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LOCAL PROJECTION FINITE ELEMENT STABILIZATION FOR DARCY FLOW

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Abstract. Local projection based stabilized finite element methods for the solution of Darcy flow offer several advantages as compared to mixed Galerkin methods. In particular, the avoidance of stability conditions between finite element spaces, the efficiency in solving the reduced linear algebraic system, and the convenience of using equal order continuous approximations for all variables. In this paper we analyze the pressure gradient method for Darcy flow and investigate its stability and convergence properties.

Key Words. Stabilized finite elements, Darcy equations, convergence, error estimates.

1. Introduction

Numerical methods for Darcy equations are traditionally-based on a primal single field formulation for the pressure or on the mixed two field velocity-pressure formulation. It is well known that the choice of the finite element spaces, for the mixed formulation, is subject to the inf-sup stability condition ([10]). This has lead to the use of classical mixed Raviart-Thomas and Brezzi-Douglas-Marini finite elements ([10]). This approach though giving good accuracy for both velocity and pressure ([20]) has its draw back complexity.

It has been a few years since stabilized finite element methods have been extended to the Darcy equations (see, [23], [5], [6], and [12]). Despite the fact that such methods are well established for fluid flow problems based on Stokes-like operator (see, [19], [17], [32], [7], [3], [16], [21], and [22]). In [23] a term based on the residual of Darcy law is added to the classical Galerkin formulation making the formulation stable for all combination of conforming continuous velocity-pressure approximations. Another class of stabilized methods has been derived using Galerkin methods enriched with bubble functions (see, [1] and [2]). Alternative stabilization techniques based on a least squares formulation have been proposed by ([5]), and ([6]).

Recently, local projection methods that seem less sensitive to the choice of parameters and have better local conservation properties were proposed for Stokes problem (see, [14], and [4]). The two-level pressure gradient method with a projection onto a discontinuous finite element space of a lower degree defined on a coarser grid has been analyzed in [4], [8], [25], [26], and [12]. We note that although the two-level pressure gradient stabilization method gives a slightly bigger discretisation stencil, the drawback is not severe because the pressure-gradient unknowns can be eliminated locally.

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In this paper we analyze the pressure gradient stabilization method for the Darcy equations. As in [29], [30], [27] and [28], the stability of the pressure-gradient method is proved by constructing an interpolant with additional orthogonality property with respect to the projection space. As a result, optimal rates of convergence are found for the velocity and pressure approximations.

2. Variational formulation

Let Ω be a bounded open region of $\mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$. Darcy's law for the flow of a viscous fluid in a permeable medium, and conservation of mass are written as follows

$$\mathbf{u} + \nabla p = \mathbf{0} \quad \text{in } \Omega \tag{1}$$
$$\nabla \cdot \mathbf{u} = f \quad \text{in } \Omega \tag{2}$$
$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \tag{3}$$

where, $\mathbf{u}$ is the Darcy velocity vector, p is the pressure, and $\mathbf{n}$ the outward normal vector.

Let

$$\mathbf{V} = \mathbf{H}_0(div, \Omega) = \left\{ \mathbf{v} \in \left[L^2(\Omega)\right]^2 : \nabla \cdot \mathbf{v} \in L^2(\Omega),\ \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \right\}$$
$$Q = H^1(\Omega) \cap L_0^2(\Omega)$$

where $L_0^2(\Omega)$ denotes the set of square integrable functions with null average.

Define the forms

$$A(\mathbf{u}, p; \mathbf{v}, q) = (\mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) + (q, \nabla \cdot \mathbf{u}) \tag{4}$$

and

$$F(\mathbf{v}, q) = (f, q), \tag{5}$$

for all $(\mathbf{v}, q) \in \mathbf{V} \times Q$, with $(\cdot, \cdot)$, as usual, denoting the L^2-inner product on the region Ω.

Then, the weak formulation of (1)-(3) reads in compact notation as

$$A(\mathbf{u}, p; \mathbf{v}, q) = F(\mathbf{v}, q) \ , \quad \forall (\mathbf{v}, q) \in \mathbf{V} \times Q. \tag{6}$$

A natural norm for the above problem is

$$\|(\mathbf{u}, p)\|_D = \|\mathbf{u}\|_{0,\Omega}^2 + \|\nabla . \mathbf{u}\|_{0,\Omega}^2 + \|p\|_{0,\Omega}^2 \,. \tag{7}$$

Let $\mathbf{V}_h$ and Q_h be finite dimensional subspaces of $\mathbf{V}$ and Q, respectively. Then, the classical Galerkin discrete problem reads

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that:

$$A(\mathbf{u}_h, p_h; \mathbf{v}, q) = F(\mathbf{v}, q) \ , \quad \forall (\mathbf{v}, q) \in \mathbf{V}_h \times Q_h. \tag{8}$$

Note that formulation (8) is stable and accurate only for velocity and pressure approximations satisfying the inf-sup condition (see, for example [10]). In particular, this condition rules out low equal-order C^0 approximations of the pressure and velocity.

