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PARABOLIC SINGULARLY PERTURBED PROBLEMS WITH EXPONENTIAL LAYERS: ROBUST DISCRETIZATIONS USING FINITE ELEMENTS IN SPACE ON SHISHKIN MESHES

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Dedicated to G. I. Shishkin on the occasion of his 70th birthday

Abstract. A parabolic initial-boundary value problem with solutions displaying exponential layers is solved using layer-adapted meshes. The paper combines finite elements in space, i.e., a pure Galerkin technique on a Shishkin mesh, with some standard discretizations in time. We prove error estimates as well for the θ-scheme as for discontinuous Galerkin in time.

Key Words. convection-diffusion, transient, finite element, Shishkin mesh, time discretization.

1. Introduction

We consider 1D unsteady convection-diffusion problems of the type

$$u_t + Lu = f \qquad \text{in} \quad Q = (0,1) \times (0,T], \tag{1a}$$

$$u(x,0) = u_0(x) \qquad \text{for } x \in [0,1], \tag{1b}$$

$$u(0,t) = u(1,t) = 0 \qquad \text{for } t \in (0,T], \tag{1c}$$

with $f : (0,1) \times (0,T] \to \mathbb{R}$. Here the differential operator L is given by,

$$Lu := -\varepsilon u_{xx} + bu_x + cu, \tag{2}$$

$0 < \varepsilon << 1$ is a small parameter and $b, c : (0,1) \to \mathbb{R}$ are sufficiently smooth with

$$b(x) > \beta > 0 \qquad \text{for } x \in (0,1). \tag{3}$$

By changing the dependent variable we may also assume that

$$c - \frac{1}{2} b_x \geq c_0 > 0 \qquad \text{for } x \in (0,1). \tag{4}$$

Here β and c_0 are constants. The exact solution of (1) has, in general, an exponential boundary layer at $x = 1$. Additionally, a discontinuity in the initial-boundary data at the point $x = 0$, $t = 0$ would lead to an interior layer along the subcharacteristics through that point. We assume sufficient compatibility of the data to exclude the existence of an interior layer, see [9].
In recent years many numerical methods have been developed to solve the corresponding stationary problem on layer-adapted meshes, resulting in error estimates

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that are uniform with respect to the parameter ε, see [9]. For unsteady problems, however, the situation is different.
Most existing papers deal with low order finite difference schemes, beginning with [10] and the error estimate

$$(5) \qquad | u(x_i,t_j) - u_{i,j} | \leq C(N^{-1}\ln^2 N + \tau)$$

for backward differencing in time and upwind differencing in space on a Shishkin mesh. This result was extended in [5], [1] and [4]; in the last paper defect correction in both space and time is applied to enhance the accuracy of the computed solution. Concerning finite elements in space on a Shishkin mesh, we only know the pointwise error estimates of [3] using space-time finite elements that are linear and continuous in space but discontinuous in time, while additionally the streamline diffusion stabilization in space-time is applied.
It is the aim of this paper to combine systematically finite elements in space (based on a Galerkin technique or stabilization on a Shishkin mesh) with some standard discretizations in time. First we shall study the θ-scheme which gives maximal order 2 with respect to time. As a higher order scheme we decided to choose and to analyze discontinuous Galerkin, because the analysis of higher order methods is similar to lower order versions and discontinuous Galerkin offers the possibility to investigate a posteriori error estimates based on standard ideas for Galerkin techniques. In the numerical experiments we restricted ourselves to low order methods, a careful numerical study of higher order methods is a task for subsequent studies. For simplicity, we present the results for problems one-dimensional in space but we apply only techniques which can be used in several dimensions as well.

2. The continuous problem

It is well known that for $f \in L_2(Q)$ and $u_0 \in L_2(\Omega)$ problem (1) has a unique solution $u \in L_2(0,T;H_0^1(\Omega))$ with $u' \in L_2(0,T;H^{-1}(\Omega))$ (in our case we have $\Omega = (0,1)$).
If we introduce the ε-weighted H^1-norm defined by

$$(6) \qquad \|v\|_\varepsilon^2 := \varepsilon|v|_1^2 + \|v\|_0^2 \quad \text{for } v \in H^1(\Omega),$$

where $\|\cdot\|_0$ defines the standard L_2-norm and $|\cdot|_1$ the H^1-seminorm respectively, standard arguments lead us to the stability estimate (see [7], Theorem 11.1.1)

$$(7) \quad \sup_{t\epsilon(0,T)} \| u(t) \|_0 + (\int_0^T \| u(t) \|_\varepsilon^2 \, dt)^{1/2} \leq C\, ((\int_0^T \| f(t) \|_0^2 \, dt)^{1/2} + \| u_0 \|_0).$$

Therefore it is natural that we shall later prove error estimates in "$L_\infty(L^2)$"- and "$\sqrt{\varepsilon}L^2(H^1)$"-norms or their discrete analogues.

Remark 1. *In [7], Proposition 11.1.1., we additionally can find an estimate for* $\max_{t\epsilon(0,T)} \| u(t) \|_1$. *But, in our singularly perturbed case, it seems not possible to follow the proof of Proposition 11.1.1 in such a way that the constants arising are independent of* ε *(if moreover,* $\| u(t) \|_1$ *is replaced by* $\| u(t) \|_\varepsilon$*).*□

Under certain compatibility conditions [9] there exists a classical solution of problem (1).

Theorem 1. *Let* $\alpha \in (0,1)$, $u_0 \in C^{2+\alpha}(0,1) \cap C^2[0,1]$, $u_0(0) = 0$, $u_0(1) = 0$, *additionally*

$$-\varepsilon u_0''(0) + b(0,0)u_0'(0) + c(0,0)u_0(0) = f(0,0)$$
$$-\varepsilon u_0''(1) + b(1,0)u_0'(1) + c(1,0)u_0(1) = f(1,0).$$

Let b, c *und* f *be Hölder continuous on* Q *with exponent* α. *Then (1) has exactly one solution in* $C^{2+\alpha}(Q)$.

