ERROR ESTIMATES OF MORLEY TRIANGULAR ELEMENT SATISFYING THE MAXIMAL ANGLE CONDITION

SHIPENG MAO, SERGE NICAISE, AND ZHONG-CI SHI

(Communicated by Xue-Cheng Tai)

Abstract. In this paper, we establish the convergence of a nonconforming triangular Morley element for the plate bending problem on degenerate meshes. An explicit bound for the interpolation error is derived for arbitrary triangular meshes without any assumptions. The optimal convergence rates of the moment error and rotation error are derived for triangular meshes satisfying the maximal angle condition. Our results can also be extended to the three dimensional Morley element presented recently in [41]. Finally, some numerical results are reported that confirm our theoretical results.

Key Words. Morley element, plate elements, plate bending problems, maximal angle condition

1. Introduction

It is well known that the regularity assumption on the meshes [13, 16], i.e., bounded ratio between outer and inner diameters, leads to the convergence of standard finite element methods. However the above conventional mesh condition is a severe restriction for some particular problems of recent interests. For instance, for problems for which the solution may have anisotropic behavior in some parts of the domain, that is to say, the solution varies significantly only in certain directions. Such problems are frequently encountered in singularly perturbed convection-diffusion-reaction equations where boundary or interior layers appear or problems set on domains with edges where edge singularities may occur. In such cases, regular meshes are inappropriate or may even fail to give satisfactory results, hence the use of degenerate (or anisotropic) meshes is recommended.

The early mathematical consideration of anisotropic elements goes back to the seventies [11, 21]. Since the end of the eighties, anisotropic elements have been extensively studied. In particular, we refer to Apel et al [4-9], Chen et al [15, 27-29], Durán et al [1-3, 17, 18], Formaggia et al [19, 20], Křížek [22, 23], Kunert [24, 25], Shenk [36], Ženisek [43, 44] and references therein. As applications of anisotropic finite elements, let us quote for example, the investigation of Laplace type problems in domains with edges [5, 7, 8], layers in some singularly perturbed problems [6, 18, 25], anisotropic phenomena in the solution of Stokes and Navier-Stokes problems [9], and anisotropic a posteriori error estimates [20, 24, 25]. From these papers, it is now well known that the regularity assumption on the meshes can be considerably weakened. However, all these references are mainly restricted

to second order problems. For fourth order problems, the plate bending problem for example, only some rectangular elements have been considered, see [15, 33, 29]. But as far as we know, up to now, there are no results for general anisotropic triangular plate elements.

Triangular plate elements, especially nonconforming ones are very popular. Such elements have more advantages over their rectangular counterparts since they can be better adapted to complex boundaries. The main goal of this paper is to provide error estimates of the well-known nonconforming Morley triangular element under a weak angle condition.

The Morley element is particularly attractive for fourth order problems, because of its simple structure and since it has low degrees of freedom. However, since the continuity of Morley element is very weak (non-C^0 element), even for regular meshes, error analysis is not easy, see [30, 26, 34, 10, 37]. In this paper, by using special properties of the shape function space of Morley element and Poincaré inequality (we refer to [12, 32]), we derive an explicit bound of its interpolation error for arbitrary triangular meshes. As usual, the consistency error for plate bending problems involves some boundary residual integrals. The standard arguments to bound these terms make use of scaling arguments and trace theorems, thus the regularity assumption on the mesh can not be avoided. Our essential idea in the estimate of the consistency error is to transform some boundary integrals to some element's ones, while some approximation properties are still retained. To this end, we firstly rearrange these nonconforming terms. Then motivated by the ideas from [2], we derive an optimal estimate of the consistency error (cf. section 3 for details) with the aid of the lowest order Raviart-Thomas interpolation operator [35]. Furthermore, the optimal convergence rate of the rotation error (discrete H^1-norm) is also obtained for convex polygonal domains. The analysis carried out in this paper is made for two dimensional Morley elements, the extension to three dimensional Morley elements from [41] is also valid following the same types of arguments.

The outline of the paper is as follows. In the next section, after introducing the nonconforming Morley element approximation of the plate bending problem, we derive the interpolation error for arbitrary triangular meshes. In section 3, we mainly discuss the moment error and angular error of Morley element on meshes satisfying the maximal angle condition. In order to verify the validity of our theoretical analysis, some numerical experiments are carried out in section 4.

2. Discretization of the model problem and the interpolation error

We consider the plate bending problem:

$$
\begin{cases}
\triangle^2 u = f, & \text{in } \Omega, \\
u = \dfrac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega,
\end{cases}
\tag{2.1}
$$

where Ω denotes a plane polygonal domain, $f \in L^2(\Omega)$ is the applied force, $n = (n_x, n_y)$ is the unit outward normal vector along the boundary $\partial\Omega$. The related variational form is :

$$
\begin{cases}
\text{Find } u \in H_0^2(\Omega), \text{ such that} \\
a(u, v) = (f, v), \quad \forall v \in H_0^2(\Omega),
\end{cases}
\tag{2.2}
$$

where

$$a(u,v) = \int_\Omega A(u,v)dxdy,$$

$$A(u,v) = \triangle u \triangle v + (1-\sigma)\left(2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial x^2}\right),$$

$$(f,v) = \int_\Omega fvdxdy,$$

$$H_0^2(\Omega) = \{v \in H^2(\Omega), v = \frac{\partial v}{\partial n} = 0, \text{ on } \partial\Omega\}$$

and σ is the Poisson ratio, $0 < \sigma < \frac{1}{2}$.

Clearly, the above bilinear form $a(\cdot,\cdot)$ is bounded and coercive :

$$\begin{cases} |a(v,w)| \leq (1+\sigma)|v|_{2,\Omega}|w|_{2,\Omega}, & v,w \in H_0^2(\Omega) \\ \qquad\qquad a(v,v) \geq (1-\sigma)|v|_{2,\Omega}^2, & v \in H_0^2(\Omega). \end{cases} \qquad (2.3)$$

Throughout this paper, we adopt the standard conventions for Sobolev norms and seminorms of a function v defined on an open set G:

$$\|v\|_{m,G} = \left(\int_G \sum_{|\alpha| \leq m} |D^\alpha v|^2\right)^{\frac{1}{2}},$$

$$|v|_{m,G} = \left(\int_G \sum_{|\alpha| = m} |D^\alpha v|^2\right)^{\frac{1}{2}}.$$

We shall also denote by $P_l(G)$ the space of polynomials on G of degrees no more than l.

Let $\mathcal{J}_h$ be an arbitrary triangulation of Ω, each element K being an open triangle of diameter h_K, and $h = \max_{K \in \mathcal{J}_h} h_K$. On this triangulation we construct the so-called Morley element (cf. [30]):

$$V_h = \{v_h \in L^2(\Omega) : v_h|_K \in P_2(K), v_h \text{ is continuous at each vertex,}$$
$$\int_F [\frac{\partial v_h}{\partial n}]ds = 0, \forall \ F \subset \partial K, K \in \mathcal{J}_h, v_h(a) = 0, a \in \partial\Omega\}, \qquad (2.4)$$

where we denote faces of elements by F and by $[v]$ the jump of the function v on the faces F. For boundary faces we identify $[v]$ with v.

