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THE TWO-LEVEL LOCAL PROJECTION STABILIZATION AS AN ENRICHED ONE-LEVEL APPROACH. A ONE-DIMENSIONAL STUDY

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This paper is dedicated to G.I. Shishkin on the occasion of his 70th birthday

Abstract. The two-level local projection stabilization is considered as a one-level approach in which the enrichments on each element are piecewise polynomial functions. The dimension of the enrichment space can be significantly reduced without losing the convergence order. For example, using continuous piecewise polynomials of degree $r \geq 1$, only one function per cell is needed as enrichment instead of r in the two-level approach. Moreover, in the constant coefficient case, we derive formulas for the user-chosen stabilization parameter which guarentee that the linear part of the solution becomes nodally exact.

Key Words. local projection stabilization, finite elements, Shishkin mesh, convection diffusion equation

1. Introduction

It is well-known that standard Galerkin finite element discretizations applied to convection-diffusion problems show spurious oscillations unless the mesh is adapted to the boundary layers of the solutions [21]. But even in the case of layer adapted meshes it makes sense to use stabilized finite element schemes in order to reduce sensitivities of the solutions on the choice of mesh parameters. Residual based stabilization methods like the streamline upwind Petrov-Galerkin (SUPG) stabilization, proposed in [5] and at first analyzed for a scalar convection-diffusion equation in [19], is a prominent example of stabilized schemes. They rely on adding weighted residuals to the standard Galerkin method to enhance stability without losing consistency.

Recently, local projection stabilization (LPS) [2, 3, 9, 10, 12, 13, 17, 18, 20] methods have become quite popular, in particular because of their commutation properties in optimization problems [4] and stabilization properties similar to those of the SUPG method [11]. In contrast, to residual based stabilization methods the LPS is no longer consistent. However, taking rich enough projection spaces any desired consistency order can be achieved. As shown in [17], the key issue in analyzing the error of LPS schemes is the existence of an interpolation for which the error is orthogonal to the projection space. It turns out, that a local inf-sup condition for the approximation and projection space is sufficient to modify an interpolation into the approximation space in such a way that the additional orthogonality property holds [17]. Two main approaches of LPS have been considered in the literature to fulfil the local inf-sup condition. In the one-level approach, a standard finite

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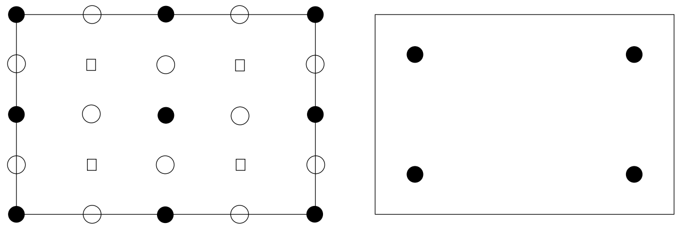

Figure 1. Degrees of freedom of two-level methods on one macro cell. Enriched biquadratic approximation spaces (left) and bilinear projection space (right).

element space is chosen as the projection space to guarantee the consistency order. Then, the approximation space is (if necessary) enriched such that the local inf-sup condition holds. In the two-level approach, a standard finite element space is chosen as the approximation space and the projection space is thinned out to a space on the next coarser mesh level to satisfy the local inf-sup condition.

The main objective of this paper is to show that the two-level variant of the LPS can be also considered as an enriched one-level method on the coarser mesh. This enables us to reduce the degrees of freedom in the two-level method without losing the convergence order. For example, on a rectangular coarse mesh 16 degrees of freedom (squares and non-filled circles) to the 9 degrees of freedom (filled circles) have to be added per macro cell to generate the full biquadratic approximation space on the next finer mesh level, see Figure 1. However, for satisfying the local inf-sup condition with respect to the associated 4-dimensional space of bilinear functions the 4 degrees of freedom indicated by squares are enough and lead to a reduced two-level method with optimal convergence order.

Here, we restrict our attention only to the one-dimensional case in which already one additional function per macro cell is sufficient. Furthermore, for constant coefficients we can chose the stabilization parameter in such a way such that the piecewise linear part of the LPS becomes nodally exact. Although such a strong result cannot be expected in the multi-dimensional case a considerable reduction of degrees of freedom in the two-level method without losing the convergence is still possible. We will address the case of higher dimensions in a forthcoming paper.

In the following, we use the standard notations for Sobolev spaces $H^k(D)$, $H_0^k(D)$, $L^2(D) = H^0(D)$ together with their norms and semi-norms $\|\cdot\|_{k,D}$, $|\cdot|_{k,D}$, and $\|\cdot\|_{0,D}$. We will drop D when $D = (0,1)$. Throughout this paper C denotes a generic positive constant that is independent of the mesh size.

2. Two Variants of Local Projection Stabilization

We consider the two-point boundary value problem

$$-\varepsilon u'' + bu' + cu = f \quad \text{in } (0,1), \qquad u(0) = u(1) = 0, \tag{1}$$

under the assumption

$$c - \frac{1}{2}b' \geq \gamma > 0, \tag{2}$$

which guarantees a unique weak solution $u \in H_0^1(0,1)$. Note that in the interesting case $0 < \varepsilon \ll 1$, the solution exhibits boundary and interior layers whose positions depend on the convection field b.

Let $0 = x_0 < x_1 < \cdots < x_N = 1$ be a decomposition $\mathcal{M}_h$ of $[0,1]$ into macro cells $M \in \mathcal{M}_h$ and h_M the diameter of $M \in \mathcal{M}_h$. In the one-level approach we set $\mathcal{T}_h = \mathcal{M}_h$, i.e. we do not distinguish between a macro cell $M \in \mathcal{M}_h$ and a cell $K \in \mathcal{T}_h$. In the two-level approach, each macro cell $M = [x_i, x_{i+1}]$ is subdivided into two son-cells $K_- = [x_i, x_{i+1/2}]$ and $K_+ = [x_{i+1/2}, x_{i+1}]$ each of diameter $h_K = h_M/2$. Then, all son-cells build the decomposition $\mathcal{T}_h$. Let $V_h \subset H_0^1(0,1)$ be a finite element space living on $\mathcal{T}_h$, D_h be a discontinuous projection space associated with the decomposition $\mathcal{M}_h$, $\pi_h : L^2(0,1) \to D_h$ be the L^2 projection, and $\kappa_h := id - \pi_h$ be a fluctuation operator. The stabilized discrete problem is:

Find $u_h \in V_h$ such that for all $v_h \in V_h$

$$\varepsilon(u_h', v_h') + (bu_h' + cu_h, v_h) + \sum_{M \in \mathcal{M}_h} \tau_M(\kappa_h(bu_h'), \kappa_h(bv_h'))_M = (f, v_h).$$

Herein, $(\cdot,\cdot)$ and $(\cdot,\cdot)_M$ denote the inner product in $L^2(0,1)$ and $L^2(M)$, respecticvely and τ_M is a user-chosen stabilization parameter. The bilinear form associated with the left-hand side is coercive with respect to the mesh-dependent norm

$$|||v||| := \left(\varepsilon |v|_1^2 + \gamma \|v\|_0^2 + \sum_{M \in \mathcal{M}_h} \tau_M \|\kappa_h(bv')\|_{0,M}^2 \right)^{1/2}.$$

In contrast to residual based stabilizations, a consistency error appears whose order depends on chosing τ_M and the projection space D_h.