Assuming still more compatibility to avoid interior layers, in [11] (see also [9], Chapter 2, Remark 2.8) there are sufficient conditions for the validity of the estimates

$$\left|\frac{\partial^{k+m}u(x,t)}{\partial x^k \partial t^m}\right| \leq C(1 + \varepsilon^{-k}e^{-\beta(1-x)/\varepsilon}) \tag{8}$$

for $k + m \leq 2$. The estimate (8) implies in the one-dimensional case as well (see [9]) the existence of an S-decomposition of the solution: $u(x,t) = S(x,t) + V(x,t)$ with

$$\left|\frac{\partial^{k+m}S(x,t)}{\partial x^k \partial t^m}\right| \leq C \quad \text{and} \quad \left|\frac{\partial^{k+m}V(x,t)}{\partial x^k \partial t^m}\right| \leq C\varepsilon^{-k}e^{-\beta(1-x)/\varepsilon}. \tag{9}$$

For the solution decomposition in case of a higher-dimensional parabolic problem in space see [10].
It is well known [9] that the existence of an S-decomposition in the stationary case allows us in a relatively simple way to estimate both interpolation errors and the error of finite element methods on S-meshes. In our analysis the solution decomposition is used to derive estimates for the error of the Ritz projection, see Section 3.

3. The θ-scheme for the discretization in time

For the discretization in space of (1) we use linear finite elements on a Shishkin mesh. Denoting the corresponding finite element space by $V^N \subset H_0^1(\Omega)$, the semidiscrete problem is: Find $u^N : (0,T] \to V^N$ such that

$$\begin{aligned} (\frac{du^N(t)}{dt}, v) + a(u^N(t), v) = (f(t), v) \quad \forall v \in V^N, \quad \forall t \in (0,T] \\ u^N(0) = u_0^N, \end{aligned} \tag{10}$$

where u_0^N is an approximation of the initial condition in V^N and will be specified later. Here, the bilinear form $a(\cdot,\cdot)$ is

$$a(w,v) := \varepsilon(w_x, v_x) + (bw_x + cw, v)$$

with $a(v,v) \geq \omega\|v\|_\varepsilon^2$ for all $v \in H_0^1(\Omega)$, $\omega := \min\{1, c_0\}$. The mesh is piecewise uniform in $[0, 1-\sigma]$ and $[1-\sigma, 1]$ with the same number of mesh points in these two subintervals and the definition

$$\sigma = \sigma_0 \varepsilon \ln N, \tag{11}$$

with σ_0 a constant. For the discretization of the stationary problem related to (1)

$$\begin{aligned} Lu_S = f^* \quad \text{in} \quad (0,1), \\ u_S(0) = u_S(1) = 0, \end{aligned}$$

$f^* : (0,1) \to \mathbb{R}$, with linear finite elements on our Shishkin mesh, the following error estimates are well known (see [9] for a detailed discussion and references):

$$\text{(13)} \qquad \text{If } \sigma_0 \geq 2, \text{ then } \| u_S - u_S^N \|_\varepsilon \leq C N^{-1} \ln N,$$

$$\text{(14)} \qquad \text{If } \sigma_0 \geq \frac{5}{2}, \text{ then } \| u_S^I - u_S^N \|_\varepsilon \leq C (N^{-1} \ln N)^2.$$

Here u_S^I denotes the linear interpolant of the stationary solution u_S. The estimate (14) implies in particular the L_2 error estimate

$$\text{(15)} \qquad \| u_S - u_S^N \|_0 \leq C (N^{-1} \ln N)^2.$$

For the analysis of the time discretization it is useful to introduce the Ritz projection Ru of u defined by $Ru : [0,T] \to V^N$ and

$$\text{(16)} \qquad a(Ru(t), v) = a(u(t), v) \quad \forall v \in V^N, t \in [0,T].$$

Then, the error $u(t) - Ru(t)$ satisfies for every t estimates of the type (13)-(15), i.e.

$$\text{(17)} \qquad \begin{aligned} \| u(t) - Ru(t) \|_\varepsilon &\leq C N^{-1} \ln N \\ \| u^I(t) - Ru(t) \|_\varepsilon &\leq C (N^{-1} \ln N)^2 \\ \| u(t) - Ru(t) \|_0 &\leq C (N^{-1} \ln N)^2. \end{aligned}$$

The proof easily follows the same argumentation as the one for (13)-(15)(see e.g. [9]) and is based on the solution decomposition proposed.
Introduce a mesh in time that is equidistant for simplicity, with mesh width τ and $\tau \cdot M = T$. Then the θ-scheme is: Find $U^m \in V^N, m = 1, \ldots, M$ such that

$$\text{(18)} \qquad (\frac{U^m - U^{m-1}}{\tau}, v) + a(U^{m-\theta}, v) = (f^{m-\theta}, v) \ \ \forall v \in V^N, \ \ U^0 = u_0^N$$

with some $0 \leq \theta \leq 1$ and the abbreviation

$$g^{m-\theta} := \theta g^m + (1-\theta) g^{m-1}$$

for some g and $g^m = g(m\tau)$, moreover $f^m = f(\cdot, m\tau)$. To analyze the θ-scheme, let us introduce $\psi := U - Ru$. Then ψ satisfies the error equation

$$\text{(19)} \qquad (\frac{\psi^m - \psi^{m-1}}{\tau}, v) + a(\psi^{m-\theta}, v) = (W^m, v) \quad \forall v \in V^N$$

with

$$\text{(20)} \qquad W^m := \frac{(Ru)^m - (Ru)^{m-1}}{\tau} - (u')^{m-\theta}.$$

Here and in the following u' denotes the derivative in time. Next, in (19) we set $v = \psi^{m-\theta}$. Further, we use for $\theta \geq 1/2$ the inequality

$$\text{(21)} \qquad (\psi^m - \psi^{m-1}, \psi^{m-\theta}) \geq (\| \psi^m \|_0 - \| \psi^{m-1} \|_0) \| \psi^{m-\theta} \|_0$$

and get

$$\| \psi^m \|_0 + \omega\tau \| \psi^{m-\theta} \|_\varepsilon \leq \| \psi^{m-1} \|_0 + \tau \| W^m \|_0 .$$

