theta_\epsilon\|^2 dt + \epsilon^2\bigg)$$

Using (7), we arrive at

$$a_\epsilon \longrightarrow a \text{ strongly in } L^\infty(I, L^2(\Omega)). \tag{15}$$

From (14), using (7), (15), we obtain as $\epsilon \longrightarrow 0$

$$f_\epsilon(\theta_\epsilon, a_\epsilon) \longrightarrow f_+(\theta, a). \tag{16}$$

Now letting $\epsilon \to 0$ in (8)-(11) and using (7)-(9), (15), (16), we obtain the existence of solution of (2)-(5). For proving uniqueness we proceed as follows. If possible, let (θ_1, a_1) and (θ_2, a_2) be two different solutions of (2)-(5). Therefore, from (4), we obtain

$$\rho c_p(\partial_t(\theta_1 - \theta_2), v) + K(\nabla(\theta_1 - \theta_2), \nabla v) = -\rho L(f_+(\theta_1, a_1) - f_+(\theta_2, a_2), v). \tag{17}$$

Setting $v = \theta_1 - \theta_2$ in (17), use Young's inequality to obtain

$$\frac{d}{dt}\|\theta_1 - \theta_2\|^2 + \|\nabla(\theta_1 - \theta_2)\|^2 \leq C\bigg(\|f_+(\theta_1, a_1) - f_+(\theta_2, a_2)\|^2 + \|\theta_1 - \theta_2\|^2\bigg) \tag{18}$$

Similarly from (2), we arrive at

$$\frac{d}{dt}\|a_1 - a_2\|^2 \leq C\bigg(\|f_+(\theta_1, a_1) - f_+(\theta_2, a_2)\|^2 + \|a_1 - a_2\|^2\bigg). \tag{19}$$

Adding (18) and (19), using Lipschitz continuity of the functions a_{eq}, f_+, integrating from 0 to T and finally using Gronwall's lemma, we obtain

$$\|\theta_1 - \theta_2\|^2 + \|a_1 - a_2\|^2 \quad \leq \quad 0,$$

which proves uniqueness.

To prove (6), subtract (8) from (2), put $w = a - a_\epsilon$, use Cauchy-Schwarz and Young's inequality to find that

$$\frac{d}{dt}\|a - a_\epsilon\|^2 \quad \leq \quad \Big(\|f_+(\theta, a) - f_\epsilon(\theta_\epsilon, a_\epsilon)\|^2 + \|a - a_\epsilon\|^2\Big). \tag{20}$$

Now integrating from 0 to t, we obtain

$$\|a(t) - a_\epsilon(t)\|^2 \quad \leq \quad \Big(\int_0^t \|f_+(\theta, a) - f_\epsilon(\theta_\epsilon, a_\epsilon)\|^2 dt + \int_0^t \|a - a_\epsilon\|^2 dt\Big). \tag{21}$$

Similarly, for a fixed $u \in U_{ad}$ and $u_\epsilon = u$, subtract (10) from (4), substitute $v = \theta - \theta_\epsilon$, integrate from 0 to t and use (20) to arrive at

$$\begin{aligned} \|\theta(t) - \theta_\epsilon(t)\|^2 \quad &+ \quad \int_0^t \|\nabla(\theta - \theta_\epsilon)\|^2 dt \\ &\leq \quad C\Big(\int_0^t \|f_+(\theta, a) - f_\epsilon(\theta_\epsilon, a_\epsilon)\|^2 dt + \int_0^t \|a - a_\epsilon\|^2 dt \\ &\qquad + \int_0^t \|\theta - \theta_\epsilon\|^2 dt\Big). \end{aligned} \tag{22}$$

Adding (21) and (22), we find that

$$\begin{aligned} \|a(t) - a_\epsilon(t)\|^2 \quad &+ \quad \|\theta(t) - \theta_\epsilon(t)\|^2 \\ &\leq \quad C\Big(\int_0^t \|f_+(\theta, a) - f_\epsilon(\theta_\epsilon, a_\epsilon)\|^2 dt + \int_0^t \|a - a_\epsilon\|^2 dt \\ &\qquad + \int_0^t \|\theta - \theta_\epsilon\|^2 dt\Big). \end{aligned} \tag{23}$$

