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A PARAMETER–UNIFORM FINITE DIFFERENCE METHOD FOR SINGULARLY PERTURBED LINEAR DYNAMICAL SYSTEMS

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Dedicated to G. I. Shishkin on his 70th birthday

Abstract. A system of singularly perturbed ordinary differential equations of first order with given initial conditions is considered. The leading term of each equation is multiplied by a small positive parameter. These parameters are assumed to be distinct and they determine the different scales in the solution to this problem. A Shishkin piecewise–uniform mesh is constructed, which is used, in conjunction with a classical finite difference discretization, to form a new numerical method for solving this problem. It is proved that the numerical approximations obtained from this method are essentially first order convergent uniformly in all of the parameters. Numerical results are presented in support of the theory.

Key Words. linear dynamical system, multiscale, initial value problem, singularly perturbed, finite difference method, parameter–uniform convergence.

1. Introduction

We consider the initial value problem for the singularly perturbed system of linear first order differential equations

$$E\vec{u}'(t) + A(t)\vec{u}(t) = \vec{f}(t), \quad t \in (0, T], \quad \vec{u}(0) \text{ given.} \tag{1}$$

Here $\vec{u}$ is a column n-vector, E and $A(t)$ are $n \times n$ matrices, $E = diag(\vec{\varepsilon})$, $\vec{\varepsilon} = (\varepsilon_1, \ \dots, \ \varepsilon_n)$ with $0 < \varepsilon_i \le 1$ for all $i = 1 \dots n$. For convenience we assume the ordering

$$\varepsilon_1 < \cdots < \varepsilon_n.$$

These n distinct parameters determine the n distinct scales in this multiscale problem. Cases with some of the parameters coincident are not considered here. We write the problem in the operator form

$$\vec{L}\vec{u} = \vec{f}, \quad \vec{u}(0) \text{ given,}$$

where the operator $\vec{L}$ is defined by

$$\vec{L} = ED + A(t) \text{ and } \mathrm{D} = \frac{\mathrm{d}}{\mathrm{dt}}.$$

We assume that, for all $t \in [0, T]$, the components $a_{ij}(t)$ of $A(t)$ satisfy the inequalities

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$$a_{ii}(t) > \sum_{\substack{j\neq i\\ j=1}}^{n} |a_{ij}(t)| \ \text{ for } \ 1 \le \mathrm{i} \le \mathrm{n}, \ \text{ and } \ \mathrm{a_{ij}(t)} \le 0 \ \text{ for } \ \mathrm{i} \neq \mathrm{j}. \tag{2}$$

We take α to be any number such that

$$0 < \alpha < \min_{\substack{t\in(0,1]\\ 1\le i\le n}} (\sum_{j=1}^{n} a_{ij}(t)). \tag{3}$$

We also assume that $T \ge 2\max_i(\varepsilon_i)/\alpha$, which ensures that the solution domain contains all of the layers. This condition is fulfilled if, for example, $T \ge 2/\alpha$. We introduce the norms $\| \vec{V} \| = \max_{1\le k\le n} |V_k|$ for any n-vector $\vec{V}$, $\| y \| = \sup_{0\le t\le T} |y(t)|$ for any scalar-valued function y and $\| \vec{y} \| = \max_{1\le k\le n} \| y_k \|$ for any vector-valued function $\vec{y}$. Throughout the paper C denotes a generic positive constant, which is independent of t and of all singular perturbation and discretization parameters. Furthermore, inequalities between vectors are understood in the componentwise sense.

The plan of the paper is as follows. In the next section both standard and novel bounds on the smooth and singular components of the exact solution are obtained. The sharp estimates in Lemma 2.4 are proved by mathematical induction, while an interesting ordering of the points $t_{i,j}$ is established in Lemma 2.6. In Section 3 the appropriate piecewise-uniform Shishkin meshes are introduced, the discrete problem is defined and the discrete maximum principle and discrete stability properties are established. In Section 4 an expression for the local truncation error is found and two distinct standard estimates are stated. In Section 5 parameter-uniform estimates for the local truncation error of the smooth and singular components are obtained in a sequence of lemmas. The section culminates with the statement and proof of the parameter-uniform error estimate, which is the main theoretical result of the paper. In the final section numerical results are presented in support of the theory.

The initial value problems considered here arise in many areas of applied mathematics; see for example [1]. Parameter uniform numerical methods for simpler problems of this kind, when all the singular perturbation parameters are equal, were considered in [4]. A special case of the present problem, with $n = 3$, was considered in [3]. However, the proof of the parameter uniform error estimate for general n, which is the main goal of the present paper, is significantly more difficult. A general introduction to parameter uniform numerical methods is given in [2] and [7].

2. Analytical results

The operator $\vec{L}$ satisfies the following maximum principle

Lemma 2.1. *Let $A(t)$ satisfy (2) and (3). Let $\vec{\psi}(t)$ be any function in the domain of $\vec{L}$ such that $\vec{\psi}(0) \ge 0$. Then $\vec{L}\vec{\psi}(t) \ge 0$ for all $t \in (0, T]$ implies that $\vec{\psi}(t) \ge 0$ for all $t \in [0, T]$.*

Proof. Let i^*, t^* be such that $\psi_{i^*}(t^*) = \min_{i,t} \psi_i(t)$ and assume that the lemma is false. Then $\psi_{i^*}(t^*) < 0$. From the hypotheses we have $t^* \neq 0$ and $\psi'_{i^*}(t^*) \le 0$.

Thus

$$(\vec{L}\vec{\psi}(t^*))_{i^*} = \varepsilon_{i^*}\psi'_{i^*}(t^*) + a_{i^*,i^*}(t^*)\psi_{i^*}(t^*) + \sum_{j=1,\ j\neq i^*}^{n} a_{i^*,j}(t^*)\psi_j(t^*)$$
$$\leq \psi_i^*(t^*)\sum_{j=1}^{n} a_{i^*,j}(t^*) < 0.$$

which contradicts the assumption and proves the result for $\vec{L}$. □

Remark. Let $\tilde{A}(t)$ be any principal sub-matrix of $A(t)$ and $\tilde{\vec{L}}$ the corresponding operator. To see that any $\tilde{\vec{L}}$ satisfies the same maximum principle as $\vec{L}$, it suffices to observe that the elements of $\tilde{A}(t)$ satisfy *a fortiori* the same inequalities as those of $A(t)$.

