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A UNIFORM NUMERICAL METHOD FOR A BOUNDARY-SHOCK PROBLEM

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This paper is dedicated to G. I. Shishkin.

Abstract. A singularly perturbed quasilinear boundary-value problem is considered in the case when its solution has a boundary shock. The problem is discretized by an upwind finite-difference scheme on a mesh of Shishkin type. It is proved that this numerical method has pointwise accuracy of almost first order, which is uniform in the perturbation parameter.

Key Words. Boundary-value problem, singular perturbation, boundary shock, finite differences, Shishkin mesh, uniform convergence.

1. Introduction

Consider the problem of finding a $C^2(0,1)$-function $u = u(x)$ which solves the following singularly perturbed boundary-value problem:

$$(1) \qquad -\varepsilon u'' - ub(u)u' + uc(x,u) = 0, \quad x \in (0,1), \quad u(0) = 0, \quad u(1) = B,$$

where $' = \mathrm{d}/\mathrm{d}x$, ε is a small positive perturbation parameter, and B is a positive constant. It is assumed that the functions b and c are sufficiently smooth and satisfy certain conditions, the main ones being $b > 0$ and $c_u \geq 0$. All the assumptions are specified in section 2. They are exactly the same as in [13] and they guarantee that there exists a unique solution u of problem (1) and that u has an exponential boundary layer at $x = 0$.

In [13], (1) is solved numerically by applying a layer-resolving transformation which renders the derivatives of the transformed solution bounded uniformly in ε. The transformed problem is then solved using finite-difference schemes on an equidistant discretization mesh. The layer-resolving transformation corresponds to mesh-generating functions used to create special meshes, dense in the boundary layer, for discretizing the problem (1) directly, cf. [16]. Numerical results obtained by this method show pointwise ε-uniform convergence. However, only L^1 ε-uniform convergence is proved in [13]. The same result is obtained in [18], but for an exponentially-fitted equidistant finite-difference scheme and for a special case ($b \equiv 1$) of problem (1). This special case has been recently considered in [17], where a robust error estimate in the maximum norm is derived. This is achieved by applying the approach in which the differential equation

$$(2) \qquad -\varepsilon u'' - \frac{1}{2}(u^2)' + uc(x,u) = 0$$

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is integrated from x to 1 and then the integral $\int_x^1 u(t)c(t,u(t))\mathrm{d}t$ is approximated using the solution of the corresponding reduced problem, cf. [6, 5]. After the described transformation, equation (2) becomes a Riccati equation, which is solved by the method from [11]. This method uses the simple backward scheme on a Shishkin-type mesh. The error of the approximate solution obtained in this way can be estimated at each mesh point by

$$M[\varepsilon + N^{-1}(\ln N)^2], \tag{3}$$

where N is the number of mesh steps and M is a positive constant independent of both ε and N. Since it often holds in practice that $\varepsilon \ll 1/N$, this error estimate gives accuracy of almost first order (*almost* means here that the accuracy is diminished by $\ln N$ factors). Nevertheless, strictly speaking, (3) does not mean convergence uniform in ε. This result can still be used to achieve ε-uniform convergence, but the method has to be combined with some classical method for solving differential equations, see the discussion in [6, 5]. However, the order of ε-uniform convergence resulting from the combination is lowered since the error can be estimated by $MN^{-\omega}$ with $0 < \omega < 1$. The goal of the present paper is to prove that ε-uniform convergence of order almost 1 can be achieved.

In the numerical method considered here, contrary to [13, 17], the only transformation of the problem is to its conservation form which is then discretized by an upwind finite-difference scheme on a Shishkin piecewise equidistant mesh. There is nothing new about this numerical method, but its analysis is new. Crucial in this is the technique from [9] used to discuss the stability of the discretization scheme. It is originally applied in [9] to a semilinear singular perturbation problem with a boundary turning point. The technique is here adjusted to the quasilinear problem and relies heavily on the Shishkin mesh used. The result is the pointwise error-estimate of the form $MN^{-1}(\ln N)^3$.

The problem (1) can be referred to as a *boundary-shock problem* in contrast to interior-shock problems for which the boundary condition at $x = 0$ is $u(0) = A < 0$, see [4] and [10] for instance. The difficulty in trying to obtain ε-uniform pointwise accuracy for interior-shock problems lies in the fact that the interior shock of the numerical solution is shifted from the original location. The method of the present paper can be applied to interior-shock problems only if the position of the shock is known; then the interior-shock problem can be broken down to two problems of type (1).

The rest of the paper is organized as follows. Properties of the continuous solution are given in section 2, which is based on [13]. The numerical method is described in section 3 and the main result is also proved there. Finally, section 4 contains some numerical results which illustrate the previously presented theory.