The key idea in the analysis of the LPS lies in the existence of a special interpolant $j_h : H_0^1(0,1) \to V_h$ that displays the usual interpolation properties and satisfies in addition the orthogonality property

$$(w - j_h w, q_h) = 0 \qquad \forall w \in H_0^1(0,1),\ \forall q_h \in D_h.$$

Using the coercivity of the underlying bilinear form, a rich enough projection space D_h, and the properties of the interpolant j_h, we end up with the error estimate

$$|||u - u_h||| \le C\,(\varepsilon^{1/2} + h^{1/2})\, h^r\, |u|_{r+1} \tag{3}$$

for $\tau_M \sim h_M$ [1, 3, 16, 17]. The existence of an interpolation j_h with additional orthogonality properties is guaranteed by the following result (adjusted to 1d):

Theorem 2.1 ([17])**.** *Let the local inf-sup condition*

$$\inf_{q_h \in D_h(M)} \sup_{v_h \in Y_h(M)} \frac{(v_h, q_h)_M}{\|v_h\|_{0,M}\, \|q_h\|_{0,M}} \ge \beta_1 > 0, \qquad \forall M \in \mathcal{M}_h \tag{4}$$

with $Y_h(M) = \{w_h|_M : w_h \in V_h, w_h = 0 \text{ on } (0,1) \setminus M\}$ *and* $D_h(M) = \{r_h|_M : r_h \in D_h\}$ *be satisfied. Then there is an interpolation* $j_h : H_0^1(0,1) \to V_h$ *with the orthogonality property*

$$(w - j_h w, q_h) = 0, \quad \forall q_h \in D_h,\ \forall w \in H_0^1(0,1)$$

and the usual interpolation error estimates.

In order to fulfil all assumptions of the convergence analysis, two different requirements for the pair (V_h, D_h) have to be reconciled:

- D_h has to be rich enough to guarantee a certain order of consistency,
- D_h should be small enough w.r.t. V_h to guarantee $j_h u - u \perp D_h$.

From this two different approaches can be derived:

$$\text{one-level } (V_h^+, D_h) \quad \Leftrightarrow \quad \text{two-level } (V_h, D_{2h}).$$

In the one-level approach, a standard finite element space is chosen as the projection space D_h to guarantee the consistency order. Then, the approximation space V_h is (if necessary) enriched to V_h^+ such that the assumptions of Theorem 2.1 are fulfilled. In the two-level approach, a standard finite element space is chosen as the approximation space V_h and the projection space D_h is thinned out to a space D_{2h} on the next coarser mesh level.

3. Two-level LPS as an Enriched One-level LPS

3.1. Discrete Problem. For some $r \in \mathbb{N}$, let the solution and projection spaces of the two-level LPS be defined by

$$V_h := \left\{ v_h \in H_0^1(0,1) : v_h\big|_K \in P_r(K) \quad \forall K \in \mathcal{T}_h \right\},$$
$$D_{2h} := \left\{ q_{2h} \in L^2(0,1) : q_{2h}\big|_M \in P_{r-1}(M) \quad \forall M \in \mathcal{M}_h \right\}.$$

Our stabilized two-level method is:

Find $u_h \in V_h$ such that for all $v_h \in V_h$

$$a(u_h, v_h) + S(u_h, v_h) = (f, v_h) \tag{5}$$

where the bilinear form a and the stabilizing term S are given by

$$a(u_h, v_h) = \varepsilon(u_h', v_h') + (bu_h' + cu_h, v_h),$$
$$S(u_h, v_h) = \sum_{M \in \mathcal{M}_h} \tau_M \left(\kappa_{2h}(bu_h'), \kappa_{2h}(bv_h')\right)_M.$$

Herein, $\pi_{2h} : L^2(0,1) \to D_{2h}$ denotes the L^2 projection and $\kappa_{2h} := id - \pi_{2h}$ the fluctuation operator. Since the pair (V_h, D_{2h}) of spaces satisfies the assumption of Theorem 2.1 we conclude the error estimate

$$|||u - u_h||| \le C(\varepsilon^{1/2} + h^{1/2})h^r |u|_{r+1} \tag{6}$$

for the solution u_h of (5) in the mesh-dependent norm

$$|||v||| := \left(\varepsilon |v|_1^2 + \gamma \|v\|_0^2 + \sum_{M \in \mathcal{M}_h} \tau_M \|\kappa_{2h}(bv')\|_{0,M}^2 \right)^{1/2}.$$

As already mentioned, the keypoint in the error analysis is the existence of an interpolation satisfying an additional orthogonality property. We will see that such an interpolation can be already constructed for a subspace of V_h which allows to create a method with less degrees of freedom than the two-level method but with the same convergence rate. We will construct such a subspace and the associated interpolation on the reference macro in the next subsection.