Lemma 2.2. *Let $A(t)$ satisfy (2) and (3). If $\vec{\psi}(t)$ is any function in the domain of $\vec{L}$ then*

$$\| \vec{\psi}(t) \| \leq \max\left\{\| \vec{\psi}(0) \|, \frac{1}{\alpha} \| \vec{L}\vec{\psi} \|\right\}, \qquad t \in [0,T]$$

Proof. Define the two functions

$$\vec{\theta}^{\pm}(t) = \max\{||\vec{\psi}(0)||, \frac{1}{\alpha}||\vec{L}\vec{\psi}||\}\vec{e} \pm \vec{\psi}(t),$$

where $\vec{e} = (1,\ \ldots\ ,1)'$ is the unit column vector. Using the properties of A it is not hard to verify that $\vec{\theta}^{\pm}(0) \geq 0$ and $\vec{L}\vec{\theta}^{\pm}(t) \geq 0$. It follows from Lemma 2.1 that $\vec{\theta}^{\pm}(t) \geq 0$ for all $t \in [0,T]$. □

The Shishkin decomposition of the solution $\vec{u}$ of (1) is given by $\vec{u} = \vec{v} + \vec{w}$ where $\vec{v}$ is the solution of $\vec{L}\vec{v} = \vec{f}$ on $(0,T]$ with $\vec{v}(0) = A^{-1}(0)\vec{f}(0)$ and $\vec{w}$ is the solution of $\vec{L}\vec{w} = \vec{0}$ on $(0,T]$ with $\vec{w}(0) = \vec{u}(0) - \vec{v}(0)$. Here $\vec{v}$, $\vec{w}$ are, respectively, the smooth and singular components of $\vec{u}$.
We define the layer functions $B_i, 1 \leq i \leq n$, associated with the solution $\vec{u}$ by

$$B_i(t) = e^{-\alpha t/\varepsilon_i},\ t \in [0,\infty).$$

The following elementary properties of these layer functions, for all $1 \leq i < j \leq n$, should be noted:
(i) $B_i(t) < B_j(t)$, for all $t > 0$.
(ii) $B_i(s) > B_i(t)$, for all $0 \leq s < t < \infty$.
(iii) $B_i(0) = 1$ and $0 < B_i(t) < 1$ for all $t > 0$.
The smooth component $\vec{v}$ of $\vec{u}$ and its derivatives are estimated the following lemma, which gives bounds showing the explicit dependence on the inhomogeneous term and the initial condition.

Lemma 2.3. *Let $A(t)$ satisfy (2) and (3). Then there exists a constant C, independent of $\varepsilon, \vec{u}(0)$ and $\vec{f}$, such that*

$$\| \vec{v} \| \leq C \| \vec{f} \|,\ \| \vec{v}' \| \leq C(\| \vec{f} \| + \| \vec{f}' \|)$$

and, for all $1 \leq i \leq n$,

$$\| \varepsilon_i v''_i \| \leq C(\| \vec{f} \| + \| \vec{f}' \|)$$

Proof. We introduce the two functions $\vec{\psi}^{\pm}(t) = C||\vec{f}||\vec{e} \pm \vec{v}(t)$ where $\vec{e}$ is the unit column vector. Noting that $\vec{v}(0) = A^{-1}(0)\vec{f}(0)$, it is not hard to see that $\vec{\psi}^{\pm}(0) \geq 0$ and $\vec{L}\vec{\psi}^{\pm}(t) \geq 0$. It follows from Lemma 2.1 that $\vec{\psi}^{\pm}(t) \geq 0$ for all $t \in [0,T]$ and so $\| \vec{v} \| \leq C \| \vec{f} \|$. To estimate the derivative we now define the two functions $\vec{\phi}^{\pm}(t) = C(||\vec{f}|| + ||\vec{f}'||)\vec{e} \pm \vec{v}'(t)$. Since $\vec{v}'(0) = 0$ and $\vec{L}\vec{v}' = \vec{f}' - A'\vec{v}$, it may be verified that $\vec{\phi}^{\pm}(0) \geq 0$ and $\vec{L}\vec{\phi}^{\pm}(t) \geq 0$. Again by Lemma 2.1 we have $\vec{\phi}^{\pm}(t) \geq 0$, which proves the result. Finally, differentiating the equation $\varepsilon_i v_i' + (A\vec{v})_i = f_i$ and using the estimates of $\vec{v}$ and $\vec{v}'$, we obtain the required bound on $\varepsilon_i v_i''$. $\square$

Bounds on the singular component $\vec{w}$ of $\vec{u}$ and its derivatives are contained in

Lemma 2.4. *Let $A(t)$ satisfy (2) and (3).Then there exists a constant C, such that, for each $t \in [0,T]$ and $i = 1, \ldots, n$,*

$$|w_i(t)| \leq CB_n(t), \quad |w_i'(t)| \leq C\sum_{q=i}^{n}\frac{B_q(t)}{\varepsilon_q}, \quad |\varepsilon_i w_i''(t)| \leq C\sum_{q=1}^{n}\frac{B_q(t)}{\varepsilon_q}.$$

Proof. First we obtain the bound on $\vec{w}$. We define the two functions $\vec{\psi}^{\pm} = CB_n\vec{e} \pm \vec{w}$. Then clearly $\vec{\psi}^{\pm}(0) \geq 0$ and $L\vec{\psi}^{\pm} = CL(B_n\vec{e})$. Then, for $i = 1, \ldots, n$, $(L\vec{\psi}^{\pm})_i = C(\sum_{j=1}^{n} a_{i,j} - \alpha\frac{\varepsilon_i}{\varepsilon_n})B_n > 0$. By Lemma 2.1 $\vec{\psi}^{\pm} \geq 0$, which leads to the required bound on $\vec{w}$.

To establish the bound on $\vec{w}'$ we begin with the n^{th} equation in $\vec{L}\vec{w} = 0$, namely

$$\varepsilon_n w_n' + a_{n,1}w_1 + \cdots + a_{n,n}w_n = 0,$$

from which the bound for $i = n$ follows. We now bound w_i' for $1 \leq i \leq n-1$. We define $\vec{p} = (w_1, \ldots, w_{n-1})$ and, taking the first $n-1$ equations satisfied by $\vec{w}$, we get

