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A ROBUST FINITE DIFFERENCE METHOD FOR A SINGULARLY PERTURBED DEGENERATE PARABOLIC PROBLEM, PART I

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This paper is dedicated to Grisha Shishkin, on the occasion of his 70th birthday

Abstract. A singularly perturbed degenerate parabolic problem in one space dimension is considered. Bounds on derivatives of the solution are proved; these bounds depend on the two data parameters that determine how singularly perturbed and how degenerate the problem is. A tensor product mesh is constructed that is equidistant in time and of Shishkin type in space. A finite difference method on this mesh is proved to converge; the rate of convergence obtained depends on the degeneracy parameter but is independent of the singular perturbation parameter. Numerical results are presented.

Key Words. singularly perturbed, degenerate parabolic problem, Shishkin mesh

1. Introduction

Consider the singularly perturbed initial-boundary value problem

$$Lu(x,t) := \varepsilon u_{xx}(x,t) - x^\alpha u_t(x,t) = x^\alpha f(x,t) \quad \text{for } (x,t) \in \Omega, \tag{1a}$$

subject to the Dirichlet initial and boundary conditions

$$u(0,t) = \varphi_L(t) \quad \text{for } 0 < t \le T, \tag{1b}$$

$$u(x,0) = \varphi_0(x) \quad \text{for } 0 \le x \le 1, \tag{1c}$$

$$u(1,t) = \varphi_R(t) \quad \text{for } 0 < t \le T, \tag{1d}$$

where $\Omega := (0,1) \times (0,T]$ for some fixed $T > 0$, the small parameter $\varepsilon \in (0,1]$ and $\alpha > 0$ is a positive constant. The function f is smooth and the functions φ are continuous; further hypotheses will be placed on them later.

The differential operator L of (1) degenerates at the boundary $x = 0$ of $\bar\Omega$ and consequently its properties are not described by the standard theory of parabolic partial differential equations, even for fixed $\varepsilon > 0$. Thus (1) suffers from *two* distinct difficulties: its singularly perturbed nature (caused by the small parameter ε) and its degenerate nature (induced by the coefficient x^α of u_t).

At the boundary $x = 1$ the solution $u(x,t)$ displays a parabolic layer of width $O(\varepsilon^{1/2})$, as in the non-degenerate case, but a more complicated layer of width $O(\varepsilon^{1/(2+\alpha)})$ appears at the boundary $x = 0$. See Figure 1.

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Shishkin [6] studied the initial-boundary problem

$$Lu(x,t) = \varepsilon^{\alpha/(2+\alpha)} f_1(x,t) + x^\alpha f_2(x,t) \tag{2}$$

with the above Dirichlet data, where f_1 and f_2 are smooth. In a later paper [7] we shall consider this more general problem, which requires many changes in the analysis presented here. As mentioned in [6], problems like this arise when one models the transfer of heat over a rectangle in a medium moving with velocity x^α along the x-axis and conducting heat only across the flow; see also [5].

To solve (2) numerically, in [6] the author constructs a tensor product mesh with N_x points in the x direction and N_t points in the t direction. The x-mesh is a modified Shishkin-type mesh with three transition points while the t-mesh is equidistant. On this (x,t)-mesh a standard finite difference scheme is employed: central differencing in the x direction with backward differencing in the t direction. Writing u^N for the numerical solution, it is shown in [6] that the maximum nodal error in $u - u^N$, measured uniformly in ε, are

$$\mathcal{O}(N_x^{-1}\ln N_x + N_t^{-1}) \quad \text{for } 1 \le \alpha \le 2,$$
$$\mathcal{O}(N_x^{-1}\ln N_x + N_x^{-4/(2+\alpha)} + N_t^{-1}) \quad \text{for } \alpha > 2,$$

but the presentation is very concise and consequently some arguments are unclear.

When (2) is replaced by the simpler problem (1), an inspection of [6] shows that one of the mesh transition points can be omitted and the maximum nodal error in $u - u^N$, measured uniformly in ε, is now $\mathcal{O}(N_x^{-1}\ln N_x + N_t^{-1})$ for all $\alpha \ge 1$.

In the present paper we sharpen this result of [6] by showing that in fact for (1) the maximum nodal error, measured uniformly in ε, is $\mathcal{O}(N_x^{-2}(\ln N_x)^2 + N_t^{-1})$ if $\alpha = 1$ or $\alpha \ge 2$. For completeness we also prove the bound $\mathcal{O}(N_x^{-1}\ln N_x + N_t^{-1})$ for $1 < \alpha < 2$. All our arguments are given in detail. Numerical results will be presented to illustrate the accuracy of the numerical method.

Notation. We use C to denote a generic constant that is independent of ε and of any mesh used. Thus C can take different values in different places, even in the same calculation. Set $S(\Omega) = \bar{\Omega} \setminus \Omega$; this is the set of points where the initial and boundary conditions are prescribed. The space of continuous functions defined on any measurable subset ω of Ω is $C(\omega)$ and the $L_\infty(\omega)$ norm on $C(\omega)$ is denoted by $\|\cdot\|_\omega$, except that when $\omega = \Omega$ we simply write $\|\cdot\|$. For non-negative integers m, k and measurable $\omega \subset \Omega$, a function g is said to lie in $C^{m,k}(\omega)$ if $\partial^{i+j} g/\partial x^i \partial t^j \in C(\omega)$ for $0 \le i \le m$ and $0 \le j \le k$.

Finally, set

$$\nu = \frac{1}{2+\alpha} \quad \text{and} \quad \gamma = \frac{\alpha}{2+\alpha}\,.$$

Note that $\gamma = \alpha\nu$ and $2\nu + \gamma = 1$.

2. Properties of the solution u of (1)

By a standard argument [3, Section 2.1] one sees that the differential operator L satisfies the usual maximum principle:

Lemma 1. *Let $\Psi \in C(\bar{\Omega}) \cap C^{2,1}(\Omega)$ with $\Psi \ge 0$ on $S(\Omega)$. If $L\Psi \le 0$ on Ω then $\Psi \ge 0$ on $\bar{\Omega}$.*

This lemma can be used to bound u via a barrier function Φ:

Lemma 2. *Assume that u and Φ lie in $C(\bar{\Omega}) \cap C^{2,1}(\Omega)$ with $\Phi \ge |u|$ on $S(\Omega)$ and $L\Phi \le -Lu$ on Ω. Then $|u| \le \Phi$ on $\bar{\Omega}$.*

Proof. Apply Lemma 1 to the functions $\Phi \pm u$ to get $\Phi \pm u \ge 0$ on $\bar{\Omega}$. □

Lemma 2 clearly implies that (1) has at most one solution in $C(\bar{\Omega}) \cap C^{2,1}(\Omega)$. For existence of a solution to (1), we have the following result.

Lemma 3. *Assume that* $f \in C(\bar{\Omega})$, $\varphi_0 \in C[0,1] \cap C^2(0,1)$, $\varphi_L, \varphi_R \in C^1(0,T]$, *and that*

$$\lim_{t\to 0^+} \varphi_L(t) = \varphi_0(0), \quad \lim_{t\to 0^+} \varphi_R(t) = \varphi_0(1). \tag{3}$$

Then (1) *has a solution* $u \in C(\bar{\Omega}) \cap C^{2,1}(\Omega)$ *and* $\|u\| \le C$.

Proof. Existence of a solution does not follow from the classical theory of parabolic differential equations presented in [3, 4] because of the degenerate nature of the differential operator. Instead, [1, Theorem 3] implies existence of a solution to (1).

Consider the function

$$\Phi(x,t) = \max\{\|\varphi_0\|, \|\varphi_L\|, \|\varphi_R\|\} + \|f\|t \quad \text{for } (x,t) \in \bar{\Omega}.$$

Clearly $\Phi(x,t) \ge u(x,t)$ on $S(\Omega)$ and $L\Phi(x,t) = -x^\alpha\|f\| \le -|x^\alpha f(x,t)|$, so by Lemma 2 we get $|u(x,t)| \le \Phi(x,t) \le C$. □

In this lemma the equations (3) are zero-order corner compatibility conditions on the data of (1). See [4, p.319] for a discussion of such conditions.

During the rest of the paper we shall assume that the hypotheses of Lemma 3 are satisfied so that (1) has a unique solution in $C(\bar{\Omega}) \cap C^{2,1}(\Omega)$. Of course the smoothness of the solution carries over to $S(\Omega)$ provided one stays away from the corners $(0,0)$ and $(1,0)$.

Set $f^*(x) = f(x,0)$ for $0 \le x \le 1$. For $0 < x < 1$, the differential equation (1a) implies that

$$u_t(x,0) = x^{-\alpha}[-x^\alpha f(x,0) + \varepsilon u_{xx}(x,0)] = -f^*(x) + \varepsilon x^{-\alpha}\varphi_0''(x) \tag{4}$$

and

$$\begin{aligned} u_{tt}(x,0) &= x^{-\alpha}[-x^\alpha f_t(x,0) + \varepsilon u_{xxt}(x,0)] \\ &= -f_t(x,0) + \varepsilon x^{-\alpha}[-f^*(x) + \varepsilon x^{-\alpha}\varphi_0''(x)]'' \end{aligned} \tag{5}$$

By applying $\partial/\partial t$ to (1a) one sees that $u_t(x,t)$ is a solution of the equation

$$\varepsilon(u_t(x,t))_{xx} - x^\alpha(u_t(x,t))_t = x^\alpha f_t(x,t)$$

with initial-boundary data $u_t(x,0)$, $\varphi_L'(t)$, and $\varphi_R'(t)$.

Lemma 4. *Assume that* $f, f_t \in C(\bar{\Omega})$, $f^* \in C^2(0,1)$, $\varphi_0 \in C^2[0,1] \cap C^4(0,1)$, $\varphi_L, \varphi_R \in C^2(0,T)$, *that* (3) *is satisfied and that*

$$\lim_{t\to 0^+} \varphi_L'(t) = -f^*(0) + \varepsilon \lim_{x\to 0^+} x^{-\alpha}\varphi_0''(x), \qquad \lim_{t\to 0^+} \varphi_R'(t) = -f^*(1) + \varepsilon\varphi_0''(1). \tag{6}$$

where all the limits are finite. Then $u, u_t \in C(\bar{\Omega}) \cap C^{2,1}(\Omega)$ *and* $\|u_t\| \le C$.