Using (14), we now obtain

$$\|a(t) - a_\epsilon(t)\|^2 + \|\theta(t) - \theta_\epsilon(t)\|^2 \quad \leq \quad C(\Omega, T)\Big(\epsilon^2 + \int_0^t (\|\theta - \theta_\epsilon\|^2 + \|a - a_\epsilon\|^2) dt\Big).$$

Using Gronwall's lemma, we arrive at

$$\|a(t) - a_\epsilon(t)\| + \|\theta(t) - \theta_\epsilon(t)\| \quad \leq \quad C(\Omega, T)\epsilon. \tag{24}$$

This completes the proof. $\square$

Remark 3.1. *Using* (24) *in* (22)*, we obtain*

$$\|\theta - \theta_\epsilon\|_{L^2(I, H^1(\Omega))} \quad \leq \quad C(\Omega, T)\epsilon.$$

Below, we discuss existence of solution to the optimal control problem (1)-(5). For $u^* \in U_{ad}$, let (θ^*, a^*) be a solution of (2)-(5). Now, the existence of a unique solution to the state equations (2)-(5) ensures the existence of a control-to-state mapping $u \mapsto (\theta, a) = (\theta(u), a(u))$ through (2)-(5). By means of this mapping, we introduce the reduced cost functional $j : U_{ad} \longrightarrow \mathbb{R}$ as

$$j(u) = J(\theta(u), a(u), u). \tag{25}$$

Then the optimal control problem can be equivalently reformulated as

$$\min_{u\in U_{ad}} j(u) \text{ subject to the dynamical system } (2)-(5). \tag{26}$$

Theorem 3.2. (1)-(5) *has at least one solution* $(\theta^*, a^*, u^*) \in X \times X \times U_{ad}$.

Proof. Let $l = \inf_{u\in U_{ad}} j(u)$ and $\{u_n\}_{n\in\mathbb{N}} \subset U_{ad}$ be a minimizing sequence such that

$$j(u_n) \longrightarrow l \text{ in } \mathbb{R}. \tag{27}$$

Since U_{ad} is bounded, the sequence $\{u_n\}$ is bounded uniformly in $L^2(I)$. Therefore, one can extract a subsequence $\{u_n\}$(say), such that

$$u_n \longrightarrow u^* \text{ weakly in } L^2(I).$$

Since the admissible space U_{ad} is a closed and convex subset of $L^2(I)$, it is weakly closed in $L^2(I)$ and hence, $u^* \in U_{ad}$. Corresponding to each u_n, we obtain $(\theta_n, a_n) \in H^{1,1} \times W^{1,\infty}(I, L^\infty(\Omega))$ satisfying (2)-(5), and $\theta_n \in L^\infty(I, H^1(\Omega))$. Therefore, we can extract a subsequence $\{(\theta_n, a_n)\}$(again say) such that

$$\begin{array}{rcl} \theta_n & \longrightarrow & \theta^* \text{ weakly in } H^{1,1} \\ \theta_n & \longrightarrow & \theta^* \text{ strongly in } C(I, L^2(\Omega)) \\ a_n & \longrightarrow & a^* \text{ weak}^* \text{ in } W^{1,\infty}(I, L^\infty(\Omega)) \\ a_n & \longrightarrow & a^* \text{ strongly in } L^\infty(I, L^2(\Omega)). \end{array}$$

