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TWO-GRID ALGORITHMS FOR AN ORDINARY SECOND ORDER EQUATION WITH AN EXPONENTIAL BOUNDARY LAYER IN THE SOLUTION

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Dedicated to G.I.Shishkin on the occasion of his 70th birthday

Abstract. This paper is concerned with the solution of the nonlinear system of equations arising from the A.M. Il'in's scheme approximation of a model semilinear singularly perturbed boundary value problem. We employ Newton and Picard methods and propose a new version of the two-grid method originated by O. Axelsson [2] and J. Xu [19]. In the first step, the nonlinear differential equation is solved on a "coarse" grid of size H. In the second step, the problem is linearized around an appropriate interpolation of the solution computed in the first step and the linear problem is then solved on a fine grid of size $h << H$. It is shown that the algorithms achieve optimal accuracy as long as the mesh sizes satisfy $h = O(H^{2^m}),\ m = 1, 2, \dots$, where m is the number of the Newton (Picard) iterations for the difference problem. We count the number of the arithmetical operations to illustrate the *computational cost* of the algorithms. Numerical experiments are discussed.

Key Words. nonlinear boundary value problem, boundary layer, Il'in scheme, nonlinear system, Newton method, Picard method, two-grid method.

1. Introduction

It is shown theoretically and experimentally that classical finite difference schemes on non-adaptive meshes have a cell Reynolds number limitation when applied to convection-dominated equations [3,6,7, 8,10,11]. For small values of the perturbation parameter, these techniques lead to spurious solutions when central differences for the advection terms are employed; on the other hand, first-order upwind methods introduce artificial diffusion that thickens the boundary layers. In order to avoid these difficulties, exponentially-fitting techniques are frequently used [1,6,7,8]. Another approach is based on the generation of layer-adapted meshes that allow resolution of the structure of the layer [3,10,11,12].

The defect correction and the Richardson extrapolation methods are used to increase the accuracy of grid solutions for singularly perturbed boundary value problems. Note that the nonlinear case has been considered in [12]. However, the Richardson procedure requires the solution of discrete nonlinear systems on each of the nested meshes.

Two-level discretizations can be dated back to Allen-Southwell [1], see also [5]. In the present paper we shall develop a new version based on the quasilinearization

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method of Belman and Kalaba [4], see also [9]. Two-grid finite element methods were proposed by O. Axelsson [2] and J. Xu [19], independently of each other. Note that the error estimates in these papers are in *weak (Sobolev-type)* discrete norms. Conversely, our errors below are measured in the *maximum* norm, which is sufficiently strong to capture layers and hence seems most appropriate for singularly perturbed problems.

We illustrate some of these concepts on the model boundary value problem

$$-\varepsilon u'' - a(x)u' + f(x,u) = 0,\ x \in \Omega \equiv (0,1);\ u(0) = A,\ u(1) = B, \tag{1}$$

where A, B are given constants, ε is a parameter in $(0,1]$, and $a(x)$ satisfies

$$|a(x)| \geq \alpha > 0,\ a \in C^2(\overline{\Omega}). \tag{2}$$

For the function $f(x,u)$ we will assume that it is twice continuously differentiable with respect to x , three times continuously differentiable with respect to u and

$$f'_u(x,u) \geq 0 \text{ on } \Omega \times R. \tag{3}$$

By these assumptions the problem (1) has the unique solution $u = u(x,\varepsilon)$ and has a boundary layer of order $O(\varepsilon)$ near $x = 0$ or $x = 1$, see for example [8,11,16].

The goal of the present paper is to construct and study theoretically and numerically two-grid interpolation algorithms for implementation of the classical Il'in's difference scheme [6] for problem (1)-(3). We begin by recalling in the next section basic properties of problem (1)-(3) and an already classical uniformly convergent result for the corresponding linear problem, Theorem 1. Then, in Section 3, we describe a Newton linearization process for the differential problem (1)-(3) in order not only to prove uniform convergence of Il'in's scheme but first of all to obtain the estimate (18) which is the key for the two-grid algorithms in the next sections. In Sections 4, 5 we employ Newton and Picard methods in the solution of the arizing systems of algebraic equations. The two-grid algorithms are formulated and their rate of convergence is established in Section 6. This strategy is motivated by the fact (Theorem 3) that the global error of the two-grid interpolation algorithm is of the order h, the same as would have been obtained if the non-linear problem had been solved directly on the fine grid. The coarse mesh can be quite coarse, (see the experiments in Section 7) and still maintain an optimal approximation.

Part of the present results was published in the conference paper [14].

Notation. Define the norm of a continuous function $f(x)$ as $\|f\| = \max_{x\in\overline{\Omega}} |f(x)|$. Throughout this paper C and $C_i,\ i \geq 0$, denote positive constants independent of H, h and ε . If $z = (z_0, \ldots, z_N) \in R^{N+1}$ is a mesh function, define its discrete norm as $\|z\|_h = \max_{0\leq i\leq N} |z_i|$. For a continuous function f defined on Ω by $[f]_{\overline{w}_h}$ we will denote it's projection on a mesh $\overline{w}_h \subset \overline{\Omega}$. In the text $u, u^{(m)}$ and $y, y^{(m)}$ denote continuous and discrete functions, respectively.