3.2. Splitting of the Approximation Space. Let $\widehat{M} = (-1, +1)$ be the reference macro, $\widehat{K}_- = (-1, 0)$, $\widehat{K}_+ = (0, +1)$, and $F_M : \widehat{M} \to M$ the affine mapping of $\widehat{M}$ onto $M \in \mathcal{M}_h$. We define the spaces

$$\widehat{P}_{r,h} = \left\{ \hat{v} \in H^1(\widehat{M}) \ : \ \hat{v}|_{\widehat{K}_-} \in P_r(\widehat{K}_-),\, \hat{v}|_{\widehat{K}_+} \in P_r(\widehat{K}_+) \right\}, \qquad \widehat{P}_{r,2h} = P_r(\widehat{M}),$$

where $\dim \widehat{P}_{r,h} = 2r+1$ and $\dim \widehat{P}_{r,2h} = r+1$. Consider the set of nodal functionals

$$N_i(\hat{v}) = \int_{-1}^{+1} \hat{v}(\xi) L_i(\xi)\, d\xi, \quad i = 0, 1, \ldots, r-1,$$

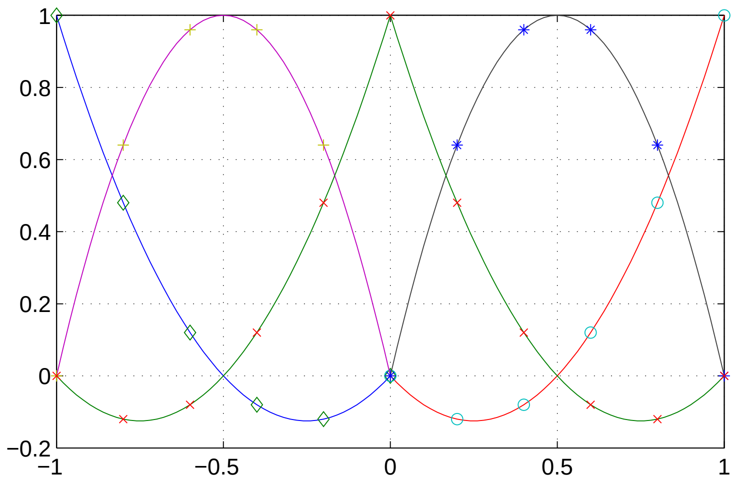

FIGURE 2. Standard nodal basis of $P_{2,h}$ on one macro element.

$$N_r(\hat{v}) = \hat{v}(-1), \qquad N_{r+1}(\hat{v}) = \hat{v}(+1)$$

where L_i, $i = 0, 1, \ldots,$ denote the Legendre polynoms of degree i on $(-1, +1)$ normalized such that $L_i(1) = 1$. The first r nodal functionals guarantee that a local interpolation $\widehat{J} : H^1(\widehat{M}) \to \widehat{P}_{r,h}$, defined by $N_i(\hat{v} - \widehat{J}\hat{v}) = 0$ for $i = 0, \ldots, r+1$, satisfies the orthogonality property

$$(\hat{v} - \widehat{J}\hat{v}, q)_{\widehat{M}} = 0, \qquad \forall q \in P_{r-1}(\widehat{M}),\ \hat{v} \in H^1(\widehat{M}).$$

The last two nodal functionals secure that the interpolation can be extended to a global continuous interpolation $j_h : H_0^1(0,1) \to V_h$ with the desired properties. However, because of dim $\widehat{P}_{r,2h} < r + 2$ we cannot hope to find an interpolation $\widehat{J} : H^1(\widehat{M}) \to \widehat{P}_{r,2h}$ into the coarse space $\widehat{P}_{r,2h}$ satisfying all $r + 2$ conditions. We will show that a suitable enrichment of $\widehat{P}_{r,2h}$ by just one additional function is enough to meet these requirements. This additional function is uniquely determined if it is orthogonal (with respect to a certain inner product) to $\widehat{P}_{r,2h}$. Let us consider the following functions

$$\hat{\varphi}_r(x) = \begin{cases} \Lambda_r(x) + \Lambda_{r-1}(x) & x \in [-1, 0] \\ \Lambda_r(-x) + \Lambda_{r-1}(-x) & x \in [0, +1] \end{cases} \qquad r \text{ odd},$$

$$\hat{\varphi}_r(x) = \begin{cases} \Lambda_r(x) - \Lambda_{r-2}(x) & x \in [-1, 0] \\ -(\Lambda_r((-x) - \Lambda_{r-2}(-x)) & x \in [0, +1] \end{cases} \qquad r \text{ even},$$

where Λ_r denotes the Legendre polynom of degree r on $(-1, 0)$ given by

$$\Lambda_r(x) = L_r(2x + 1) \quad x \in (-1, 0)$$

Furthermore, we introduce the linear mapping $\Phi : \widehat{P}_{r,h} \to \mathbb{R}^{r+2}$ given by

$$\Phi(\hat{v}) = (N_0(\hat{v}), \ldots, N_{r+1}(\hat{v})).$$

Lemma 3.1. *There is the unique splitting*

$$\widehat{P}_{r,h} = \widehat{P}_{r,2h} \oplus span\ (\hat{\varphi}_r) \oplus \ker(\Phi). \tag{7}$$

The set of nodal functionals $N_0, \dots, N_{r+1}$ *is* $\widehat{P}^+_{r,2h}$*-unisolvent, where the enriched space is given by* $\widehat{P}^+_{r,2h} = \widehat{P}_{r,2h} \oplus span\ (\hat{\varphi}_r)$*. Furthermore, the orthogonality property*

$$(\hat{\varphi}'_r, \hat{v})_{\widehat{M}} = 0 \qquad \text{for all } \hat{v} \in \widehat{P}_{r-1,2h}$$

holds true.

Proof. From the definition of the linear mapping Φ we have

$$\ker(\Phi) = \{\hat{v} \in \widehat{P}_{r,h} \ : \ N_i(\hat{v}) = 0,\ i = 0, \dots, r+1\}$$

which implies $\dim\ \ker(\Phi) \geq (2r+1) - (r+2) = r - 1$.
Now, $\widehat{P}_{r,2h} \cap \ker(\Phi) = \emptyset$, since any function $\hat{v} \in \widehat{P}_{r,2h}$ can be represented as

$$\hat{v} = \sum_{i=0}^{r} \alpha_i L_i,$$

the orthogonality of the Legendre polynoms yields $\alpha_0 = \cdots = \alpha_{r-1} = 0$, and finally we get $0 = N_{r+1}(\hat{v}) = \alpha_r$. Thus, we have $\dim\ \ker(\Phi) \leq r$.
We prove that $\dim\ \ker(\Phi) = r-1$ by showing that $\hat{\varphi}_r \notin \widehat{P}_{r,2h}$ and $\widehat{P}^+_{r,2h} \cap \ker \Phi = \emptyset$. Then, the unique splitting follows immediately. From Rodriguez' formula [6] we get the expansion with respect to powers of x

$$\Lambda_r(x) = L_r(2x+1) = \frac{(2r)!}{2^r (r!)^2}(2x+1)^r + \cdots = \frac{(2r)!}{(r!)^2} x^r + \dots,$$

showing that $\hat{\varphi}_r^{(r)}(-0) \neq \hat{\varphi}_r^{(r)}(+0)$ and thus $\hat{\varphi}_r$ cannot be a polynomial on $[-1, +1]$, i.e. $\hat{\varphi}_r \notin \widehat{P}_{r,2h}$. Now we show that $\widehat{P}^+_{r,2h} \cap \ker \Phi = \emptyset$. This is equivalent to the fact that the set of nodal functionals is $\widehat{P}^+_{r,2h}$-unisolvent. Since $\dim \widehat{P}^+_{r,2h} = r + 2$ and $r + 2$ nodal functionals are given, it suffices to show that

$$\hat{v} \in \widehat{P}^+_{r,2h}, \qquad N_i(\hat{v}) = 0, \quad i = 0, \dots, r+1, \quad \Rightarrow \quad \hat{v} = 0.$$