$$\tilde{E}\vec{p}' + \tilde{A}\vec{p} = \vec{g},$$

where $\tilde{E}, \tilde{A}$ are the matrices obtained from E, A respectively by deleting the last row and column, the components of $\vec{g}$ are $g_k = -a_{k,n}w_n$ for $1 \leq k \leq n-1$ and $\vec{\tilde{L}} = \tilde{E}D + \tilde{A}(t)$. Using the bounds already obtained for w_n, w_n' we see that $\| \vec{g(t)} \| = \max_{1\leq k\leq n} |g_k(t)|$ and $\| \vec{g'(t)} \|$ are bounded respectively by $CB_n(t)$ and $C\frac{B_n(t)}{\varepsilon_n}$. The initial condition for $\vec{p}$ is $\vec{p}(0) = \vec{u}(0) - \vec{u}^0(0)$, where $\vec{u}^0$ is the solution of the reduced problem $\vec{u}^0 = A^{-1}\vec{f}$, and is therefore bounded by $C(\| \vec{u}(0) \| + \| \vec{f}(0) \|)$. Let $\vec{q}$ and $\vec{r}$ denote, respectively, the smooth and singular components in the Shishkin decomposition of $\vec{p}$. Then

$$\vec{p} = \vec{q} + \vec{r}$$

where $\vec{\tilde{L}}\vec{q} = \vec{g}$, $\vec{q}(0) = \tilde{A}^{-1}(0)\vec{g}(0)$, $\vec{\tilde{L}}\vec{r} = \vec{0}$, $\vec{r}(0) = \vec{p}(0) - \vec{q}(0)$. Also, $\vec{q}'(0) = \vec{0}$. Introducing $\vec{\psi}^{\pm}(t) = CB_n(t)\vec{e} \pm \vec{q}(t)$, it is easy to see that $\vec{\psi}^{\pm}(0) = C\vec{e} \pm \vec{q}(0) \geq \vec{0}$ and that $\vec{\tilde{L}}\vec{\psi}^{\pm}(t) = CB_n(t)(\tilde{A} - \frac{\alpha}{\varepsilon_n}\tilde{E})\vec{e} \pm \vec{g}(t) \geq CB_n(t)diag(1 - \frac{\varepsilon_1}{\varepsilon_n}, \ \ldots, \ 1 - \frac{\varepsilon_{n-1}}{\varepsilon_n})\vec{e} \pm \vec{g}(t) > \vec{0}$ from the bound on the inhomogeneous term $\vec{g}$. Thus, from the Remark following Lemma 2.1, $\vec{\psi}^{\pm}(t) \geq \vec{0}$ or $\| \vec{q}(t) \| \leq CB_n(t)$. Defining $\vec{\theta}^{\pm}(t) = C\frac{B_n(t)}{\varepsilon_n}\vec{e} \pm \vec{q}'(t)$, a similar argument, using the bound on $\vec{g}'$, shows that $\| \vec{q}'(t) \| \leq C\frac{B_n(t)}{\varepsilon_n}$, as required.

We now use mathematical induction. We assume that Lemma 2.4 is valid for all systems with $n-1$ equations. Then Lemma 2.4 applies to $\vec{r}$ and so, for $i =$

$1, \dots, n-1$,

$$|r_i'(t)| \le C(\frac{B_i(t)}{\varepsilon_i} + \cdots + \frac{B_{n-1}(t)}{\varepsilon_{n-1}}).$$

Combining the bounds for q_i and r_i we obtain

$$|p_i'(t)| \le C(\frac{B_i(t)}{\varepsilon_i} + \cdots + \frac{B_n(t)}{\varepsilon_n}).$$

Recalling the definition of $\vec{p}$, we conclude that

$$|w_i'(t)| \le C(\frac{B_i(t)}{\varepsilon_i} + \cdots + \frac{B_n(t)}{\varepsilon_n}).$$

We have thus proved that Lemma 2.4 holds for our system with n equations. Since Lemma 2.4 is true for a system with one equation, we conclude by mathematical induction that it is true for any system of $n > 1$ equations.

Finally, to estimate the second derivative, we differentiate the i^{th} equation of the system $\vec{L}\vec{w} = 0$ to get

$$\varepsilon_k w_i'' = -(A\vec{w}_i' + A'\vec{w})_i$$

and we see that the bound on w_i'' follows easily from the bounds on $\vec{w}$ and $\vec{w}'$. □

Definition 2.5. *For each $1 \le i \ne j \le n$ we define the point $t_{i,j}$ by*

$$\frac{B_i(t_{i,j})}{\varepsilon_i} = \frac{B_j(t_{i,j})}{\varepsilon_j}. \tag{4}$$

In the next lemma it is shown that these points exist, are uniquely defined and have an interesting ordering. Sufficient conditions for them to lie in the domain $[0, T]$ are also provided. The proof is omitted, because these results are all proved in [3]

Lemma 2.6. *For all i, j with $1 \le i < j \le n$ the points $t_{i,j}$ exist, are uniquely defined and satisfy the following inequalities*

$$\varepsilon_i^{-1} B_i(t) > \varepsilon_j^{-1} B_j(t) \qquad t \in [0, t_{ij}) \tag{5}$$

and

$$\varepsilon_i^{-1} B_i(t) < \varepsilon_j^{-1} B_j(t) \qquad t \in (t_{ij}, \infty). \tag{6}$$

In addition the following ordering holds

$$t_{i,j} < t_{i+1,j}, \text{ if } \ i+1 < j \ \text{ and } \ t_{i,j} < t_{i,j+1}, \ \text{ if } \ i < j \tag{7}$$

and

$$\varepsilon_i \le \varepsilon_j/2 \ \text{ implies that } \ t_{ij} \in (0, T] \ \text{ for all } \ i < j. \tag{8}$$

3. The discrete problem

We construct a piecewise uniform mesh with N mesh-intervals and mesh-points $\{t_i\}_{i=0}^N$ by dividing the interval $[0, T]$ into $n+1$ sub-intervals as follows

$$[0, T] = [0, \sigma_1] \cup (\sigma_1, \sigma_2] \cup \dots (\sigma_{n-1}, \sigma_n] \cup (\sigma_n, T]$$

Then, on the sub-interval $[0, \sigma_1]$, a uniform mesh with $\frac{N}{2^n}$ mesh-intervals is placed, and similarly on $(\sigma_i, \sigma_{i+1}], 1 \le i \le n-1$, a uniform mesh with $\frac{N}{2^{n-i+1}}$ mesh-intervals and on $(\sigma_n, T]$ a uniform mesh with $\frac{N}{2}$ mesh-intervals. In practice it is convenient to take $N = 2^n k$ where k is some positive power of 2. The n transition points between the uniform meshes are defined by

$$\sigma_i = \min\{\frac{\sigma_{i+1}}{2}, \frac{\varepsilon_i}{\alpha} \ln N\}$$