Proof. Apply Lemma 3 to u and u_t, using (4). □

The equations satisfied by $\lim_{t\to 0^+} \varphi_L'(t)$ and $\lim_{t\to 0^+} \varphi_R'(t)$ in Lemma 4 are first-order corner compatibility conditions on the data of (1).

By applying $\partial^2/\partial t^2$ to equation (1) and invoking (5) one obtains an analogous result for $u_{tt}(x,t)$.

Lemma 5. *Assume that* $f, f_t, f_{tt} \in C(\bar\Omega)$, $f^* \in C^2[0,1] \cap C^4(0,1)$, $f_t(x,0) \in C^2(0,1)$, $\varphi_0 \in C^4[0,1] \cap C^6(0,1)$, $\varphi_L, \varphi_R \in C^3(0,T]$, *that* (3) *and* (6) *are satisfied and that*

$$\lim_{t\to 0^+} \varphi_L''(t) = -f_t(0,0) + \lim_{x\to 0^+} x^{-\alpha}\varepsilon[-f^*(x) + \varepsilon x^{-\alpha}\varphi_0''(x)]'',$$
$$\lim_{t\to 0^+} \varphi_R''(t) = -f_t(1,0) + \varepsilon[-f^{*''}(1) + \alpha(\alpha-1)\varepsilon\varphi_0''(1) - 2\alpha\varepsilon\varphi_0'''(1) + \varepsilon\varphi_0''''(1)],$$

where all the limits are finite. Then $u, u_t, u_{tt} \in C(\bar\Omega) \cap C^{2,1}(\Omega)$ *and* $\|u_{tt}\| \le C$.

The behaviour of the solution $u(x,t)$ and its derivatives near the boundary $x = 0$ is qualitatively different from the rest of the domain Ω, so in the analysis of this section we will consider (1) in each of the subdomains

$$\Omega_1 := (0,\, 3\varepsilon^\nu) \times (0,T] \quad \text{and} \quad \Omega_2 := (\varepsilon^\nu, 1) \times (0,T],$$

where it is assumed that ε is so small that $3\varepsilon^\nu < 1$.

2.1. Bounds on derivatives of u in $\bar\Omega_1$.

Lemma 6. *If the hypotheses of Lemma 4 are satisfied then* $\|u_{xx}\|_{\Omega_1} \le C\varepsilon^{-2\nu}$. *If the hypotheses of Lemma 5 are satisfied then* $\|u_{xxt}\|_{\Omega_1} \le C\varepsilon^{-2\nu}$.

Proof. Suppose that the hypotheses of Lemma 4 are satisfied. Then $\|u_t\| \le C$. Now for $(x,t) \in \Omega_1$, from (1) one gets

$$|u_{xx}(x,t)| = \varepsilon^{-1}|x^\alpha f(x,t) + x^\alpha u_t(x,t)| \le C\varepsilon^{-1}\varepsilon^\gamma[\|f\| + \|u_t\|] \le C\varepsilon^{-2\nu}.$$

Next, assume that the hypotheses of Lemma 5 are satisfied. Applying $\partial/\partial t$ to (1) then appealing to Lemma 5, for $(x,t) \in \Omega_1$, one gets similarly

$$|u_{xxt}(x,t)| = \varepsilon^{-1}|x^\alpha f_t(x,t) + x^\alpha u_{tt}(x,t)| \le C\varepsilon^{-2\nu}.$$

□

Lemma 7. *Let the hypotheses of Lemma 5 be satisfied. Then* $\|u_{xt}\|_{\Omega_1} \le C\varepsilon^{-\nu}$.

Proof. We interpolate between the bounds on $\|u_t\|$ from Lemma 4 and $\|u_{xxt}\|_{\Omega_1}$ from Lemma 6. Fix $(x,t) \in \Omega_1$. Choose an x-interval $I := (x_1, x_2)$ of length ε^ν such that $x \in I \subset (0, 3\varepsilon^\nu)$. By the mean value theorem and Lemma 4,

$$|u_{xt}(x^*,t)| = \left|\frac{u_t(x_2,t) - u_t(x_1,t)}{x_2 - x_1}\right| \le C\varepsilon^{-\nu} \quad \text{for some } x^* \in I. \tag{7}$$

Next,

$$|u_{xt}(x,t)| = \left|u_{xt}(x^*,t) + \int_{s=x^*}^{x} u_{xxt}(s,t)\, ds\right| \le C\varepsilon^{-\nu}$$

by (7), Lemma 6 and $|x - x^*| \le x_2 - x_1 = \varepsilon^\nu$. □

Lemma 8. *Let the hypotheses of Lemma 5 be satisfied and assume that* $f \in C^{2,1}(\bar\Omega_1)$. *Then*

$$\|u_{xxx}\|_{\Omega_1} \le C\varepsilon^{-3\nu} \quad \text{for } \alpha \ge 1,$$
$$\|u_{xxxx}\|_{\Omega_1} \le C\varepsilon^{-4\nu} \quad \text{if } \alpha = 1 \text{ or } \alpha \ge 2.$$

Proof. Let $(x,t) \in \Omega_1$. Applying $\partial/\partial x$ to (1) yields

$$\begin{aligned}|\varepsilon u_{xxx}(x,t)| &= \left|x^\alpha[f_x(x,t) + u_{xt}(x,t)] + \alpha x^{\alpha-1}[f(x,t) + u_t(x,t)]\right| \\ &\le C[x^\alpha(1 + \varepsilon^{-\nu}) + x^{\alpha-1}]\end{aligned}$$

by Lemmas 4 and 7. For $\alpha \geq 1$ this gives

$$|u_{xxx}(x,t)| \leq C\varepsilon^{-1}[\varepsilon^{\nu\alpha}(1+\varepsilon^{-\nu}) + \varepsilon^{\nu(\alpha-1)}] \leq C\varepsilon^{-3\nu}.$$

For the bound on u_{xxxx}, by applying $\partial^2/\partial x^2$ to (1) one obtains

$$\begin{aligned}|\varepsilon u_{xxxx}(x,t)| &= \big|x^\alpha[f_{xx}(x,t)+u_{xxt}(x,t)] + 2\alpha x^{\alpha-1}[f_x(x,t)+u_{xt}(x,t)] \\ &\qquad + \alpha(\alpha-1)x^{\alpha-2}[f(x,t)+u_t(x,t)]\big| \\ &\leq C[x^\alpha(1+\varepsilon^{-2\nu}) + x^{\alpha-1}(1+\varepsilon^{-\nu}) + (\alpha-1)x^{\alpha-2}]\end{aligned}$$

by Lemmas 6 and 7. If $\alpha = 1$ we deduce that

$$|u_{xxxx}(x,t)| \leq C\varepsilon^{-1}[\varepsilon^{\nu}(1+\varepsilon^{-2\nu}) + (1+\varepsilon^{-\nu})] \leq C\varepsilon^{-4\nu}$$

while if $\alpha \geq 2$ we get

$$|u_{xxxx}(x,t)| \leq C\varepsilon^{-1}[\varepsilon^{\nu\alpha}(1+\varepsilon^{-2\nu}) + \varepsilon^{\nu(\alpha-1)}(1+\varepsilon^{-\nu}) + \varepsilon^{\nu(\alpha-2)}] \leq C\varepsilon^{-4\nu}.$$

□

2.2. Bounds on derivatives of u in $\bar\Omega_2$. Next we consider the behaviour of u in the subdomain Ω_2. To do this we examine the more general problem

(8) $$Lw(x,t) = \varepsilon w_{xx}(x,t) - x^\alpha w_t(x,t) = \bar f(x,t) \quad \text{for } (x,t) \in \Omega_2,$$

where $\bar f(x,t)$ is some given function and Dirichlet initial-boundary conditions are specified on $S(\Omega_2) := \bar\Omega_2 \setminus \Omega_2$. Lemmas 1–5 apply, *mutatis mutandis*, to (8). We shall invoke some of these lemmas in this subsection without reminding the reader that in each case one must adjust the statement of the lemma to fit $\bar f$ and Ω_2.

Lemma 9. *Let $g \in C[\varepsilon^\nu, 1] \cap C^2(\varepsilon^\nu, 1)$ satisfy*

(9a) $$g(x) \leq g(\varepsilon^\nu) \quad \textit{for } \varepsilon^\nu \leq x \leq 1,$$

(9b) $$|g'(x)| \leq C_1 x^{-1} g(x) \quad \textit{and} \quad |g''(x)| \leq C_1 x^{-2} g(x) \quad \textit{for } \varepsilon^\nu < x < 1,$$

where $C_1 \geq 1$ is a fixed constant. Assume that the initial-boundary data of (8) and $\bar f(x,t)$ satisfy the hypotheses of Lemma 3. Assume that

$$x^\alpha|w(x,t)| \leq C_1 g(x) \ \textit{on } S(\Omega_2) \quad \textit{and} \quad |\bar f(x,t)| \leq C_1 g(x) \ \textit{on } \bar\Omega_2.$$

Then $x^\alpha|w(x,t)| \leq C_1 e^{C_2 t} g(x)$ on $\bar\Omega_2$, where $C_2 := C_1(\alpha^2+3\alpha+1)+2$.

Proof. Define the function

$$\Phi(x,t) = C_1 e^{C_2 t} x^{-\alpha} g(x) \quad \text{on } \bar\Omega_2.$$

Then

$$\begin{aligned}L\Phi(x,t) &= C_1\varepsilon e^{C_2 t}[\alpha(\alpha+1)x^{-\alpha-2}g(x) - 2\alpha x^{-\alpha-1}g'(x) + x^{-\alpha}g''(x)] \\ &\qquad - C_1C_2e^{C_2t}g(x) \\ &\leq C_1^2\varepsilon e^{C_2t}[\alpha(\alpha+1)x^{-\alpha-2}g(x) + 2\alpha x^{-\alpha-2}g(x) + x^{-\alpha-2}g(x)] \\ &\qquad - C_1C_2e^{C_2t}g(x) \\ &= C_1^2 e^{C_2t}\varepsilon x^{-\alpha-2}(\alpha^2+3\alpha+1)g(x) - C_1C_2e^{C_2t}g(x) \\ &\leq C_1e^{C_2t}[C_1(\alpha^2+3\alpha+1) - C_2]g(x) \\ &= -2C_1e^{C_2t}g(x) \\ &< -|\bar f(x,t)|\end{aligned}$$

where we used $\varepsilon x^{-\alpha-2} \leq 1$ (recall that $x \geq \varepsilon^\nu$). One can now invoke Lemma 2 to conclude that $|w(x,t)| \leq \Phi(x,t)$ on $\bar\Omega_2$. □

Let Q be some quantity that is independent of x and t.