We represent an arbitrary element of $\widehat{P}^+_{r,2h}$ as

$$\hat{v} = \sum_{i=0}^{r} \alpha_i L_i + \beta \hat{\varphi}_r.$$

Consider the case r odd first. The orthogonality property of Λ_r and Λ_{r-1} on $[-1, 0]$ and of L_k on $[-1, +1]$ imply for $k = 0, 1, \dots, r-2$

$$0 = N_k(\hat{v}) = \alpha_k + \beta \left(\int_{-1}^{0} \hat{\varphi}_r(x) L_k(x)\, dx + \int_{0}^{+1} \hat{\varphi}_r(x) L_k(x)\, dx \right) = \alpha_k.$$

Now, from $\hat{\varphi}_r(\pm 1) = 0$ we conclude

$$0 = N_r(\hat{v}) = (-1)^{r-1}(\alpha_{r-1} - \alpha_r)$$
$$0 = N_{r+1}(\hat{v}) = \alpha_{r-1} + \alpha_r,$$

consequently $\alpha_{r-1} = \alpha_r = 0$. From the orthogonality properties of Λ_k on $[-1, 0]$ we obtain for r odd

$$\begin{aligned} N_{r-1}(\hat{\varphi}_r) &= \int_{-1}^{0} \Lambda_{r-1}(x) L_{r-1}(x)\, dx + \int_{0}^{+1} \Lambda_{r-1}(-x) L_{r-1}(x)\, dx \\ &= \int_{-1}^{0} \Lambda_{r-1}(x) L_{r-1}(x)\, dx + \int_{-1}^{0} \Lambda_{r-1}(x) (-1)^{r-1} L_{r-1}(x)\, dx \\ &= 2 \int_{-1}^{0} \Lambda_{r-1}(x) L_{r-1}(x)\, dx \neq 0. \end{aligned}$$

As $N_{r-1}(\hat{\varphi}_r) \neq 0$ but $N_{r-1}(\hat{v}) = \beta N_{r-1}(\hat{\varphi}_r) = 0$ we get $\beta = 0$.
Consider now the case r even. Again, from orthogonality properties of Λ_r and Λ_{r-1} on $[-1,0]$ and of L_k on $[-1,+1]$ we get for $k = 0, 1, \ldots, r-3$

$$0 = N_k(\hat{v}) = \alpha_k + \beta \left(\int_{-1}^{0} \hat{\varphi}_r(x) L_k(x)\,dx + \int_{0}^{+1} \hat{\varphi}_r(x) L_k(x)\,dx \right) = \alpha_k.$$

Orthogonality of Λ_r to L_{r-2} on $[-1,0]$ yields

$$\begin{aligned} N_{r-2}(\hat{\varphi}_r) &= -\int_{-1}^{0} L_{r-2}(x)\Lambda_{r-2}(x)\,dx + \int_{0}^{+1} L_{r-2}(x)\Lambda_{r-2}(-x)\,dx \\ &= -\int_{-1}^{0} L_{r-2}(x)\Lambda_{r-2}(x)\,dx + \int_{-1}^{0} (-1)^{r-2} L_{r-2}(x)\Lambda_{r-2}(x)\,dx = 0. \end{aligned}$$

Thus, $N_{r-2}(\hat{v}) = 0$ implies $\alpha_{r-2} = 0$. From $\hat{\varphi}_r(\pm 1) = 0$ we get

$$\begin{aligned} 0 &= N_r(\hat{v}) = (-1)^{r-1}(\alpha_{r-1} - \alpha_r) \\ 0 &= N_{r+1}(\hat{v}) = \alpha_{r-1} + \alpha_r, \end{aligned}$$

i.e. $\alpha_{r-1} = \alpha_r = 0$. A similar computation as above but now for even r shows

$$\begin{aligned} N_{r-1}(\hat{\varphi}_r) &= -\int_{-1}^{0} \Lambda_{r-2}(x) L_{r-1}(x)\,dx + \int_{0}^{+1} \Lambda_{r-2}(-x) L_{r-1}(x)\,dx \\ &= -2\int_{-1}^{0} L_{r-2}(2x+1) L_{r-1}(x)\,dx = -\int_{-1}^{+1} L_{r-2}(t) L_{r-1}\left(\frac{t-1}{2}\right) dt \\ &= -\frac{(2r-2)!}{2^{r-1}((r-1)!)^2} \int_{-1}^{+1} \left[\left(\frac{t}{2}\right)^{r-1} - \frac{r-1}{2}\left(\frac{t}{2}\right)^{r-2} \right] L_{r-2}(t)\,dt \neq 0. \end{aligned}$$

Here, we used the orthogonality properties of the Legendre polynomial Λ_r on $[-1,0]$ and L_{r-2} on $[-1,+1]$, respectively. Now $\beta = 0$ follows from $N_{r-1}(\hat{\varphi}_r) \neq 0$.