Now letting $n \to \infty$ in the following problem

$$\begin{array}{rcl} (\partial_t a_n, w) & = & (f_+(\theta_n, a_n), w) \ \forall w \in V, \\ a_n(0) & = & 0 \\ \rho c_p(\partial_t \theta_n, v) + K(\nabla\theta_n, \nabla v) & = & -\rho L(\partial_t a_n, v) + (\alpha u_n, v) \ \forall v \in V, \\ \theta_n(0) & = & \theta_0, \end{array}$$

we obtain (θ^*, a^*) as a unique solution of (2)-(5) corresponding to the control $u^* \in U_{ad}$ and hence, (θ^*, a^*, u^*) is an admissible solution. Now we claim that it is an optimal solution. Since j is lower semi-continuous,

$$j(u^*) \leq \liminf_{n\to\infty} j(u_n)$$

and using (27), we obtain

$$j(u^*) \leq l.$$

Thus, u^* is a minimizer of the cost functional j and (θ^*, a^*, u^*) is an optimal solution. This completes the rest of the proof. □

3.1. Convergence of the Control Function.

Theorem 3.3. *Let* u^*_ϵ *be the optimal control of* (7)-(11), *for* $0 < \epsilon < 1$. *Then,* $\lim_{\epsilon\to 0} u^*_\epsilon = u^*$ *exists in* $L^2(I)$ *and* u^* *is an optimal control of* (1)-(5).

Proof: Since u^*_ϵ is an optimal control, we obtain

$$\|u^*_\epsilon\|_{L^2(I)} \leq M, \ 0 < \epsilon < 1,$$

that is, $\{u^*_\epsilon\}_{0<\epsilon<1}$ is uniformly bounded in $L^2(I)$. Thus, it is possible to extract a subsequence say $\{u^*_\epsilon\}_{0<\epsilon<1}$ in $L^2(I)$ such that

$$u^*_\epsilon \longrightarrow u^* \text{ weakly in } L^2(I). \tag{28}$$

Since the admissible space U_{ad} is a closed and convex subset of $L^2(I)$, it is weakly closed in $L^2(I)$ and hence, $u^* \in U_{ad}$. Now corresponding to each u^*_ϵ, there exists solution $(\theta^*_\epsilon, a^*_\epsilon)$ to (8)-(11). Also from Theorem 3.1, we observe that

$$\theta^*_\epsilon \longrightarrow \theta^* \text{ weakly in } H^{1,1}, \tag{29}$$

$$\theta^*_\epsilon \longrightarrow \theta^* \text{ strongly in } C(I, L^2(\Omega)), \tag{30}$$

$$a^*_\epsilon \longrightarrow a^* \text{ weak}^* \text{ in } W^{1,\infty}(I, L^\infty(\Omega)), \tag{31}$$

$$a^*_\epsilon \longrightarrow a^* \text{ strongly in } L^\infty(I, L^2(\Omega)). \tag{32}$$

Using the arguments as in the proof of Theorem 3.1, we note that

$$f_\epsilon(\theta^*_\epsilon, a^*_\epsilon) \longrightarrow f_+(\theta^*, a^*) \text{ strongly in } L^\infty(I, L^2(\Omega)). \tag{33}$$

Now passing limit as $\epsilon \to 0$ and using (29)-(32), (33) in the following problem;

$$\begin{aligned} (\partial_t a^*_\epsilon, w) &= (f_\epsilon(\theta^*_\epsilon, a^*_\epsilon), w) \ \forall w \in L^2(\Omega), \text{ a.e. in } I, \\ a^*_\epsilon(0) &= 0, \\ \rho c_p(\partial_t \theta^*_\epsilon, v) + K(\nabla \theta^*_\epsilon, \nabla v) &= -\rho L(\partial_t a^*_\epsilon, v) + (\alpha u^*_\epsilon, v) \ \forall v \in H^1(\Omega), \text{ a.e. in } I, \\ \theta^*_\epsilon(0) &= \theta_0, \end{aligned}$$

we obtain that (u^*, θ^*, a^*) is an admissible solution for the optimal control problem (1)-(5). It now remains to show that (u^*, θ^*, a^*) is an optimal solution.