3. Local projection stabilization

Let ζ_h be a shape regular partition of the region Ω into quadrilateral elements K (see, for example [9]). Denote by h_K the diameter of element K and by h the maximum diameter of the elements $K \in \zeta_h$. The coarser mesh partition ζ_{2h} of macro-elements M is obtained by grouping sets of neighbouring four elements of ζ_h. In order to guarantee stability and converge of the following method, we assume that for elements $K \subset M \in \zeta_{2h}$ we have $h_K \sim h_M$.
We then define the equal order continuous finite element spaces

$$\mathbf{V}_h = \mathbf{V} \cap (Q_h^k)^2 \tag{9}$$

and

$$Q_h = Q \cap Q_h^k \ , \tag{10}$$

where, Q_h^k denotes the standard continuous isoparametric finite element functions defined by means of a mapping from a reference element. On the reference quadrilateral the approximation functions are polynomials of degree less than or equal to k in each variable. We shall also use P_h^k to denote the space of polynomials of degree less than or equal to k over ζ_h.
Additionally, we define the pressure-gradient finite element space by

$$\mathbf{Y}_{2h} = Y_{2h}^2 = \bigoplus_{M \in \zeta_{2h}} (Q_{2h}^{k-1}(M))^2.$$

where, $Y_{2h} = Q_{2h}^{k-1,disc}$ (respectively $P_{2h}^{k-1,disc}$) denote the finite element spaces of discontinuous functions across elements of ζ_{2h}.
Define the local projection operator $\pi_M : L^2(M) \to Q_{2h}^{k-1}(M)$ by

$$(w - \pi_M w, \phi)_M = 0, \ \forall \phi \in Q_{2h}^{k-1}(M) \tag{11}$$

which generates the global projection $\pi_h : L^2(\Omega) \to Y_{2h}$ defined by

$$(\pi_h w) \mid_M = \pi_M (w \mid_M), \ \forall M \in \zeta_{2h} \ , \forall w \in L^2(\Omega). \tag{12}$$

The fluctuation operator $\kappa_h : L^2(\Omega) \to L^2(\Omega)$ is given by

$$\kappa_h = I - \pi_h \tag{13}$$

where, I denotes the identity operator on $L^2(\Omega)$. For simplicity, we shall use the same notation I, π_M, π_h, and κ_h for vector-valued functions. Thus, $\kappa_h \nabla p$ is to be inderstood as acting on each component of ∇p seperately.
Now, we are ready to introduce the stabilizing term

$$S(p,q) = \sum_{K \in \zeta_h} \alpha_K (\kappa_h \nabla p, \nabla q)_{0,K} = \sum_{K \in \zeta_h} \alpha_K (\kappa_h \nabla p, \kappa_h \nabla q)_{0,K}, \tag{14}$$

where, α_K are element parameters that depend on the local mesh size.
Thus, our stabilized discrete problem reads as:

Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that:

$$A_h(\mathbf{u}_h, p_h; \mathbf{v},q)) = F(\mathbf{v}, q) \ , \quad \forall (\mathbf{v}, q) \in \mathbf{V}_h \times Q_h. \tag{15}$$

with

$$A_h(\mathbf{u}, p; \mathbf{v}, q) = A(\mathbf{u}, p; \mathbf{v},q) + S(p,q). \tag{16}$$

In order to investigate the properties of the bilinear form $A_h(\mathbf{u}, p; \mathbf{v},q)$ on the product space $\mathbf{V}_h \times Q_h$, we introduce the mesh dependent norm

$$\|(\mathbf{v},q)\|_{D_h}^2 = \|\mathbf{v}\|_{0,\Omega}^2 + \|\nabla . \mathbf{v}\|_{0,\Omega}^2 + \|q\|_{0,\Omega}^2 + S(q,q). \tag{17}$$

4. Stability

The main idea in the analysis of local projection methods is the construction of an interpolation operator $j_h : H^1(\Omega) \to Y_{2h}$ with $j_h v \in H_0^1(\Omega)$ for all $v \in H_0^1(\Omega)$, satisfying the usual approximation property

(18)
$$\|v - j_h v\|_{0,K} + h_K |v - j_h v|_{1,K} \le C h_K^s \|v\|_{s,w(K)}, \quad \forall v \in H^s(w(K)),\ 1 \leqslant s \leqslant k+1$$

where, $w(K)$ denotes a certain local neighbourhood of K.
With the additional orthogonal property

$$(v - j_h v, \phi) = 0 \quad , \quad \forall \phi \in Y_{2h},\ \forall v \in H^1(\Omega). \tag{19}$$