Summation leads to

$$\text{(22)} \qquad \| \psi^M \|_0 + \omega \sum_m \tau \| \psi^{m-\theta} \|_\varepsilon \leq \| \psi^0 \|_0 + \tau \sum_m \| W^m \|_0 .$$

Here $\psi^0 = u_0^N - Ru_0$ is zero if we choose $u_0^N = Ru_0$. We can write W^m in the form

$$W^m = \frac{(Ru - u)^m - (Ru - u)^{m-1}}{\tau} + \left(\frac{u^m - u^{m-1}}{\tau} - (u')^{m-\theta} \right)$$

or

$$W^m = \frac{1}{\tau}\int_{t_{m-1}}^{t_m} ((R-I)u(s))' ds + \frac{1}{\tau}\int_{t_{m-1}}^{t_m} \left[u'(s) - (u')^{m-\theta}\right] ds.$$

This gives us

$$\| W^m \|_0 \le \begin{cases} C((N^{-1}\ln N)^2 + \tau) & \text{for } \frac{1}{2} < \theta \le 1, \\ C((N^{-1}\ln N)^2 + \tau^2) & \text{for } \theta = \frac{1}{2} \end{cases}$$

and consequently

Theorem 2. *Set $u_0^N = Ru_0$. Then, the error $\psi = U - Ru$ of the θ-scheme satisfies*

$$\| \psi^M \|_0 + \omega \sum_{m=1}^{M} \tau \| \psi^{m-\theta} \|_\varepsilon \le \begin{cases} C((N^{-1}\ln N)^2 + \tau^2) & \textit{for } \theta = \frac{1}{2}, \\ C((N^{-1}\ln N)^2 + \tau) & \textit{for } \theta > \frac{1}{2}. \end{cases} \tag{23}$$

For the error $U - u$ itself we get a poor result because
(24)

$$\| (u-Ru)^M \|_0 + \omega \sum_{1}^{M} \tau \| (u-Ru)^{m-\theta} \|_\varepsilon \le \begin{cases} C(N^{-1}\ln N + \tau^2) & \text{for } \theta = \frac{1}{2}, \\ C(N^{-1}\ln N + \tau) & \text{for } \theta > \frac{1}{2}. \end{cases}$$

Remark 2. *Let us assume that instead of (18) we use*

$$\Big(\frac{U^m - U^{m-1}}{\tau}, v\Big) + a_s(U^{m-\theta}, v) = (f_s^{m-\theta}, v),$$

where we replace the Galerkin scheme by some stabilization. Then a consistent stabilization, i.e., where the exact solution u satisfies

$$(\frac{du}{dt}, v) + a_s(u, v) = (f_s, v), \tag{25}$$

allows the same kind of error estimation if the stabilization term is time-independent. Again we have (19) with (20), if Ru now denotes the Ritz projection with respect to the stabilized bilinear form.
That means, for instance, that CIP stabilization can be handled without problems but SDFEM is less easy to deal with.□

Remark 3. *In the non-singularly perturbed case it is well-known that for a problem with irregular initial values (u_0 is not very smooth) the Crank-Nicolson scheme is not strongly A-stable which leads to non-physical oscillations. A possible alternative is the strategy to apply first two implicit Euler steps with step size $\tau/2$. This damped Crank-Nicolson method has better properties then the original scheme (see* [8]*, for instance), but allows error estimates of the same type as the original scheme.*□

4. Discontinuous Galerkin in time

First we describe the combination of a dG method in time with a Galerkin finite element method in space to discretize the problem

$$(\frac{du}{dt}, v) + a(u, v) = (f, v) \quad \forall v \in V = H_0^1(\Omega), \;\; u(0) = u_0 \in L_2(\Omega).$$

In the time interval (t_{m-1}, t_m) we use a finite element space $V_{h,m} \subset H_0^1(\Omega)$ of linear elements for the discretization in space (thus on every time interval we could use a different mesh). Moreover we define

$$S_{h,\tau}^q = \left\{\varphi \in L^2(Q) : \varphi|_{(t_{m-1},t_m)} \in P_q \quad \text{with coefficients from } V_{h,m}\right\},$$

where P_q is the space of polynomials of degree q.
For the discontinuous functions in time we introduce the jumps at t_m by

$$[\varphi]_m := \lim_{t\to t_m+}\varphi(t) - \lim_{t\to t_m-}\varphi(t) = \varphi_m^+ - \varphi_m^-.$$

Then our discretization is given by: Find $U \in S^q_{h,\tau}$ with

$$\begin{aligned}\sum_m \int_{t_{m-1}}^{t_m} ((U',\varphi)+a(U,\varphi))dt + \sum_2^M([U]_{m-1},\varphi_{m-1}^+) + (U_0^+,\varphi_0^+)\\ = \int_0^{t_M}(f,\varphi)dt + (u_{0,h},\varphi_0^+)\end{aligned} \tag{26}$$

for all $\varphi \in S^q_{h,\tau}$. $u_{0,h}$ is an approximation of u_0 that will be defined later.
If one introduces

$$B(u,v) := \sum_m \int_{t_{m-1}}^{t_m} ((u',v)+a(u,v))dt + \sum_2^M([u]_{m-1},v_{m-1}^+) + (u_0^+,v_0^+), \tag{27}$$

then integration by parts results in

$$B(u,v) := \sum_m \int_{t_{m-1}}^{t_m} \big(-(u,v')+a(u,v)\big)dt - \sum_1^{M-1}(u_m^-,[v]_m) + (u_M^-,v_M^-). \tag{28}$$