2. The continuous problem

The problem (1) is discretized in its conservation form,

$$Tu := -\varepsilon u'' - f(u)' + g(x,u) = 0, \quad x \in (0,1), \quad Ru := (u(0), u(1)) = (0, B), \tag{4}$$

where $B > 0$,

$$f(u) = \int_0^u tb(t)\,\mathrm{d}t, \quad \text{and} \quad g(x,u) = uc(x,u).$$

Although usually ε is small, a wider range of ε values is considered, $\varepsilon \in (0,1]$.

Detailed conditions on b and c follow, cf. [13]. Let $X = [0,1]$ and $U = [0,B]$. It is assumed that $b \in C^2(U)$ and $c \in C^2(X \times U)$ since this is needed for the proof of the main result; other results stated in the paper may require weaker smoothness assumptions. It is further assumed that

$$c^* \geq c(x,u) \geq c_* \geq 0, \quad x \in X, \quad u \in U, \tag{5}$$

and

$$g_u(x,u) \geq 0, \quad x \in X, \quad u \in U. \tag{6}$$

Using (5) we get $TB \geq 0 = T0$. Thus, B and 0 are respectively upper and lower solutions of problem (4), which therefore has a solution $u \in C^4(0,1)$ satisfying $0 \leq u(x) \leq B$, $x \in X$. This solution is unique because (6) implies that (T,R) is an inverse-monotone operator.

Let also

$$b^* \geq b(u) \geq b_* > 0, \quad u \in U, \tag{7}$$

and

$$B > \frac{b^*c^* + \sqrt{b^*c^*(b^*c^* - b_*c_*)}}{b^*b_*}. \tag{8}$$

Conditions (7) and (8) guarantee that u has a boundary layer at $x = 0$. Condition (8) looks technical, but it reduces to $B > c/b$ in the constant-coefficient case $b = b^* = b_*$, $c = c^* = c_*$ and there is no layer at $x = 0$ if $B \leq c/b$, [2].

The smoothness conditions and the conditions (5), (6), (7), and (8) are assumed throughout the paper. Also used throughout the paper are generic constants M and m which are positive numbers independent of ε (and later on of N, the number of mesh steps). They may have different values in different occurrences. M is used in upper-bound estimates and m in the lower-bound ones. Some specific values of M and m are subscripted.

It is proved in [13] that for $k = 1, 2, 3$,

$$|u^{(k)}(x)| \leq M\left(1 + \varepsilon^{-k}e^{-m_*x/\varepsilon}\right), \quad x \in X, \tag{9}$$

where

$$0 < m_* < b_*m_0, \tag{10}$$

with

$$m_0 = m_1e^{-b^*B}, \quad m_1 = \frac{1}{2}b_*B^2 - c^*B + \frac{c^*c_*}{2b^*}.$$

Note that m_1 is positive because of (8).

Estimates (9) and (10) are crucial in the error analysis of section 3. Their proofs can be found in [13], but some details are provided below for completeness. The proofs presented here differ a little from those in [13]. Moreover, closer attention is paid here to the constant m_* since its upper estimate is only implicitly contained in the proofs in [13], this being unimportant there.

Lemma 1. *The solution u of problem (4) satisfies*

$$\frac{m_1}{\varepsilon} \leq u'(0) \leq \frac{M}{\varepsilon}. \tag{11}$$

Proof. If $u'(x) < 0$ at some points in $(0,1)$, then there exists a point $\tilde{x} \in (0,1)$ such that $u'(\tilde{x}) = 0$ and $u''(\tilde{x}) < 0$. This is a contradiction because the differential equation in (1) cannot be satisfied at $x = \tilde{x}$. Therefore, $u'(x) \geq 0$ for $x \in X$.

Let

$$u^*(x) = \frac{c_*}{b^*}(x-1) + B.$$

From (8) we have that $B > c^*/b_* \geq c_*/b^*$ and therefore $u^*(x) \geq u^*(0) > 0$, $x \in X$. This means that $u^*(0) > u(0)$ while $u^*(1) = u(1)$. Since for $x \in X$,

$$Tu^*(x) = u^*(x) \left[-b(u^*(x)) \frac{c_*}{b^*} + c(x, u^*(x)) \right] \geq 0,$$

inverse monotonicity implies that

$$u(x) \leq u^*(x), \quad x \in X. \tag{12}$$

It should be noted that the proof of (12) is more complicated in [13] because $g_u \geq 0$ is introduced there later and inverse monotonicity cannot be used at this stage.