2. Preliminary analysis

In the following we will consider the problem (1)-(3) in the case $a(x) \geq \alpha > 0$. The other case $a(x) \leq \alpha < 0$ can be put into the form of the first case by the change of the independent variable from x to $1 - x$.

At first, we get the estimate for the solution of the problem (1)-(3):

$$\|u\| \leq l = \alpha^{-1}\|f(x,0)\| + \max\{|A|, |B|\}. \tag{4}$$

Let us introduce the linear operator:

$$Gz(x) = -\varepsilon z''(x) - a(x)z'(x) + b(x)z(x),$$

where $z(x) \in C^2(\overline{\Omega})$,

$$b(x) = \begin{cases} (f(x,u(x)) - f(x,0))/u(x), & \text{if } u(x) \neq 0, \\ f'_u(x,0), & \text{if } u(x) = 0. \end{cases}$$

Because of (3), we have $b(x) \geq 0$. It is known, for example see [8], that the maximum principle is valid for the operator $G : z(0) \geq 0,\ z(1) \geq 0,\ Gz(x) \geq 0,\ x \in [0,1] \to z(x) \geq 0,\ x \in [0,1]$. Define $z(x) = \alpha^{-1}\|f(x,0)\|(1-x) + \max\{|A|,|B|\} \pm u(x)$. It is clear that the requirements of the maximum principles are fulfilled. Therefore $z(x) \geq 0$, $x \in [0,1]$ which implies (4).

Make a decomposition of the solution $u(x)$:

$$u(x) = V(x) + p(x), \tag{5}$$

where

$$V(x) = r\exp(-a_0\varepsilon^{-1}x),\ r = -\varepsilon u'(0)/a_0,\ a_0 = a(0).$$

Lemma 1. *There exists a constant $C > 0$ such that:*

$$|p'(x)| \leq C. \tag{6}$$

Proof. We get using (1), (5):

(7)

$$\varepsilon p''(x) + a(x)p'(x) = F(x),\ F(x) = f(x,u(x)) + \frac{a_0 r}{\varepsilon}(a(x) - a(0))\exp(-a_0\varepsilon^{-1}x).$$

Using (4) we find the estimate

$$|F(x)| \leq C. \tag{8}$$

Next, rewrite the equation (7) in the form:

$$\Big(\varepsilon p'(x)\exp\Big(\int_0^x \varepsilon^{-1}a(r)\,dr\Big)\Big)' = F(x)\exp\Big(\int_0^x \varepsilon^{-1}a(r)\,dr\Big). \tag{9}$$

An integration of (9) from 0 to x and taking into account $p'(0) = 0$ implies:

$$p'(x) = \frac{1}{\varepsilon}\int_0^x F(s)\exp\Big(-\int_s^x \varepsilon^{-1}a(r)\,dr\Big)\,ds.$$

Finally, recalling (8), we get (6). $\diamondsuit$

According to [13], the following estimates of the derivatives are correct:

$$|u^{(j)}(x)| \leq C\Big[\frac{1}{\varepsilon^j}\exp(-\alpha\varepsilon^{-1}x) + 1\Big],\ 0 < j \leq 4. \tag{10}$$

The estimates (10) point out that the solution of the problem (1)-(3) *has a boundary layer near to the boundary* $x = 0$.

Let us introduce the uniform mesh:

$$\overline{w}_h = \{x_i = ih, i = 0,1,\ldots,N,\ x_0 = 0, x_N = 1\}, \overline{w}_h = w_h \cup \{x_0\} \cup \{x_N\}.$$

We discretize (1), taking into account the boundary layer component $V(x)$ in the solution $u(x)$ and using the difference scheme [6]:

$$\begin{gathered} T_i^h y^h = -\varepsilon_i^h \lambda_{xx}^h y_i^h - a_i\lambda_x^h y_i^h + f(x_i, y_i^h) = 0,\ i = 1,2,\ldots,N-1, \\ y_0^h = A,\ y_N^h = B, \end{gathered} \tag{11}$$

where

$$x_i \in w_h,\ a_i = a(x_i),\ \varepsilon_i^h = \frac{a_i h}{2}\mathrm{cth}\frac{a_i h}{2\varepsilon},$$

$$\lambda_x^h y_i^h = \frac{y_{i+1}^h - y_{i-1}^h}{2h},\ \lambda_{xx}^h y_i^h = \frac{y_{i+1}^h - 2y_i^h + y_{i-1}^h}{h^2}.$$

Using the maximum principle, we can prove that for any mesh functions u^h, v^h :

$$||u^h - v^h||_h \le \frac{1}{\alpha} \max_{0<i<N} |T_i u^h - T_i v^h| + |u_0^h - v_0^h| + |u_N^h - v_N^h|.$$

It follows that the problem (11) has an unique solution.

In the linear case $f(x,u) = f_0(x)u + g(x),\ f_0(x) \ge 0$, the scheme (11) coincides with the famous Il'in-Allen-Southwell scheme [1,6,7,8,11,16]. The literature contains many different techniques for analysing Il'in and related schemes for two-point boundary value problems. In the following we shall make use of the result originated in [6].