Lemma 10. *Assume that the initial-boundary data of* (8) *and* $\bar{f}(x,t)$ *satisfy the hypotheses of Lemmas 3–5. Assume also that for* $\varepsilon^{\nu} < x < 1$ *and* $0 < t \le T$ *one has*

$$w(x,0) = 0, \qquad x|\bar{f}_x(x,0)| + x^2|\bar{f}_{xx}(x,0)| \le CQ,$$

and

$$\left|\frac{\partial^j}{\partial t^j}\bar{f}(x,t)\right| \le CQ, \quad \left|\frac{\partial^j}{\partial t^j}w(\varepsilon^{\nu},t)\right| \le CQ\varepsilon^{-\gamma}, \quad \left|\frac{\partial^j}{\partial t^j}w(1,t)\right| \le CQ$$

for $j = 0,1,2$. *Then for* $j = 0,1,2$ *it follows that*

$$\left|\frac{\partial^j}{\partial t^j}w(x,t)\right| \le CQx^{-\alpha} \quad \text{on } \Omega_2. \tag{10}$$

Furthermore,

$$|w_{xt}(x,t)| \le CQ\varepsilon^{-\gamma-\nu} \quad \text{and} \quad |w_{xxt}(x,t)| \le CQ\varepsilon^{-1} \quad \text{on } \bar{\Omega}_2. \tag{11}$$

Proof. First we prove (10). For $j = 0$ the result is immediate from Lemma 9 with $g(x) = Q$. For $j = 1$, observe that w_t satisfies $\varepsilon(w_t(x,t))_{xx} - x^{\alpha}(w_t(x,t))_t = \bar{f}_t(x,t)$ and

$$|w_t(x,0)| = |-x^{-\alpha}\bar{f}(x,0) + \varepsilon x^{-\alpha}w_{xx}(x,0)| = |x^{-\alpha}\bar{f}(x,0)| \le CQx^{-\alpha}$$

so the result follows from Lemma 9 applied to w_t with $g(x) = Q$. For $j = 2$ we see that $w_{tt}(x,t)$ satisfies $\varepsilon(w_{tt}(x,t))_{xx} - x^{\alpha}(w_{tt}(x,t))_t = \bar{f}_{tt}(x,t)$ and from (8) one has

$$\begin{aligned}|w_{tt}(x,0)| &= |-x^{-\alpha}\bar{f}_t(x,0) + \varepsilon x^{-\alpha}w_{xxt}(x,0)| \\ &= |x^{-\alpha}\bar{f}_t(x,0) + \varepsilon x^{-\alpha}[x^{-\alpha}\bar{f}(x,0)]_{xx}| \\ &\le CQx^{-\alpha} + CQ\varepsilon x^{-2\alpha-2} \le CQx^{-\alpha},\end{aligned}$$

where we used $w(x,0) \equiv 0$ and (8) to infer that $w_{xxt}(x,0) = -[x^{-\alpha}\bar{f}(x,0)]_{xx}$. The bound (10) for $j = 2$ then follows from Lemma 9 applied to w_{tt} with $g(x) = Q$.

Next, consider (11). Now

$$|w_{xxt}(x,t)| = \varepsilon^{-1}|f_t(x,t) + x^{\alpha}w_{tt}| \le CQ\varepsilon^{-1}$$

by (10). Interpolating (as in the proof of Lemma 7) between this inequality and the bound $|w_t(x,t)| \le CQx^{-\alpha} \le CQ\varepsilon^{-\gamma}$ (which comes from (10) and $x \ge \varepsilon^{\nu}$), one obtains the desired bound on $|w_{xt}(x,t)|$. □

The next lemma sharpens the bound (11) on w_{xxt}.

Lemma 11. *Let all the hypotheses of Lemma 10 be satisfied. Assume also that*

$$\begin{aligned}x|\bar{f}_{xt}(x,t)| + x^2|\bar{f}_{xxt}(x,t)| &\le CQ \quad \text{on } \Omega_2, \\ |w_{xxt}(1,t)| &\le CQ \quad \text{for } 0 \le t \le T.\end{aligned}$$

Then $|w_{xxt}(x,t)| \le CQx^{-\alpha-2}$ *on* $\bar{\Omega}_2$.

Proof. Set $z(x,t) = x^{-\alpha}w_{xxt}(x,t)$ and $\tilde{f}(x,t) = [x^{-\alpha}\bar{f}_t(x,t)]_{xx}$ for $(x,t) \in \Omega_2$. From (8) we have

$$\begin{aligned}\varepsilon[x^{-\alpha}w_{xx}(x,t)]_{xxt} - w_{xxtt}(x,t) &= [x^{-\alpha}\bar{f}(x,t)]_{xxt}\,, \\ \text{i.e.,} \quad \varepsilon z_{xx}(x,t) - x^{\alpha}z_t(x,t) &= \tilde{f}(x,t).\end{aligned}$$

Recalling our hypotheses, we see that $|\tilde{f}(x,t)| \le CQx^{-\alpha-2}$. Set $\tilde{g}(x) := Qx^{-\alpha-2}$.

As the corner compatibility conditions for (8) are satisfied up to the second order, it follows that $w_{tt} \in C(\bar{\Omega}_2)$. But $\varepsilon w_{xxt}(x,t) - x^{\alpha}w_{tt}(x,t) = f_t(x,t)$, so

$w_{xxt} \in C(\bar{\Omega}_2)$. Thus $z \in C(\bar{\Omega}_2)$ and the compatibility condition of zero order for $z(x,t)$ is satisfied. Invoking (11), we get $|z(\varepsilon^\nu, t)| \le CQ\varepsilon^{-\gamma-1} = C\varepsilon^{-\nu\alpha}\tilde{g}(\varepsilon^\nu)$. Now

$$|z(x,0)| = x^{-\alpha}|w_{xxt}(x,0)| = |x^{-\alpha}[x^{-\alpha}\bar{f}(x,0)]_{xx}| \le CQx^{-2\alpha-2} = Cx^{-\alpha}\tilde{g}(x).$$

Finally, $|z(1,t)| = |w_{xxt}(1,t)| \le CQ = C\tilde{g}(1)$.

Thus we can invoke Lemma 9 to conclude that $|z(x,t)| \le Cx^{-\alpha}\tilde{g}(x)$. Hence $|w_{xxt}(x,t)| = |x^\alpha z(x,t)| \le C\tilde{g}(x) = CQx^{-\alpha-2}$. □

Lemma 12. *Let the hypotheses of Lemma 11 be satisfied. Then*

$$|w_{xt}(x,t)| \le CQx^{-\alpha-1} \quad \text{on } \Omega_2.$$

Proof. Lemmas 10 and 11 give $|w_t(x,t)| + x^2|w_{xxt}(x,t)| \le CQx^{-\alpha}$. Interpolating between these two bounds as in the proof of Lemma 7 while using an x-interval of length $x/2$, one obtains the desired result. □

Lemma 13. *Let the hypotheses of Lemma 11 be satisfied. Assume also that*

$$x|\bar{f}_x(x,t)| + x^2|\bar{f}_{xx}(x,t)| \le CQ \quad \text{on } \Omega_2.$$

Then

$$x|w_{xxx}(x,t)| + x^2|w_{xxxx}(x,t)| \le CQ\varepsilon^{-1} \quad \text{on } \Omega_2.$$

Proof. Applying $\partial/\partial x$ and $\partial^2/\partial x^2$ to (8) gives

$$\varepsilon w_{xxx}(x,t) - x^\alpha w_{xt}(x,t) - \alpha x^{\alpha-1}w_t(x,t) = \bar{f}_x(x,t)$$

and

$$\varepsilon w_{xxxx}(x,t) - x^\alpha w_{xxt}(x,t) - 2\alpha x^{\alpha-1}w_{xt}(x,t) - \alpha(\alpha-1)x^{\alpha-2}w_t(x,t) = \bar{f}_{xx}(x,t).$$

Invoking Lemmas 10–12 and our hypotheses, we get

$$|w_{xxx}(x,t)| \le C\varepsilon^{-1}\big[x^\alpha|w_{xt}(x,t)| + x^{\alpha-1}|w_t(x,t)| + |\bar{f}_x(x,t)|\big] \le CQ\varepsilon^{-1}x^{-1}$$

and

$$\begin{aligned}|w_{xxxx}(x,t)| &\le C\varepsilon^{-1}\big[x^\alpha|w_{xxt}(x,t)| + x^{\alpha-1}|w_{xt}(x,t)| + x^{\alpha-2}|w_t(x,t)| + |\bar{f}_{xx}(x,t)|\big]\\ &\le CQ\varepsilon^{-1}x^{-2},\end{aligned}$$

as desired. □

We now return to the behaviour on Ω_2 of the solution u of (1). Define the regular component U^0 of u by

$$U^0(x,t) = \varphi_0(x) - \int_0^t f(x,s)ds \quad \text{for } (x,t) \in \Omega_2. \tag{12}$$

Then decompose u as $u = U^0 + v + V^L + V^R$, where the function v is defined by

$$\varepsilon v_{xx}(x,t) - x^\alpha v_t(x,t) = -\varepsilon U^0_{xx}(x,t) \quad \text{for } (x,t) \in \Omega_2, \tag{13a}$$

$$v(\varepsilon^\nu, t) = a_1 t + \frac{1}{2}a_2 t^2 \quad \text{for } 0 < t \le T, \tag{13b}$$

$$v(x,0) = 0 \quad \text{for } \varepsilon^\nu \le x \le 1, \tag{13c}$$

$$v(1,t) = \int_0^t \varepsilon U^0_{xx}(1,s)ds + \frac{1}{2}b_2 t^2 \quad \text{for } 0 < t \le T. \tag{13d}$$