We note that V_h is not a subspace of $H^1(\Omega)$ (non C^0 conforming element). The discrete problem of (2.2) then reads as

$$\begin{cases} \text{Find } u_h \in V_h, \text{ such that} \\ a_h(u_h,v_h) = (f,v_h), \quad \forall v_h \in V_h, \end{cases} \qquad (2.5)$$

where $a_h(u_h,v_h) = \sum_{K \in \mathcal{J}_h} \int_K A(u_h,v_h)dxdy.$

The discrete norms are defined as

$$\|\cdot\|_{m,h} = \left(\sum_{K \in \mathcal{J}_h} |\cdot|_{m,K}^2\right)^{\frac{1}{2}}.$$

It is easy to prove that $\|\cdot\|_{2,h}$ is a norm of V_h, so the discrete problem (2.5) has a unique solution by the Lax-Milgram Lemma.

Let u and u_h be the solutions of (2.1) and (2.5), respectively, by the second Strang's Lemma [13,16], we have

$$\|u - u_h\|_{2,h} \leq C \left(\inf_{v_h \in V_h} \|u - v_h\|_{2,h} + \sup_{v_h \in V_h, v_h \neq 0} \frac{|a_h(u, v_h) - (f, v_h)|}{\|v_h\|_{2,h}} \right), \quad (2.6)$$

where the first term is the approximation error and the second one is the consistency error. Throughout this paper, the positive constant C will be used as a generic constant, which is independent of the diameter h_K and of the aspect ratio $\frac{h_K}{\rho_K}$, where ρ_K is the supremum diameter of the inscribed circle of K. In this section we only consider the approximation error, the consistency error will be discussed in the next section.

The Morley's interpolant $\Pi_h : H^2(\Omega) \longrightarrow V_h$ is defined by $\Pi_h|_K = \Pi_K, \forall K \in \mathcal{J}_h$ with

$$\begin{cases} \Pi_K u(a) = u(a), & \forall \ \text{vertex} \ a \in K, \\ \int_F \frac{\partial \Pi_K u}{\partial n} ds = \int_F \frac{\partial u}{\partial n} ds, & \forall F \subset \partial K. \end{cases} \quad (2.7)$$

The following result is the classical Poincaré inequality that can be found in [12, 32].

Lemma 2.1 *Let G be a bounded convex domain and let $w \in H^1(G)$ be a function with vanishing average, then*

$$\|w\|_{0,G} \leq \frac{d}{\pi} |w|_{1,G} \quad (2.8)$$

where d is the diameter of G.

Remark 2.1 It is very interesting to remark that the constant in the Poincaré inequality can be taken explicitly and independent of the shape (i.e., depending only on the diameter) for a general convex domain. However, the proof in [32] contains a mistake, and recently [12] gave a modified proof, fortunately, the optimal constant $\frac{d}{\pi}$ in the Poincaré inequality remains valid.

Now, we will derive the optimal interpolation error estimate under arbitrary triangular meshes.

Theorem 2.1 *Let $u \in H^3(\Omega)$, then there holds*

$$\inf_{v_h \in V_h} \|u - v_h\|_{2,h} \leq \|u - \Pi_h u\|_{2,h} \leq \frac{2}{\pi} h |u|_{3,\Omega}. \quad (2.9)$$

Proof. We only need to prove the following result

$$|u - \Pi_K u|_{2,K} \leq \frac{2}{\pi} h_K |u|_{3,K}, \forall K \in \mathcal{J}_h. \quad (2.10)$$

Firstly, let us consider $\alpha = (2, 0)$, denote the unit tangent vector to ∂K by s. Since $D^\alpha \Pi_K u$ is constant, then by Green's formula, the relation

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} n_x & -n_y \\ n_y & n_x \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial n} \\ \frac{\partial}{\partial s} \end{pmatrix},$$

and the definition of Morley's interpolant, we have

$$\begin{aligned}
D^\alpha \Pi_K u &= \frac{1}{|K|} \int_K D^\alpha \Pi_K u \, dx dy = \frac{1}{|K|} \sum_{F \subset \partial K} \int_F \frac{\partial \Pi_K u}{\partial x} n_x ds \\
&= \frac{1}{|K|} \sum_{F \subset \partial K} \int_F \left(\frac{\partial \Pi_K u}{\partial n} n_x - \frac{\partial \Pi_K u}{\partial s} n_y \right) n_x ds \\
&= \frac{1}{|K|} \sum_{F \subset \partial K} \int_F \left(\frac{\partial u}{\partial n} n_x - \frac{\partial u}{\partial s} n_y \right) n_x ds \\
&= \frac{1}{|K|} \sum_{F \subset \partial K} \int_F \frac{\partial u}{\partial x} n_x ds \\
&= \frac{1}{|K|} \int_K D^\alpha u \, dx dy.
\end{aligned} \tag{2.11}$$

Therefore, $(D^\alpha u - D^\alpha \Pi_K u)$ has vanishing mean value on the element K, it follows from Lemma 2.1 that

$$\|D^\alpha u - D^\alpha \Pi_K u\|_{0,K} \le \frac{h_K}{\pi} |D^\alpha u|_{1,K}. \tag{2.12}$$

By the same argument, we can obtain the same result as (2.11) for $\alpha = (0,2)$ and $\alpha = (1,1)$, which implies (2.10). The proof of the theorem is completed.

3. Error estimates

In this section, we will focus on the ideas for the estimation of the consistency error. Let the triangulation $\mathcal{J}_h$ be a union of triangles satisfying the maximal angle condition. That is to say, there is a constant $\sigma < \pi$ (independent of h and K) such that the maximal interior angle θ of any element $K \in \mathcal{J}_h$ is bounded by σ, i.e., $\theta < \sigma$.

In view of (2.6), our aim is to derive an estimate for

$$\sup_{v_h \in V_h} \frac{|a_h(u, v_h) - (f, v_h)|}{\|v_h\|_{2,h}}.$$

If we start in the usual way, the well known result [26, 37, 38] gives

$$a_h(u, v_h) = - \sum_{K \in \mathcal{J}_h} \int_K \nabla \triangle u \cdot \nabla v_h \, dx dy + E_1(u, v_h) + E_2(u, v_h), \tag{3.1}$$

where

$$\begin{cases}
E_1(u, v_h) = \sum_{K \in \mathcal{J}_h} \int_{\partial K} \left(\triangle u - (1 - \sigma) \frac{\partial^2 u}{\partial s^2} \right) \frac{\partial v_h}{\partial n} ds, \\
E_2(u, v_h) = \sum_{K \in \mathcal{J}_h} \int_{\partial K} (1 - \sigma) \frac{\partial^2 u}{\partial s \partial n} \frac{\partial v_h}{\partial s} ds.
\end{cases} \tag{3.2}$$

The classical method to estimate $E_1(u, v_h)$ and $E_2(u, v_h)$ [26, 37, 38] is based on the estimate of the following identity:

$$\int_F (v - P_{0,F} v)(w - P_{0,F} w) ds, \quad F \subset \partial K, v, w \in H^1(K), \tag{3.3}$$

where $P_{0,F} v = \frac{1}{|F|} \int_F v \, ds$, together with the coordinate transformation, through $\partial K \to \partial \hat{K} \to \hat{K} \to K$, interpolation theory and trace theorem. But with this method, the regularity assumption on the mesh can not be avoided. Hence we rearrange the nonconforming term $a_h(u, v_h) - (f, v_h)$ in the following way:

Lemma 3.1 Let $v \in H_0^1(\Omega)$, then we have

$$a_h(u, v_h) - (f, v_h) = \sum_{i=2}^{4} R_i(u, v_h) + R_1(u, v_h, v) + R_5(f, v_h, v), \qquad (3.4)$$

where

$$
\begin{cases}
R_1(u, v_h, v) = -\sum_{K \in \mathcal{J}_h} \int_K \nabla \triangle u \cdot \nabla(v_h - v) dx dy, \\[2mm]
R_2(u, v_h) = -(1 - \sigma) \sum_{K \in \mathcal{J}_h} \int_{\partial K} \mathbf{curl}\left(\frac{\partial u}{\partial x}\right) \cdot n \frac{\partial v_h}{\partial y} ds, \\[2mm]
R_3(u, v_h) = (1 - \sigma) \sum_{K \in \mathcal{J}_h} \int_{\partial K} \mathbf{curl}\left(\frac{\partial u}{\partial y}\right) \cdot n \frac{\partial v_h}{\partial x} ds, \\[2mm]
R_4(u, v_h) = \sum_{K \in \mathcal{J}_h} \int_{\partial K} \triangle u \nabla v_h \cdot n ds, \\[2mm]
R_5(f, v_h, v) = -(f, v_h - v),
\end{cases}
\qquad (3.5)
$$

with the **curl** operator defined as

$$\mathbf{curl}\phi = \left(-\frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial x}\right).$$

Proof. The proof is based on Green's formula. By the equation (2.1) we can derive

$$\int_\Omega f v dx dy = \int_\Omega \triangle^2 u v dx dy = -\int_\Omega \nabla \triangle u \cdot \nabla v dx dy, \quad \forall v \in H_0^1(\Omega), \qquad (3.6)$$

together with

$$\int_K \triangle u \triangle v_h = -\int_K \nabla \triangle u \cdot \nabla v_h dx dy + \int_{\partial K} \triangle u \nabla v_h \cdot n ds, \qquad (3.7)$$

and

$$
\begin{aligned}
&\int_K \left(2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v_h}{\partial x \partial y} - \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v_h}{\partial y^2} - \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v_h}{\partial x^2}\right) dx dy \\
&= \int_{\partial K} \frac{\partial^2 u}{\partial x \partial y}\frac{\partial v_h}{\partial y} n_x ds - \int_{\partial K} \frac{\partial^2 u}{\partial x^2}\frac{\partial v_h}{\partial y} n_y ds \\
&\quad + \int_{\partial K} \frac{\partial^2 u}{\partial x \partial y}\frac{\partial v_h}{\partial x} n_y ds - \int_{\partial K} \frac{\partial^2 u}{\partial y^2}\frac{\partial v_h}{\partial x} n_x ds \\
&= -\int_{\partial K} \mathbf{curl}\left(\frac{\partial u}{\partial x}\right) \cdot n \frac{\partial v_h}{\partial y} ds + \int_{\partial K} \mathbf{curl}\left(\frac{\partial u}{\partial y}\right) \cdot n \frac{\partial v_h}{\partial x} ds,
\end{aligned}
\qquad (3.8)
$$

we can complete the proof.

The following theorem contains the main results of this paper.

Theorem 3.2 Let $\mathcal{J}_h$ satisfy the maximal angle condition and suppose that $f \in L^2(\Omega)$ and that the solution $u \in H_0^2(\Omega)$ of (2.2) has the additional regularity $u \in H^3(\Omega)$, then we have

$$\|u - u_h\|_{2,h} \le Ch\left(|u|_{3,\Omega} + h\|f\|_{0,\Omega}\right). \qquad (3.9)$$

Moreover if the domain Ω is a convex polygonal domain, then for a data $f \in L^2(\Omega)$, the solution $u \in H_0^2(\Omega)$ of (2.2) satisfies $u \in H^3(\Omega)$ and we further have

$$\|u - u_h\|_{1,h} \le Ch^2\left(|u|_{3,\Omega} + h\|f\|_{0,\Omega}\right). \qquad (3.10)$$

Proof. We firstly prove (3.9). In view of (2.6) and (3.4), one only needs to bound the different terms of (3.5). Taking $v = \Pi_1 v_h$ in (3.4), where Π_1 is the usual interpolant of conforming linear finite element, we have

$$R_1(u, v_h, v) \leq \sum_{K \in \mathcal{J}_h} \|\nabla \triangle u\|_{0,K} |v_h - \Pi_1 v_h|_{1,K} \leq Ch|u|_{3,\Omega} \|v_h\|_{2,h} \qquad (3.11)$$

and

$$R_5(f, v_h, v) \leq \sum_{K \in \mathcal{J}_h} \|f\|_{0,K} \|v_h - \Pi_1 v_h\|_{0,K} \leq Ch^2 \|f\|_{0,\Omega} \|v_h\|_{2,h}. \qquad (3.12)$$

Note that we have used the anisotropic interpolation result of linear conforming element (cf. [4], [11], [21]) in the inequality (3.11).

Now we come to the estimate of $R_2(u, v_h)$, which involves some boundary integrals. The first objective of the method is to avoid the usage of some trace theorems. The basic idea is to subtract a constant function in every boundary integral and then transform them to some elementwise integrals, while some approximation properties are still retainable. To this end, and motivated by [2], we introduce the interpolant of the lowest order $H(\mathrm{div})$ element $\mathbf{RT}$ (cf. [35]), which is defined as

$$\int_F (\mathbf{v} - \mathbf{RT}(\mathbf{v})) \cdot n \, ds = 0, \quad \forall F \subset \partial K, K \in \mathcal{J}_h \qquad (3.13)$$

on every element K and F are the three edges of K. It is clear that $\mathbf{RT}\left(\mathbf{curl}\left(\frac{\partial u}{\partial x}\right)\right)$ make sense. Moreover, from the definition of $\mathbf{RT}\left(\mathbf{curl}\left(\frac{\partial u}{\partial x}\right)\right)$, we know that $\mathbf{RT}\left(\mathbf{curl}\left(\frac{\partial u}{\partial x}\right)\right) \cdot n$ is a constant vector along each edge, which is continuous across the edges of elements and vanishes on the boundary of Ω, together with the fact that $\int_F [\frac{\partial v_h}{\partial y}] ds = \int_F \left(\left[\frac{\partial v_h}{\partial n}\right] n_y + \left[\frac{\partial v_h}{\partial s}\right] n_x \right) ds = 0$, so we can derive

$$R_2(u, v_h)$$

$$= -(1 - \sigma) \sum_{K \in \mathcal{J}_h} \int_{\partial K} \left[\mathbf{curl}\left(\frac{\partial u}{\partial x}\right) - \mathbf{RT}\left(\mathbf{curl}\left(\frac{\partial u}{\partial x}\right)\right) \right] \cdot n \frac{\partial v_h}{\partial y} ds$$

$$= -(1 - \sigma) \sum_{K \in \mathcal{J}_h} \int_{\partial K} \left[\mathbf{curl}\left(\frac{\partial u}{\partial x}\right) - \mathbf{RT}\left(\mathbf{curl}\left(\frac{\partial u}{\partial x}\right)\right) \right] \cdot n \left(\frac{\partial v_h}{\partial y} - P_{0,K} \frac{\partial v_h}{\partial y} \right) ds$$