for $i = 1, \dots, n-1$ and

$$\sigma_n = \min\{\frac{T}{2}, \frac{\varepsilon_n}{\alpha}\ln N\}.$$

Clearly

$$0 < \sigma_1 < \cdots < \sigma_n \leq \frac{T}{2}.$$

This construction leads to a class of 2^n piecewise uniform Shishkin meshes $M_{\vec{b}}$, where $\vec{b}$ denotes an n–vector with $b_i = 0$ if $\sigma_i = \frac{\sigma_{i+1}}{2}$ and $b_i = 1$ otherwise. Note that $M_{\vec{0}}$ is a classical uniform mesh. Writing $\delta_j = t_j - t_{j-1}$ we remark that

$$\delta_j \leq CN^{-1}, \text{ for any } j,\ 1 \leq j \leq N. \tag{9}$$

$$\frac{\sigma_i}{\varepsilon_i} \leq C\ln N, \text{ for any } i,\ 1 \leq i \leq n. \tag{10}$$

$$B_i(\sigma_i) = N^{-1} \text{ if } b_i = 1. \tag{11}$$

$$\sigma_i = 2^{-(j-i+1)}\sigma_{j+1}, \text{ for } i \leq j, \text{ if } b_k = 0,\ \forall\ k \ni i \leq k \leq j. \tag{12}$$

On any $M_{\vec{b}}$ we now consider the discrete solutions defined by the backward Euler finite difference scheme

$$ED^{-}\vec{U} + A(t)\vec{U} = \vec{f}, \qquad \vec{U}(0) = \vec{u}(0),$$

or in operator form

$$\vec{L}^N\vec{U} = \vec{f}, \qquad \vec{U}(0) = \vec{u}(0),$$

where

$$\vec{L}^N = ED^{-} + A(t)$$

and D^{-} is the backward difference operator

$$D^{-}\vec{U}(t_j) = \frac{\vec{U}(t_j) - \vec{U}(t_{j-1})}{\delta_j}.$$

We have the following discrete maximum principle analogous to the continuous case.

Lemma 3.1. *Let $A(t)$ satisfy (2) and (3). Then, for any mesh function $\vec{\Psi}$, the inequalities $\vec{\Psi}(0) \geq \vec{0}$ and $\tilde{\mathrm{L}}^{\mathrm{N}}\tilde{\Psi}(\mathrm{t_j}) \geq \tilde{0}$ for $1 \leq j \leq N$, imply that $\vec{\Psi}(t_j) \geq \vec{0}$ for $0 \leq j \leq N$.*

Proof. Let i^*, j^* be such that $V_{i^*}(t_{j^*}) = \min_{i,j} V_i(t_j)$ and assume that the lemma is false. Then $V_{i^*}(t_{j^*}) < 0$. From the hypotheses we have $j^* \neq 0$ and $V_{i^*}(t_{j^*}) - V_{i^*}(t_{j^*-1}) \leq 0$. Thus

$$(\vec{L^N}\vec{V}(t_{j^*}))_{i^*} = \varepsilon_{i^*}\frac{V_{i^*}(t_{j^*}) - V_{i^*}(t_{j^*-1})}{\delta_{j^*}} + a_{i^*,i^*}(t_{j^*})V_{i^*}(t_{j^*}) + \sum_{k=1\ k\neq i^*}^{n} a_{i^*,k}(t_{j^*})V_k(t_{j^*})$$

$$\leq V_{i^*}(t_{j^*})\sum_{k=1}^{n} a_{i^*,k}(t_{j^*}) < 0,$$

which contradicts the assumption, as required. $\square$

An immediate consequence of this is the following discrete stability result.

Lemma 3.2. *Let $A(t)$ satisfy (2) and (3). Then, for any mesh function $\vec{\Psi}$,*

$$\parallel \vec{\Psi}(t_j) \parallel \leq \max\left\{\parallel \vec{\Psi}(0) \parallel, \frac{1}{\alpha} \parallel \vec{L}^N\vec{\Psi} \parallel\right\}, 0 \leq j \leq N$$

Proof. Define the two functions

$$\vec{\Theta}^{\pm}(t) = \max\{||\vec{\Psi}(0)||, \frac{1}{\alpha}||\vec{L^N}\vec{\Psi}||\}\vec{e} \pm \vec{\Psi}(t)$$

where $\vec{e} = (1, \ldots, 1)$ is the unit vector. Using the properties of A it is not hard to verify that $\vec{\Theta}^{\pm}(0) \geq 0$ and $\vec{L^N}\vec{\Theta}^{\pm}(t_j) \geq 0$. It follows from Lemma 3.1 that $\vec{\Theta}^{\pm}(t_j) \geq 0$ for all $0 \leq j \leq N$. □

4. The local truncation error

From Lemma 3.2, we see that in order to bound the error $\parallel \vec{U} - \vec{u} \parallel$ it suffices to bound $\vec{L}^N(\vec{U} - \vec{u})$. But this expression satisfies

$$\vec{L}^N(\vec{U} - \vec{u}) = \vec{L}^N(\vec{U}) - \vec{L}^N(\vec{u}) = \vec{f} - \vec{L}^N(\vec{u}) = \vec{L}(\vec{u}) - \vec{L}^N(\vec{u})$$
$$= (\vec{L} - \vec{L}^N)\vec{u} = -E(D^- - D)\vec{u},$$

which is the local truncation of the first derivative. We have

$$E(D^- - D)\vec{u} = E(D^- - D)\vec{v} + E(D^- - D)\vec{w}$$

and so, by the triangle inequality,

$$\parallel \vec{L}^N(\vec{U} - \vec{u}) \parallel \leq \parallel E(D^- - D)\vec{v} \parallel + \parallel E(D^- - D)\vec{w} \parallel . \tag{13}$$

Thus, we can treat the smooth and singular components of the local truncation error separately. In view of this we note that, for any smooth function ψ, we have the following two distinct estimates of the local truncation error of its first derivative

$$|(D^- - D)\psi(t_j)| \leq 2 \max_{s \in I_j} |\psi'(s)| \tag{14}$$

and

$$|(D^- - D)\psi(t_j)| \leq \frac{\delta_j}{2} \max_{s \in I_j} |\psi''(s)|, \tag{15}$$

where $I_j = [t_{j-1}, t_j]$.

5. Error estimate

We now establish the error estimate by generalizing the approach based on Shishkin decompositons used in [3]. For a reaction-diffusion boundary value problem in the special case $n = 2$ a parameter uniform numerical method was analyzed in [6] by a similar technique and in the general case in [5] using discrete Green's functions.

We estimate the smooth component of the local truncation error in the following lemma.