Theorem 1. *Let $u(x)$ be the solution of (1)-(3) in the linear case $f(x,u) = f_0(x)u + g(x)$ and y^h- the solution of the corresponding discrete problem (11). Then*

$$\|y^h - [u]_{\overline{w}_h}\|_h \le Ch.$$

3. Uniform convergence

To solve (1), we use a quasilinearization process [4,9] and obtain the Newton sequence $\{u^{(m)}\}_0^\infty$ for an initial guess $u^{(0)}(x)$ with $u^{(0)}(0) = A,\ u^{(0)}(1) = B$:

$$Lu^{(m+1)} = -\varepsilon \frac{d^2 u^{(m+1)}}{dx^2} - a(x) \frac{du^{(m+1)}}{dx} + f'_u(x, u^{(m)}) u^{(m+1)} = f'_u(x, u^{(m)}) u^{(m)} -$$

$$-f(x, u^{(m)}),\quad u^{(m+1)}(0) = A,\ u^{(m+1)}(1) = B,\ m = 0,1,2\ldots \tag{12}$$

Let us first consider the convergence of the process (12). Suppose that

$$\|u^{(0)} - u\| \le \rho. \tag{13}$$

Let l be as in (4) and introduce

$$\theta = \max_{x \in \overline{\Omega}, |\xi| \le l + 2\rho} \|f''_{uu}(x, \xi)\|.$$

Lemma 2. *Assume that $\alpha^{-1}\theta\rho < 1$. Then*

$$\|u^{(m)} - u\| \le \alpha\theta^{-1}(\alpha^{-1}\theta\rho)^{2^m},\ m = 0,1,2,\ldots \tag{14}$$

Proof. The boundary value problem for $v^{(m+1)} = u^{(m+1)} - u$ reads as follows:

$$Lv^{(m+1)}(x) = F^{(m)}(x),\ x \in \Omega,\ v^{(m+1)}(0) = 0,\ v^{(m+1)}(1) = 0, \tag{15}$$

where

$$F^{(m)}(x) = f(x, u^{(m)}(x)) - f(x, u(x)) + f'_u(x, u^{(m)}(x))(u(x) - u^{(m)}(x)).$$

We will prove by induction that for all $k \ge 0,\ \|u^{(k)} - u\| \le \rho$. For $k = 0$ this inequality is obvious. Suppose that it holds for $k = m$. Using the mean value theorem, we easily obtain $\|F^{(m)}\| \le \theta \|u^{(m)} - u\|^2$. The maximum principle applied to problem (15) implies:

$$\|u^{(m+1)} - u\| \le \alpha^{-1}\theta \|u^{(m)} - u\|^2. \tag{16}$$

In view of the assumptions $\|u^{(m)} - u\| \le \rho$ and $\alpha^{-1}\theta\rho < 1$ we reach to the next induction step. So, for all $m \ge 0$ we have $\|u^{(m)} - u\| \le \rho$ which implies (16) for $m = 0,1,2,\ldots$ The inequality (14) follows from (16). ♢

Let us consider the Newton iterative method for the scheme (11):

$$L_i^h y^{(m+1)} = -\varepsilon_i^h \lambda_{xx}^h y_i^{(m+1)} - a_i \lambda_x^h y_i^{(m+1)} + f'_u(x_i, y_i^{(m)}) y_i^{(m+1)} = f'_u(x_i, y_i^{(m)}) y_i^{(m)} -$$

$$-f(x_i, y_i^{(m)}),\ i = 1, \cdots, N-1,\ y_0^{(m+1)} = A,\ y_N^{(m+1)} = B,\quad m = 0, 1, 2, \ldots \tag{17}$$

The next lemma is a keystone for the construction of the two-grid algorithms.

Lemma 3. *Let* $y^{(0)} = [u^{(0)}]_{\bar{w}_h}$ *in (17). There are constants* h_0 *and* ρ_0*, independent of* ε*, such that for* $h \le h_0$ *and* $\|u^{(0)} - u\| \le \rho \le \rho_0$

$$\|y^{(m)} - [u]_{\overline{w}_h}\|_h \le C_0 \Big[h + (\alpha^{-1}\theta\rho)^{2^m} \Big], m = 0, 1, 2, \ldots \tag{18}$$

Proof. We suppose that ρ_0 is sufficiently small such that $\alpha^{-1}\theta\rho < 1$, and introduce the auxiliary iterative process:

$$\begin{gathered} -\varepsilon_i^h \lambda_{xx}^h \tilde{y}_i^{(m+1)} - a_i \lambda_x^h \tilde{y}_i^{(m+1)} + f'_u(x_i, u^{(m)}(x_i)) \tilde{y}_i^{(m+1)} \\ = f'_u(x_i, u^{(m)}(x_i)) u^{(m)}(x_i) - f(x_i, u^{(m)}(x_i)),\ i = 1, 2, \cdots, N-1, \\ \tilde{y}_0^{(m+1)} = A,\ \tilde{y}_N^{(m+1)} = B,\ m = 0, 1, 2, \ldots \end{gathered} \tag{19}$$

An application of Theorem 1 provides the estimate

$$\|\tilde{y}^{(m+1)} - [u^{(m+1)}]_{\overline{w}_h}\|_h \le C_1 h,\quad m = 0, 1, 2, \cdots \tag{20}$$