Here

$$a_1 := \varepsilon^{2\nu}U^0_{xx}(\varepsilon^\nu, 0) = \varepsilon^{2\nu}\varphi_0''(\varepsilon^\nu), \quad a_2 := \varepsilon^{2\nu}[-f(x,0) + \varepsilon x^{-\alpha}\varphi_0''(x)]''_{x=\varepsilon^\nu},$$

$$b_2 := \varepsilon^2[x^{-\alpha}\varphi_0''(x)]''_{x=1};$$

the terms $a_1 t$ and $a_2 t^2/2$ are present in $v(\varepsilon^\nu, t)$ to yield compatibility at the corner $(\varepsilon^\nu, 0)$ of $\bar{\Omega}_2$; the term $\int_0^t \varepsilon U^0_{xx}(x,t)ds$ is added to $v(1,t)$ to ensure that no layer appears along the boundary $x = 1$; finally, the term $b_2 t^2/2$ ensures compatibility of the data at the corner $(1,0)$ of $\bar{\Omega}_2$. The functions V^L and V^R are defined by

(14a) $\varepsilon V^L_{xx}(x,t) - x^\alpha V^L_t(x,t) = 0 \quad \text{for } (x,t) \in \Omega_2,$

(14b) $V^L(\varepsilon^\nu, t) = u(\varepsilon^\nu, t) - \varphi_0(\varepsilon^\nu) + \int_0^t f(\varepsilon^\nu, s)ds - a_1 t - \frac{1}{2}a_2 t^2 \text{ for } 0 < t \le T,$

(14c) $V^L(x,0) = 0 \quad \text{for } \varepsilon^\nu \le x \le 1,$

(14d) $V^L(1,t) = 0 \quad \text{for } 0 < t \le T,$

and

(15a) $\varepsilon V^R_{xx}(x,t) - x^\alpha V^R_t(x,t) = 0 \quad \text{for } (x,t) \in \Omega_2,$

(15b) $V^R(\varepsilon^\nu, t) = 0 \quad \text{for } 0 < t \le T,$

(15c) $V^R(x,0) = 0 \quad \text{for } \varepsilon^\nu \le x \le 1,$

(15d) $V^R(1,t) = \varphi_R(t) - \varphi_0(1) - \int_0^t \varepsilon U^0_{xx}(1,s)ds + \int_0^t f(1,s)ds - \frac{1}{2}b_2 t^2$

for $0 < t \le T$.

Observe that (13a) implies that

$$v_{xx}(1,t) = \varepsilon^{-1}[-\varepsilon U^0_{xx}(1,t) + v_t(1,t)] = \varepsilon^{-1} b_2 t$$

by (13d). Hence

(16) $|v_{xxt}(1,t)| = |\varepsilon^{-1} b_2| = \left|\varepsilon[x^{-\alpha}\varphi_0''(x)]''_{x=1}\right| \le C\varepsilon \quad \text{and} \quad |v_{xx}(1,t)| \le C\varepsilon.$

Lemma 14. *For each component of the solution in the decomposition (13), (14) and (15), the compatibility conditions are satisfied up to second order.*

Proof. This result can be verified by direct calculation. □

Set $U = U^0 + v$.

Lemma 15. *Assume that $f \in C^{4,2}(\bar{\Omega}_2)$ and that the hypotheses of Lemmas 3–5 are satisfied. Then there exists a constant C such that*

$$|U_{tt}(x,t)| + x|U_{xxx}(x,t)| + x^2|U_{xxxx}(x,t)| \le C \quad \text{on } \Omega_2.$$

Proof. The hypotheses imply that

$$\|U^0_{tt}\|_{\Omega_2} + \|U^0_{xx}\|_{\Omega_2} + \|U^0_{xxx}\|_{\Omega_2} + \|U^0_{xxxx}\|_{\Omega_2} \le C.$$

Now (8) holds true with $w = v$ and $\bar{f} = -\varepsilon U^0_{xx}(x,t)$. Set $Q = C\varepsilon$. One can verify that the hypotheses of Lemma 13 are satisfied and it follows that

$$x|v_{xxx}(x,t)| + x^2|v_{xxxx}(x,t)| \le C \quad \text{on } \Omega_2.$$

Furthermore, by virtue of Lemma 10 we get $|v_{tt}(x,t)| \le C\varepsilon x^{-\alpha} \le C$ on Ω_2. The desired result now follows from $U = U^0 + v$. □

Lemma 16. *The function V^L satisfies*

$$|V^L_{tt}(x,t)| + \varepsilon^{2\nu}|V^L_{xx}(x,t)| \le Ce^{-x/\varepsilon^\nu}$$

and

$$x^2|V^L_{xxxx}(x,t)| \le C\varepsilon^{-2\nu}$$

on Ω_2.

Proof. Set $\Phi(x,t) = Ce^t e^{-x/\varepsilon^\nu}$ where C is chosen so that $\Phi(x,t) \ge V^L(x,t)$ on $S(\Omega_2)$; this can be done by (14). Then

$$L\Phi(x,t) = C\varepsilon^\gamma e^t e^{-x/\varepsilon^\nu} - Cx^\alpha e^t e^{-x/\varepsilon^\nu} \le 0 \quad \text{in } \Omega_2.$$

Invoking Lemma 2, we see that $|V^L(x,t)| \le \Phi(x,t) \le Ce^{-x/\varepsilon^\nu}$ on Ω_2. One can show similarly that

$$|V_t^L(x,t)| + |V_{tt}^L(x,t)| \le Ce^{-x/\varepsilon^\nu}$$

(observe that $V_t^L(x,0) = V_{tt}^L(x,0) = 0$ for all x). Then from (14a) we obtain

$$|V_{xx}^L(\varepsilon^\nu,t)| = \varepsilon^{-1}(\varepsilon^\nu)^\alpha |V_t^L(x,t)| \le C\varepsilon^{-2\nu}. \tag{17}$$

Set $z(x,t) = x^{-\alpha}V_{xx}^L(x,t)$. On $S(\Omega_2)$ we have $z(x,0) \equiv 0$, $z(1,t) \equiv 0$ and $|z(\varepsilon^\nu,t)| \le C\varepsilon^{-1}$ from (17). By virtue of (14a) we see that

$$Lz(x,t) = \varepsilon[x^{-\alpha}V_{xx}^L(x,t)]_{xx} - x^\alpha[x^{-\alpha}V_{xx}^L(x,t)]_t = 0.$$

Set $\Phi_1(x,t) = C_3\varepsilon^{-2\nu}x^{-\alpha}e^{Mt}e^{-x/\varepsilon^\nu}$ where $M = \alpha^2+3\alpha+2$ and the fixed constant C_3 is chosen such that $\Phi_1(x,t) \ge V^L(x,t)$ on $S(\Omega_2)$. Then

$$\begin{aligned} L\Phi_1(x,t) &= C_3\varepsilon^{1-2\nu}[\alpha(\alpha+1)x^{-\alpha-2} - 2\alpha x^{-\alpha-1}\varepsilon^{-\nu} + x^{-\alpha}\varepsilon^{-2\nu}]e^{Mt}e^{-x/\varepsilon^\nu} \\ &\qquad - C_3\varepsilon^{-2\nu}Me^{Mt}e^{-x/\varepsilon^\nu} \\ &\le C_3[\varepsilon\varepsilon^{-2\nu}(\alpha^2+3\alpha+1)\varepsilon^{-1} - \varepsilon^{-2\nu}M]e^{Mt}e^{-x/\varepsilon^\nu} < 0. \end{aligned}$$

Applying Lemma 2 with the barrier function $\Phi_1(x,t)$, we obtain $|z(x,t)| \le \Phi_1(x,t)$, whence $|V_{xx}^L(x,t)| \le C_3\varepsilon^{-2\nu}e^{-x/\varepsilon^\nu}$.

Finally, equation (8) holds true with $w \equiv V^L(x,t)$ and $\bar{f} \equiv 0$. Consequently Lemma 13, with $Q = C\varepsilon^\gamma$, yields $x^2|V_{xxxx}^L(x,t)| \le CQ\varepsilon^{-1} = C\varepsilon^{-2\nu}$. □

Lemma 17. *The function $V^R(x,t)$ satisfies*

$$|V_{tt}^R(x,t)| + |V_{xx}^R(x,t)| \le C\varepsilon^{-1}x^\alpha e^{-(1-x)/\varepsilon^{1/2}} \quad \text{on } \Omega_2$$

and $|V_{xxxx}^R(x,t)| \le C\varepsilon^{-2}$ *for* $x \ge 1/2$ *and* $0 < t \le T$.

Proof. By (15d) one has $|V^R(1,t)| \le C$. Invoking Lemma 2 with barrier function $\Phi(x,t) := Ce^{[2(\alpha+2)^2+1]t}e^{-(1-x^{\alpha+2})/\varepsilon^{1/2}}$ leads to $|V^R(x,t)| \le \Phi(x,t) \le Ce^{-(1-x)/\varepsilon^{1/2}}$. One can derive bounds for $|V_t^R(x,t)|$ and $|V_{tt}^R(x,t)|$ in a similar manner.

Set $z(x,t) = \varepsilon x^{-\alpha}V_{xx}^R(x,t)$. On $S(\Omega_2)$ we have $z(\varepsilon^\nu,t) = 0$, $z(x,0) = 0$ and $|z(1,t)| \le C$. From (15a) it follows that

$$Lz(x,t) = \varepsilon[\varepsilon x^{-\alpha}V_{xx}^R(x,t)]_{xx} - x^\alpha[\varepsilon x^{-\alpha}V_{xx}^R(x,t)]_t = 0.$$

Again appealing to Lemma 2 with the above barrier function $\Phi(x,t)$, we obtain $|z(x,t)| \le \Phi(x,t) \le Ce^{-(1-x)/\varepsilon^{1/2}}$, whence $|V_{xx}^R(x,t)| \le C\varepsilon^{-1}x^\alpha e^{-(1-x)/\varepsilon^{1/2}}$.