Lemma 5.1. *Let $A(t)$ satisfy (2) and (3). Then, for each $i = 1, \ldots, n$ and $j = 1, \ldots, N$, we have*

$$|\varepsilon_i(D^- - D)v_i(t_j)| \leq CN^{-1}.$$

Proof. Using (15), Lemma 2.3 and (9) we obtain

$$|\varepsilon_i(D^- - D)v_i(t_j)| \le C\delta_j \max_{s\epsilon I_j} |\varepsilon_i v_i''(s)| \le C\delta_j \le CN^{-1}$$

as required. □

For the singular component we obtain a similar estimate, but in the proof we must distinguish between the different types of mesh. We need the following preliminary lemmas.

Lemma 5.2. *Let $A(t)$ satisfy (2) and (3). Then, for each $i = 1, \dots, n$ and $j = 1, \dots, N$, on each mesh $M_{\vec{b}}$, we have the estimate*

$$|\varepsilon_i(D^- - D)w_i(t_j)| \le C\frac{\delta_j}{\varepsilon_1}.$$

Proof. From (15) and Lemma 2.4, we have

$$\begin{array}{rcl} |\varepsilon_i(D^- - D)w_i(t_j)| & \le & C\delta_j \max_{s\epsilon I_j} |\varepsilon_i w_i''(s)| \\ & \le & C\delta_j \sum_{q=1}^{n} \frac{B_q(t_{j-1})}{\varepsilon_q} \\ & \le & C\frac{\delta_j}{\varepsilon_1} \end{array}$$

as required. □

In what follows we make use of second degree polynomials of the form

$$p_{i;\theta} = \sum_{k=0}^{2} \frac{(t-t_\theta)^k}{k!} w_i^{(k)}(t_\theta),$$

where θ denotes a pair of integers separated by a comma.

Lemma 5.3. *Let $A(t)$ satisfy (2) and (3). Then, for each $i = 1, \dots, n$, $j = 1, \dots, N$ and $k = 1, \dots, n-1$, on each mesh $M_{\vec{b}}$ with $b_k = 1$, there exists a decomposition*

$$w_i = \sum_{m=1}^{k+1} w_{i,m},$$

for which we have the following estimates for each m, $1 \le m \le k$,

$$|\varepsilon_i w_{i,m}'(t)| \le CB_m(t), \quad |\varepsilon_i w_{i,m}''(t)| \le C\frac{B_m(t)}{\varepsilon_m}$$

and

$$|\varepsilon_i w_{i,k+1}''(t)| \le C \sum_{q=k+1}^{n} \frac{B_q(t)}{\varepsilon_q}.$$

Furthermore

$$|\varepsilon_i(D^- - D)w_i(t_j)| \le C(B_k(t_{j-1}) + \frac{\delta_j}{\varepsilon_{k+1}}).$$

Proof. Since $b_k = 1$ we have $\varepsilon_k \le \varepsilon_{k+1}/2$, and so, by Lemma 2.6, $t_{r,r+1} \in (0, T]$ for $r = 1, \dots, k$. Therefore, we can define the decomposition

$$w_i = \sum_{m=1}^{k+1} w_{i,m},$$

where the components $w_{i,m}$, $1 \le m \le k+1$, are given by

$$w_{i,k+1} = \begin{cases} p_{i;k,k+1} & \text{on } [0, t_{k,k+1}) \\ w_i & \text{otherwise} \end{cases}$$

and for each m, $k \geq m \geq 2$,

$$w_{i,m} = \begin{cases} p_{i;m-1,m} & \text{on } [0, \mathrm{t_{m-1,m}}) \\ w_i - \sum_{q=m+1}^{k+1} w_{i,q} & \text{otherwise} \end{cases}$$

and finally

$$w_{i,1} = w_i - \sum_{q=2}^{k+1} w_{i,q} \text{ on } [0, \mathrm{T}].$$

From the above expressions we note that for each m, $1 \leq m \leq k$, $w_{i,m} = 0$ on $[\mathrm{t_{m,m+1}}, \mathrm{T}]$. To establish the bounds on the second derivatives we observe that:

in $[t_{k,k+1}, T]$, using Lemma 2.4 and $t \geq t_{k,k+1}$, we obtain

$$|\varepsilon_i w''_{i,k+1}(t)| = |\varepsilon_i w''_i(t)| \leq C \sum_{q=1}^{n} \frac{B_q(t)}{\varepsilon_q} \leq C \sum_{q=k+1}^{n} \frac{B_q(t)}{\varepsilon_q};$$

in $[0, t_{k,k+1}]$, using Lemma 2.4 and $t \leq t_{k,k+1}$, we obtain

$$|\varepsilon_i w''_{i,k+1}(t)| = |\varepsilon_i w''_i(t_{k,k+1})| \leq \sum_{q=1}^{n} \frac{B_q(t_{k,k+1})}{\varepsilon_q} \leq \sum_{q=k+1}^{n} \frac{B_q(t_{k,k+1})}{\varepsilon_q} \leq \sum_{q=k+1}^{n} \frac{B_q(t)}{\varepsilon_q};$$

and for each $m = k, \ \ldots \ , 2$, we see that

in $[t_{m,m+1}, T]$, $w''_{i,m} = 0$;

in $[t_{m-1,m}, t_{m,m+1}]$, using Lemma 2.4, we obtain

$$|\varepsilon_i w''_{i,m}(t)| \leq |\varepsilon_i w''_i(t)| + \sum_{q=m+1}^{k+1} |\varepsilon_i w''_{i,q}(t)| \leq C \sum_{q=1}^{n} \frac{B_q(t)}{\varepsilon_q} \leq C \frac{B_m(t)}{\varepsilon_m};$$

in $[0, t_{m-1,m}]$, using Lemma 2.4 and $t \leq t_{m-1,m}$, we obtain

$$|\varepsilon_i w''_{i,m}(t)| = |\varepsilon_i w''_i(t_{m-1,m})| \leq C \sum_{q=1}^{n} \frac{B_q(t_{m-1,m})}{\varepsilon_q} \leq C \frac{B_m(t_{m-1,m})}{\varepsilon_m} \leq C \frac{B_m(t)}{\varepsilon_m};$$

in $[t_{1,2}, T]$, $w''_{i,1} = 0$;

in $[0, t_{1,2}]$, using Lemma 2.4,

$$|\varepsilon_i w''_{i,1}(t)| \leq |\varepsilon_i w''_i(t)| + \sum_{q=2}^{k+1} |\varepsilon_i w''_{i,q}(t)| \leq C \sum_{q=1}^{n} \frac{B_q(t)}{\varepsilon_q} \leq C \frac{B_1(t)}{\varepsilon_1}.$$