Define $v^{(m+1)} = y^{(m+1)} - \tilde{y}^{(m+1)}$ and estimate $||v^{(m+1)}||_h$. We subtract (19) from (17), make transformations and obtain the problem:

$$L_i^h v^{(m+1)} = F_i^{(m)},\ i = 1, \cdots, N-1,\ v_0^{(m+1)} = 0,\ v_N^{(m+1)} = 0, \tag{21}$$

where $F_i^{(m)}$ has the form:

$$\begin{aligned} F_i^{(m)} &= -(y_i^{(m)} - u^{(m)}(x_i)) \Big[f''_{uu}(x_i, \xi_i^{(4)})(\xi_i^{(1)} - \xi_i^{(3)}) + \\ &+ \tilde{y}_i^{(m+1)}(f''_{uu}(x_i, \xi_i^{(2)}) - f''_{uu}(x_i, \xi_i^{(3)})) + f''_{uu}(x_i, \xi_i^{(3)})(\tilde{y}_i^{(m+1)} - \xi_i^{(3)}) \Big], \end{aligned}$$

all $\xi_i^{(j)}$, $j = 1, 2, 3, 4$, lie between $y_i^{(m)}$ and $u^{(m)}(x_i)$. We estimate $\tilde{y}_i^{(m+1)} - \xi_i^{(3)}$ by $\widetilde{y}_i^{(m+1)} - \xi_i^{(3)} = (\widetilde{y}_i^{(m+1)} - u^{(m+1)}(x_i)) + (u^{(m+1)}(x_i) - u^{(m)}(x_i)) + (u^{(m)}(x_i) - \xi_i^{(3)})$.

We use that continuous functions f''_{uu}, f'''_{uuu} with given arguments are bounded and obtain that there are some constants C_2, C_3, C_4 such that:

$$\Big\| F^{(m)} \Big\|_h \le \Big(C_2 \Big\| y^{(m)} - [u^{(m)}]_{\overline{w}_h} \Big\|_h + C_3 \Big\| u^{(m+1)} - u^{(m)} \Big\| + C_4 h \Big) \Big\| y^{(m)} - [u^{(m)}]_{\overline{w}_h} \Big\|_h.$$

Applying the maximum principle to the problem (21), using the last inequality and (20), we get

$$\begin{aligned} &\Big\| y^{(m+1)} - [u^{(m+1)}]_{\overline{w}_h} \Big\|_h \le \alpha^{-1} \Big(C_2 \Big\| y^{(m)} - [u^{(m)}]_{\overline{w}_h} \Big\|_h \\ &+ C_3 \Big\| u^{(m+1)} - u^{(m)} \Big\| + C_4 h \Big) \Big\| y^{(m)} - [u^{(m)}]_{\overline{w}_h} \Big\|_h + C_1 h. \end{aligned} \tag{22}$$

It follows from (22) that for enough small values of h and ρ, defined in (13),

$$\Big\| y^{(m+1)} - [u^{(m+1)}]_{\overline{w}_h} \Big\|_h \le \frac{1}{2} \Big\| y^{(m)} - [u^{(m)}]_{\overline{w}_h} \Big\|_h + C_1 h,\ m = 0, 1, 2 \ldots$$

Because of $y^{(0)} = [u^{(0)}]_{\overline{w}_h}$, we get that for any $m \ge 0$

$$\Big\| y^{(m)} - [u^{(m)}]_{\overline{w}_h} \Big\|_h \le 2 C_1 h.$$

Using (14) we complete the proof. $\diamondsuit$

Now we are in a position to prove the ε-uniform convergence of the scheme (11).

Theorem 2. *Let $u(x)$ be the solution of the problem (1) and y^h be the solution of the scheme (11). Then the following estimate holds:*

$$\|y^h - [u]_{\overline{w}_h}\|_h \le C_0 h.$$

Proof. We suppose that ρ is small enough, $\rho \le \rho_0$ and take $h \le h_0$, where ρ_0 and h_0 satisfy the requirements given in the proof of Lemma 3. Similarly as in Lemma 2 one can prove that

$$\|y^{(m)} - y^h\|_h \le \alpha\theta^{-1}(\alpha^{-1}\theta\|y^{(0)} - y^h\|_h)^{2^m},\ m = 0, 1, 2, \dots \tag{23}$$

Hence, $y^{(m)} \to y^h$, as $m \to \infty$, if $\alpha^{-1}\theta\|y^{(0)} - y^h\|_h < 1$. Let $m \to \infty$, then from (18) we get the required estimate. To complete the proof the case $h > h_0$ must be considered. The maximum principle implies $\|y^h\| \le l$ (l corresponds to (4)). Therefore

$$\|y^h - [u]_{\overline{w}_h}\|_h \le \|y^h\|_h + \|[u]_{\overline{w}_h}\|_h \le C_0 h,\ C_0 = 2lh_0^{-1}.$$

$\diamondsuit$

4. Newton's method

In this section Newton's method is discussed for solving the system of nonlinear algebraic equations (11). To compute the solution of the scheme (11) we consider the Newton iterative method (17). Let $y^{(0)}$ be an initial guess such that $||y^{(0)} - y^h||_h \le \delta$, where δ is a given constant.