$$= -(1 - \sigma) \sum_{K \in \mathcal{J}_h} \int_K -\mathrm{div}\left[\mathbf{RT}\left(\mathbf{curl}\left(\frac{\partial u}{\partial x}\right)\right) \right] \left(\frac{\partial v_h}{\partial y} - P_{0,K} \frac{\partial v_h}{\partial y} \right) dxdy$$

$$- (1 - \sigma) \sum_{K \in \mathcal{J}_h} \int_K \left[\mathbf{curl}\left(\frac{\partial u}{\partial x}\right) - \mathbf{RT}\left(\mathbf{curl}\left(\frac{\partial u}{\partial x}\right)\right) \right] \nabla \left(\frac{\partial v_h}{\partial y} \right) dxdy,$$

$$(3.14)$$

where the div operator is defined as $\mathrm{div}\mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}$ and $P_{0,K} : L^1(K) \longrightarrow P_0(K)$ is the averaging operator on the element K which preserves polynomials of degree zero and is defined as $P_{0,K}v = \frac{1}{|K|} \int_K v dx dy$.

Thanks to the fact that $\text{div}\left[\mathbf{RT}\left(\mathbf{curl}\left(\frac{\partial u}{\partial x}\right)\right)\right]$ is constant function and the error estimate of the interpolation of $\mathbf{RT}$ has been derived in [2], we have

$$
\begin{aligned}
R_2(u, v_h) &= C \sum_{K \in \mathcal{J}_h} \int_K \left[\mathbf{curl}\left(\frac{\partial u}{\partial x}\right) - \mathbf{RT}\left(\mathbf{curl}\left(\frac{\partial u}{\partial x}\right)\right) \right] \nabla \left(\frac{\partial v_h}{\partial y}\right) dxdy \\
&\le C \sum_{K \in \mathcal{J}_h} \left\| \mathbf{curl}\left(\frac{\partial u}{\partial x}\right) - \mathbf{RT}\left(\mathbf{curl}\left(\frac{\partial u}{\partial x}\right)\right) \right\|_{0,K} \left| \frac{\partial v_h}{\partial y} \right|_{1,K} \\
&\le C \sum_{K \in \mathcal{J}_h} h_K \left| \mathbf{curl}\left(\frac{\partial u}{\partial x}\right) \right|_{1,K} |v_h|_{2,K} \\
&\le Ch|u|_{3,\Omega} \|v_h\|_{2,h}.
\end{aligned}
$$

(3.15)

Similarly, we can prove that

$$
R_3(u, v_h) \le Ch|u|_{3,\Omega} \|v_h\|_{2,h}. \tag{3.16}
$$

Concerning $R_4(u, v_h)$, since the Morley interpolant reproduces both the normal derivative (c.f. (2.4)) and the tangential derivative as used in (2.11), following the same idea, it can be estimated as

$$
\begin{aligned}
R_4(u, v_h) &= \sum_{K \in \mathcal{J}_h} \int_{\partial K} \triangle u (\nabla v_h - \mathbf{RT}(\nabla v_h)) \cdot n ds \\
&= \sum_{K \in \mathcal{J}_h} \int_{\partial K} (\triangle u - P_{0,K} \triangle u)(\nabla v_h - \mathbf{RT}(\nabla v_h)) \cdot n ds \\
&= \sum_{K \in \mathcal{J}_h} \int_K (\triangle u - P_{0,K} \triangle u)(\triangle v_h - P_{0,K}(\triangle v_h)) dxdy \\
&\quad + \sum_{K \in \mathcal{J}_h} \int_K \nabla \triangle u (\nabla v_h - \mathbf{RT}(\nabla v_h)) dxdy \\
&= \sum_{K \in \mathcal{J}_h} \int_K \nabla \triangle u (\nabla v_h - \mathbf{RT}(\nabla v_h)) dxdy \\
&\le Ch|u|_{3,\Omega} \|v_h\|_{2,h},
\end{aligned}
$$

(3.17)

where we have used the fact that $\int_F [\mathbf{RT}(\nabla v_h)) \cdot n] ds = 0$ on the first line of (3.17).

A collection of (3.11)-(3.17) and Theorem 2.1 implies (3.9).

Now, we are in a position to prove (3.10). To this end, we adopt the technique developed in [37]. Set $e = u - u_h$ and let

$$
g = -\triangle(\Pi_1(\Pi_h e)), \quad \text{in} \ \ H^{-1}(\Omega). \tag{3.18}
$$

Consider the following auxiliary variational problem:

$$
\begin{cases}
\quad \text{Find} \ \ \phi \in H_0^2(\Omega), \ \ \text{such that} \\
a(\phi, v) = < g, v >, \quad \forall v \in H_0^2(\Omega).
\end{cases} \tag{3.19}
$$

When Ω is a convex polygonal domain, we have the regularity $\phi \in H^3(\Omega)$ as well as the estimate [14, Theorem 2]

$$
\|\phi\|_{3,\Omega} \le C\|g\|_{-1,\Omega}. \tag{3.20}
$$

Let $\mathcal{D}(\Omega)$ be the linear space of infinitely differentiable functions, with compact support in Ω. By Green's formula, there holds

$$
\begin{aligned}
< g, v > &= - < \triangle(\Pi_1(\Pi_h e)), v > \\
&= \int_\Omega \nabla(\Pi_1(\Pi_h e)) \nabla v \, dx dy, \quad \forall v \in \mathcal{D}(\Omega).
\end{aligned} \tag{3.21}
$$

Since $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, the above Green's formula is still valid for any $v \in H_0^1(\Omega)$. Noticing that $\Pi_1(\Pi_h e) \in H_0^1(\Omega)$, we have

$$
| < g, v > | \le |\Pi_1(\Pi_h e)|_{1,\Omega} |v|_{1,\Omega}, \tag{3.22}
$$

$$
< g, \Pi_1(\Pi_h e) > = |\Pi_1(\Pi_h e)|_{1,\Omega}^2. \tag{3.23}
$$

Then

$$
\begin{aligned}
\|\phi\|_{3,\Omega} &\le C\|g\|_{-1,\Omega} \\
&= C \sup_{v \in H_0^1(\Omega), v \ne 0} \frac{< g, v >}{|v|_{1,\Omega}} \le C|\Pi_1(\Pi_h e)|_{1,\Omega}.
\end{aligned} \tag{3.24}
$$

On the other hand,

$$
\begin{aligned}
|\Pi_1(\Pi_h e)|_{1,\Omega}^2 &= < g, \Pi_1(\Pi_h e) > = < \triangle^2 \phi, \Pi_1(\Pi_h e) > \\
&= - \int_\Omega \nabla \triangle \phi \cdot \nabla(\Pi_1(\Pi_h e)) dx dy \\
&= \sum_{K \in \mathcal{J}_h} \int_K \nabla \triangle \phi \cdot \nabla(\Pi_h e - \Pi_1(\Pi_h e)) dx dy \\
&\quad + \sum_{K \in \mathcal{J}_h} \int_K \nabla \triangle \phi \cdot \nabla(\Pi_h e) dx dy \\
&= A_1 + A_2.
\end{aligned} \tag{3.25}
$$