For the bounds on the first derivatives we observe that for each m, $1 \leq m \leq k$:

in $[t_{m,m+1}, T]$, $w'_{i,m} = 0$;

in $[0, t_{m,m+1}]$ $\int_t^{t_{m,m+1}} \varepsilon_i w''_{i,m}(s)ds = \varepsilon_i w'_{i,m}(t_{m,m+1}) - \varepsilon_i w'_{i,m}(t) = -\varepsilon_i w'_{i,m}(t)$ and so

$$|\varepsilon_i w'_{i,m}(t)| \leq \int_t^{t_{m,m+1}} |\varepsilon_i w''_{i,m}(s)|ds \leq \frac{C}{\varepsilon_m} \int_t^{t_{m,m+1}} B_m(s)ds \leq CB_m(t).$$

Finally, since

$$|\varepsilon_i(D^- - D)w_i(t_j)| \leq \sum_{m=1}^{k} |\varepsilon_i(D^- - D)w_{i,m}(t_j)| + |\varepsilon_i(D^- - D)w_{i,k+1}(t_j)|,$$

using (15) on the last term and (14) on all other terms on the right hand side, we obtain

$$|\varepsilon_i(D^- - D)w_i(t_j)| \le C(\sum_{m=1}^{k} \max_{s\in I_j} |\varepsilon_i w'_{i,m}(s)| + \delta_j \max_{s\in I_j} |\varepsilon_i w''_{i,k+1}(s)|).$$

The desired result follows by applying the bounds on the derivatives in the first part of this lemma. □

Lemma 5.4. *Let $A(t)$ satisfy (2) and (3). Then, for each $i = 1, \ldots, n$ and $j = 1, \ldots, N$, on each mesh $M_{\vec{b}}$, we have the estimate*

$$|\varepsilon_i(D^- - D)w_i(t_j)| \le CB_n(t_{j-1}).$$

Proof. From (14) and Lemma 2.4, for each $i = 1, \ldots, n$ and $j = 1, \ldots, N$, we have

$$\begin{aligned} |\varepsilon_i(D^- - D)w_i(t_j)| &\le C \max_{s\epsilon I_j} |\varepsilon_i w'_i(s)| \\ &\le C\varepsilon_i \textstyle\sum_{q=i}^{n} \frac{B_q(t_{j-1})}{\varepsilon_q} \\ &\le CB_n(t_{j-1}) \end{aligned}$$

as required. □

Using the above preliminary lemmas on appropriate subintervals we obtain the desired estimate of the singular component of the local truncation error in the following lemma.

Lemma 5.5. *Let $A(t)$ satisfy (2) and (3). Then, for each $i = 1, \ldots, n$ and $j = 2, \ldots, N$, we have the estimate*

$$|\varepsilon_i(D^- - D)w_i(t_j)| \le CN^{-1}\ln N.$$

Proof. Stepping out from the origin we consider each subinterval separately.
First, in the subinterval $(0, \sigma_1]$ we have $\delta_j \le CN^{-1}\sigma_1$ and the result follows on any mesh $M_{\vec{b}}$ from (10) and Lemma 5.2.
Secondly, in the subinterval $(\sigma_1, \sigma_2]$ we have $\sigma_1 \le t_{j-1}$ and $\delta_j \le CN^{-1}\sigma_2$. We divide the 2^{n+1} possible meshes into 2 subclasses. On the meshes $M_{\vec{b}}$ with $b_1 = 0$ the result follows from (10), (12) and Lemma 5.2. On the meshes $M_{\vec{b}}$ with $b_1 = 1$ the result follows from (10), (11)and Lemma 5.3.
Thirdly, in a general subinterval $(\sigma_m, \sigma_{m+1}]$ for $2 \le m \le n-1$ we have $\sigma_m \le t_{j-1}$ and $\delta_j \le CN^{-1}\sigma_{m+1}$. We divide $M_{\vec{b}}$ into 3 subclasses: $M^0_{\vec{b}} = \{M_{\vec{b}} : b_1 = \cdots = b_m = 0\}$, $M^r_{\vec{b}} = \{M_{\vec{b}} : b_r = 1,\ b_{r+1} = \cdots = b_m = 0 \text{ for some } 1 \le r \le m-1\}$ and $M^m_{\vec{b}} = \{M_{\vec{b}} : b_m = 1\}$. On $M^0_{\vec{b}}$ the result follows from (10), (12) and Lemma 5.2; on $M^r_{\vec{b}}$ from (10), (11), (12) and Lemma 5.3; on $M^m_{\vec{b}}$ from (10), (11) and Lemma 5.3.
Finally, in the subinterval $(\sigma_n, T]$ we have $\sigma_n \le t_{j-1}$ and $\delta_j \le CN^{-1}$. We divide $M_{\vec{b}}$ into 3 subclasses: $M^0_{\vec{b}} = \{M_{\vec{b}} : b_1 = \cdots = b_n = 0\}$, $M^r_{\vec{b}} = \{M_{\vec{b}} : b_r = 1,\ b_{r+1} = \cdots = b_n = 0 \text{ for some } 1 \le r \le n-1\}$ and $M^n_{\vec{b}} = \{M_{\vec{b}} : b_n = 1\}$. On $M^0_{\vec{b}}$ the result follows from (10), (12) and Lemma 5.2; on $M^r_{\vec{b}}$ from (10), (11), (12) and Lemma 5.3; on $M^n_{\vec{b}}$ from (10), (11) and Lemma 5.3. □

Let $\vec{u}$ denote the exact solution of (1) and $\vec{U}$ the discrete solution. Then, the main result of this paper is the following ε-uniform error estimate

Theorem 5.6. *Let $A(t)$ satisfy (2) and (3). Then there exists a constant C such that*

$$\| \vec{U} - \vec{u} \| \le CN^{-1}\ln N,$$

for all $N > 1$

Proof. This follows immediately by applying Lemmas 5.1 and 5.5 to (13) and using Lemma 3.2. $\square$

6. Numerical results

In order to validate the theoretical results of this paper, the numerical method constructed above is used in this section to compute approximate solutions of two examples of initial value problems for singularly perturbed systems of first order ordinary differential equations. For each case the computed order of $\vec{\varepsilon}$–uniform convergence p^* and the $\vec{\varepsilon}$–uniform error constant $C^*_{p^*}$ are found using the general methodology in [2] and [7].