Lemma 1. *Let $i_h : H^1(\Omega) \to V_h$ be an interpolation operator such that $i_h v \in H_0^1(\Omega)$ for all $v \in H_0^1(\Omega)$ with the error estimate*

$$\|v - i_h v\|_{0,K} + h_K |v - i_h v|_{1,K} \le C h_K^s \|v\|_{s,w(K)}, \quad \forall v \in H^s(\Omega),\ 1 \leqslant s \leqslant k+1 \tag{20}$$

Further, assume that the local inf-sup condition

$$\inf_{q_h \in Y_{2h}(K)} \sup_{v_h \in V_h(K)} \frac{(v_h, q_h)_K}{\|v_h\|_{0,K} \|q_h\|_{0,K}} \geqslant \beta_1 \tag{21}$$

holds for all $K \in \zeta_{2h}$, with a positive constant β_1 independent of the mesh size. Then, there exists an interpolation operator $j_h : H^1(\Omega) \to Y_{2h}$ with the properties (18) and (19).

For the construction of the interpolation operator j_h we refer to Theorem 2.2 in ([24]).

Remark 2. *Note that condition (21) can be checked using Stenberg's technique on macro-elements $M \in \zeta_{2h}$ which are equivalent to a reference element $\hat{M}$. The* $\inf - \sup$ *condition holds if the the null space N_M is such that*

$$N_M = \left\{ q_h \in Y_{2h}(M) : (v_h, q_h)_M = 0,\ \forall v_h \in V_h(M) \cap H_0^1(M) \right\} = \{0\}. \tag{22}$$

Note also that the fluctuation operator κ_h satisfies the approximation property

$$\|\kappa_h q\|_{0,M} \le C h_M^l |q|_{l,M}, \ \forall q \in H^l(M), \forall M \in \zeta_{2h},\ 0 \leqslant l \leqslant k. \tag{23}$$

Since, The L^2- local projection $\pi_M : L^2(M) \to Y_{2h}(M)$ becomes the identity for the space $Q^{k-1}(M) \subset H^l(M)$, and the kernel of κ_h contains $P^{k-1}(M) \subset Q^{k-1}(M)$. Then, the Bramble-Hilbert Lemma gives the approximation properties stated in assumption (23).

Remark 3. *The justification that the pair $(V_h, Y_{2h}) = (Q_h^k, Q_{2h}^{k-1,disc})$, for $k \geqslant 1$, satisfy (21) follows from (22) using the one-to-one property of the mapping $F_M : \hat{M} \to M$ combined with a positive bilinear function corresponding to the central node of $\hat{M}$ (see,* [24] *and* [18]*). Further, using the same argument we can show that $(V_h, Y_{2h}) = (Q_h^k, P_{2h}^{k-1,disc})$ gives also a stable approximation.*

Assume that for elements $K \subset M \in \zeta_{2h}$ we have $h_K \sim h_M$. Then, the following theorem guaranties stability and converge of the method.

Theorem 4. *Let properties (18), (19), and (23) hold and the parameters α_K be such that $\alpha_K = C h_K^2$, for each element $K \in \zeta_h$. Then, the bilinear form of the local projection stabilized method satisfies*

$$\sup_{\substack{(\mathbf{z},r) \in V_h \times Q_h \\ (\mathbf{z},r) \neq 0}} \frac{A_h((\mathbf{v},q)\,;(\mathbf{z},r))}{\|(\mathbf{z},r)\|_{D_h}} \ge \beta \|(\mathbf{v},q)\|_{D_h}$$

for some positive constant β independent of the mesh parameter h.

Proof. The spaces $\mathbf{V}_h$ and Q_h do not satisfy the inf-sup condition. However, by coarsening the mesh or taking polynomials of lower degree (see, ([31]), and ([11])), we obtain $\tilde{Q}_h \subset Q_h$ such that the velocity pressure pair $(\mathbf{V}_h, \tilde{Q}_h)$ is stable. Hence, Fortin's interpolant $\Pi_{\mathcal{F}}\mathbf{v} \in \mathbf{V}_h$ exists for all $q \in \tilde{Q}_h$, and all $\mathbf{v} \in (H_0^1(\Omega))^2$ such that

$$(\nabla \cdot \mathbf{v}, q) \quad = \quad (\nabla \cdot \Pi_{\mathcal{F}}\mathbf{v}, q),\ \textit{and}\ \|\Pi_{\mathcal{F}}\mathbf{v}\|_{1,\Omega} \leqslant \|\mathbf{v}\|_{1,\Omega}\,. \tag{24}$$