Now assume that $x \ge 1/2$. From (15a) and the bound already proved for V_{tt}^R, one gets $|V_{xxt}^R(x,t)| = |\varepsilon^{-1}x^\alpha V_{tt}^R(x,t)| \le C\varepsilon^{-1}$. Interpolating between this bound and $|V_{tt}^R(x,t)| \le C$ gives $|V_{xt}^R(x,t)| \le C\varepsilon^{-1/2}$. Then by (15a) we have

$$|V_{xxxx}^R(x,t)| = \varepsilon^{-1}|x^\alpha V_{xxt}^R(x,t) + 2\alpha x^{\alpha-1}V_{xt}^R(x,t) + \alpha(\alpha-1)x^{\alpha-2}V_t^R(x,t)| \le C\varepsilon^{-2}.$$

□

3. The numerical method

3.1. The mesh. On $\bar{\Omega}$ we use a tensor product mesh $\omega = \omega_x \times \omega_t$ which in space is of Shishkin type with two transition points and is equidistant in time.

Denote by N_x and N_t the numbers of mesh intervals in space and time respectively. Then $\omega_t = \{t_0, t_1, ..., t_{N_t}\}$ where $t_j = jT/N_t$ for $j = 0, 1, ..., N_t$. Set

$$\sigma_1 = 2q\varepsilon^\nu \ln N_x \quad \text{and} \quad \sigma_2 = 2\varepsilon^{1/2} \ln N_x,$$

where q will be specified later. The Shishkin mesh transition points are σ_1 and $1-\sigma_2$. We assume that $\max\{\sigma_1, \sigma_2\} \le 1/4$; if this is not the case, then ε is large relative to N_x^{-1} and the analysis can be carried out using classical techniques. The spatial mesh $\omega_x = \{x_0, x_1, ..., x_{N_x}\}$ is piecewise equidistant with

$$x_i = \begin{cases} \frac{4\sigma_1 i}{N_x} & \text{for } i = 0, 1, \dots, \frac{N_x}{4}, \\ \sigma_1 + \frac{2(1-\sigma_1-\sigma_2)}{N_x}(i - \frac{N_x}{4}) & \text{for } i = \frac{N_x}{4}+1, \dots, \frac{3N_x}{4}, \\ 1 - \sigma_2 + \frac{4\sigma_2}{N_x}(i - \frac{3N_x}{4}) & \text{for } i = \frac{3N_x}{4}+1, \dots, N_x. \end{cases}$$

Set $h_i = x_i - x_{i-1}$ for $1 \le i \le N_x$ and $\tau = T/N_t$. Then

$$h_i = \begin{cases} \frac{4\sigma_1}{N_x} = \frac{8q\varepsilon^\nu \ln N_x}{N_x} & \text{for } i = 1, \dots, \frac{N_x}{4}, \\ \frac{2(1-\sigma_1-\sigma_2)}{N_x} & \text{for } i = \frac{N_x}{4}+1, \dots, \frac{3N_x}{4}, \\ \frac{4\sigma_2}{N_x} = \frac{8\varepsilon^{1/2} \ln N_x}{N_x} & \text{for } i = \frac{3N_x}{4}+1, \dots, N_x. \end{cases}$$

The parameter q in the definition of σ_1 is chosen such that the point $x = \varepsilon^\nu$ is a mesh point in ω_x: set $\bar{\sigma}_1 = 2\varepsilon^\nu \ln N_x$ then divide the interval $[0, \overline{\sigma}_1]$ into $N_x/4$ uniform intervals by the points

$$y_i := \frac{4\bar{\sigma}_1 i}{N_x} \quad \text{for } 0 \le i \le N_x/4.$$

Denote by k the maximum index such that $y_k \le \varepsilon^\nu$. Then set $q = \varepsilon^\nu / y_k$.

To ensure that q is well defined we demonstrate that $k > 0$. Assume without loss of generality that $N_x > 26$. Then

$$y_1 = \frac{4\bar{\sigma}_1}{N_x} = \varepsilon^\nu \frac{8 \ln N_x}{N_x} < \varepsilon^\nu$$

so $k = 0$ is impossible.

The definition of k implies that

$$y_k \le \varepsilon^\nu < y_{k+1}, \quad \text{i.e., } 1 \le \frac{\varepsilon^\nu}{y_k} < \frac{y_{k+1}}{y_k}, \quad \text{so } 1 \le q < \frac{k+1}{k} \le 2.$$

Now $x_k = \varepsilon^\nu$ and

$$x_k + \frac{4\sigma_1}{N_x} = \varepsilon^\nu + \frac{4\sigma_1}{N_x} = \sigma_1 \left(\frac{1}{2q \ln N_x} + \frac{4}{N_x} \right) < \sigma_1.$$

That is, $x_{k+1} < \sigma_1$ and $h_{k+1} = 4\sigma_1/N_x$. Also for $2 \le i \le N_x - 1$ and $x_i \ne \sigma_1$ we have $x_{i+1}/x_i \le 2$.

Denote by $S(\omega)$ the meshpoints of ω that lie in $S(\Omega)$, and by ω_I the interior points of the mesh, i.e., $\omega_I = \omega \setminus S(\omega)$.

Notation. For each function r defined on the mesh ω (including functions obtained by restricting a $C(\Omega)$ function to ω), for convenience we often write r_{ij} instead of $r(x_i, t_j)$.

3.2. The difference scheme. Let L^N denote the difference operator obtained by applying backward Euler differencing in time with a standard second-order difference approximation in space on the mesh ω:

$$L^N r_{ij} = \frac{2\varepsilon}{h_i + h_{i+1}} \left(\frac{r_{i+1,j} - r_{ij}}{h_{i+1}} - \frac{r_{ij} - r_{i-1,j}}{h_i} \right) - x_i^\alpha \frac{r_{ij} - r_{i,j-1}}{\tau} \tag{18}$$

for each mesh function r and all $(x_i, t_j) \in \omega_I$.

To solve (1) numerically, define u^N on ω by $L^N u_{ij}^N = f_{ij}$ for $(x_i, t_j) \in \omega_I$ with initial-boundary conditions $u^N = u$ on $S(\omega)$.

The operator L^N satisfies a discrete maximum principle analogous to Lemma 2.

Lemma 18. *Let Ψ be any function defined on ω that satisfies $L^N \Psi_{ij} < 0$ on ω_I and $\Psi_{ij} \geq 0$ on $S(\omega)$. Then $\Psi_{ij} \geq 0$ on ω.*

Proof. Suppose that the result is false. Then Ψ^N attains a negative minimum on ω. Since $\Psi(x_i, t_j) \geq 0$ on $S(\omega)$, this minimum must be at some point $(x_{i_0}, t_{j_0}) \in \omega_I$. Hence $\Psi(x_{i_0-1}, t_{j_0}) \geq \Psi(x_{i_0}, t_{j_0})$, $\Psi(x_{i_0+1}, t_{j_0}) \geq \Psi(x_{i_0}, t_{j_0})$ and $\Psi(x_{i_0}, t_{j_0-1}) \geq \Psi(x_{i_0}, t_{j_0})$; these imply that $L^N \Psi(x_{i_0}, t_{j_0}) \geq 0$ which contradicts our hypotheses. It follows that the result is true. □

Recall that $x_k = \varepsilon^\nu$. The next lemma is a useful source of barrier functions later.

Lemma 19. *Define Φ on ω by*

$$\Phi_{ij} = \begin{cases} 2K(1 + t_j)\varepsilon^{-\nu} x_i (2 - \varepsilon^{-\nu} x_i) & \text{for } 0 \leq i \leq k, \\ 2K(1 + t_j) & \text{for } k + 1 \leq i \leq N_x, \end{cases}$$

where $j = 0, \ldots, N_t$ and K is any quantity that is independent of i and j. Then $L^N \Phi(x_i, t_j) < -K(\varepsilon^\gamma + x_i^\alpha)$ for all $(x_i, t_j) \in \omega_I$.

Proof. First consider the case $1 \leq i \leq k - 1$. Then

$$\begin{aligned} L^N \Phi_{ij} &< \frac{2\varepsilon}{h_i + h_{i+1}} \left(\frac{\Phi_{i+1,j} - \Phi_{ij}}{h_{i+1}} - \frac{\Phi_{ij} - \Phi_{i-1,j}}{h_i} \right) \\ &= -4\varepsilon^{1-2\nu} K(1 + t_j) \\ &< -K(\varepsilon^\gamma + x_i^\alpha) \end{aligned}$$

since $x_i^\alpha < \varepsilon^{\alpha\nu} = \varepsilon^\gamma$. Next, suppose that $k + 1 \leq i \leq N_x - 1$. Then

$$L^N \Phi_{ij} = -2K x_i^\alpha \leq -K(\varepsilon^\gamma + x_i^\alpha).$$

All that remains is the case $i = k$. As the function $x(2 - \varepsilon^{-\nu} x)$ attains its maximum at the point $x = \varepsilon^\nu$, we have $\Phi_{k-1,j} \leq \Phi_{kj}$ and $\Phi_{k+1,j} \leq \Phi_{kj}$, which implies that

$$L^N \Phi_{kj} \leq -x_k^\alpha \frac{\Phi_{kj} - \Phi_{k,j-1}}{\tau} < -2K x_k^\alpha = -K(\varepsilon^\gamma + x_k^\alpha).$$

This completes the proof. □

3.3. Error analysis. Denote by $E_{ij} = E(x_i, t_j) = u(x_i, t_j) - u^N(x_i, t_j)$ the error between the true and numerical solutions. Now

$$L^N E_{ij} = (L^N - L) u(x_i, t_j) \quad \text{for } (x_i, t_j) \in \omega. \tag{19}$$

Clearly $E_{ij} = 0$ for all $(x_i, t_j) \in S(\omega)$.