Let $(\bar{u}^*, \bar{\theta}^*, \bar{a}^*)$ be another optimal solution of (1)-(5). Now, consider the auxiliary problem:

$$(\partial_t a_\epsilon, w) = (f_\epsilon(\theta_\epsilon, a_\epsilon), w), \tag{34}$$

$$a_\epsilon(0) = 0, \tag{35}$$

$$\rho c_p(\partial_t \theta_\epsilon, v) + K(\nabla \theta_\epsilon, \nabla v) = -\rho L(\partial_t a_\epsilon, v) + (\alpha \bar{u}^*, v), \tag{36}$$

$$\theta_\epsilon(0) = \theta_0, \tag{37}$$

for all $(w, v) \in L^2(\Omega) \times H^1(\Omega)$ and a.e. in I. Then by Theorem 2.1, there exists a solution to (34)-(37), say $(\bar{\theta}_\epsilon, \bar{a}_\epsilon) \in H^{1,1} \times W^{1,\infty}(I, L^\infty(\Omega))$. Similar to (29)-(32), we arrive at

$$\bar{\theta}_\epsilon \longrightarrow \bar{\theta} \text{ weakly in } H^{1,1}, \tag{38}$$

$$\bar{\theta}_\epsilon \longrightarrow \bar{\theta} \text{ strongly in } C(I, L^2(\Omega)), \tag{39}$$

$$\bar{a}_\epsilon \longrightarrow \bar{a} \text{ weakly in } W^{1,\infty}(I, L^\infty(\Omega)), \tag{40}$$

$$\bar{a}_\epsilon \longrightarrow \bar{a} \text{ strongly in } L^\infty(I, L^2(\Omega)). \tag{41}$$

Now letting $\epsilon \to 0$ in (34)-(37), we obtain that $(\bar{\theta}, \bar{a})$ is a unique solution of (2)-(5) with respect to the control $\bar{u}^*$. Since the solution to (2)-(5) for a fixed control is unique, we find that $\bar{\theta} = \bar{\theta}^*$ and $\bar{a} = \bar{a}^*$.

Since u^*_ϵ is the optimal control for (7)-(11), we have

$$j(u^*_\epsilon) \le j(\bar{u}^*). \tag{42}$$

Now letting $\epsilon \to 0$ in (42) and using (28), we obtain

$$j(u^*) \le j(\bar{u}^*). \tag{43}$$

(43) indicates that if $\bar{u}^*$ is another optimal control, then $j(\bar{u}^*)$ will be greater than or equal to $j(u^*)$, which shows that u^* is an optimal solution. The equality sign in (43) shows the possibility of non- unique optimal control.

Next we need to show that $\lim\limits_{\epsilon \to 0} \|u^*_\epsilon - u\|_{L^2(I)} = 0$. Since $u^*_\epsilon \longrightarrow u^*$ weakly in $L^2(\Omega)$, it is enough show that $\lim\limits_{\epsilon \to 0} \|u^*_\epsilon\|_{L^2(I)} = \|u^*\|_{L^2(I)}$.

Using Theorem 3.1 and (28), we find that

$$\begin{aligned}\lim_{\epsilon\to 0}\frac{\beta_3}{2}\|u_\epsilon^*\|_{L^2(I)}^2 &= \lim_{\epsilon\to 0}\left(J(\theta_\epsilon^*,a_\epsilon^*,u_\epsilon^*)-\frac{\beta_1}{2}\|a_\epsilon^*(T)-a_d\|^2-\frac{\beta_2}{2}\|[\theta_\epsilon^*-\theta_m]_+\|_{I,\Omega}^2\right)\\ &= J(\theta^*,a^*,u^*)-\frac{\beta_1}{2}\|a^*(T)-a_d\|^2-\frac{\beta_2}{2}\|[\theta^*-\theta_m]_+\|_{I,\Omega}^2\\ &= \frac{\beta_3}{2}\|u^*\|_{L^2(I)}^2.\end{aligned}$$

Therefore, we have $\lim_{\epsilon\to 0}\|u_\epsilon^*\|_{L^2(I)}=\|u^*\|_{L^2(I)}$ and $\lim_{\epsilon\to 0}\|u_\epsilon^*-u^*\|=0$. This completes the rest of the proof. □

4. Numerical Experiment

In this section, we carry out a numerical experiment for the optimal control problem using the non-linear conjugate gradient method [12].