First we study the convergence of the iterative process (17). Letting $z^{(m)} = y^{(m)} - y^h$, we have from (11), (17):

$$-\varepsilon_i^h \lambda_{xx}^h z_i^{(m+1)} - a_i \lambda_x^h z_i^{(m+1)} + f_u'(x_i, y_i^{(m)}) z_i^{(m+1)} = f_{uu}''(x_i, r_i^m)(y_i^{(m)} - s_i^m) z_i^{(m)},$$

$$z_0^{(m+1)} = 0,\ z_N^{(m+1)} = 0, \tag{24}$$

where r_i^m and s_i^m are between $y_i^{(m)}$ and y_i^h. The application of the maximum principle to the problem (24) implies

$$||y^{(m+1)} - y^h||_h \le \alpha^{-1}\theta||y^{(m)} - y^h||_h^2. \tag{25}$$

It follows from (25) that the Newton method converges if

$$\alpha^{-1}\theta\delta < 1. \tag{26}$$

Also (25) implies

$$||y^{(m)} - y^h||_h \le \alpha\theta^{-1}(\alpha^{-1}\theta\delta)^{2^m},\ \ m \ge 0. \tag{27}$$

Further we will find a lower bound for the necessary number m_h of iterations in order to fulfill the following estimate: $||y^{(m_h)} - y^h||_h \le h$. ¿From (27) we get:

$$m_h \ge \log_2 \frac{\ln(\alpha^{-1}\theta h)}{\ln(\alpha^{-1}\theta\delta)}. \tag{28}$$

Now we will calculate the number of arithmetical operations. Suppose that in each iteration of method (17) we need about $d\,N$ operations. Note that for Gauss elimination one needs about $8N$ operations. Then for m_h iterations we need

$$N_h \approx d\,N \log_2 \frac{\ln(\alpha^{-1}\theta h)}{\ln(\alpha^{-1}\theta\delta)} \tag{29}$$

operations. To reduce the number of operations we will develop a new version of the two-grid Newton method [2, 19]. Introduce the coarse grid $\overline{w}_H$:

$$\overline{w}_H = \{X_i = iH, i = 0, 1, \dots, n,\ X_0 = 0, X_n = 1\},\ \overline{w}_H = w_H \cup \{X_0\} \cup \{X_n\}$$

and write Il'in's scheme on $\overline{w}_H$:

$$-\varepsilon_i^H \lambda_{xx}^H y_i^H - a_i \lambda_x^H y_i^H + f(x_i, y_i^H) = 0,\ 0 < i < n,\ y_0^H = A, y_n^H = B. \tag{30}$$

Then, in view of Theorem 2, we have

$$||y^H - [u]_{\overline{w}_H}||_H \le CH.$$

To compute the solution of (30), we use the Newton method:

$$-\varepsilon_i^H \lambda_{xx}^H y_i^{(m+1)} - a_i \lambda_x^H y_i^{(m+1)} + f(x_i, y_i^{(m)}) + f_u'(x_i, y_i^{(m)})(y_i^{(m+1)} - y_i^{(m)}) = 0,$$

$$0 < i < n, \quad y_0^{(m+1)} = A,\ y_n^{(m+1)} = B. \tag{31}$$

We will count the iterations to achieve $\max_i |y_i^{(m)} - y_i^H| \le H$. If m_H is the necessary number of iterations, as in (28), we have

$$m_H \ge \log_2 \frac{\ln(\alpha^{-1}\theta H)}{\ln(\alpha^{-1}\theta\delta)}. \tag{32}$$

Denote by $y^{(m_H)}$ the approximate solution of scheme (31).

Further we will investigate, how to interpolate the solution $y^{(m_H)}$ from nodes of a coarse grid to nodes of a fine grid. It was shown in [18] for a problem with a boundary layer that the linear interpolation on a uniform mesh leads to significant errors. We use for this purpose exponential interpolation of the function $u(x)$:

$$Int([u]_{\overline{w}_H}, x) = (u_i - u_{i-1}) \frac{\exp(-a_0\varepsilon^{-1}x) - \exp(-a_0\varepsilon^{-1}X_i)}{\exp(-a_0\varepsilon^{-1}X_i) - \exp(-a_0\varepsilon^{-1}X_{i-1})} + u_i \tag{33}$$

for $x \in [X_{i-1}, X_i],\ \ i = 1, 2, \dots, n$. It is proved in [17] that for a function $u(x)$ satisfying (5), (6) the following estimate,

$$|u(x) - Int([u]_{\overline{w}_H}, x)| \le 2CH,$$

holds true. Formula (33) is stable with respect to the perturbation of $[u]_{\overline{w}_H}$, therefore

$$|u(x) - Int(y^{(m_H)}, x)| \le C_1 H. \tag{34}$$

Note that, in order to decrease the number of calculations for an interpolation, outside the boundary layer we can use a formula of linear interpolation:

$$u_L(x) = \frac{x - X_{i-1}}{H}(u_i - u_{i-1}) + u_{i-1}.$$

If $\varepsilon^2 \ge H$, then it holds for some constant C that $|u(x) - u_L(x)| \le CH$ for any mesh interval $[X_{i-1}, X_i]$. Otherwise, if $\varepsilon^2 < H$, then $|u(x) - u_L(x)| \le CH$ for $x \in [X_{i-1}, X_i]$, provided that $X_{i-1} \ge -2\varepsilon\alpha^{-1} \ln\left(\varepsilon/\sqrt{H}\right)$.

Let $y_H^I = [Int(y^{(m_H)}, x)]_{\overline{w}_h}$. Taking into account Theorem 2 and the estimate (34), we obtain $||y_H^I - y^h||_h \le CH$.