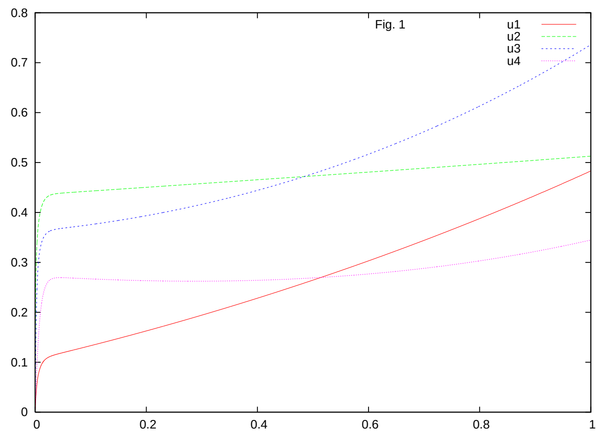

FIGURE 1. *Numerical solution of Example 6.2 for* $\varepsilon_4 = \frac{2^{-4}}{3}, \varepsilon_3 = \frac{2^{-4}}{5}, \varepsilon_2 = \frac{2^{-4}}{7}, \varepsilon_1 = \frac{2^{-4}}{9}$ *and* $N = 2^{11}$ *points*

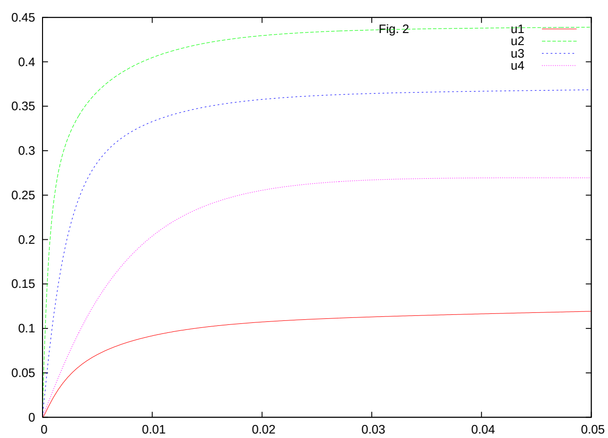

FIGURE 2. *Blowup of Figure 1 for t in the subdomain* $[0.00,\ 0.05]$

TABLE 1. *Values of* $D_\varepsilon^N, D^N, p^N, p^*, C_{p^*}^N$ *and* $C_{p^*}^*$ *for Example 6.1 with* $\varepsilon_1 = \frac{r}{16}$, $\varepsilon_2 = \frac{r}{4}$, $\varepsilon_3 = r$, *for various values of* r *and* N.

	Number of mesh points N								
r	256	512	1024	2048	4096	8192	16384	32768	65536
2^{-1}	0.903-3	0.429-3	0.235-3	0.134-3	0.814-4	0.408-4	0.204-4	0.102-4	0.511-5
2^{-2}	0.104-2	0.560-3	0.292-3	0.152-3	0.839-4	0.419-4	0.229-4	0.124-4	0.668-5
2^{-3}	0.106-2	0.566-3	0.296-3	0.154-3	0.803-4	0.423-4	0.231-4	0.125-4	0.675-5
2^{-4}	0.106-2	0.569-3	0.297-3	0.155-3	0.809-4	0.425-4	0.232-4	0.126-4	0.678-5
2^{-5}	0.106-2	0.570-3	0.298-3	0.156-3	0.811-4	0.426-4	0.233-4	0.126-4	0.679-5
2^{-6}	0.106-2	0.570-3	0.298-3	0.156-3	0.812-4	0.427-4	0.233-4	0.126-4	0.680-5
2^{-7}	0.107-2	0.571-3	0.299-3	0.156-3	0.813-4	0.427-4	0.233-4	0.126-4	0.681-5
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.107-2	0.571-3	0.299-3	0.156-3	0.813-4	0.427-4	0.233-4	0.126-4	0.681-5
D^N	0.107-2	0.571-3	0.299-3	0.156-3	0.839-4	0.427-4	0.233-4	0.126-4	0.681-5
p^N	0.873+0	0.935+0	0.937+0	0.894+0	0.974+0	0.873+0	0.884+0	0.893+0	
$C_{0.873}^N$	0.162+0	0.159+0	0.153+0	0.146+0	0.144+0	0.134+0	0.134+0	0.133+0	0.131+0
The computed order of convergence $p^* = 0.873$									
The computed error constant $C_{p^*}^* = 0.162$									

TABLE 2. *Values of* $D_\varepsilon^N, D^N, p^N, p^*, C_{p^*}^N$ *and* $C_{p^*}^*$ *for Example 6.1 with* $\varepsilon_1 = \frac{r}{7}$, $\varepsilon_2 = \frac{r}{5}$, $\varepsilon_3 = \frac{r}{3}$, *for various values of* r *and* N.

	Number of mesh points N								
r	256	512	1024	2048	4096	8192	16384	32768	65536
2^{-0}	0.125-2	0.645-3	0.327-3	0.165-3	0.827-4	0.414-4	0.207-4	0.104-4	0.519-5
2^{-1}	0.182-2	0.123-2	0.672-3	0.341-3	0.172-3	0.862-4	0.432-4	0.216-4	0.108-4
2^{-3}	0.188-2	0.114-2	0.665-3	0.382-3	0.214-3	0.118-3	0.646-4	0.350-4	0.189-4
2^{-5}	0.189-2	0.114-2	0.669-3	0.384-3	0.215-3	0.119-3	0.649-4	0.352-4	0.190-4
2^{-7}	0.189-2	0.114-2	0.670-3	0.384-3	0.215-3	0.119-3	0.650-4	0.353-4	0.190-4
2^{-9}	0.189-2	0.115-2	0.671-3	0.384-3	0.215-3	0.119-3	0.650-4	0.353-4	0.190-4
2^{-11}	0.189-2	0.115-2	0.671-3	0.384-3	0.215-3	0.119-3	0.651-4	0.353-4	0.190-4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-19}	0.189-2	0.115-2	0.671-3	0.384-3	0.215-3	0.119-3	0.651-4	0.353-4	0.190-4
D^N	0.189-2	0.123-2	0.672-3	0.384-3	0.215-3	0.119-3	0.651-4	0.353-4	0.190-4
p^N	0.617+0	0.875+0	0.806+0	0.836+0	0.856+0	0.871+0	0.883+0	0.892+0	
$C_{0.617}^N$	0.109+0	0.109+0	0.907-1	0.796-1	0.684-1	0.579-1	0.486-1	0.404-1	0.334-1
The computed order of convergence $p^* = 0.617$									
The computed error constant $C_{p^*}^* = 0.109$									

Example 6.1.

$$\left.\begin{aligned} \varepsilon_1 u_1'(t) + 4u_1(t) - u_2(t) - u_3(t) &= t \\ \varepsilon_2 u_2'(t) - u_1(t) + 4u_2(t) - u_3(t) &= 1 \\ \varepsilon_3 u_3'(t) - u_1(t) - u_2(t) + 4u_3(t) &= 1 + t^2 \end{aligned}\right\} \qquad \forall\, t \in (0,1]$$

$$u_1(0) = 0, \quad u_2(0) = 0, \quad u_3(0) = 0.$$

This is solved for $\alpha = 1.9$, two different sets of values of $\varepsilon_1, \ \varepsilon_2, \ \varepsilon_3$ and $N = 2^q, \ q = 7, \cdots, 16$. The results are presented in Table 1 and Table 2.