By virtue of (3.9) and the interpolation result (2.9), we get that

$$
\begin{aligned}
\|\Pi_h e\|_{2,h} &= \|\Pi_h u - u_h\|_{2,h} \\
&\le \|u - u_h\|_{2,h} + \|\Pi_h u - u\|_{2,h} \\
&\le Ch \left(|u|_{3,\Omega} + h\|f\|_{0,\Omega} \right).
\end{aligned} \tag{3.26}
$$

Then A_1 can be estimated as

$$
\begin{aligned}
A_1 &= \sum_{K \in \mathcal{J}_h} \int_K \nabla \triangle \phi \cdot \nabla(\Pi_h e - \Pi_1(\Pi_h e)) dx dy \\
&\le \sum_{K \in \mathcal{J}_h} \|\nabla \triangle \phi\|_{0,K} \|\nabla(\Pi_h e - \Pi_1(\Pi_h e))\|_{0,K} \\
&\le Ch|\phi|_{3,\Omega} \|\Pi_h e\|_{2,h} \\
&\le Ch^2 |\phi|_{3,\Omega} \left(|u|_{3,\Omega} + h\|f\|_{0,\Omega} \right).
\end{aligned} \tag{3.27}
$$

In view of (3.7) and (3.8), and after some rearrangements, A_2 can be written as

$$
\begin{aligned}
A_2 &= \sum_{K \in \mathcal{J}_h} \int_K \nabla \triangle \phi \cdot \nabla(\Pi_h e) dx dy \\
&= -a_h(\phi, \Pi_h e) + \sum_{i=2}^4 R_i(\phi, \Pi_h e).
\end{aligned} \tag{3.28}
$$

The terms $R_i(\phi, \Pi_h e), i = 2, 3, 4$ can be bounded in the same way as (3.15)-(3.17), which leads to

$$\left| \sum_{i=2}^{4} R_i(\phi, \Pi_h e) \right| \leq Ch|\phi|_{3,\Omega} \|\Pi_h e\|_{2,h}$$

$$\leq Ch^2|\phi|_{3,\Omega} \left(|u|_{3,\Omega} + h\|f\|_{0,\Omega} \right). \tag{3.29}$$

Now, let us consider $a_h(\phi, \Pi_h e)$, which is decomposed as

$$a_h(\phi, \Pi_h e) = a_h(\phi, \Pi_h u - u) + a_h(\phi, u - u_h)$$
$$= a_h(\phi, \Pi_h u - u) + a_h(\phi - \Pi_h \phi, u - u_h) + a_h(\Pi_h \phi, u - u_h) \tag{3.30}$$
$$= A_{21} + A_{22} + A_{23}.$$

By the definition of $a_h(\cdot, \cdot)$, and thanks to the fact that (see (2.11))

$$D^\alpha \Pi_h u = P_{0,K}(D^\alpha u), \quad \forall \alpha, \quad |\alpha| = 2,$$

we can derive that

$$A_{21} = a_h(\phi, \Pi_h u - u) = -\sum_{K \in \mathcal{J}_h} \left[\int_K (\triangle \phi - P_{0,K} \triangle \phi)(\triangle u - P_{0,K} \triangle u) dxdy \right.$$

$$+ (1 - \sigma) \int_K \left(2 \left(\frac{\partial^2 \phi}{\partial x \partial y} - P_{0,K} \frac{\partial^2 \phi}{\partial x \partial y} \right) \left(\frac{\partial^2 u}{\partial x \partial y} - P_{0,K} \frac{\partial^2 u}{\partial x \partial y} \right) \right.$$

$$- \left(\frac{\partial^2 \phi}{\partial x^2} - P_{0,K} \frac{\partial^2 \phi}{\partial x^2} \right) \left(\frac{\partial^2 u}{\partial y^2} - P_{0,K} \frac{\partial^2 u}{\partial y^2} \right) \tag{3.31}$$

$$\left. - \left(\frac{\partial^2 \phi}{\partial y^2} - P_{0,K} \frac{\partial^2 \phi}{\partial y^2} \right) \left(\frac{\partial^2 u}{\partial x^2} - P_{0,K} \frac{\partial^2 u}{\partial x^2} \right) \right) dxdy \right]$$

$$\leq Ch^2|\phi|_{3,\Omega}|u|_{3,\Omega}.$$

For A_{22}, it can be estimated as

$$A_{22} = a_h(\phi - \Pi_h \phi, u - u_h)$$
$$\leq C\|\phi - \Pi_h \phi\|_{2,h}\|u - u_h\|_{2,h} \tag{3.32}$$
$$\leq Ch^2|\phi|_{3,\Omega} \left(|u|_{3,\Omega} + h\|f\|_{0,\Omega} \right).$$

Due to the symmetry of $a_h(\cdot, \cdot)$, (2.2) and (3.4), A_{23} can be written as

$$A_{23} = a_h(\Pi_h \phi, u - u_h)$$
$$= a_h(u, \Pi_h \phi) - a_h(u_h, \Pi_h \phi)$$
$$= a_h(u, \Pi_h \phi) - (f, \Pi_h \phi) \tag{3.33}$$
$$= \sum_{i=2}^{4} R_i(u, \Pi_h \phi) + R_1(u, \Pi_h \phi, v) + R_5(f, \Pi_h \phi, v),$$

for all $v \in H_0^1(\Omega)$. Since

$$\sum_{i=2}^{4} R_i(u, \phi) + R_1(u, \phi, v) + R_5(f, \phi, v) = a(u, \phi) - (f, \phi) = 0,$$

then we have

$$A_{23} = \sum_{i=2}^{4} R_i(u, \Pi_h \phi - \phi) + R_1(u, \Pi_h \phi - \phi, v) + R_5(f, \Pi_h \phi - \phi, v)$$
$$\leq Ch \left(|\phi|_{3,\Omega} + h\|f\|_{0,\Omega} \right) \|\Pi_h \phi - \phi\|_{2,h}$$
$$\leq Ch^2 |\phi|_{3,\Omega} \left(|u|_{3,\Omega} + h\|f\|_{0,\Omega} \right).$$
(3.34)

A collection of (3.30)-(3.34) gives

$$a_h(\phi, \Pi_h \phi) \leq Ch^2 |\phi|_{3,\Omega} \left(|u|_{3,\Omega} + h\|f\|_{0,\Omega} \right). \tag{3.35}$$

Substituting the above inequality and (3.29) into (3.28) yields

$$A_2 \leq Ch^2 |\phi|_{3,\Omega} \left(|u|_{3,\Omega} + h\|f\|_{0,\Omega} \right). \tag{3.36}$$

Putting (3.27) and (3.36) into (3.25), by (3.24), we get

$$|\Pi_1(\Pi_h e)|_{1,\Omega} \leq Ch^2 \left(|u|_{3,\Omega} + h\|f\|_{0,\Omega} \right). \tag{3.37}$$

Then recalling (3.26), we can derive

$$\|\Pi_h e\|_{1,h} \leq \|\Pi_h e - \Pi_1(\Pi_h e)\|_{1,h} + |\Pi_1(\Pi_h e)|_{1,\Omega}$$
$$\leq Ch\|\Pi_h e\|_{2,h} + Ch^2 \left(|u|_{3,\Omega} + h\|f\|_{0,\Omega} \right)$$
$$\leq Ch^2 \left(|u|_{3,\Omega} + h\|f\|_{0,\Omega} \right).$$
(3.38)