The combination of (27) and (28) allows the estimate

$$B(v,v) \geq \ \|v\|_{dG}^2 \quad \text{with} \tag{29}$$

$$\|v\|_{dG}^2 := \ \omega \ \sum_m \int_{t_{m-1}}^{t_m} \|v\|_\varepsilon^2 dt + \frac{1}{2}\|v_0^+\|_0^2 + \frac{1}{2}\sum_1^{M-1}\| \ [v]_m\|_0^2 \ + \ \frac{1}{2}\| \ v_M^-\|_0^2. \tag{30}$$

Next, denote by $\pi u \in S^q_{h,\tau}$ some interpolant of u in space and time that will be defined later. We are interested in estimating $U - \pi u$ because then to bound the error itself we have only to estimate the interpolation error. The exact solution u satisfies

$$\sum_m \int_{t_{m-1}}^{t_m} \{(u',\varphi)+a(u,\varphi)\}dt = \int_0^{t_M}(f,\varphi)dt,$$

or (again integration by parts)

$$\begin{aligned}\sum_m \int_{t_{m-1}}^{t_m} \big\{(-u,\varphi')+a(u,\varphi)\big\}dt + (u_M,\varphi_M^-) - \sum_m(u_m^-,[\varphi]_m)\\ - (u_0,\varphi_0^+) = \int_0^{t_M}(f,\varphi)dt.\end{aligned}$$

It follows that we have the error equation

$$\begin{aligned}B(U-\pi u,\xi) = \sum_m \int_{t_{m-1}}^{t_m} &\{-(u-\pi u,\xi')+a(u-\pi u,\xi)\}dt\\ &+ ((u-\pi u)_M^-,\xi_M^-) - \sum_m((u-\pi u)_m^-,[\xi]_m)\\ &+ (u_{0,h}-u_0,\xi_0^+).\end{aligned} \tag{31}$$

Based on (29) the choice $\xi := U - \pi u$ allows us to estimate $\|U - \pi u\|_{dG}$ if one is able to bound the right-hand-side of (31) by some suitable quantity multiplied by $\|\xi\|_{dG}$.

Remark 4. *(A duality trick)*
A second possibility for error estimation is the following (see [12]*): Let* $\psi = (U - \pi u)_M$ *and define the auxiliary function* $Z \in S^q_{h,\tau}$ *such that*

$$B(\varphi, Z) = (\varphi_M, \psi) \tag{32}$$

for all $\varphi \in S^q_{h,\tau}$*. With* $\varphi := U - \pi u$ *we get for the* L_2 *error at the time* $t = t_M = T$

$$\|(U - \pi u)_M\|_0^2 = B(U - \pi u, Z); \tag{33}$$

again we can reformulate $B(U - \pi u, Z)$ *as in (31). The final error estimate is the result of two steps:*

- *estimate* $B(U-\pi u, Z)$ *by terms containing the interpolation error multiplied by some norm* $|||\cdot|||$ *of* Z
- *prove the a priori estimate* $|||Z||| \leq C\|\psi\|_0$ *for the solution of* (32) *with respect to the norm* $|||\cdot|||$*.*

We remark that Z *solves a backward homogeneous problem in time.*□

If now $u_{0,h}$ is the L_2 projection of u_0, one term of (31) vanishes. Therefore we assume this for the rest of the paper. Next we have to answer the crucial question: How to choose the interpolant of u in space and time?
Let us first study the case $q = 0$, i.e., piecewise constant approximation in time. If we now denote by Ru the Ritz projection with respect to $a(\cdot,\cdot)$, then the choice

$$\pi u\big|_{(t_{m-1},t_m)} = \frac{1}{\tau}\int_{t_{m-1}}^{t_m} (Ru)(t)dt \tag{34}$$

leads to the simplified error equation

$$B(U - \pi u, \xi) = ((u - \pi u)_M^-, \xi_M^-) - \sum_{m=1}^{M-1}((u - \pi u)_m^-\, [\xi]_m). \tag{35}$$

The definition (30) of the dG norm, inequality (29) and Cauchy-Schwarz result in

$$\|U - \pi u\|_{dG} \leq C\big\{\|(u - \pi u)_M^-\|_0^2 + \sum_{m=1}^{M-1}\|(u - \pi u)_m^-\|_0^2\big\}^{1/2}. \tag{36}$$

Remark 5. *The duality trick also leads to* (36) *(but we get an error estimate only for* $\|(U - \pi u)_M\|_0$*) because* Z *satisfies*

$$\|Z_M^-\|_0^2 + \sum_{m=1}^{M-1}\|[Z]_m\|_0^2 \leq \|\psi\|_0^2. \tag{37}$$

For sharpened estimates for Z *see the next remark.*□

To estimate the right-hand side of (36) we observe that

$$u(t_m^-) - (\pi u)(t_m^-) = u(t_m^-) - u(\tilde t_m) + u(\tilde t_m) - Ru(\tilde t_m) \tag{38}$$

because (34) implies $\pi u|_{(t_{m-1},t_m)} = Ru(\tilde t_m)$ for $\tilde t_m \in (t_{m-1}, t_m)$.
On a Shishkin mesh with linear elements we obtain consequently

Theorem 3. *Set* $u_{0,h}$ *to be the* L_2 *projection of* u_0*. Then, the error* $U - \pi u$ *of our discretization method can be estimated by*

$$\|U - \pi u\|_{dG} \leq C\big\{\tau^2 + (N^{-1}\ln N)^4 + \frac{(N^{-1}\ln N)^4 + \tau^2}{\tau}\big\}^{1/2}. \tag{39}$$

Here we used

$$\sum_m \int_\Omega (u(t_m^-) - u(\tilde{t}_m))^2 = \sum_m \int_\Omega (\int_{\tilde{t}_m}^{t_m^-} u_t)^2 \le \tau \int_\Omega \int_0^T u_t^2.$$