Since $q \in L_0^2(\Omega)$, then there exists $\mathbf{w} \in (H_0^1)(\Omega))^2$ such that

$$\nabla \cdot \mathbf{w} \quad = \quad q \tag{25}$$

$$\textit{and}$$

$$\|\mathbf{w}\|_{1,\Omega} \quad \leqslant \quad \|q\|_{0,\Omega}\,. \tag{26}$$

Then, using the linearity of $A_h(\cdot,\cdot)$ we get

$$A_h(\mathbf{v}, q; \mathbf{v}, q) = \|\mathbf{v}\|_{0,\Omega}^2 + \sum_{K\in\zeta_h} \alpha_K \|\kappa_h \nabla q\|_{0,K}^2\,. \tag{27}$$

$$\begin{aligned} A_h(\mathbf{v}, q; -\Pi_{\mathcal{F}}\mathbf{w}, q) \quad &= \quad (\mathbf{v}, -\Pi_{\mathcal{F}}\mathbf{w}) - (q, \nabla \cdot -\Pi_{\mathcal{F}}\mathbf{w}) \\ &\quad +(q, \nabla \cdot \mathbf{v}) + \sum_{K\in\zeta_h} \alpha_K \|\kappa_h \nabla q\|_{0,K}^2\,. \end{aligned} \tag{28}$$

$$\begin{aligned} A_h(\mathbf{v}, q; \mathbf{v}, \nabla \cdot \mathbf{v}) \quad &= \quad \|\mathbf{v}\|_{0,\Omega}^2 - (q, \nabla \cdot \mathbf{v}) + (\nabla \cdot \mathbf{v}, \nabla \cdot \mathbf{v}) \\ &\quad + \sum_{K\in\zeta_h} \alpha_K (\kappa_h \nabla q, \kappa_h \nabla(\nabla \cdot \mathbf{v}))_{0,K}. \end{aligned} \tag{29}$$

$$\begin{aligned} A_h(\mathbf{v}, q; -\Pi_{\mathcal{F}}\mathbf{w}, \nabla \cdot \mathbf{v}) \quad &= \quad (\mathbf{v}, -\Pi_{\mathcal{F}}\mathbf{w}) - (q, \nabla \cdot -\Pi_{\mathcal{F}}\mathbf{w}) + (\nabla \cdot \mathbf{v}, \nabla \cdot \mathbf{v}) \\ &\quad + \sum_{K\in\zeta_h} \alpha_K (\kappa_h \nabla q, \kappa_h \nabla(\nabla \cdot \mathbf{v}))_{0,K}. \end{aligned} \tag{30}$$

Setting $(\mathbf{z}, r) = (\mathbf{v} - \delta\Pi_{\mathcal{F}}\mathbf{w}, q + \delta\nabla \cdot \mathbf{v})$ we obtain

$$\begin{aligned} A_h(\mathbf{v}, q; \mathbf{z}, r) \quad &= \quad A_h(\mathbf{v}, q; \mathbf{v}, q) + \delta A_h(\mathbf{v}, q; -\Pi_{\mathcal{F}}\mathbf{w}, q) \\ &\quad +\delta A_h(\mathbf{v}, q; \mathbf{v}, \nabla \cdot \mathbf{v}) + \delta^2 A_h(\mathbf{v}, q; -\Pi_{\mathcal{F}}\mathbf{w}, \nabla \cdot \mathbf{v}). \end{aligned} \tag{31}$$

Hence,

$$\begin{aligned} A_h(\mathbf{v}, q; \mathbf{z}, r) = \|\mathbf{v}\|_{0,\Omega}^2 &+ \sum_{K\in\zeta_h} \|\kappa_h \nabla q\|_{0,K}^2 + \delta[(\mathbf{v}, -\Pi_{\mathcal{F}}\mathbf{w}) + (q, \nabla \cdot \Pi_{\mathcal{F}}\mathbf{w}) \\ &+ \sum_{K\in\zeta_h} \|\kappa_h \nabla q\|_{0,K}^2 + \|\mathbf{v}\|_{0,\Omega}^2 + (\nabla \cdot \mathbf{v}, \nabla \cdot \mathbf{v}) \\ &+ \sum_{K\in\zeta_h} (\kappa_h \nabla q, \kappa_h \nabla(\nabla \cdot \mathbf{v}))_{0,K}] + \delta^2[(\mathbf{v}, -\Pi_{\mathcal{F}}\mathbf{w}) + (q, \nabla \cdot \Pi_{\mathcal{F}}\mathbf{w}) \\ &+ (\nabla \cdot \mathbf{v}, \nabla \cdot \mathbf{v}) + \sum_{K\in\zeta_h} (\kappa_h \nabla q, \kappa_h \nabla(\nabla \cdot \mathbf{v}))_{0,K}]. \end{aligned}$$

i.e.