So, using iterations on a coarse grid and exponential interpolation, we got the initial guess y_H^I for the method (17) on a fine grid with accuracy $O(H)$. Then we continue iterations (17) to find y^h with accuracy $O(h)$. Let us count the number of arithmetical operations for the two-grid method:

$$N_{hH} \approx d\, n \log_2 \frac{\ln(\alpha^{-1}\theta H)}{\ln(\alpha^{-1}\theta\delta)} + d\, N \log_2 \frac{\ln(\alpha^{-1}\theta h)}{\ln(\alpha^{-1}\theta H)} + I_H, \tag{35}$$

where I_H is the number of operations for an interpolation. Note that we make an interpolation only once.

Suppose that $H \ll \alpha^{-1}\theta$. Then

$$N_h \approx d\,N\log_2\frac{\ln(h)}{\ln(\alpha^{-1}\theta\delta)},\quad N_{hH} \approx d\,n\log_2\frac{\ln(H)}{\ln(\alpha^{-1}\theta\delta)} + d\,N\log_2\frac{\ln(h)}{\ln(H)} + I_H$$

and we can estimate the economy of the operations as follows:

$$N_h - N_{hH} \approx d(N-n)\log_2\frac{\ln(H)}{\ln(\alpha^{-1}\theta\delta)} - I_H. \tag{36}$$

Consider the case $h = H^2$. Then (25) shows that we have to perform only one iteration (17) on a fine grid to find the solution of the scheme (11) with accuracy $O(h)$. In the case $h = H^4$ we need two iterations (17) to find y^h with accuracy $O(h)$.

5. Picard method

Suppose that instead of the condition (3) we have stronger restriction:

$$\beta \geq f'_u(u,x) \geq \gamma > 0 \text{ on } \Omega \times R. \tag{37}$$

To find the solution of the scheme (11) let us consider the Picard iterative method:

$$-\varepsilon_i^h\lambda_{xx}^h y_i^{(m+1)} - a_i\lambda_x^h y_i^{(m+1)} + \beta y_i^{(m+1)} = \beta y_i^{(m)} - f(x_i, y_i^{(m)}),\ 0 < i < N,$$

$$y_0^{(m+1)} = A,\ y_N^{(m+1)} = B. \tag{38}$$

Using the maximum principle, we can prove that

$$||y^{(m+1)} - y^h||_h \leq \Big(1 - \frac{\gamma}{\beta}\Big)||y^{(m)} - y^h||_h. \tag{39}$$

Therefore the Picard method converges for any initial guess.

Let us count the necessary number of iterations to achive the accuracy

$$||y^{(m)} - y^h||_h \leq h.$$

To obtain accuracy of order h for m_h iterations we must have: $\delta\Big(1-\gamma/\beta\Big)^{m_h} \leq h$, which implies $m_h \approx \ln\frac{h}{\delta}/\ln(1-\frac{\gamma}{\beta})$.

To reduce the number of calculations let us consider the iterative method for the scheme (30) on a coarse grid $\overline{w}_H$:

$$-\varepsilon_i^H\lambda_{xx}^H y_i^{(m+1)} - a_i\lambda_x^H y_i^{(m+1)} + \beta y_i^{(m+1)} = \beta y_i^{(m)} - f(x_i, y_i^{(m)}),$$

$$y_0^{(m+1)} = A,\ y_n^{(m+1)} = B. \tag{40}$$

We use the iterations (40) to obtain the accuracy $||y^{(m)} - y^H||_H \leq H$. We need about m_H iterations,

$$m_H \approx \ln\frac{H}{\delta}/\ln(1-\frac{\gamma}{\beta}). \tag{41}$$

Suppose that we got $y^{(m_H)}$ after m_H iterations (40). Then

$$||y^{(m_H)} - [u]_{\overline{w}_H}||_H \leq (C_0+1)H.$$

Let $y_H^I = [Int(y^{(m_H)}, x)]_{\overline{w}_h}$. Then

$$||y_H^I - y^h||_h \leq C_1 H,\quad C_1 = 3(C_0+1).$$

Now we perform iterations (38) on a fine grid, using y_H^I as initial guess to find y^h with the accuracy $O(h)$. Let us count the number of arithmetic operations for the Picard method on a fine grid and for the two-grid Picard method:

$$N_h \approx \frac{dN \ln \frac{h}{\delta}}{\ln(1-\frac{\gamma}{\beta})}, \quad N_{hH} \approx \frac{dn \ln \frac{H}{\delta}}{\ln(1-\frac{\gamma}{\beta})} + \frac{dN \ln \frac{h}{H}}{\ln(1-\frac{\gamma}{\beta})} + I_H. \tag{42}$$

Therefore, an economy of operations can be realized , if we use two-grid Picard method:

$$N_h - N_{hH} \approx \frac{d(N-n)}{\ln(1-\frac{\gamma}{\beta})} \ln \frac{H}{\delta} - I_H.$$

Remark. As it is known, Newton method requires an initial guess, close to the exact solution. According to (26), the closeness can be expressed by the inequality $\alpha^{-1}\theta||y^{(0)} - y^H||_H < 1$. We can perform several initial iterations using the Picard method to achieve the inequality $\alpha^{-1}\theta||y^{(m)} - y^H||_H < 1$ and then to continue with Newton iterations.