Repeat applications of triangle inequality and using the fact that $\Pi_1 \Pi_h e = \Pi_1 e$ yields

$$\|e\|_{1,h} \leq \|e - \Pi_h e\|_{1,h} + \|\Pi_h e\|_{1,h}$$
$$\leq \|e - \Pi_1 e\|_{1,h} + \|\Pi_h e - \Pi_1(\Pi_h e)\|_{1,h} + \|\Pi_h e\|_{1,h}$$
$$\leq Ch\|e\|_{2,h} + Ch\|\Pi_h u - u_h\|_{2,h} + \|\Pi_h e\|_{1,h}$$
$$\leq Ch^2 \left(|u|_{3,\Omega} + h\|f\|_{0,\Omega} \right),$$
(3.39)

where we have used the estimate that

$$\|e - \Pi_1 e\|_{1,h} = \left(\sum_{K \in \mathcal{J}_h} |e - \Pi_1 e|_{1,K}^2 \right)^{\frac{1}{2}}$$
$$\leq C \left(\sum_{K \in \mathcal{J}_h} h_K^2 |e|_{2,K}^2 \right)^{\frac{1}{2}}$$
$$\leq Ch\|e\|_{2,h},$$

then we obtain the desired estimate (3.10).

4. Numerical Experiments

In order to examine the numerical performance of the Morley element for narrow triangular meshes, we carry out numerical tests to the following model problems set on the unit square $\Omega = [0, 1] \times [0, 1]$.

Model problem 1. The classical unit square plate bending problem with clamped boundaries under a uniform load $f = 1$ and the Poisson ratio $\sigma = 0.3$ (cf. [40]). The analytic value of deflection at the center is 0.00126532, the analytic value of bending moment at the center is 0.022905. This experiment is to investigate the convergence for the classical plate bending problem with anisotropic meshes.

Model problem 2. The biharnomic differential equation with $f(x, y) = 8\pi^4 \cos(2\pi x) \cos(2\pi y) - 8\pi^4 \cos(2\pi x) \sin^2(\pi y) - 8\pi^4 \cos(2\pi y) \sin^2(\pi x) \in L^2(\Omega)$. It can be verified

that the exact solution of problem (4.2) is $u(x, y) = \sin^2(\pi x)\sin^2(\pi y)$. This experiment is to investigate the optimal convergence property for the standard biharmonic problem with anisotropic meshes.

Model problem 3. The following fourth order singular perturbation problems taken from [31, 42]:

$$
\begin{cases}
-\varepsilon^2 \triangle^2 u + \triangle u = f, & \text{in } \Omega, \\
u = g_1, \ \ \dfrac{\partial u}{\partial n} = g_2, & \text{on } \partial\Omega,
\end{cases}
$$

with the right hand side $f(x, y) = 0$ and g_1, g_2 chosen such that $u(x, y) = (1 - e^{-x(1-x)/\varepsilon})^2(1 - e^{-y(1-y)/\varepsilon})^2$ (we refer to Figures 3 and 4 for some illustrations) is the exact solution. This solution presents significant boundary layers for small values of ε.

For the two first examples, the unit square $\Omega = [0, 1] \times [0, 1]$ is subdivided in the following two fashions:

mesh 1: Each edge of Ω is divided into n segments with $n + 1$ points $(1 - \cos(\frac{i\pi}{n}))/2, i = 0, 1, ..., n$. The mesh obtained in this way for $n = 16$ is illustrated at Figure 1, and the anisotropic triangular mesh is obtained by dividing each rectangular into two triangles.

mesh 2: Each edge of Ω is divided into n segments with $n+1$ points $\sin(\frac{i\pi}{n})/2, i = 0, 1, \cdots, n/2, (1-\cos(\frac{i\pi}{n} - \frac{\pi}{2}))/2, i = n/2+1, \cdots, n$. The mesh obtained in this way for $n = 16$ is shown at Figure 2. Then the anisotropic triangular mesh is obtained by dividing each rectangular into two triangles.

For model problem 1, the error of the deflection $|(u - u_h)(O)|$ and the error of bending moment $|(M - M_h)(O)|$ at the center of the unit square are shown in Table 4.1 and Table 4.2, from which the good convergence of the element for non regular subdivisions can be seen.

For model problem 2, the numerical results of $\|u - u_h\|_{1,h}$ and $\|u - u_h\|_{2,h}$ are presented in Table 4.3 and Table 4.4, which validate the optimal order of convergence rate of Morley elements with anisotropic meshes.

For model problem 3, we adopt the discretization method developed in [42]. The energy norm $E_h(u - u_h)$ (cf. [42]) for regular triangular meshes (the triangular mesh is obtained from a square partition built as the tensor product of uniform $1 - d$ meshes) are computed and compared to that for Shishkin type meshes (cf. [39]). Figure 5 shows that anisotropic meshes are more attractive than regular meshes. This experiment is made in order to present the advantages of the use of anisotropic meshes over regular meshes. It shows that it is worthy to give a detailed analysis for singular perturbed problems, we refer to future works.

Table 4.1. The errors $|(u - u_h)(O)|$ and $|(M - M_h)(O)|$ (*mesh 1*)

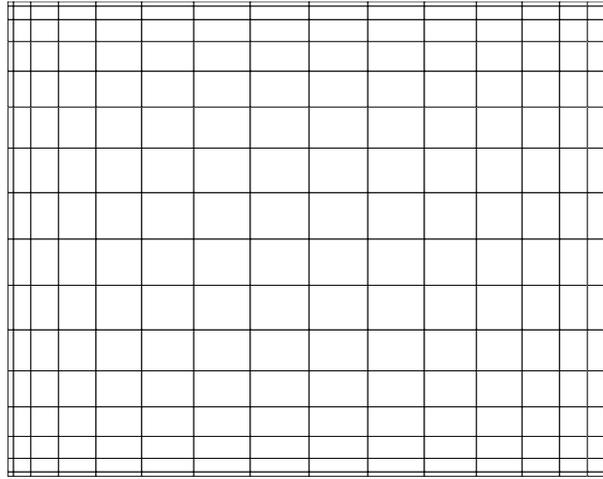| $n \times n$	8×8	16×16	32×32	64×64	128×128
$|(u - u_h)(O)|$	0.001927	0.001475	0.001316	0.001278	0.001269
$|(M - M_h)(O)|$	0.021239	0.022532	0.022816	0.022883	0.022810
$\max\limits_{K \in \mathcal{J}_h} h_K$	0.27060	0.13795	0.06931	0.03470	0.01735
$\max\limits_{K \in \mathcal{J}_h}\{h_K/\rho_K\}$	7.10973	14.35875	28.7869	57.6087	115.235