$$
\begin{aligned}
A_h(\mathbf{v}, q; \mathbf{z}, r) &= (1+\delta)[\|\mathbf{v}\|_{0,\Omega}^2 + \sum_{K\in\zeta_h} \alpha_K \|\kappa_h \nabla q\|_{0,K}^2] \\
&\quad + \delta(1+\delta)[-(\mathbf{v}, \Pi_{\mathcal{F}}\mathbf{w}) + (q, \nabla\cdot\Pi_{\mathcal{F}}\mathbf{w}) \\
&\quad + (\nabla\cdot\mathbf{v}, \nabla\cdot\mathbf{v}) + \sum_{K\in\zeta_h} (\kappa_h\nabla q, \kappa_h\nabla(\nabla\cdot\mathbf{v}))_{0,K}]. \qquad (32)
\end{aligned}
$$

The sixth term of (32) is estimated by taking $\alpha_K = Ch_K^2$ and using the continuity of κ_h and the inverse inequality.

$$
\begin{aligned}
\|S(q, \nabla\cdot\mathbf{v})\| &\leqslant \left(\sum_{K\in\zeta_h} \alpha_K \|\kappa_h\nabla q\|_{0,K}^2\right)^{\frac{1}{2}} \left(\sum_{K\in\zeta_h} \alpha_K \|\kappa_h\nabla(\nabla.\mathbf{v})\|_{0,K}^2\right)^{\frac{1}{2}} \\
&\leqslant \left(\sum_{K\in\zeta_h} \alpha_K \|\kappa_h\nabla q\|_{0,K}^2\right)^{\frac{1}{2}} \left(\sum_{K\in\zeta_h} \alpha_K C_1^2 h_K^{-2} \|\kappa_h(\nabla.\mathbf{v})\|_{0,K}^2\right)^{\frac{1}{2}} \\
&\leqslant C^{\frac{1}{2}} C_1 \left(\sum_{K\in\zeta_h} \alpha_K \|\kappa_h\nabla q\|_{0,K}^2\right)^{\frac{1}{2}} \|\kappa_h(\nabla\cdot\mathbf{v})\|_{0,\Omega} \\
&\leqslant C^{\frac{1}{2}} C_1 C_2 \|\nabla\cdot\mathbf{v}\|_{0,\Omega} \left(\sum_{K\in\zeta_h} \alpha_K \|\kappa_h\nabla q\|_{0,K}^2\right)^{\frac{1}{2}}.
\end{aligned}
$$

where C_1 is the inverse inequality constant and C_2 the continuity constant of κ_h. i.e.

$$
|S(q, \nabla.\mathbf{v})| \leqslant C_3 \|\nabla.\mathbf{v}\|_{0,\Omega} \left(\sum_{K\in\zeta_h} \alpha_K \|\kappa_h\nabla q\|_{0,K}^2\right)^{\frac{1}{2}} \qquad (33)
$$

Thus, using Young's inequality we obtain

$$
S(q, \nabla\cdot\mathbf{v}) \geqslant -C_3 \left(\frac{1}{2\epsilon_1} \|\nabla\cdot\mathbf{v}\|_{0,\Omega}^2 \frac{\epsilon_1}{2} \sum_{K\in\zeta_h} \alpha_K \|\kappa_h\nabla q\|_{0,K}^2\right). \qquad (34)
$$

Using (26) and (30) we also have

$$
-(\mathbf{v}, \Pi_{\mathcal{F}}\mathbf{w}) \geqslant -\|\mathbf{v}\|_{0,\Omega} \|\Pi_{\mathcal{F}}\mathbf{w}\|_{1,\Omega} \geqslant -\|\mathbf{v}\|_{0,\Omega} \|q\|_{0,\Omega}, \qquad (35)
$$

$$
\textit{and}\ \ (q, \nabla\cdot\Pi_{\mathcal{F}}\mathbf{w}) = (q, \nabla\cdot\mathbf{w}) = (q, q) = \|q\|_{0,\Omega}^2.
$$