Remark 6. *: In the case of a symmetric bilinear form, in* [12] *we can find a stability estimate that sharpens* (37)*, namely*

$$\sum_{m=1}^{M-1} \|[Z]_m\|_0 \le \; C\, L\, \|\psi\|_0 \tag{40}$$

(here L depends logarithmically on the mesh in time). Inequality (40) allows us to estimate the L_2 *error of* $(U - \pi u)_M$*) by*

$$\sup_m \|(U - \pi u)_m^-\|_0,$$

consequently for a symmetric problem on a standard mesh the resulting L_2 *error is proportional to* $\tau + h^2$ *instead of* $\{(\tau^2 + h^4)/\tau\}^{1/2}$ *which we got before. For our non-symmetric problem it is an open question whether or not improved estimates can be derived.* □

To estimate the error $U - u$ we use the estimate (39) and have, additionally, to estimate $u - \pi u$. The second part of the norm (30) is easily bounded with the help of (38) and the error estimates for the Ritz projection

$$\begin{aligned} \frac{1}{2}\|(u - \pi u)_0^+\|_0^2 + \frac{1}{2}\sum_1^{M-1} \| \, [u - \pi u]_m\|_0^2 \; + \; \frac{1}{2}\| \, (u - \pi u)_M^-\|_0^2 \\ \le \; C\big\{\tau + (N^{-1}\ln N)^4 + \frac{(N^{-1}\ln N)^4}{\tau}\big\}. \end{aligned}$$

We have still to bound

$$\begin{aligned} \sum_m \int_{t_{m-1}}^{t_m} \|u - \pi u\|_\varepsilon^2 \, dt \le 2 \sum_m \int_{t_{m-1}}^{t_m} (\|u(t) - u(\tilde{t}_m)\|_\varepsilon^2 \\ + \|u(\tilde{t}_m) - Ru(\tilde{t}_m)\|_\varepsilon^2) dt. \end{aligned}$$

This gives an error contribution of the order

$$O(\tau^{1/2} + N^{-1}\ln N + \frac{1}{\tau^{1/2}}(N^{-1}\ln N)^2). \tag{41}$$

Now we start to consider dG methods with $q > 0$. We define our interpolant πu now in two steps (see [12, 2]):

- $\tilde{u}$ is the piecewise polynomial in t of degree q with
$$\tilde{u}(t_m^-) = u(t_m), \qquad \int_{t_{m-1}}^{t_m} (\tilde{u}(t) - u(t))t^l dt = 0 \;\text{ for }\; l \le q - 1.$$
- πu is the L_2-projection of $\tilde{u}$ onto our finite element space.

Then, we get the error equation

$$B(\xi, \xi) = \sum_m \int_{t_{m-1}}^{t_m} a(u - \pi u, \xi) dt - \sum_{m=1}^{M-1} ((u - \pi u)_m^-, [\xi]_m). \tag{42}$$

We use the splitting

$$u - \pi u = u - \tilde{u} + \tilde{u} - \Pi\tilde{u} \;\; (\Pi \text{ denotes the } L_2 \text{ projection in space}).$$

It is known that $\tilde{u}$ approximates u with an accuracy of order $O(\tau^{q+1})$, but we have to keep in mind (compare (8)) that derivatives of $\tilde{u}$ with respect to x behave like derivatives of u.
What about the L_2 projection of $\tilde{u}$ on a Shishkin mesh? First, L_2 stability shows that

$$\|u - \Pi u\|_0 \le C\,\|u - u^I\|_0 \le \; C(N^{-1}\ln N)^2 \tag{43}$$

Moreover, in 1D the L_2 projection is $L_\infty - L_\infty$ stable:

$$\|u - \Pi u\|_\infty \le C\,\|u - u^I\|_\infty \le C\,(N^{-1}\ln N)^2, \tag{44}$$

(for the two-dimensional case see [6]).
The error in the H^1-seminorm satisfies

$$\varepsilon^{1/2} \mid u - \Pi u \mid_1 \le C\,N^{-1}(\ln N)^{3/2}. \tag{45}$$

For the proof we introduce the standard interpolant u^I of u and use the triangle inequality

$$\varepsilon^{1/2} \mid u - \Pi u \mid_1 \le \varepsilon^{1/2} \mid u - u^I \mid_1 + \varepsilon^{1/2} \mid u^I - \Pi u \mid_1 .$$

The first term is already of the desired order. The second term is estimated using an inverse inequality on both the coarse and the fine parts of the mesh and introducing the L_∞-norm:

$$|u^I - \Pi u|_{1,\tilde{\Omega}} \le \frac{C}{h_{\tilde{\Omega}}}(meas\,\tilde{\Omega})^{1/2}\|u^I - \Pi u\|_{\infty,\tilde{\Omega}}.$$

Applying the triangle inequality again the estimate (45) follows.
Therefore, in the one-dimensional case we have all the ingredients needed to estimate the right-hand side of (42). First we get

$$|\sum_{m=1}^{M-1}((u-\pi u)_m^-,\ [\xi]_m)| \le C(\frac{(N^{-1}\ln N)^2}{\tau^{1/2}} + \tau^{q+1/2})\|\xi\|_{dG}. \tag{46}$$

Next we have to estimate

$$\sum_m \int_{t_{m-1}}^{t_m} a(u - \tilde{u}, \xi)dt \ \text{ and } \ \sum_m \int_{t_{m-1}}^{t_m} a(\tilde{u} - \Pi\tilde{u}, \xi)dt. \tag{47}$$

In the first term we use the smallness of $u - \tilde{u}$ but have difficulties with the convective term. Integration by parts yields on (t_{m-1}, t_m)

$$\mid (u - \tilde{u}, \bigtriangledown\xi) \mid \le \; C(N\tau^{q+1} + \tau^{q+1}\ln^{1/2} N)\|\xi\|_\varepsilon. \tag{48}$$

In the estimate of the second term we use the approximation properties of the L_2-projection and the standard arguments on Shishkin meshes:

$$\mid a(\tilde{u} - \Pi\tilde{u}, \xi) \mid \le C\; N^{-1}(\ln N)^{3/2}\;\|\xi\|_\varepsilon. \tag{49}$$

The final estimate follows from (46), (48), (49).