Both estimates in (11) can be proved after integrating $Tu = 0$ from 0 to some point $\zeta \in (0, 1]$. Expressing $\varepsilon u'(0)$, we get

$$\varepsilon u'(0) = \varepsilon u'(\zeta) + f(u(\zeta)) - \int_0^\zeta g(x, u(x))\, \mathrm{d}x. \tag{13}$$

To prove the lower estimate in (11), set $\zeta = 1$ and use $u'(1) \geq 0$ in (13) to get

$$\begin{aligned} \varepsilon u'(0) &\geq f(B) - \int_0^1 g(x, u(x))\, \mathrm{d}x \geq \frac{1}{2} b_* B^2 - c^* \int_0^1 u^*(x)\, \mathrm{d}x \\ &= \frac{1}{2} b_* B^2 - c^* B + \frac{c^* c_*}{2b^*} = m_1. \end{aligned}$$

For the upper estimate, choose ζ so that

$$\varepsilon u'(\zeta) = u(\varepsilon) - u(0) = u(\varepsilon) \leq M, \quad \zeta \in (0, \varepsilon).$$

Then (13) implies that $\varepsilon u'(0) \leq M$. □

Lemma 2. *The solution u of problem (4) satisfies*

$$u(x) \geq m_0 x/\varepsilon, \quad x \in [0, \varepsilon], \tag{14}$$

$$u(x) \geq m_0, \quad x \in [\varepsilon, 1]. \tag{15}$$

Proof. Let $\beta(x) = \int_0^x u(t) b(u(t)) \mathrm{d}t$. The differential equation in (1) can be expressed in the form

$$\varepsilon \left(e^{\beta(x)/\varepsilon} u'(x) \right)' = g(x, u(x)) e^{\beta(x)/\varepsilon},$$

from where

$$u'(x) = \left[\varepsilon^{-1} \int_0^x g(t, u(t)) e^{\beta(t)/\varepsilon}\, \mathrm{d}t + u'(0) \right] e^{-\beta(x)/\varepsilon}. \tag{16}$$

(We can see now that $u'(x) > 0$ for all $x \in X$.) It further follows that

$$u'(x) \geq u'(0) e^{-\beta(x)/\varepsilon} \geq \frac{m_1}{\varepsilon} e^{-b^* B x/\varepsilon}. \tag{17}$$

If $x = 0$, the inequality in (14) is trivially satisfied. If $x \in (0, 1]$, there exists $\xi \in (0, x)$ such that

$$u(x) = x u'(\xi) \geq \frac{m_1}{\varepsilon} x e^{-b^* B \xi/\varepsilon} \geq \frac{m_1}{\varepsilon} x e^{-b^* B x/\varepsilon}, \tag{18}$$

where we have used (17). This immediately implies (14). As for (15), this inequality follows from the fact that $u(x) \geq u(\varepsilon)$ when $x \in [\varepsilon, 1]$ and from (18) with $x = \varepsilon$. □

Theorem 1. *The solution u of problem (4) satisfies (9) with m_* like in (10).*

Proof. It follows from (16) that

$$\begin{aligned} u'(x) &\le \left[\varepsilon^{-1}\frac{c^*}{b_*}\int_0^x u(t)b(u(t))e^{\beta(t)/\varepsilon}\,\mathrm{d}t + u'(0)\right]e^{-\beta(x)/\varepsilon} \\ &= \left[\frac{c^*}{b_*}\left(e^{\beta(x)/\varepsilon}-1\right)+u'(0)\right]e^{-\beta(x)/\varepsilon} \\ &\le M\left[1+\varepsilon^{-1}e^{-\beta(x)/\varepsilon}\right], \quad x\in X, \end{aligned}$$

where Lemma 1 is used in the last step. Since (14) and (15) imply that

$$\beta(x) \ge \begin{cases} b_* m_0 x^2/(2\varepsilon) & \text{for } x\in[0,\varepsilon], \\ b_* m_0 x & \text{for } x\in[\varepsilon,1], \end{cases}$$

and since

$$e^{-b_* m_0 x^2/(2\varepsilon)} \le e^{b_* m_0(1-x/\varepsilon)} \le M e^{-b_* m_0 x/\varepsilon}, \quad x\in[0,\varepsilon],$$

it follows that

$$u'(x) \le M\left(1+\varepsilon^{-1}e^{-b_* m_0 x/\varepsilon}\right), \quad x\in X.$$

This proves (9) for $k=1$.