It follows that

$$
\begin{aligned}
A_h(\mathbf{v}, q; \mathbf{w}, r) &\geqslant (1+\delta)[\|\mathbf{v}\|_{0,\Omega}^2 + \sum_{K\in\zeta_h} \alpha_K \|\kappa_h\nabla q\|_{0,K}^2 - \delta \|\mathbf{v}\|_{0,\Omega} \|q\|_{0,\Omega} \\
&\quad + \delta \|q\|_{0,\Omega}^2 + \delta(\nabla\cdot\mathbf{v}, \nabla\cdot\mathbf{v}) + \delta \sum_{K\in\zeta_h} (\kappa_h\nabla q, \kappa_h\nabla(\nabla\cdot\mathbf{v}))_{0,K}],
\end{aligned}
$$

which by Young's inequality gives

$$
\begin{aligned}
A_h(\mathbf{v}, q; \mathbf{z}, r) &\geqslant (1+\delta)[(1 - \frac{\delta}{2\epsilon_2}) \|\mathbf{v}\|_{0,\Omega}^2 + (1 - \frac{\delta\epsilon_1 C_3}{2}) \sum_{K\in\zeta_h} \|\kappa_h\nabla q\|_{0,K}^2 \\
&\quad + \delta(1 - \frac{\epsilon_2}{2}) \|q\|_{0,\Omega}^2 + \delta(1 - \frac{C_3}{2\epsilon_1}) \|\nabla\cdot\mathbf{v}\|_{0,\Omega}^2]. \qquad (36)
\end{aligned}
$$

Where, δ, ϵ_1, and ϵ_2 are choosen such that $\epsilon_1 > \frac{C_3}{2}$, $\epsilon_2 < 2$, and $\delta < \min\left\{2\epsilon_2, \frac{2}{\epsilon_1 C_3}\right\}$.
Thus, for $(\mathbf{v}, q) \in V_h \times Q_h$ we have found $(\mathbf{z}, r)$ such that

$$A_h(\mathbf{v}, q; \mathbf{z}, r) \quad \geqslant \quad C \left\|(\mathbf{v}, q)\right\|_{D_h}^2 . \tag{37}$$

where, C is a positive constant defined by:
$C = (1+\delta)\min\{1 - \frac{\delta}{2\epsilon_2}, 1 - \frac{\delta\epsilon_1 C_3}{2}, \delta(1 - \frac{\epsilon_2}{2}), \delta(1 - \frac{C_3}{2\epsilon_1})\}$.

The norm of $(\mathbf{z}, r) = (\mathbf{v} - \delta\Pi_{\mathcal{F}}\mathbf{w}, q + \delta\nabla\cdot\mathbf{v})$ is estimated by

$$\begin{aligned} \left\|(\mathbf{z}, r)\right\|_{D_h}^2 \quad \leqslant \quad & (\left\|\mathbf{v}\right\|_{0,\Omega} + \delta\left\|\Pi_{\mathcal{F}}\mathbf{w}\right\|_{0,\Omega})^2 + (\left\|q\right\|_{0,\Omega} \\ & +\delta\left\|\nabla\cdot\mathbf{v}\right\|_{0,\Omega})^2 + (\left\|\nabla\cdot\mathbf{v}\right\|_{0,\Omega} + \delta\left\|\nabla\cdot\Pi_{\mathcal{F}}\mathbf{w}\right\|_{0,\Omega})^2 \\ & +(\sum_{K\in\zeta_h}\alpha_K\left\|\kappa_h\nabla q\right\|_{0,K} + \delta\left\|\kappa_h\nabla(\nabla\cdot\mathbf{v}\right\|_{0,\Omega})^2. \end{aligned} \tag{38}$$

Hence, Young's inequality with the continuity of κ_h and the inverse inequality as used in (33) give

$$\begin{aligned} \left\|(\mathbf{z}, r)\right\|_{D_h}^2 \quad \leqslant \quad & 2(1+\delta)[\left\|\mathbf{v}\right\|_{0,\Omega}^2 + \left\|\Pi_{\mathcal{F}}\mathbf{w}\right\|_{0,\Omega}^2 + \left\|q\right\|_{0,\Omega}^2 + \left\|\nabla\cdot\mathbf{v}\right\|_{0,\Omega}^2 \\ & + \left\|\nabla\cdot\Pi_{\mathcal{F}}\mathbf{w}\right\|_{0,\Omega}^2 + \sum_{K\in\zeta_h}\alpha_K\left\|\kappa_h\nabla q\right\|_{0,K}^2 + C_3^2\left\|\nabla\cdot\mathbf{v}\right\|_{0,\Omega}^2]. \end{aligned} \tag{39}$$

which by (35), and (36), implies

$$\begin{aligned} \left\|(\mathbf{z}, r)\right\|_{D_h}^2 \quad \leqslant \quad & 2(1+\delta)[\ \left\|\mathbf{v}\right\|_{0,\Omega}^2 + 3\left\|q\right\|_{0,\Omega}^2 + (1 + C_3^2)\left\|\nabla\cdot\mathbf{v}\right\|_{0,\Omega}^2 \\ & + \sum_{K\in\zeta_h}\alpha_K\left\|\kappa_h\nabla q\right\|_{0,K}^2]. \end{aligned}$$