Theorem 4. *Set $u_{0,h}$ to be the L_2 projection of u_0. Then, the error $U - \pi u$ of our discretization method can be estimated by*

$$\|U - \pi u\|_{dG} \le \; C\big(N^{-1}(\ln N)^{3/2} + \frac{(N^{-1}\ln N)^2}{\tau^{1/2}} + \tau^{q+1/2} + N\tau^{q+1}\big).$$

Remark that for a stabilization technique instead of pure Galerkin one can hope to replace the last term $N\tau^{q+1}$ by the expression $N^{1/2}\tau^{q+1}$.

5. Numerical experiments

We consider the initial-boundary value problem

$$
\begin{aligned}
u_t(x,t) - \varepsilon u_{xx}(x,t) + u_x(x,t) + u(x,t) &= f(x,t) \quad \text{in } \Omega := (0,1)\times(0,1] \\
u(0,t) &= te^{-1/\varepsilon} + 1 + t^2 \\
u(1,t) &= t + t^2 \\
u(x,0) &= 1 - x^2
\end{aligned} \tag{50}
$$

with the right-hand side

$$f(x,t) = 2\varepsilon - 2x + 2t + 1 - x^2 + t^2 + te^{-(1-x)/\varepsilon} + e^{-(1-x)/\varepsilon}$$

and the exact solution

$$u(x,t) = te^{-(1-x)/\varepsilon} + 1 - x^2 + t^2.$$

The same example was studied in [3] using a streamline diffusion scheme, here we will use linear finite elements on a Shishkin mesh in space and the θ-scheme and respectively the discontinuous Galerkin scheme for the discretization on an equidistant mesh in time.
We solve the problem for various ε and N, the number of intervals in space. M depends in our calculations on N. The convergence rates p_N^ε are computed by

$$p_N^\varepsilon = (\ln E^\varepsilon_{N,M(N)} - \ln E^\varepsilon_{2N,M(2N)})/\ln 2, \tag{51}$$

where $E^\varepsilon_{N,M(N)}$ is the discretization error in the corresponding norm as described below.

5.1. The θ-scheme. We will first examine the implicit Euler scheme and set $\theta = 1$. For the error one gets from (24) and for some N

$$E^\varepsilon_{N,M(N)} = \| (U-u)^M \|_0 + \omega \sum_1^M \tau \| (U-u)^{m-\theta} \|_\varepsilon \le C(N^{-1}\ln N + M^{-1}). \tag{52}$$

We set $\omega = 1$ and $\tau = 1/M$. We choose M to be the greatest integer lower bound of $N/\ln N$ and compute the order of convergence in space according to (51). The errors and the corresponding convergence rates are displayed in Tables 1 and 2.

ε	N=8	N=16	N=32	N=64	N=128	N=256
1.56250e-2	2.7937e-1	1.8944e-1	1.1626e-1	7.1401e-2	4.1964e-2	2.4020e-2
3.90625e-3	2.8258e-1	1.9168e-1	1.1764e-1	7.2304e-2	4.2508e-2	2.4334e-2
9.76562e-4	2.8345e-1	1.9240e-1	1.1801e-1	7.2518e-2	4.2637e-2	2.4409e-2
2.44141e-4	2.8367e-1	1.9261e-1	1.1815e-1	7.2583e-2	4.2669e-2	2.4428e-2
6.10352e-5	2.8373e-1	1.9266e-1	1.1819e-1	7.2611e-2	4.2681e-2	2.4433e-2
1.52588e-5	2.8374e-1	1.9268e-1	1.1820e-1	7.2621e-2	4.2686e-2	2.4435e-2
3.81470e-6	2.8375e-1	1.9268e-1	1.1821e-1	7.2623e-2	4.2688e-2	2.4436e-2
9.53674e-7	2.8375e-1	1.9268e-1	1.1821e-1	7.2624e-2	4.2689e-2	2.4436e-2
2.38419e-7	2.8375e-1	1.9268e-1	1.1821e-1	7.2624e-2	4.2689e-2	2.4437e-2
5.96046e-8	2.8375e-1	1.9268e-1	1.1821e-1	7.2624e-2	4.2689e-2	2.4437e-2
1.49012e-8	2.8375e-1	1.9268e-1	1.1821e-1	7.2624e-2	4.2689e-2	2.4437e-2
3.72529e-9	2.8375e-1	1.9268e-1	1.1821e-1	7.2624e-2	4.2689e-2	2.4437e-2

TABLE 1. Errors for the implicit Euler scheme

ε	N=8	N=16	N=32	N=64	N=128
1.56250e-2	5.6043e-1	7.0441e-1	7.0334e-1	7.6678e-1	8.0494e-1
3.90625e-3	5.5994e-1	7.0435e-1	7.0221e-1	7.6634e-1	8.0476e-1
9.76562e-4	5.5900e-1	7.0520e-1	7.0249e-1	7.6621e-1	8.0468e-1
2.44141e-4	5.5855e-1	7.0508e-1	7.0289e-1	7.6643e-1	8.0466e-1
6.10352e-5	5.5843e-1	7.0497e-1	7.0286e-1	7.6661e-1	8.0477e-1
1.52588e-5	5.5839e-1	7.0494e-1	7.0281e-1	7.6660e-1	8.0485e-1
3.81470e-6	5.5838e-1	7.0493e-1	7.0280e-1	7.6658e-1	8.0484e-1
9.53674e-7	5.5838e-1	7.0492e-1	7.0280e-1	7.6658e-1	8.0483e-1
2.38419e-7	5.5838e-1	7.0492e-1	7.0279e-1	7.6657e-1	8.0483e-1
5.96046e-8	5.5838e-1	7.0492e-1	7.0279e-1	7.6657e-1	8.0483e-1
1.49012e-8	5.5838e-1	7.0492e-1	7.0279e-1	7.6657e-1	8.0483e-1
3.72529e-9	5.5838e-1	7.0492e-1	7.0279e-1	7.6657e-1	8.0483e-1

TABLE 2. Convergence rates in space for the implicit Euler scheme

As stated in (52) we get almost first order convergence in space and first order convergence in time because of the choice of M.
For the Crank Nicolson scheme, i.e. $\theta = 1/2$, we have

$$(53)\quad E^{\varepsilon}_{N,M(N)} = \| (U-u)^M \|_0 + \omega \sum_1^M \tau \| (U-u)^{m-\theta} \|_\varepsilon \leq C(N^{-1}\ln N + M^{-2}).$$

Now we choose M to be the greatest integer lower bound of $\sqrt{N/\ln N}$. The results are displayed in Tables 3 and 4.