By Taylor expansions one sees easily that if $h_i = h_{i+1}$ then

$$\begin{aligned} |(L^N - L) u(x_i, t_j)| \leq & C \big[\varepsilon \min\{ h_i^2 \|u_{xxxx}(x, t_j)\|_{[x_{i-1}, x_{i+1}]}, \|u_{xx}(x, t_j)\|_{[x_{i-1}, x_{i+1}]} \} \\ & + \tau x_i^\alpha \|u_{tt}(x_i, t)\|_{[t_{j-1}, t_j]} \big] \end{aligned} \tag{20}$$

while if $h_{i+1} \neq h_i$ then

$$|(L^N - L)u(x_i, t_j)| \\ \leq C\big[\varepsilon \min\{(h_i + h_{i+1})\|u_{xxx}(x, t_j)\|_{[x_{i-1},x_{i+1}]}, \|u_{xx}(x, t_j)\|_{[x_{i-1},x_{i+1}]}\} \\ + \tau x_i^\alpha \|u_{tt}(x_i, t)\|_{[t_{j-1},t_j]}\big], \tag{21}$$

for $1 \leq i \leq N_x - 1$ and $1 \leq j \leq N_t$. Here we use the notation $\|\cdot\|_{[a,b]}$ for the maximum over an interval for a function of one variable defined on that interval.

We will now obtain bounds for $|(L^N - L)u(x_i, t_j)|$. First we consider points (x_i, t_j) for $1 \leq i \leq k$.

Lemma 20. *Let the hypotheses of Lemma 8 be satisfied. Then for $\alpha \in (1, 2)$ one has*

$$|(L^N - L)u(x_i, t_j)| \leq C\varepsilon^\gamma [N_x^{-1} \ln N_x + N_t^{-1}] \quad \text{for } 1 \leq i \leq k,\ 1 \leq j \leq N_t, \tag{22}$$

and for $\alpha = 1$ or $\alpha \geq 2$ one has

$$|(L^N - L)u(x_i, t_j)| \leq C\varepsilon^\gamma [N_x^{-2} (\ln N_x)^2 + N_t^{-1}] \quad \text{for } 1 \leq i \leq k,\ 1 \leq j \leq N_t. \tag{23}$$

Proof. Lemma 5 implies that $\tau x_i^\alpha |u_{tt}(x_i, t_j)| \leq C\varepsilon^\gamma N_t^{-1}$ for $1 \leq i \leq k$.

Recall that $k+1 < N_x/4$ and $h_i = 4\sigma_1/N_x \leq C\varepsilon^\nu N_x^{-1} \ln N_x$ for $i = 1, \ldots, N_x/4$. We also get $(x_{k+1}, t_j) \in \bar{\Omega}_1$. For $\alpha = 1$ or $\alpha \geq 2$, Lemma 8 yields $|u_{xxxx}| \leq C\varepsilon^{-4\nu}$ in $\bar{\Omega}_1$, from which follows

$$h_i^2 |\varepsilon u_{xxxx}(x_i, t_j)| \leq C\varepsilon^{1+2\nu-4\nu} N_x^{-2} (\ln N_x)^2 = C\varepsilon^\gamma N_x^{-2} (\ln N_x)^2, \quad 1 \leq i \leq k.$$

Substituting these bounds into (20) gives (23).

For $\alpha \in (1, 2)$, Lemma 8 yields $|u_{xxx}| \leq C\varepsilon^{-3\nu}$ in $\bar{\Omega}_1$, from which follows similarly

$$(h_i + h_{i+1}) |\varepsilon u_{xxx}(x_i, t_j)| \leq C\varepsilon^\gamma N_x^{-1} \ln N_x, \quad 1 \leq i \leq k.$$

Now (21) gives (22). □

To obtain bounds on $|(L^N - L)u(x_i, t_j)|$ at points (x_i, t_j) for $i > k$ we will consider the earlier decomposition of the continuous solution $u = U^0 + v + V^L + V^R = U + V^L + V^R$. Now we have

$$|L^N E(x_i, t_j)| = |(L^N - L)u(x_i, t_j)| \\ = |(L^N - L)U(x_i, t_j) + (L^N - L)V^L(x_i, t_j) + (L^N - L)V^R(x_i, t_j)| \\ \leq |(L^N - L)U(x_i, t_j)| + |(L^N - L)V^L(x_i, t_j)| + |(L^N - L)V^R(x_i, t_j)|. \tag{24}$$

We will find bounds for each term in (24) separately. Notice that for $k < i \leq N_x$ one has $x_{i-1} \geq \varepsilon^\nu$ so $\varepsilon^\nu x_{i-1}^{-1} \leq 1$.

Lemma 21. *Function V^L satisfies*

$$|(L^N - L)V^L(x_i, t_j)| \\ \leq C x_i^\alpha [N_x^{-2} (\ln N_x)^2 + N_t^{-1}] \quad \text{for } k + 1 \leq i \leq N_x - 1, \quad 1 \leq j \leq N_t.$$

Proof. Lemma 16 gives us $x_i^\alpha \|V_{tt}^L(x_i, t)\|_{[t_{j-1},t_j]} \leq C x_i^\alpha e^{-x_i/\varepsilon^\nu} \leq C x_i^\alpha$,

$$|V_{xxxx}^L(x, t)| \leq C\varepsilon^{-2\nu} x^{-2} \leq C\varepsilon^{-4\nu} \quad \text{and} \quad |V_{xx}^L(x, t)| \leq C\varepsilon^{-2\nu} e^{-x/\varepsilon^\nu}.$$

For $k < i < N_x/4$ we have $h_i = h_{i+1} = 4\sigma_1/N_x \le C\varepsilon^\nu N_x^{-1}\ln N_x$. Recalling (20), one gets

$$\begin{aligned}|(L^N - L)V^L(x_i,t_j)| &\le C\big[\varepsilon h_i^2\|V^L_{xxxx}(x,t_j)\|_{[x_{i-1},x_{i+1}]} + \tau x_i^\alpha\|V^L_{tt}(x_i,t)\|_{[t_{j-1},t_j]}\big]\\ &\le C[\varepsilon^{1+2\nu-4\nu}N_x^{-2}(\ln N_x)^2 + N_t^{-1}x_i^\alpha]\\ &\le Cx_i^\alpha[N_x^{-2}(\ln N_x)^2 + N_t^{-1}].\end{aligned}$$

For $N_x/4 < i \le N_x$ we have $x_{i-1} \ge \sigma_1$, so (20) yields

$$\begin{aligned}|(L^N - L)V^L(x_i,t_j)| &\le C\big[\varepsilon\|V^L_{xx}(x,t_j)\|_{[x_{i-1},x_{i+1}]} + \tau x_i^\alpha\|V^L_{tt}(x_i,t)\|_{[t_{j-1},t_j]}\big]\\ &\le C[\varepsilon^\gamma e^{-x_{i-1}/\varepsilon^\nu} + N_t^{-1}x_i^\alpha]\\ &\le Cx_i^\alpha[N_x^{-2} + N_t^{-1}].\end{aligned}$$

For $i = N_x/4$ we proceed as in the previous case — the only difference is that now $e^{-x_{i-1}/\varepsilon^\nu} \le e^{-(\sigma_1-4\sigma_1/N_x)/\varepsilon^\nu} \le CN_x^{-2}$. This completes the proof. □

Lemma 22. *The function V^R satisfies*

$$\begin{aligned}&|(L^N - L)V^R(x_i,t_j)|\\ &\qquad\le Cx_i^\alpha[N_x^{-2}(\ln N_x)^2 + N_t^{-1}] \quad \textit{for } k+1 \le i \le N_x - 1, \quad 1 \le j \le N_t.\end{aligned}$$

Proof. For $3N_x/4 < i \le N_x$ we have $h_i = h_{i+1} = 4\sigma_2/N_x \le C\varepsilon^{1/2}N_x^{-1}\ln N_x$. Recalling (20) and the bounds of Lemma 17, one has

$$\begin{aligned}|(L^N - L)V^R(x_i,t_j)| &\le C\big[\varepsilon h_i^2\|V^R_{xxxx}(x,t_j)\|_{[x_{i-1},x_{i+1}]} + \tau x_i^\alpha\|V^R_{tt}(x_i,t)\|_{[t_{j-1},t_j]}\big]\\ &\le C[\varepsilon^{1+1-2}N_x^{-2}(\ln N_x)^2 + N_t^{-1}x_i^\alpha]\\ &\le Cx_i^\alpha[N_x^{-2}(\ln N_x)^2 + N_t^{-1}]\end{aligned}$$

since $x_i \ge 1/2$.

For $k < i \le 3N_x/4$ with $i \ne N_x/4$, the mesh satisfies $x_{i+1} \le 1 - \sigma_2 + 4\sigma_2/N_x$ and $x_{i+1}/x_i < 2$, so (20) and Lemma 17 yield

$$\begin{aligned}|(L^N - L)V^R(x_i,t_j)| &\le C\big[\varepsilon\|V^R_{xx}(x,t_j)\|_{[x_{i-1},x_{i+1}]} + \tau x_i^\alpha\|V^R_{tt}(x_i,t)\|_{[t_{j-1},t_j]}\big]\\ &\le C[x_{i+1}^\alpha e^{-(1-x_{i+1})/\varepsilon^{1/2}} + N_t^{-1}x_i^\alpha]\\ &\le Cx_i^\alpha[(x_{i+1}/x_i)^\alpha e^{-\sigma_2/\varepsilon^{1/2}}e^{4\sigma_2/N_x} + N_t^{-1}]\\ &\le Cx_i^\alpha[N_x^{-2} + N_t^{-1}].\end{aligned}$$

For $i = N_x/4$ we have $x_i = \sigma_1$, $x_{i+1} \le \sigma_1 + 2/N_x$ and $1 - \sigma_1 \ge 1/2 + \sigma_2$ so we get

$$\begin{aligned}\varepsilon\|V^R_{xx}(x,t_j)\|_{[x_{i-1},x_{i+1}]} &\le x_{i+1}^\alpha e^{-(1-x_{i+1})/\varepsilon^{1/2}}\\ &\le Cx_i^\alpha\left(\frac{x_{i+1}}{x_i}\right)^\alpha e^{-(1-\sigma_1+2/N_x)/\varepsilon^{1/2}}\\ &\le Cx_i^\alpha\left(\frac{1}{\sigma_1}\right)^\alpha e^{-(1/2+\sigma_2+2/N_x)/\varepsilon^{1/2}}\\ &\le Cx_i^\alpha\varepsilon^{-\gamma}N_x^{-2}e^{-1/(2\varepsilon^{1/2})}\\ &\le Cx_i^\alpha N_x^{-2},\end{aligned}$$

where we used the definitions of σ_1 and σ_2 in the calculation. Hence $|(L^N - L)V^R(x_i,t_j)| \le Cx_i^\alpha[N_x^{-2} + N_t^{-1}]$ for $i = N_x/4$. □

Recall that $U = U^0 + v$.