FIGURE 1. The initial rectangular mesh of *mesh* 1 for case $n = 16$
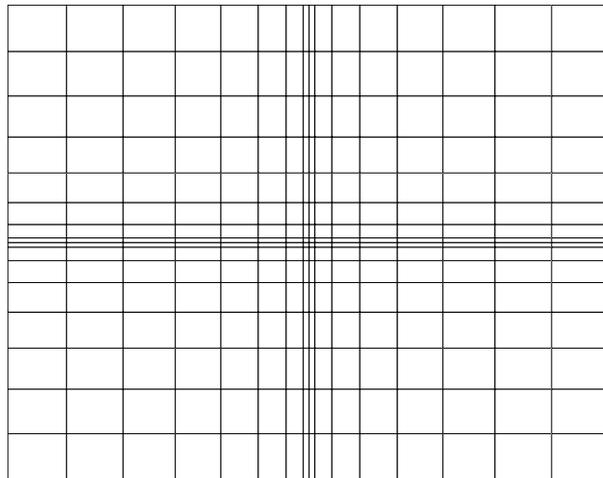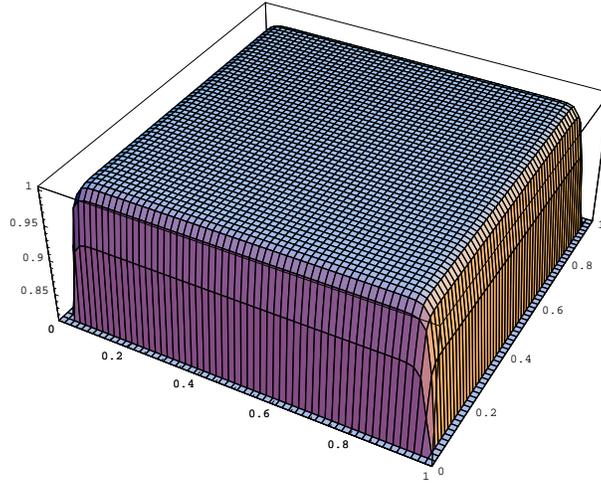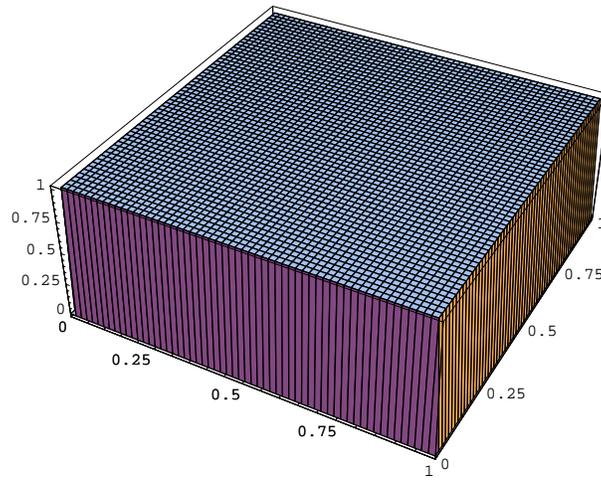
FIGURE 2. The initial rectangular meshes of *mesh* 2 for case $n = 16$

Table 4.2. The errors $|(u - u_h)(O)|$ and $|(M - M_h)(O)|$ (*mesh 2*)

$n \times n$	8×8	16×16	32×32	64×64	128×128
$|(u - u_h)(O)|$	0.001855	0.001420	0.001304	0.001275	0.001268
$|(M - M_h)(O)|$	0.021641	0.022622	0.022837	0.0228883	0.022901
$\max_{K \in \mathcal{J}_h} h_K$	0.27060	0.13795	0.06931	0.03470	0.01735
$\max_{K \in \mathcal{J}_h} \{h_K/\rho_K\}$	7.10973	14.35875	28.7869	57.6087	115.235

FIGURE 3. The exact solution u of model 3 for case $\varepsilon = 0.01$

FIGURE 4. The exact solution u of model 3 for case $\varepsilon = 0.00001$

Table 4.3: The errors $\|u - u_h\|_{2,h}$ and $\|u - u_h\|_{1,h}$ for model 2 (*mesh 1*)

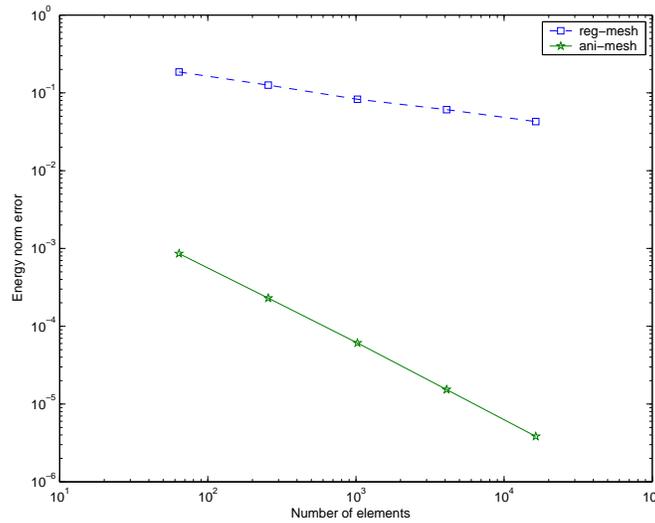| $n \times n$	8×8	16×16	32×32	64×64	128×128
$\|u - u_h\|_{1,h}$	0.139968	0.036008	0.009060	0.002267	0.000567
α	\	1.958691	1.990684	1.998735	1.999862
$\|u - u_h\|_{2,h}$	3.896537	1.918543	0.955667	0.477393	0.238585
α	\	1.022181	1.005432	1.001330	1.000675

FIGURE 5. The energy norm $E_h(u - u_h)$ for case $\varepsilon = 0.00001$

Table 4.4: The errors $\|u - u_h\|_{2,h}$ and $\|u - u_h\|_{1,h}$ for model 2 (*mesh 2*)

$n \times n$	8×8	16×16	32×32	64×64	128×128
$\|u - u_h\|_{1,h}$	0.139002	0.035949	0.009061	0.002270	0.000568
α	\	1.951098	1.988121	1.99720	1.99956
$\|u - u_h\|_{2,h}$	3.805938	1.874348	0.933891	0.466539	0.233167
α	\	1.021863	1.005063	1.001256	1.000637

5. Conclusion

We have analyzed the nonconforming triangular Morley element on general triangular meshes for the approximation of fourth order problems. An explicit bound for the interpolation error is derived for arbitrary triangular meshes without any assumptions. Optimal convergence rates of the moment error and angular error are both derived for any triangular meshes satisfying the maximal angle condition. This result, which is obtained for plate problems, agrees with the results obtained with triangular elements for second order problems. Note that our analysis can be extended to three dimensional Morley element [41] following the same types of arguments. In a forthcoming paper we will analyze the use of anisotropic Morley elements to some fourth order elliptic singular perturbed problems [31,42].

Acknowledgments. The authors would like to thank the anonymous referees for their valuable suggestions and corrections, which contribute significantly to the improvement of the paper.

References

[1] G. Acosta and R. G. Durán, Error estimates for Q_1 isoparametric elements satisfying a weak angle condition, SIAM J. Numer. Anal., 38(2000), 1073-1088.
[2] G. Acosta and R. G. Durán, The maximum angle condition for mixed and nonconforming elements: Application to the Stokes equations, SIAM J. Numer. Anal., 37(1999), 18-36.
[3] G. Acosta, Lagrange and average interpolation over 3D anisotropic meshes, J. Comp. Appl. Math., 135(2001), 91-109.