It remains to show the orthogonality property. Since $\hat{\varphi}_r \in H_0^1(\widehat{M})$, we obtain by integration by parts

$$(\hat{\varphi}_r', \hat{v})_{\widehat{M}} = -(\hat{\varphi}_r, \hat{v}')_{\widehat{M}}.$$

For r odd, the right hand side vanishes due to $\hat{v}' \in \widehat{P}_{r-2,2h}$ and the orthogonality of Λ_r and Λ_{r-1} to $\widehat{P}_{r-2,2h}$ on $[-1,0]$, respectively. For even r, we have

$$\begin{aligned} (\hat{\varphi}_r', \hat{v})_{\widehat{M}} = -(\hat{\varphi}_r, \hat{v}')_{\widehat{M}} &= \int_{-1}^{0} \Lambda_{r-2}(x)\hat{v}'(x)\,dx - \int_{0}^{+1} \Lambda_{r-2}(-x)\hat{v}'(x)\,dx \\ &= A\int_{-1}^{0} \Lambda_{r-2}(x)\,x^{r-2}\,dx + A\int_{0}^{-1} \Lambda_{r-2}(x)\,x^{r-2}\,dx = 0. \end{aligned}$$

where we assumed an expansion in the form $\hat{v}'(x) = Ax^{r-2} + \ldots$. □

For the case $r = 2$ the kernel of Φ is represented by the function

$$\hat{w}(x) = \begin{cases} \frac{3}{2}(1+x)(1+3x) & \text{if } x \in [-1,0], \\ \frac{3}{2}(1-x)(1-3x) & \text{if } x \in [0,+1], \end{cases}$$

and the additional function $\hat{\varphi}_2$ for enriching $\widehat{P}_{2,2h}$ is

$$\hat{\varphi}_2(x) = \begin{cases} 6x(1+x) & \text{if } x \in [-1,0], \\ 6x(1-x) & \text{if } x \in [0,+1]. \end{cases}$$

For $\widehat{P}_{2,h}$ the standard nodal basis and a basis corresponding to the splitting of Lemma 3.1 are shown in Figures 2 and 3 .

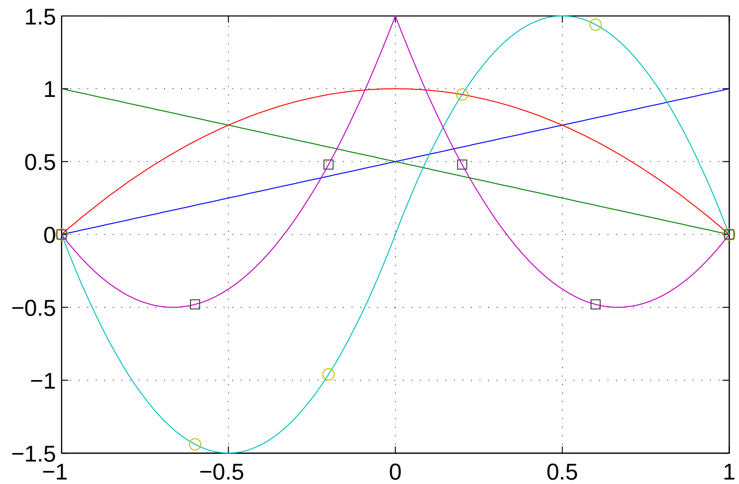

FIGURE 3. The splitting $\widehat{P}_{2,h} = \widehat{P}_{2,2h} \oplus \mathrm{span}(\hat{\varphi}_2) \oplus \ker\Phi$. The circles and squares indicate the functions $\hat{\varphi}_2$ and $\hat{w} \in \ker\Phi$.

Remark 3.1. *We could replace $\hat{\varphi}_r$ by any function from $\hat{\varphi}_r + \widehat{P}_{r,2h}$ leading to the same enriched space $\widehat{P}^+_{r,2h}$. However, the $L^2(\widehat{M})$-orthogonality of $\hat{\varphi}'_r$ to $\widehat{P}_{r-1,2h}(\widehat{M})$ defines the function to be added to $\widehat{P}_{r,2h}(\widehat{M})$ uniquely.*

3.3. Reduced Two-level Approach. The results of the previous section motivate us to consider a subspace of V_h for the approximation space combined with the unchanged projection space D_{2h}. Since there is only one function per macro cell added to the same space on the next coarser mesh we consider this type of reduced two-level approach as a one-level approach on $\mathcal{M}_h$ with enrichments of piecewise polynomial functions. Thus, we define the approximation and projection space by

$$V^+_{2h} := \left\{ v_h \in H^1_0(0,1) : v_h\big|_M \in P^+_{r,2h}(M) \quad \forall M \in \mathcal{M}_h \right\},$$
$$D_{2h} := \left\{ q_{2h} \in L^2(0,1) : q_{2h}\big|_M \in P_{r-1}(M) \quad \forall M \in \mathcal{M}_h \right\}$$

where $P^+_{r,2h}(M)$ is just the mapped finite element space $\widehat{P}^+_{r,2h}$ introduced in Lemma 3.1. Our reduced two-level discretization is

Find $u^+_{2h} \in V^+_{2h}$ such that for all $v^+_{2h} \in V^+_{2h}$

$$a(u^+_{2h}, v^+_{2h}) + S(u^+_{2h}, v^+_{2h}) = (f, v^+_{2h}). \tag{8}$$

Essential for the error estimation of a solution u^+_{2h} of (8) is the existence of the special interpolation into the reduced finite element space $V^+_{2h} \subset V_h$.

Lemma 3.2. *There is an interpolation operator $j_h : H^1_0(0,1) \to V^+_{2h}$ such that*

$$(j_h w - w, q_{2h}) = 0 \qquad \forall q_{2h} \in D_{2h},\ w \in H^1_0(0,1) \tag{9}$$

$$|j_h w - w|_{m,M} \le C\, h_M^{\ell+1-m}\, \|w\|_{\ell+1,M} \qquad \forall w \in H^{\ell+1}(M),\ M \in \mathcal{M}_h \tag{10}$$

for $\ell = 0,\ldots,r$, $m = 0,1$.

Proof. On each $M = (x_i, x_{i+1})$ a local interpolant $j_h^M w \in P_{r,2h}(M)$ is uniquely defined by the $r+2$ conditions

$$j_h^M w(x_i) = w(x_i), \quad j_h^M w(x_{i+1}) = w(x_{i+1}), \quad (j_h^M w - w, q)_M = 0, \quad \forall q \in P_{r-1}(M)$$

due to Lemma 3.1. The global interpolant $j_h w$ defined by

$$j_h w|_M = j_h^M (w|_M) \qquad \forall M \in \mathcal{M}_h$$

belongs to V_{2h}^+ by construction. Since $j_h^M w = w$ for all $w \in P_{r,2h}(M)$, we obtain (10) by means of the Bramble-Hilbert-Lemma. Using $q_{2h}|_M \in P_{r-1}(M)$, we have $(j_h^M w - w, q_{2h})_M = 0$ for all $M \in \mathcal{M}_h$ from which (9) follows by summation. □

Theorem 3.3. *Let u be the weak solution of* (1) *and u_{2h}^+ the the solution of the reduced two-level method* (8) *for $\tau_M \sim h_M$, respectively. Then, the error estimate*

$$|||u - u_{2h}^+||| \le C \left(\sum_{M \in \mathcal{M}_h} (\varepsilon + h_M)\, h_M^{2r} \|u\|_{r+1,M}^2 \right)^{1/2}$$

holds provided that $u \in H_0^1(0,1) \cap H^{r+1}(0,1)$.