Lemma 23. *Suppose that the hypotheses of Lemma 15 are satisfied. Then for* $1 \le j \le N_t$ *one has*

$$|(L^N - L)U(x_i, t_j)| \le \begin{cases} C[\varepsilon^{\gamma} N_x^{-2} + x_i^{\alpha} N_t^{-1}] & \text{for } k < i \le N_x,\ x_i \ne \sigma_1,\ x_i \ne 1 - \sigma_2, \\ C[\varepsilon^{\gamma+\nu} N_x^{-1} + x_i^{\alpha} N_t^{-1}] & \text{for } x_i = \sigma_1,\ x_i = 1 - \sigma_2. \end{cases}$$

Proof. Suppose first that $k < i \le N_x$ with $i \ne N_x/4,\ 3N_x/4$. From (20) and Lemma 15 we get

$$\begin{aligned} |(L^N - L)U(x_i, t_j)| &\le C(\varepsilon h_i^2 \|U_{xxxx}(x, t_j)\|_{[x_{i-1}, x_{i+1}]} + \tau x_i^{\alpha} \|U_{tt}(x_i, t)\|_{[t_{j-1}, t_j]}) \\ &\le C[N_x^{-2} \varepsilon x_{i-1}^{-2} + N_t^{-1} x_i^{\alpha}] \\ &= C[\varepsilon^{\gamma} N_x^{-2} (\varepsilon^{\nu} x_{i-1}^{-1})^2 + x_i^{\alpha} N_t^{-1}] \\ &\le C[\varepsilon^{\gamma} N_x^{-2} + x_i^{\alpha} N_t^{-1}]. \end{aligned}$$

In the case $x_i = \sigma_1$ or $x_i = 1 - \sigma_2$, from (21) and Lemma 15 we get

$$\begin{aligned} |(L^N - L)U(x_i, t_j)| &\le C\big[\varepsilon(h_i + h_{i+1}) \|U_{xxx}(x, t_j)\|_{[x_{i-1}, x_{i+1}]} + \tau x_i^{\alpha} \|U_{tt}(x_i, t)\|_{[t_{j-1}, t_j]}\big] \\ &\le C[N_x^{-1} \varepsilon x_{i-1}^{-1} + x_i^{\alpha} N_t^{-1}] \\ &= C[\varepsilon^{\gamma+\nu} N_x^{-1} \varepsilon^{\nu} x_{i-1}^{-1} + x_i^{\alpha} N_t^{-1}] \\ &\le C[\varepsilon^{\gamma+\nu} N_x^{-1} + x_i^{\alpha} N_t^{-1}]. \end{aligned}$$

□

Theorem 1. *Let the hypotheses of Lemmas 15 and 20 be satisfied. Assume that* $\alpha \in (1, 2)$. *Then*

$$\max_{(x_i, t_j) \in \omega} |(u - u^N)(x_i, t_j)| \le C[N_x^{-1} \ln N_x + N_t^{-1}].$$

Proof. Lemmas 20–23 imply that

$$|L^N E_{ij}| \le C(\varepsilon^{\alpha\nu} + x_i^{\alpha})[N_x^{-1} \ln N_x + N_t^{-1}]$$

for $(x_i, t_j) \in \omega_I$. Applying Lemma 19 with $K = C[N_x^{-1} \ln N_x + N_t^{-1}]$, we get

$$L^N \Phi_{ij} < -|L^N E_{ij}|, \quad (x_i, t_j) \in \omega_I.$$

Clearly $\Phi_{ij} \ge 0 = E_{ij}$ for $(x_i, t_j) \in S(\omega)$. Thus we can invoke Lemma 18 to obtain

$$|E_{ij}| < \Phi_{ij} \le CK \le C[N_x^{-1} \ln N_x + N_t^{-1}] \quad \text{for } (x_i, t_j) \in \omega.$$

□

Theorem 2. *Let the hypotheses of Lemmas 15 and 20 be satisfied. Assume that* $\alpha = 1$ *or* $\alpha \ge 2$. *Then*

$$\max_{(x_i, t_j) \in \omega} |(u - u^N)(x_i, t_j)| \le C[N_x^{-2} (\ln N_x)^2 + N_t^{-1}].$$

Proof. Lemmas 20–23 yield

$$|L^N E(x_i, t_j)| \le C(\varepsilon^{\alpha\nu} + x_i^{\alpha})[N_x^{-2} (\ln N_x)^2 + N_t^{-1}]$$

for $k + 1 \le i \le N_x$, $i \ne N_x/4$, $i \ne 3N_x/4$ and $1 \le j \le N_t + 1$, while

$$|L^N E(x_i, t_j)| \le C(\varepsilon^{\alpha\nu} + x_i^{\alpha})[N_x^{-2} (\ln N_x)^2 + N_t^{-1}] + C\varepsilon^{\gamma+\nu} N_x^{-1}$$

for $i = N_x/4$ and $i = 3N_x/4$.

Set $K = C[N_x^{-2}(\ln N_x)^2 + N_t^{-1}]$ and define the function Φ_2 by

$$\Phi_2(x_i,t_j) = \begin{cases} Kx_i/\sigma_1 & \text{for } 0 \le i \le N_x/4,\ 0 \le j \le N_t, \\ K & \text{for } N_x/4 \le i \le 3N_x/4,\ 0 \le j \le N_t, \\ K(1-x_i)/\sigma_2 & \text{for } 3N_x/4 \le i \le N_x+1,\ 0 \le j \le N_t. \end{cases}$$

For $i = N_x/4$ we get

$$\begin{aligned} L^N\Phi_2(\sigma_1,t_j) &< -\varepsilon\frac{2}{h_{i+1}+h_i}\cdot\frac{\Phi_2(x_i,t_j)-\Phi_2(x_{i-1},t_j)}{h_i} \\ &\le -C\varepsilon N_x K/\sigma_1 \\ &\le -C\varepsilon^{\gamma+\nu}N_x^{-1}. \end{aligned}$$

For $i = 3N_x/4$ we get

$$\begin{aligned} L^N\Phi_2(\sigma_1,t_j) &< \varepsilon\frac{2}{h_{i+1}+h_i}\cdot\frac{\Phi_2(x_{i+1},t_j)-\Phi_2(x_i,t_j)}{h_i} \\ &\le -C\varepsilon N_x K/\sigma_2 \\ &\le -C\varepsilon^{1/2}N_x^{-1}. \end{aligned}$$

Now define the function Φ_3 by $\Phi_3 = \Phi_2 + \Phi$ where Φ is the function of Lemma 19 with $K = C[N_x^{-2}(\ln N_x)^2 + N_t^{-1}]$. For $i \ne N_x/4,\ 3N_x/4$ we have

$$L^N\Phi_3(x_i,t_j) = L^N\Phi_2(x_i,t_j) + L^N\Phi(x_i,t_j) = L^N\Phi(x_i,t_j) < -|L^N E(x_i,t_j)|.$$

In each case below we use Lemmas 19–23 to bound $|L^N E(x_i,t_j)|$. For $i = N_x/4$ we get

$$\begin{aligned} L^N\Phi_3(x_i,t_j) &= L^N\Phi_2(x_i,t_j) + L^N\Phi(x_i,t_j) \\ &< -C\varepsilon^{\gamma+\nu}N_x^{-1} - (\varepsilon^\gamma + x_i^\alpha)[N_x^{-2}(\ln N_x)^2 + N_t^{-1}] \\ &< -|L^N E(x_i,t_j)|. \end{aligned}$$

For $i = 3N_x/4$ we get

$$\begin{aligned} L^N\Phi_3(x_i,t_j) &= L^N\Phi_2(x_i,t_j) + L^N\Phi(x_i,t_j) \\ &< -\varepsilon^{1/2}CN_x^{-1} - (\varepsilon^\gamma + x_i^\alpha)[N_x^{-2}(\ln N_x)^2 + N_t^{-1}] \\ &= -\varepsilon^{-\gamma/2}\varepsilon^{\gamma+\nu}CN_x^{-1} - (\varepsilon^\gamma + x_i^\alpha)[N_x^{-2}(\ln N_x)^2 + N_t^{-1}] \\ &< -|L^N E(x_i,t_j)|. \end{aligned}$$

We have shown that $L^N\Phi_3(x_i,t_j) < -|L^N E(x_i,t_j)|$ for all $(x_i,t_j) \in \omega_I$. Clearly $\Phi_3(x_i,t_j) > 0$ on $S(\omega)$. Thus we can invoke Lemma 18 to deduce that

$$|E(x_i,t_j)| \le \Phi_3(x_i,t_j) \le C[N_x^{-2}(\ln N_x)^2 + N_t^{-1}]$$

for all $(x_i,t_j) \in \omega$. □

4. Numerical results

We give numerical results for two examples. Since we shall vary N_x and N_t, the mesh ω of Section 3 is now written as $\omega^{N_x,N_t} = \omega_x^{N_x} \times \omega_t^{N_t}$. Let u^{N_x,N_t} denote the numerical solution computed on this mesh.