For $k=2$, differentiate once the differential equation in (1) and express $u''(x)$ the way this is done in (16) with $u'(x)$. From there it follows that, [13, Lemma 4],

$$|u''(x)| \le M\left(1+\varepsilon^{-3}xe^{-\beta(x)/\varepsilon}\right), \quad x\in X.$$

Then, using the above arguments,

$$|u''(x)| \le M\left(1+\varepsilon^{-3}xe^{-b_* m_0 x/\varepsilon}\right) \le M\left(1+\varepsilon^{-2}e^{-m_* x/\varepsilon}\right), \quad x\in X.$$

The estimate for $k=3$ can be proved analogously. □

3. The numerical method

Let X^N be the discretization mesh with points $0=x_0<x_1<\cdots<x_N=1$. Let also $h_i=x_i-x_{i-1}$, $i=1,2,\ldots,N$ and $\chi_i=\theta h_i+(1-\theta)h_{i+1}$, $i=1,2,\ldots,N-1$, for a fixed θ in $[0,1]$. Mesh functions on X^N are denoted by $v^N=(v_i)=(v(x_i))$, $w^N=(w_i)=(w(x_i))$, etc. In particular, $u^N=(u(x_i))$ is the discretization of the continuous solution onto X^N. Let

$$\|v^N\|_{X^N} = \max_{x\in X^N}|v(x)|.$$

The problem (4) is discretized as follows:

$$\begin{gathered} T^N w_i := -\varepsilon D'' w_i - D_\theta f(w_i) + g(x_i,w_i) = 0, \quad i=1,2,\ldots,N-1, \\ w_0=0, \quad w_N=B, \end{gathered} \tag{19}$$

where

$$D_\theta w_i = \frac{w_{i+1}-w_i}{\chi_i} \quad\text{and}\quad D''w_i = \frac{1}{\chi_i}\left(D_0 w_i - D_0 w_{i-1}\right).$$

Because of the assumption (5), the discrete problem (19) has a unique solution $w^N=(w_i)$ satisfying $w_i\in U$, $i=1,2,\ldots,N-1$.

In the discussion of the stability of the discrete operator T^N, the following linear discrete operator is needed:

$$\Lambda^N v_i := -\varepsilon D'' v_i + p_i\frac{v_i-v_{i-1}}{\chi_i} + q_i v_i. \tag{20}$$

Let (G_i^j) be the discrete Green's function associated with Λ^N at the point x_j, $j = 1, 2, \ldots, N-1$. Thus,

$$\Lambda^N G_i^j = \frac{\delta_{ij}}{\chi_i}, \quad i = 1, 2, \ldots, N-1, \quad G_0^j = G_N^j = 0,$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Lemma 3. *Consider the operator Λ^N with $p_i > 0$ and $q_i \geq 0$, $i = 1, 2, \ldots, N-1$. Then its discrete Green's function satisfies*

$$0 \leq G_i^j \leq Q_0^j, \quad i, j = 0, 1, \ldots, N, \tag{21}$$

where Q_0^j is defined below in (22).

Proof. The proof uses the technique from [9] with appropriate modifications. The operator Λ^N is inverse monotone and $G_i^j \geq 0$ for all i and j. Q_0^j is defined in the following more general formula:

$$Q_i^j = \begin{cases} \dfrac{h_j R_j^j + h_{j-1} R_{j-1}^j + \cdots + h_{i+1} R_{i+1}^j}{\varepsilon + p_j h_j} & \text{for } i = 0, 1, \ldots, j-1, \\ 0 & \text{for } i = j, j+1, \ldots, N, \end{cases} \tag{22}$$

with

$$R_i^j = (\sigma_i \sigma_{i+1} \cdots \sigma_{j-1})^{-1}, \quad i = 1, 2, \ldots, j-1, \quad \sigma_\nu = 1 + \frac{p_\nu h_\nu}{\varepsilon},$$

and

$$R_j^j := 1.$$

The estimate (21) is proved using the inverse monotonicity of Λ^N and the barrier function

$$B_i^j = \begin{cases} Q_0^j - Q_i^j & \text{for } i = 0, 1, \ldots, j-1, \\ Q_0^j & \text{for } i = j, j+1, \ldots, N. \end{cases}$$

For any fixed j, it holds that

$$\frac{B_i^j - B_{i-1}^j}{h_i} = \begin{cases} \dfrac{R_i^j}{\varepsilon + p_j h_j} & \text{for } i = 1, 2, \ldots, j, \\ 0 & \text{for } i = j+1, j+2, \ldots, N, \end{cases}$$

and it can be verified that

$$\Lambda^N B_i^j \geq \frac{\delta_{ij}}{\chi_i} = \Lambda^N G_i^j, \quad i = 1, 2, \ldots, N-1.$$

Therefore, $G_i^j \leq B_i^j$ for $i = 1, 2, \ldots, N-1$ and (21) follows because $B_i^j \leq Q_0^j$ for all i. □