It follows that

$$\left\|(\mathbf{z}, r)\right\|_{D_h}^2 \quad \leqslant \quad C_4\left\|(\mathbf{v}, q)\right\|_{D_h}^2 . \tag{40}$$

Where, $C_4 = 2(1+\delta)\max\{3, 1 + C_3^2\}$.
Thus, (37) and (40) yield the required stability result

$$\sup_{\substack{(\mathbf{z},r)\in V_h\times Q_h \\ (\mathbf{z},r)\neq 0}} \frac{A_h(\mathbf{v}, q; \mathbf{z}, r)}{\left\|(\mathbf{z}, r)\right\|_{D_h}} \geq \beta\left\|(\mathbf{v}, q)\right\|_{D_h} . \tag{41}$$

□

Note that the above theorem guaranties unique solvability of the stabilized discrete problem (15). However, unlike the residual-based stabilization schemes ([19], [17]), here, we do not have Galerkin orthogonality. As a consequence a consistency estimate is given by the following lemma (see, [18], [29], and [30]).

5. Error Analysis

5.1. consistency.

Lemma 5. *Assume that the fluctuation operator κ_h satisfies (23). Let $(\mathbf{u}, p) \in \mathbf{V} \times (Q \cap H^{l+1}(\Omega))$ be the solution of the Darcy problem (6) and $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$*

the solution of the stabilized problem (15). Then, the consistency error can be estimated by

$$A(\mathbf{u}-\mathbf{u}_h, p-p_h; \mathbf{v}, q) \leqslant C \left(\sum_{K\in\zeta_h} \alpha_K h_K^{2l} \left|p\right|_{l,K}^2 \right)^{\frac{1}{2}}$$

for all $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$.

5.2. Error estimates. As a consequence of the above stability and consistency results we obtain the following error estimates.

Theorem 6. *Assume that the solution* $(\mathbf{u}, p)$ *of (8) belongs to* $\mathbf{V} \cap (H^{s+1}(\Omega))^2 \times (Q \cap H^{l+1}(\Omega))$, $1 \leq s, l \leq k$. *Then, the following error estimate holds*

$$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} + \|\nabla\cdot(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega} + \|p-p_h\|_{0,\Omega} \quad \leq \quad C(h^{s+1}\|\mathbf{u}\|_{s+1,\Omega} + h^{l+1}\|p\|_{l+1,\Omega}).$$

Where, C is a positive constant independent of h.

Proof. Let $\tilde{\mathbf{u}}_h = j_h\mathbf{u}$ and $\tilde{p}_h = i_h p$ be the interpolants of the velocity and pressure, respectively. Then, Theorem 1 implies the existence of $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$ such that

$$\|(\mathbf{v}, q)\|_{D_h} \quad \leqslant \quad C \tag{42}$$

with

$$\|\tilde{\mathbf{u}}_h-\mathbf{u}_h\|_{0,\Omega} + \|\nabla\cdot(\tilde{\mathbf{u}}_h-\mathbf{u}_h)\|_{0,\Omega} + \|\tilde{p}_h-p_h\|_{0,\Omega} \leqslant \|(\tilde{\mathbf{u}}_h-\mathbf{u}_h, \tilde{p}_h-p_h)\|_{D_h}. \tag{43}$$

The right hand side satisfies

$$\begin{aligned} \|(\tilde{\mathbf{u}}-\mathbf{u}_h, \tilde{p}_h-p_h)\|_{D_h} &\leqslant \frac{1}{\beta}\frac{A_h(\tilde{\mathbf{u}}_h-\mathbf{u}_h, \tilde{p}_h-p_h; \mathbf{v}, q) + S(\tilde{p}_h-p_h, q)}{\|(\mathbf{v}, q)\|_{D_h}} \\ &\leqslant \frac{1}{\beta}\frac{A(\tilde{\mathbf{u}}_h-\mathbf{u}, \tilde{p}_h-p; \mathbf{v}, q) + S(\tilde{p}_h-p, q)}{\|(\mathbf{v}, q)\|_{D_h}} \\ &\quad + \frac{1}{\beta}\frac{A(\mathbf{u}-\mathbf{u}_h, p-p_h; \mathbf{v}, q) + S(p-p_h, q)}{\|(\mathbf{v}, q)\|_{D_h}}. \end{aligned} \tag{44}$$

Consequently, the consistency estimate of Lemma 1 for the method implies

$$\frac{A(\mathbf{u}-\mathbf{u}_h, p-p_h; \mathbf{v}, q) + S(p-p_h, q)}{\|(\mathbf{v}, q)\|_{D_h}} \quad \leqslant \quad Ch^l\|p\|_{l,\Omega}. \tag{45}$$