6. Two-grid algorithms of high accuracy

The results obtained in the previous sections can be used for formulation of high order accuracy two-grid algorithms. The estimate (18) plays a key role in the construction of the algorithms.

Let y^H be the solution of the nonlinear discrete problem (30) on a coarse mesh with a step H. If in the iterative process (17) on a fine mesh with the step $h = H^2$ we take an initial guess $y^{(0)} = [Int(y^H, x)]_{\overline{w}_h}$, then $||y^h - y^{(0)}||_h \le \rho = CH$, in view of the estimate (18) we have

$$||y^{(1)} - [u]_{\overline{w}_h}||_h \le C_1 h = C_1 H^2.$$

Further, if we take $h = H^4$, then according to (18)

$$||y^{(2)} - [u]_{\overline{w}_h}||_h \le C_2 h = C_2 H^4.$$

On the base of this discussion we can formulate algorithms of high accuracy.

Algorithm 1.

1. Solve the nonlinear difference scheme (30) and let y^H be the solution of the difference scheme. Compute y^H with the accuracy $O(H)$ using Newton or Picard method.
2. Interpolate the mesh function y^H, using exponential interpolation (33), $u_H^I(x) = Int(y^H, x)$.
3. Let $h = H^2$, $\quad y_H^I = [u_H^I]_{\overline{w}_h}$.
4. Solve on the fine mesh w_h the linear problem

$$-\varepsilon_i^h \lambda_{xx}^h y_i^h - a_i \lambda_x^h y_i^h + f(x_i, (y_H^I)_i) + f'_u(x_i, (y_H^I)_i)(y_i^h - (y_H^I)_i) = 0,$$

$$0 < i < N, \quad y_0^h = A, \ y_N^h = B.$$

5. Let $u_h^I = Int(y^h, x)$.

Combining results of sections 3,4 it is easy to prove the following theorem.

Theorem 3. *Let u be the solution of the problem (1)-(3) and let u_h^I be the interpolant as defined in Algorithm 1. Then the following error estimate holds true:*

$$||u_h^I - u|| \le CH^2. \tag{43}$$

Algorithm 2.
Steps 1 and 2 are the same as in Algorithm 1.
3. Let $h = H^4$, $\quad y_H^I = [u_H^I]_{\overline{w}_h}$.
4. Solve on the fine mesh w_h the linear problem

$$-\varepsilon_i^h \lambda_{xx}^h y_i^{hH} - a_i \lambda_x^h y_i^{hH} + f(x_i, (y_H^I)_i) + f'_u(x_i, (y_H^I)_i)(y_i^{hH} - (y_H^I)_i) = 0,$$
$$0 < i < N, \quad y_0^{hH} = A,\ y_N^{hH} = B.$$

5. Solve the linear problem on the mesh w_h :

$$-\varepsilon_i^h \lambda_{xx}^h y_i^h - a_i \lambda_x^h y_i^h + f(x_i, y_i^{hH}) + f'_u(x_i, y_i^{hH})(y_i^h - y_i^{hH}) = 0,$$
$$0 < i < N, \quad y_0^h = A,\ y_N^h = B.$$

6. Let $u_{hH}^I = Int(y^h, x)$.
Again, combining results from Sections 3 and 4 one can prove the next theorem.

Theorem 4. *Let u be the solution of the problem (1)-(3) and let u_{hH}^I be the interpolant as defined in Algorithm 2. Then the following error estimate holds true:*

$$||u_{hH}^I - u|| \le CH^4. \tag{44}$$

It is clear that we can continue Algorithm 2 to obtain on the fine mesh w_h, with $h = H^8$ the accuracy CH^8.

7. Numerical experiments

We present here some numerical results in the following situation:

$$\varepsilon u'' - u' + \exp(-u) + g(x) = 0,\ u(1/2) = A,\ u(1) = B,$$

where $A,\ B,\ g(x)$ correspond to the exact solution

$$u(x) = \exp\left((x-1)/\varepsilon\right) + \ln x$$

with a boundary layer near the point $x = 1$, and according to (1) $f(x,u) = -[\exp(-u) + g(x)]$. The following estimates hold:

$$\beta \ge f'_u \ge \gamma > 0 \ \text{ with } \beta = 2,\ \gamma = e^{-1}.$$

The initial iteration for the iterative methods in all the experiments is chosen as the straight line between boundary conditions: $y_i^{(0)} = A + (2x_i - 1)(B - A)$.

At first, compare the necessary number of iterations for one-grid and for two-grid methods to achieve a given accuracy. We make iterations on the coarse grid and on the fine grid, accordingly, to achieve: $||y^{(m)} - y^{(m-1)}||_H \le rH$, $||y^{(m)} - y^{(m-1)}||_h \le rh$, where m is the number of iterations and r is a given parameter. We use $r = 0.02, \varepsilon = 0.\ 1$.

In Table 1 (left) the number of iterations for the Newton method is presented for different values of N. The upper number in each cell is the number of iterations for the one-grid method and the lower numbers are the number of iterations on the fine grid and the number (parenthesized) of iterations on the coarse grid with n nodes.

Table 1 (right) shows the analogous numbers for the Picard method with parameter $\beta = 2$. Recall that $nH = Nh = 0.5$.

Note that results for other values of r and ε are similar. We have a significant economy of iterations on the fine grid of the two-grid method in comparison to the one-grid method on a fine grid.