From this point on, a special discretization mesh, dense in the layer, is needed. A Shishkin-type mesh, introduced for the first time in [12], is used. It consists of two equidistant parts: the fine part with J subintervals and the coarse part with $N - J$ intervals. J is here less than N and such that $\kappa := J/N$ is a fixed constant. The transition point between the two parts of the mesh is

$$\tau = \min\{\kappa, \lambda\} \quad \text{with} \quad \lambda = \frac{a\varepsilon L}{m_*},$$

where a is a positive constant, m_* is from (10), and $L = L(N)$ denotes a real number satisfying

$$L \leq \ln N \quad \text{and} \quad e^{-L} \leq \frac{L}{N}.$$

Therefore, the fine and coarse mesh steps are $h = \tau/J$ and $H = (1-\tau)/(N-J)$, respectively, and the mesh points are $x_i = ih$ for $i = 0, 1, \ldots, J$ and $x_i = \tau + (i-J)H$ for $i = J+1, J+2, \ldots, N$. Let this mesh be denoted by $S^N(L)$. The standard Shishkin mesh uses $\ln N$ instead of L, typically with $\kappa = \frac{1}{2}$, see [12]. When $N \to \infty$, L behaves like $\ln N$, see [15]. However, there is a practical reason for using values of L which are less than $\ln N$: if L is smaller, the mesh is denser in the layer and more accurate numerical results can be expected. The smallest possible L is the solution L_* of the equation $e^{-L_*} = L_*/N$.

Linß [7] generalizes the results from [9] while making use of the same kind of stability inequality as in [9]. This stability inequality, based on a result analogous to the above Lemma 3, is used differently in [7] than in [9]. In order to obtain further estimates which are needed in the error-analysis, Linß derives a sharper estimate of the quantity corresponding to the present Q_0^j. This sharper estimate takes the Shishkin discretization mesh into account. The same is done here, but the analysis is different. The case $\tau = \lambda$ is considered initially.

Lemma 4. *Let $p_i \geq mu(x_i)$, $i = 1, 2, \ldots, N-1$, where u is the solution of (4). Let also $\tau = \lambda$ and let N be sufficiently large independently of ε. Then the estimate*

$$Q_0^j \leq ML, \tag{23}$$

is satisfied on $S^N(L)$ for all $j = 1, 2, \ldots, N-1$.

Proof. Consider $j = 1, 2, \ldots, J$. According to (22),

$$Q_0^j = \frac{h(1 + R_{j-1}^j + \cdots + R_1^j)}{\varepsilon + p_j h} \leq \frac{jh}{\varepsilon}$$

since $R_i^j < 1$ for $i = 1, 2, \ldots, j-1$ and since p_j is positive because of (14) and (15). Then $h = \lambda/J$ implies (23) in this case.

Let now $j = J+1, J+2, \ldots, N-1$. Since N is large enough, $\lambda \geq \varepsilon$ and therefore $x_j \geq \varepsilon$. Then because of (15) there exists a constant m_2 such that $p_j \geq m_2$. Q_0^j can now be estimated as follows:

$$Q_0^j \leq \frac{H\rho^j + Jh}{\varepsilon + m_2 H},$$

where

$$\rho^j = R_j^j + R_{j-1}^j + \cdots + R_{J+1}^j.$$

Since

$$\frac{Jh}{\varepsilon + m_2 H} \leq \frac{Jh}{\varepsilon} \leq ML,$$

(23) follows from

$$P := \frac{H\rho^j}{\varepsilon + m_2 H} \leq M.$$

To see this, set $\sigma = 1 + m_2 H/\varepsilon$ and use the fact that

$$R_\nu^j \leq \sigma^{\nu - j} \quad \text{for} \quad \nu = J+1, J+2, \ldots, j.$$

Then

$$\rho^j \leq 1 + \sigma^{-1} + \sigma^{-2} + \cdots + \sigma^{J+1-j} \leq \frac{1}{1 - \sigma^{-1}},$$

which implies

$$P \leq \frac{H}{\varepsilon\sigma} \cdot \frac{1}{1 - \sigma^{-1}} = \frac{H}{\varepsilon} \cdot \frac{1}{\sigma - 1} = \frac{1}{m_2}.$$

∎

It should be noted that the simplicity of the Shishkin mesh is beneficial in the above analysis. Bakhvalov discretization meshes [1] are also well-known in the numerical analysis of singular perturbation problems as layer-resolving meshes which are used in ε-uniform methods. Being smoother than Shishkin meshes, they typically give more accurate results, see [14] for instance. However, the proof of Lemma 4 would be much more complicated on a mesh of Bakhvalov type.