TABLE 3. *Values of* $D_\varepsilon^N, D^N, p^N, p^*, C_{p^*}^N$ *and* $C_{p^*}^*$ *for Example 6.2 with* $\varepsilon_1 = \dfrac{r}{64}, \varepsilon_2 = \dfrac{r}{16}, \varepsilon_3 = \dfrac{r}{4}, \varepsilon_4 = r$, *for various values of* r *and* N.

	Number of mesh points N								
r	256	512	1024	2048	4096	8192	16384	32768	65536
2^{-1}	0.937-3	0.462-3	0.232-3	0.141-3	0.764-4	0.383-4	0.192-4	0.960-5	0.480-5
2^{-2}	0.105-2	0.591-3	0.319-3	0.170-3	0.843-4	0.400-4	0.217-4	0.118-4	0.634-5
2^{-3}	0.104-2	0.584-3	0.316-3	0.164-3	0.837-4	0.424-4	0.215-4	0.116-4	0.627-5
2^{-4}	0.104-2	0.581-3	0.314-3	0.163-3	0.832-4	0.421-4	0.213-4	0.116-4	0.623-5
2^{-5}	0.103-2	0.579-3	0.313-3	0.163-3	0.829-4	0.420-4	0.213-4	0.115-4	0.621-5
2^{-6}	0.103-2	0.578-3	0.313-3	0.162-3	0.828-4	0.419-4	0.213-4	0.115-4	0.621-5
2^{-7}	0.103-2	0.578-3	0.312-3	0.162-3	0.827-4	0.419-4	0.212-4	0.115-4	0.620-5
2^{-8}	0.103-2	0.578-3	0.312-3	0.162-3	0.827-4	0.418-4	0.212-4	0.115-4	0.620-5
2^{-9}	0.103-2	0.577-3	0.312-3	0.162-3	0.827-4	0.418-4	0.212-4	0.115-4	0.620-5
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-20}	0.103-2	0.577-3	0.312-3	0.162-3	0.827-4	0.418-4	0.212-4	0.115-4	0.620-5
D^N	0.105-2	0.591-3	0.319-3	0.170-3	0.843-4	0.424-4	0.217-4	0.118-4	0.634-5
p^N	0.831+0	0.890+0	0.910+0	0.101+1	0.992+0	0.967+0	0.883+0	0.893+0	
$C_{0.831}^N$	0.135+0	0.135+0	0.130+0	0.123+0	0.109+0	0.972-1	0.884-1	0.853-1	0.817-1
The computed order of convergence $p^* = 0.831$									
The computed error constant $C_{p^*}^* = 0.135$									

TABLE 4. *Values of* $D_\varepsilon^N, D^N, p^N, p^*, C_{p^*}^N$ *and* $C_{p^*}^*$ *for Example 6.2 with* $\varepsilon_1 = \dfrac{r}{9}, \varepsilon_2 = \dfrac{r}{7}, \varepsilon_3 = \dfrac{r}{5}, \varepsilon_4 = \dfrac{r}{3}$, *for various values of* r *and* N.

	Number of mesh points N								
r	256	512	1024	2048	4096	8192	16384	32768	65536
2^{-0}	0.107-2	0.557-3	0.284-3	0.144-3	0.721-4	0.361-4	0.181-4	0.905-5	0.453-5
2^{-2}	0.272-2	0.194-2	0.104-2	0.542-3	0.277-3	0.140-3	0.702-4	0.352-4	0.176-4
2^{-4}	0.264-2	0.167-2	0.103-2	0.612-3	0.348-3	0.194-3	0.107-3	0.582-4	0.314-4
2^{-6}	0.263-2	0.166-2	0.103-2	0.611-3	0.347-3	0.194-3	0.107-3	0.580-4	0.313-4
2^{-8}	0.263-2	0.166-2	0.103-2	0.611-3	0.347-3	0.194-3	0.107-3	0.580-4	0.313-4
2^{-10}	0.263-2	0.166-2	0.103-2	0.611-3	0.347-3	0.194-3	0.107-3	0.580-4	0.313-4
2^{-12}	0.263-2	0.166-2	0.103-2	0.610-3	0.347-3	0.194-3	0.107-3	0.580-4	0.313-4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-19}	0.263-2	0.166-2	0.103-2	0.610-3	0.347-3	0.194-3	0.107-3	0.580-4	0.313-4
D^N	0.272-2	0.194-2	0.104-2	0.614-3	0.349-3	0.195-3	0.107-3	0.583-4	0.324-4
p^N	0.488+0	0.894+0	0.765+0	0.816+0	0.841+0	0.861+0	0.879+0	0.847+0	
$C_{0.488}^N$	0.101+0	0.101+0	0.764-1	0.631-1	0.503-1	0.394-1	0.304-1	0.232-1	0.181-1
The computed order of convergence $p^* = 0.488$									
The computed error constant $C_{p^*}^* = 0.101$									

Example 6.2.

$$\left.\begin{array}{rcl} \varepsilon_1 {u_1}'(t) + (5+e^{-t})u_1(t) - tu_2(t) - u_3(t) - u_4(t) & = & t \\ \varepsilon_2 {u_2}'(t) - u_1(t) + (4+t^2)u_2(t) - u_3(t) - u_4(t) & = & 1 \\ \varepsilon_3 {u_3}'(t) - u_1(t) - u_2(t) + 5u_3(t) - (1+t)u_4(t) & = & 1+t^2 \\ \varepsilon_4 {u_4}'(t) - tu_1(t) - tu_2(t) - u_3(t) + 5u_4(t) & = & 1-t \end{array}\right\} \quad \forall\, t \in (0,1]$$

$$u_1(0) = 0, \quad u_2(0) = 0, \quad u_3(0) = 0, \quad u_4(0) = 0.$$

This is solved for $\alpha = 0.9$, two different sets of values of $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3,\ \varepsilon_4$ and $N = 2^q,\ \ q = 7, \cdots, 16$. The results are presented in Table 3 and Table 4 and plots of the numerical solution for a particular choice of the set of values of $\varepsilon_1,\ \varepsilon_2,\ \varepsilon_3,\ \varepsilon_4$ and of N are given in the two figures.

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