The computational domain is chosen as $\Omega=(0,5)\times(-1,0)$ and T is chosen as 5.25. In (8)-(11), we consider the physical data as $\rho c_p = 4.91\frac{J}{cm^3K}, k = 0.64\frac{J}{cmKs}$ and $\rho L = 627.9\frac{J}{cm^3}$ [12]. The regularized monotone function $\mathcal{H}_\epsilon$ is chosen as

$$\mathcal{H}_\epsilon(s)=\begin{cases}1 & s\geq\epsilon\\ 10(\frac{s}{\epsilon})^6-24(\frac{s}{\epsilon})^5+15(\frac{s}{\epsilon})^4 & 0\leq s<\epsilon\\ 0 & s<0\end{cases}$$

The initial temperature θ_0 and the melting temperature θ_m are chosen as 20 and 1800, respectively. The pointwise data for $a_{eq}(\theta)$ and $\tau(\theta)$ are given by

θ	730	830	840	930
$a_{eq}(\theta)$	0	0.91	1	1
$\tau(\theta)$	1	0.2	0.18	0.05

We use a cubic spline interpolation to obtain approximations for the functions $a_{eq}(\theta)$ and $\tau(\theta)$.The shape function $\alpha(x,y,t)=\frac{4k_1A}{\pi D^2}exp(-\frac{2(x-vt)^2}{D^2})exp(k_1y)$, where $D=0.47cm$, $k_1=60/cm$, $A=0.3cm$ and $v=1cm/s$. In the nonlinear conjugate gradient method, tolerance is chosen as 10^{-7}.Also, we choose $\beta_1=5000,\beta_2=1000$ and $\beta_3=10^{-3}$.The main aim of this experiment is to achieve a constant hardening depth of 1mm, see Figure 3, with expected order of convergence $O(\epsilon)$ for the approximation of (θ,a) and u. While applying the non-linear conjugate method for the optimal control problem, we choose the initial control u_0 as 1200. The finite

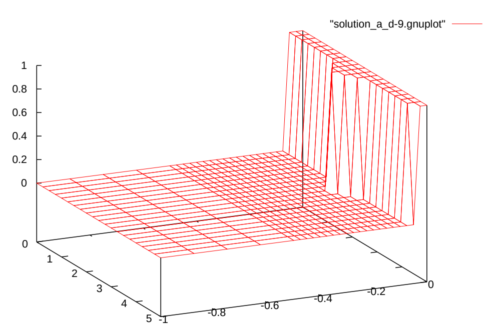

FIGURE 3. Goal a_d to be achieved for the volume fraction of austenite

element *a priori* estimates developed in [10], yields the order of convergence

$$\|\theta_\epsilon - \theta_{\epsilon,hk}\| + \|a_\epsilon - a_{\epsilon,hk}\| = \mathcal{O}(h^2 + k),$$

where $(\theta_{\epsilon,hk}, a_{\epsilon,hk})$ is the solution to (8)-(11) obtained after a finite element discretization, h and k being the space and time discretization parameters respectively. Therefore using Theorem 3.1, we have

$$\|\theta - \theta_{\epsilon,hk}\| + \|a - a_{\epsilon,hk}\| = \mathcal{O}(h^2 + k + \epsilon),$$

The mesh used for space discretization is much more refined near the area, where hardness is desired. With the initial control as u_0, we find that $\|a^0_{\epsilon,hk}(T) - a_d\| = 0.239547$, where $a^0_{\epsilon,hk} = 0.397440$ corresponds to the austenite value for initial control u_0, which is being reduced to $\|a^{optimal}_{\epsilon,hk}(T) - a_d\| = 0.069105$ after applying non-linear conjugate method. A comparison of Figure 3 and Figure 4 shows that the goal of uniform hardening depth is nearly achieved. Also, the state constraint that $\|\theta_{\epsilon,hk}\|_{L^\infty(I,L^\infty(\Omega))} < 1800$ is satisfied, since $\|\theta_{\epsilon,hk}\|_{L^\infty(I,L^\infty(\Omega))} < 1000$, see Figure 5. Figure 6 shows the evolution of control variable (laser energy) with