[4] T. Apel and M. Dobrowolski, Anisotropic interpolation with applications to the finite element method, Computing, 47(1992), 277-293.

[5] T. Apel, Anisotropic finite element: local estimates and applications, Stuttgart Teubner, 1999.

[6] T. Apel, G. Lube, Anisotropic mesh refinement in stabilized Galerkin methods, Numer. Math., 74(1996), 261-282.

[7] T. Apel, S. Nicaise, The finite element method with anisotropic mesh grading for elliptic problems in domains with corners and edges, Math. Methods Appl. Sic., 21(1998), 519-549.

[8] T. Apel, S. Nicaise, Joachim Schöberl, Crouzeix-Raviart type finite elements on anisotropic meshes, Numer. Math., 89(2001),193-223.

[9] T. Apel, S. Nicaise, J. Schöberl, A nonconforming finite elements method with anisotropic mesh grading for the Stokes problem in domains with edges, IMA J. Numer. Anal., 21(2001), pp. 843-856.

[10] D. N. Arnold and F. Brezzi, Mixed and nonconforming finite element methods: Implementation, postprocessing and error estimates, M^2AN, 19(1985),7-32.

[11] I. Babuska and A. K. Aziz, On the angle condition in the finite element method, SIAM J. Numer.Anal., 13(1976), 214-226.

[12] M. Bebendorf, A note on the Poincaré inequality for convex domains, Zeitschrift für Analysis und ihre Anwendungen, 22(2003), No. 4, 751–756.

[13] S. C. Brenner, L. R. Scott, The mathematical theory of finite element methods, New York, Springer-Verlag, 1994.

[14] H. Blum and R. Rannacher, On the boundary value problem for the biharmonic operator on domains with angular corners, Math. Mech. Appl. Sci., 2 (1980), No. 4, 556–581.

[15] S. C. Chen, D. Y. Shi, Y. C. Zhao, Anisotropic interpolation and quasi-Wilson element for narrow quadrilateral meshes, IMA J. Numer. Anal., 24(2004), 77-95.

[16] P. G. Ciarlet, The Finite Element method for Elliptic Problems, Amsterdam, North-Holland, 1978.

[17] R. G. Durán, Error estimates for narrow 3D finite elements, Math. Comp., 68 (1999), 187-199.

[18] R. G. Durán, A. L. Lombardi, Error estimates on anisotropic Q_1 elements for functions in weighted sobolev spaces, Math. Comp., 74(2005), 1679-1706.

[19] L. Formaggia and S. Perotto, New anisotropic a priori error estimates, Numer. Math., 89 (2001), 641-667.

[20] L. Formaggia and S. Perotto, Anisotropic error estimates for elliptic problems, Numer. Math., 94 (2003), 67-92.

[21] P. Jamet, Estimations d'erreur pour des *éléments* finis droits presque *dégénérés*, RAIRO Anal. *Numér*.,10(1976),46-61.

[22] M. Křížek, On semiregular families of triangulations and linear interpolation. Appl. Math., 36(1991), 223-232.

[23] M. Křížek, On the maximal angle condition for linear tetrahedral elements, SIAM J. Numer. Anal., 29(1992), 513-520.

[24] G. Kunert, R. Verfürth, Edge residuals dominate a posteriori error estimates for linear finite element methods on anisotropic triangular and tetrahedral meshes, Numer. Math., 86(2000), 283-303.

[25] G. Kunert, A posteriori error estimation for convection dominated problems on anisotropic meshes, Math. Methods Appl. Sci., 26(2003), 589-617.

[26] P. Lascaux and P. Lesaint, Some noncomforming finite element for the plate bending problem, RAIRO. Anal. *Numér*., R-1(1975), 9-53.

[27] S. P. Mao, S. C. Chen, H. X. Sun, A quadrilateral, anisotropic, superconvergent nonconforming double set parameter element, Appl. Numer. Math., 27 (2006), 19-43.

[28] S. P. Mao, S. C. Chen, Convergence analysis of a nonconforming finite element on anisotropic rectangular meshes, Electronic Transactions on Numerical Analysis, 20 (2005), 154-163.

[29] S. P. Mao, S. C. Chen, Accuracy analysis of Adinis non-conforming plate element on anisotropic meshes, Communications in Numerical Methods in Engineering, 22(2006), 433-440.

[30] L. S. D. Morley, The triangular equilibrium element in the solution of plate bending problems, Aero.Quart., 19(1968),149-169.

[31] T. K. Nilssen, X. C. Tai, and R. Winther, A robust nonconforming H^2-element, Math. Comp., 70(2000), 489-505.

[32] L. E. Payne and H. F. Weinberger: An optimal Poincaré inequality for convex domains, Arch. Rational. Mech. Anal., 5(1960), 286-292.

[33] Z. H. Qiao and S. C. Chen, Narrow Bicubic Hermite element, Journal of Zhengzhou University, 35(2003), 6-10.

[34] R. Rannacher, On nonconforming and mixed finite elements for plate bending problem, RAIRO Modél. Math. Anal. Num ér., 13(1979), 369-387.

[35] P. A. Raviart, J.M. Thomas, Primal hybrid finite element methods for 2nd order elliptic equations, Math. Comp., 31 (1977), 391-396.

[36] N. Al. Shenk, Uniform error estimates for certain narrow Lagrange finite elements, Math Comp., 63(1994), 105-119.

[37] Z. C. Shi, On the error estimates of Morley's element, Math. Numer. Sinica (in Chinese), 2(1990), 113-118.

[38] Z.C.Shi, The F-E-M-Test for convergence of nonconforming elements.Math Comp,49(1987) 391-405.

[39] G. I. Shishkin, Mesh approximations of singularly perturbed elliptic and parabolic problems. Ekaterinburg: Russ. Acad. Sci. 1992.

[40] S. Timoshenko and S. Woinowsky-Krieger, Theory of plates and shells. 2nd ed. McGrawhill, 1959.

[41] M. Wang, J. C. Xu and Y. C. Hu, The Morley element for fourth order elliptic equations in any dimensions, Numer. Math., 103(2006), 155-169.

[42] M. Wang, J. C. Xu, Modified Morley element method for a fourth order elliptic singular perturbation problem, J. Comp. Math., 24(2006), 113-120.

[43] A. Ženisek, M. Vanmaele, The interpolation theorem for narrow quadrilateral isoparametric finite elements, Numer. Math., 72(1995), 123-141.

[44] A. Ženisek, M. Vanmaele, Applicability of the Bramble-Hilbert lemma in interpolation problems of narrow quadrilateral isoparametric finite elements, J. Comp. Appl. Math., 65(1995), 109-122.

LSEC, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and System Science, Chinese Academy of Science, PO Box 2719, Beijing, 100190, China

E-mail: `maosp@lsec.cc.cn`

URL: `http://lsec.cc.ac.cn/maosp`

Universite de Valenciennes et du Hainaut Cambresis, LAMAV, ISTV, F-59313 - Valenciennes Cedex 9, France

E-mail: `snicaise@univ-valenciennes.fr`

LSEC, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and System Science, Chinese Academy of Science, PO Box 2719, Beijing, 100190, China

E-mail: `shi@lsec.cc.cn`

URL: `http://lsec.cc.ac.cn/shi`