The main result of the paper follows.

Theorem 2. *Consider the discrete problem (19) on the mesh $S^N(L)$ with $\tau = \lambda$, $a \geq 2$, and N sufficiently large but independent of ε. Then the solution w^N of (19) satisfies*

$$\|w^N - u^N\|_{S^N(L)} \leq M\frac{L^3}{N},$$

where $u^N = (u(x_i))$, u being the solution of the continuous problem (4).

Proof. Let v^N be an arbitrary mesh function with $v_0 = v_N = 0$. Define for $i = 1, 2, \ldots, N-1$

$$\Lambda^{*,N} v_i = -\varepsilon D'' v_i - D_\theta(p_i v_i) + q_i v_i,$$

where

$$p_i = \int_0^1 z_i(t) b(z_i(t))\,\mathrm{d}t, \quad z_i(t) = tu(x_i) + (1-t)w_i,$$

and

$$q_i = \int_0^1 g_u(x_i, z_i(t))\,\mathrm{d}t.$$

It holds true that

$$\Lambda^{*,N}[u(x_i) - w_i] = T^N u(x_i) - T^N w_i = T^N u(x_i).$$

The operators Λ^N (defined in (20)) and $\Lambda^{*,N}$ are adjoint in the sense that

$$\sum_{i=1}^{N-1} \chi_i v_i \Lambda^N y_i = \sum_{i=1}^{N-1} \chi_i y_i \Lambda^{*,N} v_i$$

(y^N is here another mesh function satisfying $y_0 = y_N = 0$). Therefore,

$$v_j = \sum_{i=1}^{N-1} \chi_i v_i \Lambda^N G_i^j = \sum_{i=1}^{N-1} \chi_i G_i^j \Lambda^{*,N} v_i, \quad j = 1, 2, \ldots, N-1.$$

Up to this point, the proof has followed the steps from [8]. Lemmas 3 and 4 are now used to get

$$|v_j| \leq \sum_{i=1}^{N-1} \chi_i Q_0^j |\Lambda^{*,N} v_i| \leq ML \sum_{i=1}^{N-1} \chi_i |\Lambda^{*,N} v_i|,$$

that is,

$$|u(x_j) - w_j| \leq ML \sum_{i=1}^{N-1} \chi_i |T^N u(x_i)|. \tag{24}$$

Lemma 4 can be applied since $q_i \geq 0$ and

$$p_i \geq b_* \int_0^1 [tu(x_i) + (1-t)w_i]\,\mathrm{d}t = \frac{b_*}{2}[u(x_i) + w_i] \geq mu(x_i)$$

(recall that $w_i \geq 0$). The assertion then follows from (24) if it is proved that

$$\Sigma_1^{N-1} \leq M\frac{L^2}{N}, \tag{25}$$

where

$$\Sigma_j^k := \sum_{i=j}^{k} r_i, \quad r_i := \chi_i |T^N u(x_i)|.$$

The estimate in (25) can be proved using a fairly standard technique, cf. [9] for instance. By Taylor's expansion and (9) it follows that

$$r_i \le M h_{i+1}^2 \left(1 + \varepsilon^{-2} e^{-m_* x_{i-1}/\varepsilon}\right). \tag{26}$$

Consider Σ_1^{J-1} first. Then $h_{i+1} = h \le M\varepsilon L/N$, where h is the fine-mesh step size. This implies that

$$r_i \le M \left(\frac{\varepsilon L}{N}\right)^2 (1 + \varepsilon^{-2}) \le M \frac{L^2}{N^2}.$$

Therefore,

$$\Sigma_1^{J-1} \le M \frac{L^2}{N}. \tag{27}$$

On the other hand, within Σ_{J+2}^{N-1}, $h_{i+1} = H \le M/N$, where H is the coarse-mesh step size, and $x_{i-1} \ge \lambda + H$. Then (26) gives

$$\begin{aligned} r_i \quad &\le \quad M\left[H^2 + \left(\tfrac{H}{\varepsilon}\right)^2 e^{-m_* H/\varepsilon} \cdot e^{-m_*\lambda/\varepsilon}\right] \le M\left[N^{-2} + e^{-m_*\lambda/\varepsilon}\right] \\ &\le \quad M\left[N^{-2} + \left(e^{-L}\right)^a\right] \le M\left[N^{-2} + \left(\frac{L}{N}\right)^a\right] \le M\frac{L^2}{N^2}, \end{aligned}$$

implying that

$$\Sigma_{J+2}^{N-1} \le M \frac{L^2}{N}. \tag{28}$$

Finally, let $i = J, J+1$. Then $h_{i+1} = H$ and $x_{i-1} \ge \lambda - h$, so that

$$r_i \le M\left[H^2 + \left(\tfrac{H}{\varepsilon}\right)^2 e^{-m_*\lambda/\varepsilon}\right] \le M\left[N^{-2} + \left(\frac{1}{\varepsilon N}\right)^2 \cdot \left(\frac{L}{N}\right)^a\right].$$