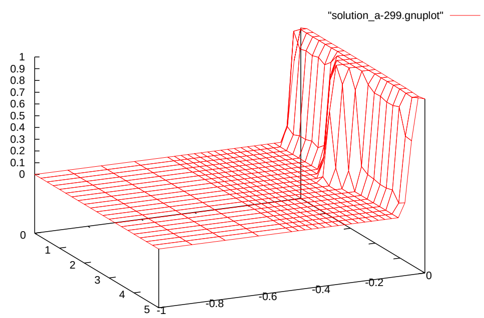

FIGURE 4. The volume fraction of the austenite at time $t = T$

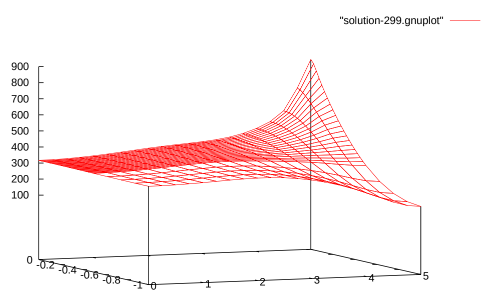

FIGURE 5. The temperature at time $t = T$

time. Figure 7 and 8 represents the L^2 errors in temperature, austenite formation and control, respectively, as a function of regularization parameter ϵ in the log-log scale. For the purpose of implementation, the values of epsilon were taken as $\{0.5, 0.10, 0.15, 0.20, 0.25\}$. The numerical results obtained confirms the theoretical results obtained in Theorem 3.1.

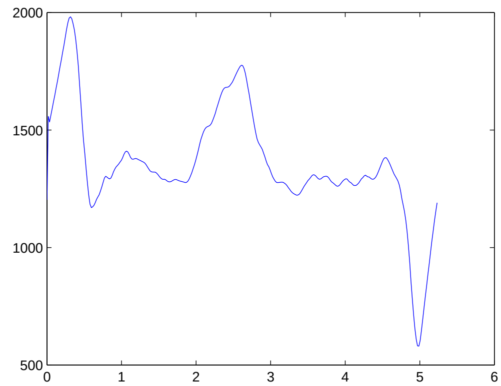

FIGURE 6. Laser energy

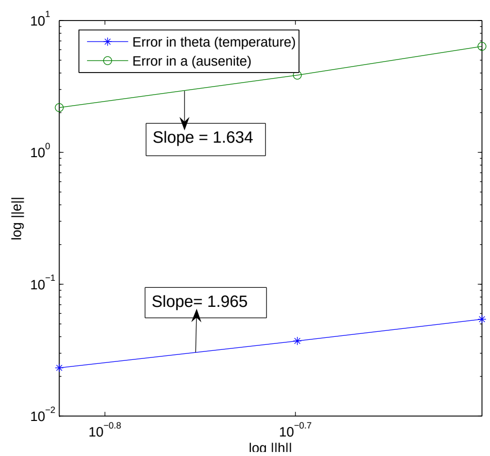

FIGURE 7. Evolution of state error as a function of ϵ.

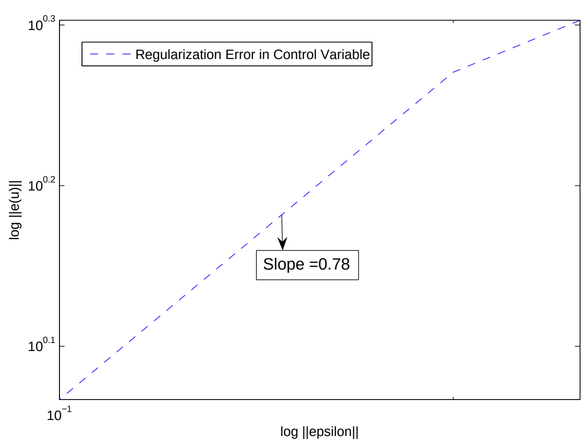

FIGURE 8. Evolution of control error as a function of ϵ.

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