Proof. The proof is analog to that given in [24, Theorem 2]. □

3.4. Elimination of Enrichments. In the following, we consider the reduced two-level approach (V_{2h}^+, D_{2h}) in the special case that $b = \mathit{const}$, $c = 0$, and f piecewise $P_{r-1,2h}$. As in [24] we want to eliminate the enrichments locally. However, in contrast to the one-level approach, we have to deal with enrichments by piecewise polynomial functions

$$\varphi_{r,M}(x) := \hat{\varphi}_r \left(\frac{2x - x_i - x_{i+1}}{x_{i+1} - x_i} \right), \qquad x \in M = [x_i, x_{i+1}],$$

which need more care compared to polynomials.

Clearly, we have $\varphi_{r,M}|_M \in H_0^1(M)$. We split the approximation space into the direct sum

$$V_{2h}^+ = V_{2h} \oplus B_h, \qquad B_h = \bigoplus_{M \in \mathcal{M}_h} \operatorname{span} \varphi_{r,M},$$

$$V_{2h} := \left\{ v_{2h} \in H_0^1(0,1) : v_{2h}\big|_M \in P_r(M) \quad \forall M \in \mathcal{M}_h \right\}.$$

The direct sum $V_{2h}^+ = V_{2h} \oplus B_h$ generates a unique splitting of the solution $u_{2h}^+ \in V_{2h}^+$ of the local projection stabilization in the same way

$$u_{2h}^+ = u_{2h} + \sum_{M \in \mathcal{M}_h} u_M \varphi_{r,M}.$$

Our aim is to reformulate the reduced two-level scheme in terms of this splitting. For this we consider first some terms appearing in the LPS approach. Taking into consideration that for any $v_{2h} \in V_{2h}$ we have $v_{2h}'|_M \in P_{r-1}(M)$, the L_2 projection becomes $\pi_{2h} v_{2h}' = v_{2h}'$ thus the fluctuation $\kappa_{2h}(v_{2h}')$ vanishes. Furthermore, we obtain

$$(b\, \varphi_{r,M}', \varphi_{r,M}) = \frac{1}{2}(b, (\varphi_{r,M}^2)') = -\frac{1}{2}(b', \varphi_{r,M}^2) = 0.$$

Now, the reduced two-level discretization (8) can be rewritten as

Find $u_{2h} \in V_{2h}$ and $\{u_M\} \in \mathbb{R}^N$ such that for all $v_{2h} \in V_{2h}$, $M \in \mathcal{M}_h$

$$\begin{aligned} &\varepsilon(u_{2h}', v_{2h}') + (b u_{2h}', v_{2h}) + \sum_{M \in \mathcal{M}_h} u_M (b\, \varphi_{r,M}', v_{2h}) = (f, v_{2h}), \\ &u_M \left[\varepsilon(\varphi_{r,M}', \varphi_{r,M}') + \tau_M(\kappa_{2h}(b\varphi_{r,M}'), \kappa_{2h}(b\varphi_{r,M}')) \right] = (f - b u_{2h}', \varphi_{r,M}). \end{aligned} \tag{11}$$

Using the orthogonality $\varphi'_{r,M} \perp P_{r-1}(M)$ stated in Lemma 3.1, we can see that

$$(\pi_{2h}\varphi'_{r,M}, q)_M = (\varphi'_{r,M}, q)_M = 0 \qquad \forall q \in P_{r-1}(M),$$

thus the projection $\pi_{2h}\varphi'_{r,M}$ vanishes and $\kappa_{2h}\varphi'_{r,M} = \varphi'_{r,M}$. Then, the second set of equations of the reduced two-level discretization (11) reads

$$u_M\left[\varepsilon + \tau_M b^2\right](\varphi'_{r,M}, \varphi'_{r,M}) = (f - bu'_{2h}, \varphi_{r,M}) \qquad \forall M \in \mathcal{M}_h. \tag{12}$$

Next we intend to apply following Lemma to the right hand side of (12).

Lemma 3.4. *There is a* $\psi_M \in H^r(M)$ *with* $\psi_M|_{K_\pm} \in P_{2r-1}(K_\pm)$ *satisfying*

$$\psi_M^{(r-1)} = \varphi_{r,M} \quad \text{in } M, \qquad \psi_M^{(j)} = 0 \quad \text{on } \partial M,\ j = 0, 1, \ldots, r-1.$$

Proof. We show that on the reference macro $\widehat{M} = [-1, +1]$ there is a function $\hat{\psi} \in H^r(\widehat{M})$ with

$$\hat{\psi}^{(r-1)} = \hat{\varphi}_r \quad \text{in } \widehat{M}, \qquad \hat{\psi}^{(j)}(\pm 1) = 0, \quad j = 0, \ldots, r-1. \tag{13}$$

Thanks to the formula of Rodriguez we derive an explicit representation. Consider the case of odd r first. Then, a careful check shows that

$$\hat{\psi}(x) = \begin{cases} \dfrac{1}{r!}\dfrac{d}{dx}\left[x(1+x)\right]^r + \dfrac{1}{(r-1)!}\left[x(1+x)\right]^{r-1} & x \in [-1, 0], \\ \dfrac{1}{r!}\dfrac{d}{dx}\left[x(1-x)\right]^r + \dfrac{1}{(r-1)!}\left[x(1-x)\right]^{r-1} & x \in [0, +1] \end{cases}$$

satisfies all requirements of the Lemma. Similarly, for even r we derive a formula starting with

$$\hat{\psi}^{(r)}(x) = \hat{v}'_r(x) = 2(2r-1)\begin{cases} L_{r-1}(1+2x) & x \in [-1, 0], \\ L_{r-1}(1-2x) & x \in [0, +1], \end{cases}$$

integrating this representation $(r-1)$-times, and using the conditions at $x = -1$. As a result we obtain

$$\hat{\psi}'(x) = \frac{2(2r-1)}{(r-1)!}\begin{cases} \left[x(1+x)\right]^{r-1} & x \in [-1, 0], \\ \left[x(1-x)\right]^{r-1} & x \in [0, +1]. \end{cases}$$

Taking into consideration that $\hat{\psi}'(-t) = -\hat{\psi}'(t)$ and setting

$$\hat{\psi}(x) := \int_{-1}^{x} \hat{\psi}'(t)\, dt$$

we have $\hat{\psi}$ satisfying (13).