Then

$$r_i \le M \frac{L^2}{N^2}, \quad i = J, J+1, \quad \text{if} \;\; \varepsilon \ge \frac{1}{N}. \tag{29}$$

It is left to consider the case $\varepsilon \le 1/N$. r_i is then estimated differently from (26):

$$r_i \le 2\varepsilon \max_{x_{i-1} \le x \le x_{i+1}} u'(x) + b^* \int_{u(x_i)}^{u(x_{i+1})} t \; \mathrm{d}t + \chi_i |g(x_i, u(x_i))|. \tag{30}$$

Since

$$\begin{aligned} \int_{u(x_i)}^{u(x_{i+1})} t \; \mathrm{d}t \quad &\le \quad M[u(x_{i+1}) - u(x_i)] = M \int_{x_i}^{x_{i+1}} u'(t) \; \mathrm{d}t \\ &\le \quad M \int_{x_i}^{x_{i+1}} \left(1 + \varepsilon^{-1} e^{-m_* t/\varepsilon}\right) \; \mathrm{d}t \le M\left(H + e^{-m_* x_i/\varepsilon}\right), \end{aligned}$$

it follows from (30) that

$$r_i \le M\left[\varepsilon + e^{-m_* x_{i-1}/\varepsilon} + H\right] \le M\left[N^{-1} + e^{-m_*(\lambda - h)/\varepsilon}\right] \le M\left[N^{-1} + \left(\frac{L}{N}\right)^2\right].$$

This means that

$$r_i \le M\frac{L^2}{N}, \quad i = J, J+1, \quad \text{if} \ \ \varepsilon \le \frac{1}{N}. \tag{31}$$

Then (25) follows from (27), (28), (29), and (31). □

The case $\tau = \kappa$ has not been discussed above. It implies that $\varepsilon \ln N \ge m$, thus N is unreasonably large. Nevertheless, it is only befitting in a paper dedicated to G. I. Shishkin to give a complete proof of ε-uniform convergence in its strictest sense. Theorem 3 below shows that the result of Theorem 2 holds true when $\tau = \kappa$.

Theorem 3. *Consider the discrete problem (19) on the mesh $S^N(L)$ with $\tau = \kappa$. Then the solution w^N of (19) satisfies*

$$\|w^N - u^N\|_{S^N(L)} \le M\frac{L^3}{N},$$

where $u^N = (u(x_i))$, u being the solution of the continuous problem (4).

Proof. Since $\tau = \kappa$, it holds true that $\varepsilon^{-1} \le ML$. Then for $p_j \ge 0$,

$$Q_0^j \le \frac{H(1 + R_{j-1}^j + \cdots + R_1^j)}{\varepsilon + p_j h} \le \frac{1}{\varepsilon} \le ML, \quad j = 1, 2, \ldots, N-1,$$

and the estimate from Lemma 4 is still valid. The estimate in (25) is also satisfied since

$$\chi_i |T^N u(x_i)| \le MN^{-2}(1 + \varepsilon^{-2}) \le M\frac{L^2}{N^2}, \quad i = 1, 2, \ldots, N-1.$$

□

4. Numerical results

The theoretical results from the previous section are now illustrated by some numerical experiments. The following example from [18] is taken for the test problem:

$$-\varepsilon u'' - uu' + u = 0 \ \text{ on } X, \quad u(0) = 0, \quad u(1) = 2. \tag{32}$$

This problem satisfies (8). Its asymptotic solution is

$$u_A(x) = x + 1 - 2 \cdot \frac{e^{-x/\varepsilon}}{1 + e^{-x/\varepsilon}}.$$

For u_A and the solution u of problem (32), it holds true that

$$|u(x) - u_A(x)| \le M\varepsilon, \quad x \in X,$$

see [18]. Therefore, when $\varepsilon \ll N^{-1}$, the numerical solution w^N can be compared to $u_A^N = (u_A(x_i))$. This is done in Tables 1–3, which show the errors

$$E^N = \|u_A^N - w^N\|_{S^N(L)}$$

and the values of the numerical order of convergence

$$\omega^N = \log_2 \frac{E^N}{E^{2N}}.$$

Different values of the scheme parameter θ produce almost identical results. The results presented here are for $\theta = \frac{1}{2}$. The values of E^N are smaller when there are more mesh points in the layer. This can be achieved by making a and L less and κ and m_* greater. Of course, the preceding theory requires that the mesh parameters satisfy $a \ge 2$, $L_* \le L \le \ln N$, $0 < \kappa < 1$, and $0 < m_* < b_* m_0$ (see (10)). In all tables, $a = 2$. In Tables 1–3, all ε values, $\varepsilon = 10^{-k}$ for $k = 4, 5, \ldots, 12$, produce identical errors. This confirms that the numerical method is uniform in ε. In Tables