ε	N=8	N=16	N=32	N=64	N=128	N=256
1.56250e-2	1.6647e-1	1.0983e-1	6.8130e-2	4.0526e-2	2.3473e-2	1.3345e-2
3.90625e-3	1.6203e-1	1.0692e-1	6.6976e-2	4.0121e-2	2.3347e-2	1.3314e-2
9.76562e-4	1.6055e-1	1.0567e-1	6.6379e-2	3.9898e-2	2.3267e-2	1.3287e-2
2.44141e-4	1.6015e-1	1.0530e-1	6.6130e-2	3.9795e-2	2.3231e-2	1.3274e-2
6.10352e-5	1.6005e-1	1.0520e-1	6.6056e-2	3.9755e-2	2.3216e-2	1.3269e-2
1.52588e-5	1.6002e-1	1.0517e-1	6.6037e-2	3.9745e-2	2.3210e-2	1.3267e-2
3.81470e-6	1.6001e-1	1.0517e-1	6.6032e-2	3.9743e-2	2.3209e-2	1.3266e-2
9.53674e-7	1.6001e-1	1.0517e-1	6.6031e-2	3.9742e-2	2.3208e-2	1.3266e-2
2.38419e-7	1.6001e-1	1.0517e-1	6.6031e-2	3.9742e-2	2.3208e-2	1.3266e-2
5.96046e-8	1.6001e-1	1.0517e-1	6.6031e-2	3.9742e-2	2.3208e-2	1.3266e-2
1.49012e-8	1.6001e-1	1.0517e-1	6.6031e-2	3.9742e-2	2.3208e-2	1.3266e-2
3.72529e-9	1.6001e-1	1.0517e-1	6.6031e-2	3.9742e-2	2.3208e-2	1.3266e-2

TABLE 3. Errors for the Crank Nicolson scheme

With the same arguments as for implicit Euler we get almost first order convergence in space but now second order in time.

5.2. The discontinuous Galerkin scheme. For the discontinuous Galerkin scheme for piecewise constant functions in time, i.e. $q = 0$, we know from (41)

$$(54)\qquad E^{\varepsilon}_{N,M(N)} = \| U-u \|_{dG} \leq C(M^{-1/2} + N^{-1}\ln N + \frac{1}{M^{-1/2}}(N^{-1}\ln N)^2).$$

ε	N=8	N=16	N=32	N=64	N=128
1.56250e-2	5.9997e-1	6.8889e-1	7.4946e-1	7.8781e-1	8.1469e-1
3.90625e-3	5.9980e-1	6.7476e-1	7.3929e-1	7.8108e-1	8.1037e-1
9.76562e-4	6.0346e-1	6.7075e-1	7.3441e-1	7.7802e-1	8.0834e-1
2.44141e-4	6.0496e-1	6.7108e-1	7.3272e-1	7.7651e-1	8.0750e-1
6.10352e-5	6.0537e-1	6.7135e-1	7.3256e-1	7.7603e-1	8.0708e-1
1.52588e-5	6.0548e-1	6.7142e-1	7.3251e-1	7.7603e-1	8.0695e-1
3.81470e-6	6.0550e-1	6.7144e-1	7.3249e-1	7.7602e-1	8.0695e-1
9.53674e-7	6.0551e-1	6.7145e-1	7.3248e-1	7.7602e-1	8.0695e-1
2.38419e-7	6.0551e-1	6.7145e-1	7.3248e-1	7.7602e-1	8.0695e-1
5.96046e-8	6.0551e-1	6.7145e-1	7.3248e-1	7.7602e-1	8.0695e-1
1.49012e-8	6.0551e-1	6.7145e-1	7.3248e-1	7.7602e-1	8.0695e-1
3.72529e-9	6.0551e-1	6.7145e-1	7.3248e-1	7.7602e-1	8.0695e-1

Table 4. Convergence rates in space for the Crank Nicolson scheme

If we now choose M to be the greatest integer lower bound of $(N/\ln N)^2$, the error is dominated by $N^{-1}\ln N$. We can see this fact in Tables 5 and 6, the latter showing the almost first order convergence in space.

ε	N=8	N=16	N=32	N=64	N=128	N=256
1.56250e-2	3.3527e-1	2.2063e-1	1.3821e-1	8.3147e-2	4.8536e-2	2.7746e-2
3.90625e-3	3.3097e-1	2.1712e-1	1.3587e-1	8.1769e-2	4.7747e-2	2.7302e-2
9.76562e-4	3.3000e-1	2.1646e-1	1.3533e-1	8.1379e-2	4.7520e-2	2.7174e-2
2.44141e-4	3.2977e-1	2.1636e-1	1.3530e-1	8.1326e-2	4.7462e-2	2.7140e-2
6.10352e-5	3.2972e-1	2.1633e-1	1.3531e-1	8.1358e-2	4.7468e-2	2.7132e-2
1.52588e-5	3.2970e-1	2.1633e-1	1.3532e-1	8.1374e-2	4.7488e-2	2.7139e-2
3.81470e-6	3.2970e-1	2.1633e-1	1.3532e-1	8.1379e-2	4.7497e-2	2.7149e-2
9.53674e-7	3.2970e-1	2.1633e-1	1.3532e-1	8.1380e-2	4.7499e-2	2.7153e-2
2.38419e-7	3.2970e-1	2.1633e-1	1.3532e-1	8.1380e-2	4.7500e-2	2.7154e-2
5.96046e-8	3.2970e-1	2.1633e-1	1.3532e-1	8.1381e-2	4.7500e-2	2.7154e-2
1.49012e-8	3.2970e-1	2.1633e-1	1.3532e-1	8.1381e-2	4.7500e-2	2.7154e-2
3.72529e-9	3.2970e-1	2.1633e-1	1.3532e-1	8.1381e-2	4.7500e-2	2.7154e-2

Table 5. Errors for dG and $q = 0$

For $q = 1$ we have

$$E^{\varepsilon}_{N,M(N)} \leq C(M^{-3/2} + NM^{-2} + N^{-1}(\ln N)^{3/2} + \frac{1}{M^{-1/2}}(N^{-1}\ln N)^2). \tag{55}$$

For M set to the greatest integer lower bound of $(N/\ln N)^2$ the results are displayed in Tables 7 and 8. The error seems to be dominated by the same term as for $q = 0$.