Now we discuss numerical results concerning the accuracy of Il'in's scheme realized with Algorithms 1 and 2. Define the error of the difference scheme

$$E_N = \|y^h - [u]_{\overline{w}_h}\|_h$$

and the numerical order of the convergence by the formula

$$CR_N = \log_2(E_N/E_{2N}).$$

In Table 2 the error E_N and convergence rate CR_N are presented depending on ε and h. In Tables we denote $10^{\pm m}$ as $e \pm m$. Results confirm the well-known estimate of the accuracy in the linear case for Il'in's scheme [8]:

$$\left\| y^h - [u]_{\overline{w}_h} \right\|_h \le Ch^2/(h+\varepsilon).$$

In Tables 3, 4 the error E_N of Algorithm 1 and Algorithm 2 and it's convergence rate CR_N are presented. Numerical experiments confirm the results of Theorems 2-4.

Table 1. Number of Newton iterations for one-grid and two-grid methods (left) and Number of Picard iterations for one-grid and two-grid methods , $\beta = 2$ (right)

n	N			
	50	500	5000	10000
3	5 4 (4)	6 5 (4)	7 6 (4)	7 6 (4)
5	5 2 (4)	6 3 (4)	7 3 (4)	7 3 (4)
10	5 2 (4)	6 2 (4)	7 3 (4)	7 3 (4)
50	5	6 2 (5)	7 2 (5)	7 2 (5)
100	5	6 2 (5)	7 2 (5)	7 2 (5)

n	N			
	50	500	5000	10000
3	5 4 (4)	6 5 (4)	7 6 (4)	7 6 (4)
5	5 4 (4)	6 5 (4)	7 6 (4)	7 6 (4)
10	5 3 (4)	6 4 (4)	7 5 (4)	7 5 (4)
50	5	6 2 (5)	7 3 (5)	7 4 (5)
100	5	6 2 (5)	7 2 (5)	7 3 (5)

Table 2. Error of Il'in's scheme and CR_N

ε	h				
	1/10	1/20	1/40	1/80	1/160
1.0	$9.71e-4$ 1.83	$2.73e-4$ 2.16	$6.08e-5$ 1.91	$1.50e-5$ 2.10	$3.78e-6$
$1.0e-1$	$5.22e-3$ 1.94	$1.36e-3$ 2.00	$3.41e-4$ 2.06	$8.17e-5$ 1.91	$2.17e-5$
$1.0e-2$	$2.46e-2$ 1.27	$1.02e-2$ 1.67	$3.21e-3$ 1.89	$8.65e-4$ 1.99	$2.18e-4$
$1.0e-3$	$2.95e-2$ 0.92	$1.56e-2$ 1.01	$7.73e-3$ 1.11	$3.58e-3$ 1.28	$1.47e-3$
$1.0e-4$	$2.99e-2$ 0.89	$1.61e-2$ 0.95	$8.32e-3$ 0.99	$4.19e-3$ 1.01	$2.08e-3$

TABLE 3. Error of Algorithm 1 and CR_N

ε	h				
	1/10	1/20	1/40	1/80	1/160
1.0	$9.72e-6$ 4.00	$6.08e-7$ 4.00	$3.80e-8$ 4.00	$2.42e-9$ 4.20	$1.20e-10$
$1.0e-1$	$9.98e-5$ 3.98	$6.34e-6$ 3.99	$3.98e-7$ 3.99	$2.49e-8$ 4.12	$1.43e-9$
$1.0e-2$	$2.03e-3$ 2.97	$2.59e-4$ 3.46	$2.34e-5$ 3.82	$1.65e-6$ 3.95	$1.07e-7$
$1.0e-3$	$4.87e-3$ 2.28	$9.97e-4$ 2.68	$1.56e-4$ 2.42	$2.92e-5$ 2.68	$4.61e-6$
$1.0e-4$	$5.48e-3$ 1.99	$1.38e-3$ 2.18	$3.03e-4$ 2.52	$5.25e-5$ 2.35	$1.03e-5$

TABLE 4. Error of Algorithm 2 and CR_N

ε	h		
	1/4	1/8	1/12
1.0	$1.46e-6$ 8.00	$5.72e-9$ 8.50	$1.77e-10$
$1.0e-1$	$8.65e-6$ 7.99	$3.41e-8$ 8.38	$1.10e-9$
$1.0e-2$	$1.02e-4$ 6.48	$1.14e-6$ 5.48	$1.24e-7$
$1.0e-3$	$7.34e-4$ 7.18	$5.06e-6$ 5.87	$4.68e-7$
$1.0e-4$	$1.30e-3$ 5.29	$3.32e-5$ 7.32	$1.71e-6$
$1.0-5$	$1.38e-3$ 4.12	$7.94e-5$ 5.01	$1.04e-5$

8. Conclusions

A two grid-method, based on the quasilinearization of Belmann and Kalaba, combined with the special interpolation and Il'in scheme, for a second order ordinary differential equation with an exponential boundary layer is constructed. It is shown that an application of two-grid method leads to a decrease in the number of arithmetical operations. The method applied in this paper is also applicable to systems of ordinary differential equations [15] and to elliptic problems. However, new techniques are required in order to theoretically justify the application of the present method to these more general problems. This is out of the scope of our paper.

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