TABLE 1. Results on $S^N(L_*)$ with $m_* = .067$; $\varepsilon = 10^{-k}$ for $k = 4, 5, \dots, 12$

	$\kappa = 1/2$		$\kappa = 3/4$	
N	E^N	ω^N	E^N	ω^N
64	1.57E−1	.58	9.03E−2	.55
128	1.05E−1	.59	6.16E−2	.66
256	6.98E−2	.70	3.91E−2	.68
512	4.29E−2	.77	2.43E−2	.76
1024	2.51E−2	—	1.44E−2	—

TABLE 2. Results on $S^N(\ln N)$ with $m_* = .067$ and $\kappa = 3/4$; $\varepsilon = 10^{-k}$ for $k = 4, 5, \dots, 12$

N	E^N	ω^N	η^N
64	1.27E−1	.68	1.43
128	7.91E−2	.56	2.30
256	5.38E−2	.67	1.91
512	3.37E−2	.75	1.63
1024	2.00E−2	—	—

1 and 2, $m_* = .067$ ($b_* m_0 = 1/(2e^2) \approx .067667$ for problem (32)). Two different values of κ are used in Table 1.

Table 2 enables a comparison between the use of L_* and $\ln N$ in the mesh construction. It also shows that the corresponding error-estimate $E^N \leq M(\ln N)^3/N$ is too rough. If it is assumed that the error is of the form

$$E^N \approx MN^{-1}(\ln N)^\eta,$$

for some positive constant η, then η can be found from

$$\eta \approx \eta^N := \frac{\ln(2E^{2N}) - \ln E^N}{\ln(\ln 2N) - \ln(\ln N)}.$$

As can be seen in Table 2, the values of η^N are well below 3 in this numerical example.

It should be noted that the theoretically safe value of m_* is small and that this spoils the accuracy to some extent. The estimate (10) is probably too conservative. From the form of u_A it can be concluded that $m_* \in (0, 1)$ may be used in this example. Table 3 shows the numerical results for $m_* = .8$. The very good values for the numerical order of convergence are beyond the theory presented in this paper.

Table 3 is extended in Table 4 to include greater values of ε. Since in Table 4 $\varepsilon \ll N^{-1}$ is not generally satisfied, u_A cannot be used to calculate the errors. The double-mesh principle [3, p. 166] is used instead. In this principle, the error and order of convergence are estimated respectively by

$$E_D^N = \|w^N - \bar{w}^{2N}\|_{S^N(L)} \quad \text{and} \quad \omega_D^N = \log_2 \frac{E_D^N}{E_D^{2N}},$$

where $\bar{w}^{2N}$ is the piecewise linear interpolant of the numerical solution w^{2N}. Since the test problem (32) is of type (2), the method from [17] can be applied to it. However, as pointed out in (3), this method is not appropriate for all $\varepsilon \in (0, 1]$.

TABLE 3. Results on $S^N(L_*)$ with $m_* = .8$ and $\kappa = 3/4$; $\varepsilon = 10^{-k}$ for $k = 4, 5, \dots, 12$

N	E^N	ω^N
64	5.09E−2	.97
128	2.59E−2	.99
256	1.31E−2	.99
512	6.58E−3	1.00
1024	3.30E−3	—

TABLE 4. Results on $S^N(L_*)$ with $m_* = .8$ and $\kappa = 3/4$

	$\varepsilon = 1$		$\varepsilon = .1$		$\varepsilon = .01$		$\varepsilon = .001$	
N	E_D^N	ω_D^N	E_D^N	ω_D^N	E_D^N	ω_D^N	E_D^N	ω_D^N
64	2.79E−3	.97	1.17E−2	1.74	4.33E−2	.96	4.70E−2	.93
128	1.42E−3	.98	3.49E−3	.98	2.22E−2	1.00	2.47E−2	.96
256	7.19E−4	.99	1.77E−3	.99	1.11E−2	1.02	1.27E−2	.98
512	3.61E−4	—	8.88E−4	—	5.49E−3	—	6.41E−3	—

Table 4 illustrates that an advantage of the method presented here is that it can be used successfully for the full range of ε values.

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