As the exact solution is unknown in our examples, the convergence is examined using a two-mesh approach as in [2]. Define the maximal nodal error E^{N_x,N_t} by

$$E^{N_x,N_t} = \|u^{N_x,N_t} - \tilde{u}^{2N_x,2N_t}\|_{\omega^{N_x,N_t}},$$

where u^{N_x,N_t} is computed on our standard mesh ω^{N_x,N_t} but $\tilde{u}^{2N_x,2N_t}$ is computed on the mesh $\tilde{\omega}^{2N_x,2N_t} := \tilde{\omega}_x^{2N_x} \times \omega_t^{2N_t}$ where $\tilde{\omega}_x^{2N_x}$ contains all points from $\omega_x^{N_x}$ and also the points $x_{i+1/2} = (x_i + x_{i+1})/2$ for $i = 0, ..., N_x$. (In other words, $\tilde{\omega}_x^{2N_x}$ has the same transition points as $\omega_x^{N_x}$ but the mesh is twice as fine everywhere.) Define the numerical order of convergence

$$p^{N_x,N_t} := \log_2(E^{N_x,N_t}/E^{2N_x,2N_t}).$$

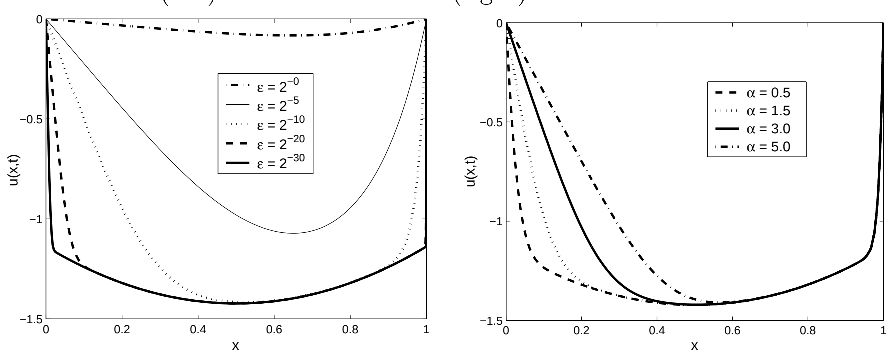
Example 1. *Consider*

$$\varepsilon u_{xx}(x,t) - x^\alpha u_t(x,t) = x^\alpha t^2[1 + x - x^2]$$

on $\Omega = (0,1) \times (0,1.5]$ *with zero boundary and initial data.*

The hypotheses of Lemmas 3-5 are satisfied for this example. In Figure 1 its numerical solution is plotted at time $t = 1.5$ for various values of ε and α. The left diagram illustrates how $\|u\|$ is uniformly bounded with respect to ε. From the right diagram one can see that varying α has little effect on the layer at $x = 1$ but strongly influences the layer at $x = 0$.

FIGURE 1. Computed solution $u^{128,128}(x,t)$ at time $t = 1.5$, fixed $\alpha = 2.5$ (left) and fixed $\varepsilon = 2^{-12}$ (right)



Example 1 is solved using the method of Section 3. Taking $\alpha = 2.5$, its errors (measured in the discrete maximum norm) and orders of convergence are presented in Table 1. The orders of convergence are close to 1, which is in accordance with Theorem 2 as the first-order convergence in N_t dominates the higher-order convergence in N_x.

To confirm the order of convergence in N_x we follow [2] by using double refinement in N_t: that is, we now compute the numerical order of convergence given by

$$\hat{p}^{N_x,N_t} := \log_2(E^{N_x,N_t}/E^{2N_x,4N_t}).$$

These results are presented in Table 2 for $\alpha = 2.5$ and in Table 3 for $\alpha = 1.5$. The computed orders of convergence in table 2 are in good accordance with Theorem 2, but those of Table 3 suggest that the result of Theorem 1 is suboptimal because one gets almost second-order convergence in N_x. We shall investigate this complicated issue in a later paper [7].

The numbers in Tables 2 and 3 are almost identical because the maximum error appears at the right boundary layer which is independent of α. To examine the behaviour of the numerical method in the left boundary layer we now consider a second example.

ε/N	32	64	128	256	512	1024
2^{-0}	0.0023E-2	0.0012E-2	0.0006E-2	0.0003E-2	0.0002E-2	0.0010E-2
	0.9521	0.9804	0.9910	0.9955	0.9976	
2^{-10}	3.2990E-2	1.6388E-2	0.8165E-2	0.4076E-2	0.2036E-2	0.1018E-2
	1.0094	1.0050	1.0025	1.0013	1.0006	
2^{-20}	3.3465E-2	1.6608E-2	0.8272E-2	0.4128E-2	0.2062E-2	0.1030E-2
	1.0108	1.0056	1.0028	1.0014	1.0007	
2^{-30}	3.5230E-2	1.6606E-2	0.8272E-2	0.4128E-2	0.2062E-2	0.1030E-2
	1.0851	1.0055	1.0028	1.0014	1.0007	
2^{-40}	3.5056E-2	1.6607E-2	0.8272E-2	0.4128E-2	0.2062E-2	0.1030E-2
	1.0778	1.0055	1.0028	1.0014	1.0007	

TABLE 1. E^{N_x,N_t} and $\hat{p}^{N_x,N_t}$ for Example 1, $\alpha = 2.5\,;\ N_x = N_t = N$.

ε/N	32	64	128	256	512	1024
2^{-10}	1.9207E-2	0.6804E-2	0.1802E-2	0.0454E-2	0.0114E-2	0.0029E-2
	1.4972	1.9168	1.9879	1.9928	1.9992	
2^{-20}	1.8171E-2	0.6804E-2	0.2459E-2	0.0816E-2	0.0259E-2	0.0080E-2
	1.4172	1.4681	1.5924	1.6527	1.6933	
2^{-30}	2.1905E-2	0.6791E-2	0.2454E-2	0.0814E-2	0.0259E-2	0.0080E-2
	1.6897	1.4682	1.5924	1.6528	1.6933	
2^{-40}	2.1906E-2	0.6790E-2	0.2454E-2	0.0814E-2	0.0259E-2	0.0080E-2
	1.6898	1.4682	1.5924	1.6528	1.6933	

TABLE 2. E^{N_x,N_t} and $\hat{p}^{N_x,N_t}$ for Example 1 with $\alpha = 2.5$, $N_x = N,\ N_t = N_x^2/64$.

ε/N	32	64	128	256	512	1024
2^{-10}	1.8769E-2	0.6628E-2	0.1754E-2	0.0442E-2	0.0111E-2	0.0029E-2
	1.5016	1.9182	1.9884	1.9928	1.9993	
2^{-20}	2.1699E-2	0.6798E-2	0.2457E-2	0.0815E-2	0.0259E-2	0.0080E-2
	1.6744	1.4681	1.5924	1.6527	1.6933	
2^{-30}	2.1696E-2	0.6790E-2	0.2454E-2	0.0814E-2	0.0259E-2	0.0080E-2
	1.6759	1.4682	1.5924	1.6528	1.6933	
2^{-40}	2.1696E-2	0.6790E-2	0.2454E-2	0.0814E-2	0.0259E-2	0.0080E-2
	1.6759	1.4682	1.5924	1.6528	1.6933	

TABLE 3. E^{N_x,N_t} and $\hat{p}^{N_x,N_t}$ for Example 1 with $\alpha = 1.5$, $N_x = N,\ N_t = N_x^2/64$.

Example 2. *Consider*

$$\varepsilon u_{xx}(x,t) - x^{\alpha} u_t(x,t) = x^{\alpha} t^2 [1 + x - x^2]$$

on $\Omega = (0,1)\times(0,1.5]$ *with* $\varphi_L(t) \equiv 0$, $\varphi_R(t) = -t^3/3$ *for* $0 < t \le T$, *and* $\varphi_0(x) \equiv 0$. *The boundary data along* $x = 1$ *are chosen in such way that no layer appears there.*

Results for this example are presented in Tables 4 and 5. They again agree with our theoretical bounds.

ε/N	32	64	128	256	512	1024
2^{-10}	0.0441E-2	0.0105E-2	0.0026E-2	0.0007E-2	0.0002E-2	0.0000E-2
	2.0656	2.0003	1.9997	2.0001	1.9999	
2^{-20}	0.7514E-2	0.2358E-2	0.0568E-2	0.0143E-2	0.0036E-2	0.0009E-2
	1.6718	2.0538	1.9895	2.0025	1.9999	
2^{-30}	2.1905E-2	0.5979E-2	0.1777E-2	0.0594E-2	0.0188E-2	0.0058E-2
	1.8733	1.7508	1.5797	1.6610	1.6980	
2^{-40}	2.1906E-2	0.5979E-2	0.1776E-2	0.0594E-2	0.0188E-2	0.0058E-2
	1.8734	1.7508	1.5796	1.6610	1.6980	

TABLE 4. E^{N_x,N_t} and $\hat{p}^{N_x,N_t}$ for Example 2 with $\alpha = 2.5$, $N_x = N,\ N_t = N_x^2/64$.

ε/N	32	64	128	256	512	1024
2^{-10}	0.1224E-2	0.0301E-2	0.0075E-2	0.0019E-2	0.0005E-2	0.0001E-2
	2.0216	2.0134	2.0039	2.0012	2.0005	
2^{-20}	2.1699E-2	0.6384E-2	0.2196E-2	0.0708E-2	0.0223E-2	0.0061E-2
	1.7651	1.5396	1.6330	1.6672	1.8640	
2^{-30}	2.1696E-2	0.6385E-2	0.2196E-2	0.0708E-2	0.0223E-2	0.0068E-2
	1.7646	1.5400	1.6329	1.6671	1.7050	
2^{-40}	2.1696E-2	0.6385E-2	0.2196E-2	0.0708E-2	0.0223E-2	0.0068E-2
	1.7646	1.5400	1.6329	1.6671	1.7050	

TABLE 5. E^{N_x,N_t} and $\hat{p}^{N_x,N_t}$ for Example 2 with $\alpha = 1.5$, $N_x = N,\ N_t = N_x^2/64$.

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References

[1] C. Y. Chan and W. Y. Chan. Existence of classical solutions for degenerate semilinear parabolic problems. *Appl. Math. Comput.*, 101(2-3):125–149, 1999.

[2] C. Clavero, J. L. Gracia, and J. C. Jorge. High-order numerical methods for one-dimensional parabolic singularly perturbed problems with regular layers. *Numer. Methods Partial Differential Equations*, 21(1):148–169, 2005.

[3] A. Friedman. *Partial differential equations of parabolic type*. Prentice-Hall Inc., Englewood Cliffs, N.J., 1964.

[4] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural′ceva. *Linear and quasilinear equations of parabolic type*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1967.

[5] H. Ockendon. Channel flow with temperature-dependent viscosity and internal viscous dissipation. *J. Fluid Mech.*, 93:737–746, 1979.

[6] G. I. Shishkin. Difference approximation of singularly perturbed parabolic equations that are degenerate on the boundary. *Zh. Vychisl. Mat. i Mat. Fiz.*, 31(10):1498–1511, 1991. Translation in Comput. Math. Math. Phys. 31 (1991), no. 10, 53–63 (1992).

[7] M. Viscor and M. Stynes. A robust finite difference method for a singularly perturbed degenerate parabolic problem, Part II. (in preparation).

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