ε	N=8	N=16	N=32	N=64	N=128
1.56250e-2	6.0367e-1	6.7481e-1	7.3310e-1	7.7661e-1	8.0677e-1
3.90625e-3	6.0819e-1	6.7630e-1	7.3259e-1	7.7615e-1	8.0641e-1
9.76562e-4	6.0836e-1	6.7770e-1	7.3370e-1	7.7613e-1	8.0633e-1
2.44141e-4	6.0808e-1	6.7725e-1	7.3437e-1	7.7694e-1	8.0637e-1
6.10352e-5	6.0798e-1	6.7694e-1	7.3397e-1	7.7733e-1	8.0696e-1
1.52588e-5	6.0795e-1	6.7685e-1	7.3374e-1	7.7700e-1	8.0721e-1
3.81470e-6	6.0794e-1	6.7682e-1	7.3367e-1	7.7683e-1	8.0695e-1
9.53674e-7	6.0794e-1	6.7682e-1	7.3365e-1	7.7677e-1	8.0682e-1
2.38419e-7	6.0794e-1	6.7681e-1	7.3365e-1	7.7676e-1	8.0678e-1
5.96046e-8	6.0794e-1	6.7681e-1	7.3364e-1	7.7676e-1	8.0676e-1
1.49012e-8	6.0794e-1	6.7681e-1	7.3364e-1	7.7676e-1	8.0676e-1
3.72529e-9	6.0794e-1	6.7681e-1	7.3364e-1	7.7676e-1	8.0676e-1

TABLE 6. Convergence rates in space for dG and $q = 0$

ε	N=8	N=16	N=32	N=64	N=128	N=256
1.56250e-2	2.6531e-1	1.7578e-1	1.1013e-1	6.6150e-2	3.8547e-2	2.2000e-2
3.90625e-3	2.6500e-1	1.7532e-1	1.0981e-1	6.5885e-2	3.8363e-2	2.1883e-2
9.76562e-4	2.6476e-1	1.7488e-1	1.0958e-1	6.5791e-2	3.8300e-2	2.1843e-2
2.44141e-4	2.6469e-1	1.7473e-1	1.0941e-1	6.5716e-2	3.8282e-2	2.1832e-2
6.10352e-5	2.6467e-1	1.7468e-1	1.0934e-1	6.5650e-2	3.8256e-2	2.1829e-2
1.52588e-5	2.6467e-1	1.7467e-1	1.0933e-1	6.5625e-2	3.8229e-2	2.1819e-2
3.81470e-6	2.6467e-1	1.7467e-1	1.0932e-1	6.5618e-2	3.8218e-2	2.1807e-2
9.53674e-7	2.6467e-1	1.7467e-1	1.0932e-1	6.5616e-2	3.8215e-2	2.1803e-2
2.38419e-7	2.6467e-1	1.7467e-1	1.0932e-1	6.5616e-2	3.8215e-2	2.1801e-2
5.96046e-8	2.6467e-1	1.7467e-1	1.0932e-1	6.5616e-2	3.8214e-2	2.1801e-2
1.49012e-8	2.6467e-1	1.7467e-1	1.0932e-1	6.5616e-2	3.8214e-2	2.1801e-2
3.72529e-9	2.6467e-1	1.7467e-1	1.0932e-1	6.5616e-2	3.8214e-2	2.1801e-2

TABLE 7. Errors for dG and $q = 1$

ε	N=8	N=16	N=32	N=64	N=128
1.56250e-2	5.9391e-1	6.7453e-1	7.3541e-1	7.7911e-1	8.0909e-1
3.90625e-3	5.9598e-1	6.7497e-1	7.3702e-1	7.8024e-1	8.0990e-1
9.76562e-4	5.9830e-1	6.7444e-1	7.3599e-1	7.8055e-1	8.1015e-1
2.44141e-4	5.9921e-1	6.7541e-1	7.3538e-1	7.7956e-1	8.1020e-1
6.10352e-5	5.9946e-1	6.7587e-1	7.3602e-1	7.7911e-1	8.0941e-1
1.52588e-5	5.9953e-1	6.7600e-1	7.3634e-1	7.7959e-1	8.0907e-1
3.81470e-6	5.9954e-1	6.7604e-1	7.3644e-1	7.7984e-1	8.0946e-1
9.53674e-7	5.9955e-1	6.7605e-1	7.3646e-1	7.7991e-1	8.0965e-1
2.38419e-7	5.9955e-1	6.7605e-1	7.3647e-1	7.7993e-1	8.0971e-1
5.96046e-8	5.9955e-1	6.7605e-1	7.3647e-1	7.7993e-1	8.0973e-1
1.49012e-8	5.9955e-1	6.7605e-1	7.3647e-1	7.7993e-1	8.0973e-1
3.72529e-9	5.9955e-1	6.7605e-1	7.3647e-1	7.7993e-1	8.0973e-1

TABLE 8. Convergence rates in space for dG and $q = 1$

We tested as well the case $N = M$, here the error is a little bit larger. It could be that different error constants influence the behaviour of the error; in any case further detailed numerical studies for $q = 1$ and $q > 1$ are a task for future studies.

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