Using the transformation $F_M : \widehat{M} \to M$, we see that the function

$$\psi_M(x) = \left(\frac{h_M}{2}\right)^{r-1} \hat{\psi}\left(\frac{2x - x_i - x_{i+1}}{x_{i+1} - x_i}\right)$$

satisfies all requirements of the Lemma. □

Now, using the fact that $f - bu'_{2h}$ is a piecewise polynomial function of degree less than or equal to $r-1$ with respect to $\mathcal{M}_h$, we can integrate by parts

$$(f - bu'_{2h}, \varphi_{r,M}) = (f - bu'_{2h}, \psi_M^{(r-1)}) = (-1)^{r-1}((f - bu'_{2h})^{(r-1)}, \psi_M)$$

and obtain from (12) the representation

$$u_M = (f - bu'_{2h})^{(r-1)}|_M \frac{(-1)^{r-1}(1, \psi_M)}{(\varepsilon + \tau_K b^2)|\varphi_{r,M}|^2_{1,M}}.$$

Using this to eliminate u_M in (11) we end up with the following method:

Find $u_{2h} \in V_{2h}$ such that for all $v_{2h} \in V_{2h}$

$$
\begin{aligned}
&\varepsilon(u'_{2h}, v'_{2h}) + (bu'_{2h}, v_{2h}) + \sum_{M\in\mathcal{M}_h} \gamma_M((bu'_{2h})^{(r-1)}, (bv'_{2h})^{(r-1)})_M \\
&\qquad = (f, v_{2h}) + \sum_{M\in\mathcal{M}_h} \gamma_M(f^{(r-1)}, (bv'_{2h})^{(r-1)})_M
\end{aligned} \tag{14}
$$

where the parameter γ_M is related to the parameter τ_M of the LPS in the following way

$$
\gamma_M = \frac{(1, \psi_M)^2}{(\varepsilon + \tau_M b^2)\, h_M\, |\varphi_{r,M}|^2_{1,M}}\,. \tag{15}
$$

For the considered case $b = const$, $c = 0$, and f piecewise polynomial of degree $r-1$, the method (14) is identical with the differentiated residual method (DRM) which has been studied already in [23, 24].

Theorem 3.5. *Assume $b = const$, $c = 0$, and f piecewise $P_{r-1,2h}$ and let the approximation space $V_{2h} = P_{r,2h} \cap H^1_0(0,1)$ on each cell $M \in \mathcal{M}_h$ be locally enriched by $\varphi_{r,M}$. Then, eliminating the enrichment in the reduced two-level LPS results in the $P_{r,2h}$-DRM. The associated stabilization parameter τ_M and γ_M are related by* (15).

Remark. It is well-known that by an appropriately chosen stabilization parameter γ_M nodal exactness of the piecewise linear part of the solution can be achieved for the DRM in the case $b = const$, $c = 0$, $f = const$. Therefore, the formula (15) allows to choose the stabilization parameters τ_M in the reduced two-level LPS is such a way that the piecewise linear part of the $P^+_{r,2h}$ solution is nodal exact. For this we compute

$$
(1, \psi_M) = \begin{cases} \dfrac{2(r-1)!}{(2r-1)!}\left(\dfrac{h_M}{2}\right)^r & r \text{ odd} \\[2ex] -\dfrac{(r-2)!}{(2r-3)!}\left(\dfrac{h_M}{2}\right)^r & r \text{ even} \end{cases}, \qquad |\varphi_{r,M}|^2_{1,M} = \begin{cases} \dfrac{16r^2}{h_M} & r \text{ odd} \\[2ex] \dfrac{16(2r-1)}{h_M} & r \text{ even} \end{cases}
$$

and use the formulas defining $\gamma_{r,M}$ from [24]:

$$
\tau_{r,M} = \frac{h_M}{\alpha_r b \Phi_r(q_M)} - \frac{\varepsilon}{b^2}, \quad q_M = \frac{bh_M}{2\varepsilon}, \quad \alpha_r = \begin{cases} 2^{2r+1} r^2 & r \text{ odd}, \\ 2^{2r+1}/(2r-1) & r \text{ even}, \end{cases}
$$

$$
\Phi_{r+1}(q) = \frac{1}{\Phi_r(q)} - \frac{2r+1}{q}, \quad \Phi_1(q) = \coth q - \frac{1}{q}\,.
$$

4. Numerical Examples

Example 1. We choose $b = 1$, $c = 0$, $f = 1$, and $\varepsilon = 10^{-7}$. Apart from an exponential layer near $x = 1$ the solution can be approximated by $u_{asymp}(x) = x$. The stabilization parameter $\tau_{r,M}$ depends on the polynomial degree $r \geq 1$ and is chosen as in Section 3.4. In Figure 4 we present the results for piecewise quadratic approximations on a uniform macro mesh with $2h = 1/N$, $N = 20$ and $N = 40$. We clearly see the nodal exactness of the linear part (marked by stars) of the computed solution and that oscillations are restricted to the boundary layer region for the reduced two-level approach (left). For the classical two-level variant (right) we have taken also the formulas for $\tau_{r,M}$ given in Section 3.4 but with a modified (experimentally fitted) $\alpha_2 = 13.856$ instead of $\alpha_2 = 32/3$. Note that for this variant nodal exactness cannot be guaranteed. The localization of oscillations on