The Galerkin terms of $A(\tilde{\mathbf{u}}_h-\mathbf{u}, \tilde{p}_h-p; \mathbf{v}, q) + S(\tilde{p}_h-p, q)$ can be estimated using the approximation properties of j_h and i_h. Hence, we get

$$(\tilde{\mathbf{u}}_h-\mathbf{u}, \mathbf{v}) \quad \leqslant \quad \|\tilde{\mathbf{u}}_h-\mathbf{u}\|_{0,\Omega}\|\mathbf{v}\|_{0,\Omega} \leqslant Ch^{s+1}|\mathbf{u}|_{s+1,\Omega}\|(\mathbf{v}, q)\|_{D_h}, \tag{46}$$

$$\|(p-\tilde{p}_h, \nabla\cdot\mathbf{v})\| \quad \leqslant \quad C\|p-\tilde{p}_h\|_{0,\Omega}\|\nabla\cdot\mathbf{v}\|_{0,\Omega} \leqslant Ch^{l+1}\|p\|_{l+1,\Omega}\|(\mathbf{v}, q)\|_{D_h}. \tag{47}$$

The divergence of the second Galerkin term is estimated by applying the orthogonality property of j_h and using $\alpha_K = Ch_K^2$.

$$\begin{aligned} |(\nabla\cdot(\tilde{\mathbf{u}}_h-\mathbf{u}), q)| \quad &= \quad |(\tilde{\mathbf{u}}_h-\mathbf{u}, \nabla q)| = |(\tilde{\mathbf{u}}_h-\mathbf{u}, \kappa_h\nabla q)| \\ &\leqslant \left(\sum_{K\in\zeta_h}\alpha_K^{-1}\|\tilde{\mathbf{u}}_h-\mathbf{u}\|_{0,K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\zeta_h}\alpha_K\|\kappa_h\nabla q\|_{0,K}^2\right)^{\frac{1}{2}} \\ &\leqslant C\left(\sum_{K\in\zeta_h}\frac{h_K^2}{\alpha_K}h_K^{2s}\|\mathbf{u}\|_{s+1,K}^2\right)^{\frac{1}{2}}\left(\sum_{K\in\zeta_h}\alpha_K\|\kappa_h\nabla q\|_{0,K}^2\right)^{\frac{1}{2}} \end{aligned}$$

i.e.

$$(48) \qquad |(\nabla.(\tilde{\mathbf{u}}_h - \mathbf{u}), q)| \leqslant Ch^{s+1} \|\mathbf{u}\|_{k+1,K} \|(\mathbf{v}, q)\|_{D_h}.$$

The stability term is estimated using the L_2-stability of the fluctuation operator κ_h, the approximation properties of i_h and $\alpha_K = Ch_K^2$, hence we obtain

$$\begin{aligned} S(\tilde{p}_h - p, q) &= \sum_{K\in\zeta_h} \alpha_K(\kappa_h\nabla(\tilde{p}_h - p), \kappa_h\nabla q) \\ &\leqslant \left(\sum_{K\in\zeta_h} \alpha_K \|\kappa_h\nabla(\tilde{p}_h - p)\|_{0,K}^2\right)^{\frac{1}{2}} \left(\sum_{K\in\zeta_h} \alpha_K \|\kappa_h\nabla q\|_{0,K}^2\right)^{\frac{1}{2}} \\ &\leqslant C_1 \left(\sum_{K\in\zeta_h} C_2 h_K^2\, h_K^{2l} \|p\|_{l+1,w(K)}^2\right)^{\frac{1}{2}} \|(\mathbf{v}, q)\|_{D_h} \end{aligned}$$

i.e.

$$(49) \qquad S(\tilde{p}_h - p, q) \quad \leqslant \quad Ch^{l+1} \|p\|_{l+1,\Omega} \|(\mathbf{v}, q)\|_{D_h}.$$

Thus, using (45), (46), (48), and (49) we obtain the required error estimate

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} + \|\nabla.(\mathbf{u} - \mathbf{u}_h)\|_{0,\Omega} + \|p - p_h\|_{0,\Omega} \leq C(h^{k+1} \|\mathbf{u}\|_{k+1,\Omega} + h^{l+1} \|p\|_{l+1,\Omega}).$$

$\square$

Remark 7. *We note that because of the stability of the $Q_h^k - P_{2h}^{k-1,disc}$ approximation (*[24]*) the stability of (15) and the above error estimates also hold for such approximation.*

Remark 8. *The above error estimates hold also for equal order stabilized methods by local projection onto a discontinuous space defined on the same mesh (see,* [18] *and* [27]*).*

6. Conclusion

In this paper we have analyzed the local projection method for solving steady Darcy equations. By constructing a special interpolant and have proved stability and convergence of the method. As a result, optimal rates of of convergence were obtained for the velocity and pressure solutions.

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