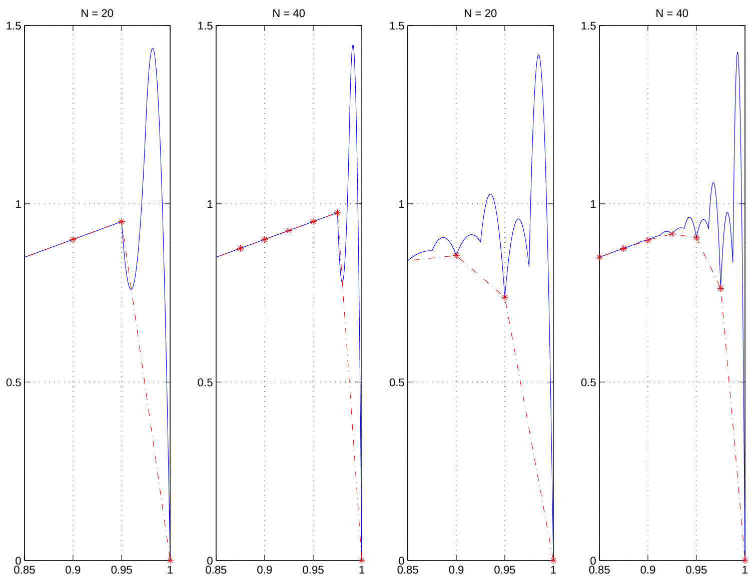

FIGURE 4. Piecewise quadratic two-level reduced (left) and two-level (right) method. The stars display the linear part of the solutions.

the boundary layer region seems to be less pronounced. For the case of piecewise cubic approximations both variants show a similar localization behaviour, however the amplitudes are much larger for the reduced two-level approach. Next we study the influence of layer adapted meshes. We computed the solution on a uniform mesh of $N = 10$ macro cells and subdivided the last element into two cells, and repeated this approach resulting into eleven and twelve macro cells, respectively. The first three diagrams of Figure 5 show the results for the piecewise cubic reduced two-level method. Oscillations are concentrated on the last macro cell and their amplitudes are almost constant. The fourth diagram shows the result on a Shishkin mesh with the same number of degrees of freedom $N = 12$. A Shishkin mesh is a piecewise uniform mesh with a transition point at $1-(3/2)\varepsilon \log N$, see [7, 8, 14, 15, 22]. Thus, the LPS is able to localize oscillations on the layer region but in order to suppress their amplitudes layer adapted meshes seem to be necessary.

Example 2. We choose $\Omega = (-1, +1)$, $b = -|x|$, $c = 1/2$, $f = 0$, $u(-1) = 1$, $u(1) = 2$. An exponential layer is located near $x = -1$, there is also an inner layer in the first derivative at $x = 0$. In Figure 6 we present the results for the two-level method for a piecewise cubic approximation on a macro mesh of $N = 80$ cells. The main characteristic of LPS, to localize oscillations on layer regions, holds true also in the case of non-constant coefficients. Moreover, no oscillation near the inner layer in the derivative are observed.

5. Conclusions

We have exposed the close relationship of the two variants of the local projection stabilization. In particular, the two-level method can be regarded as a one-level approach on the coarser mesh using piecewise polynomial enrichments. It turns out that from this point of view the two-level approach uses larger enrichments than needed to construct a certain interpolant which guarantees optimal order of

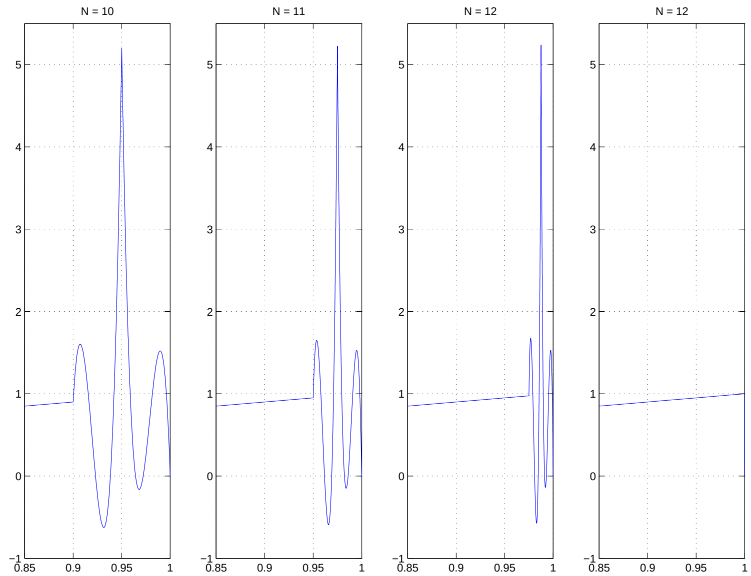

FIGURE 5. Piecewise cubic reduced two-level method on adapted refined meshes and on a Shishkin mesh (see the right most panel).

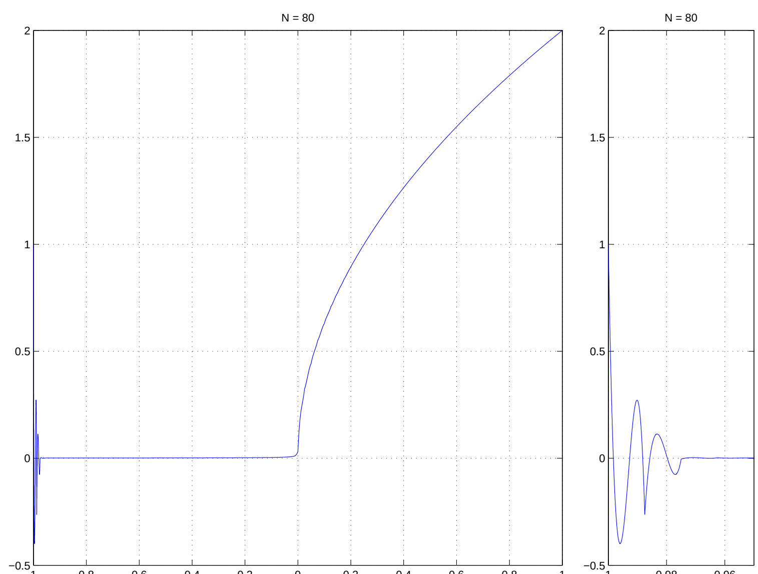

FIGURE 6. Piecewise cubic two-level method for a non-constant coefficient case and the close-up range of the layer region.

convergence. One additional degree of freedom per macro cell is sufficient which corresponds to the one-level approach studied in [24]. Eliminating the enrichments in the constant coefficient case (b, f constant, $c = 0$), we obtain the differentiated residual method [23] for which optimal choices of the stabilization parameters are known [24] leading to the nodal exactness of the linear part of the solution.

We think that these investigations provide some insight into the better understanding of the stabilizing properties of the popular local projection stabilization. The observation that the dimension of the enrichment space can be considerably reduced is not restricted to the one-dimensional case. It opens also in the multi-dimensional case the way to design reduced two-level methods which preserve the convergence order. We will address these questions in a forthcoming paper.

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