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ANALYSIS OF AN ALTERNATING DIRECTION METHOD APPLIED TO SINGULARLY PERTURBED REACTION-DIFFUSION PROBLEMS

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Abstract. We present an analysis of an *Alternating Direction Implicit* (ADI) scheme for a linear, singularly perturbed reaction-diffusion equation. By providing an expression for the error that separates the temporal and spatial components, we can use existing results for steady-state problems to give a succinct analysis for the time-dependent problem, and that generalizes for various layer-adapted meshes. We report the results of numerical experiments that support the theoretical findings. In addition, we provide a numerical comparison between the ADI and Euler techniques, as well details of the computational advantage gained by parallelizing the algorithm.

Key Words. reaction-diffusion problems, layer-adapted meshes, alternating directions, singular perturbation

1. Introduction

We consider the problem of computing a satisfactory numerical solution to a time-dependent singularly perturbed reaction-diffusion equation using an alternating direction finite difference method. The problem under consideration is

$$\partial_t u + \mathcal{L}u = f \quad \text{in} \quad Q := \Omega \times (0,T],\ \Omega = (0,1)^2, \tag{1a}$$

where $\mathcal{L}v := -\varepsilon^2 \Delta v + rv$, $f, r : \Omega \times (0,T] \to \mathbb{R}$, $r \geq \varrho^2$ on $\bar{\Omega} \times [0,T]$, $\varrho > 0$, subject to boundary and initial conditions

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0,T], \quad u(\cdot,0) = 0 \quad \text{on} \quad \bar{\Omega}. \tag{1b}$$

Solutions to (1) typically exhibit layers: narrow regions in which derivatives of the solution are large.

Miller et al. [14] give a numerical analysis of a time-dependent reaction-diffusion problem that is one-dimensional (in space), and show that the solution computed on a Shishkin mesh will converge at a rate that is almost second-order. A more general analysis based on Green's functions is in [11]. See also [17, II.3.4.3] and references provided there.

The analysis of numerical techniques for the two-dimensional steady-state analogue of (1) has received recent attention: for example, Clavero et al. [2] provide an analysis for a finite difference method on a piecewise uniform Shishkin mesh, while Kellogg et al. [8] have analysed a finite difference method for various fitted meshes used to solve coupled systems for steady reaction-diffusion problems. See also [17, Remark II.2.10] and references given there.

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The studies of two-dimensional problems mentioned above all use spatial tensor product meshes. When extending these techniques to time-dependent problems, it is very natural to consider *dimension splitting*: at each time-step, one alternately solves independent one dimensional problems in the x- and y-directions. This is because it is typically much more efficient to solve, say, N tridiagonal systems of N unknowns, than a banded system of N^2 unknowns. Furthermore, the opportunities for parallelization are easy to exploit.

Such Alternating Direction Implicit (ADI) scheme for classical problems are presented in detail in, e.g., [12, 13, 18, 21]. Of primary interest to this study is the work of Clavero et al. [5] who use an alternating directions technique to semidiscretize (1) in time first. A sequence of one-dimension problems is obtained which in turn are solved approximately by central differencing on a layer-adapted piecewise uniform mesh—a so-called Shishkin mesh. The resulting scheme is shown to be uniformly convergent with respect to the perturbation parameter ε with the maximum nodal error bounded by $C\left(\tau + N^{-1}\right)$, where τ is the maximal time-step size and N the number of mesh points used in each direction of the tensor product spatial mesh, and C is a constant that is independent of ε, τ and N.

In this paper we shall present an alternative error analysis that—in a certain sense—is simpler than that in [5]. In particular, when a Shishkin mesh is used, we show that the nodal error is bounded by $C\left(\tau + N^{-2}\ln^2 N + \varepsilon N^{-1}\right)$. Furthermore, our approach makes it possible to deduce error bounds for other fitted meshes. For example, we also show that if the graded mesh of Bakhvalov [1] is used, then the error may be bounded as $C\left(\tau + N^{-2}\right)$.

ADI schemes have also been applied to other singularly perturbed problems such as convection diffusion. For example in [4] the maximum-norm errors of a first-order ADI method combined with simple upwinding in space are shown to be bounded by $C\left(\tau + \tau^{-1}N^{-1}\ln N\right)$.

Outline. In §2 we describe the discretization of (1) both by the standard implicit Euler method and the ADI approach. Bounds on derivatives of the solution are summarized in §3, which is followed by a brief discussion of the mathematical approach of [5], and then our own numerical analysis of the technique. This leads to an expression of the error that depends on the analysis of the given method for a steady-state problem. Then in §3.5 we can derive error bounds for particular meshes.

The paper concludes with a report of detailed numerical experiments. For a specially constructed test problem where the true solution is known, we investigate separately the dependence of the error on the time and spatial discretization. We also compare the accuracies of the ADI and Euler techniques. The paper concludes by highlighting the speed-up that can be gained when the algorithm is implemented on a parallel computer.

Notation. Given arbitrary meshes $\omega_x : 0 = x_0 < x_1 < \cdots < x_N = 1$ and $\omega_y : 0 = y_0 < y_1 < \cdots < y_N = 1$ in the x- and y-directions, one may construct the tensor-product mesh $\omega := \omega_x \times \omega_y$. We write the mesh in the time variable as $\omega_t : 0 = t_0 < t_1 < \cdots < t_K = T$. The mesh intervals are denoted

$$h_i := x_i - x_{i-1}, \quad k_j := y_j - y_{j-1}, \quad \tau_n := t_n - t_{n-1}, \quad \tau := \max_{n=1,\ldots,K} \tau_n.$$

To simplify the notation we set $g^n_{i,j} := g(x_i, y_j, t_n)$ for any function $g \in C(\bar{Q})$. Similarly, $g^n := g(\cdot, \cdot, t_n)$.

In this paper, we are concerned with maximum nodal errors, so we use norms

$$\|g^n\|_\omega := \max_{(x,y)\in\omega} |g^n(x,y)| \quad \text{and} \quad \|g\|_{\omega_t\times\omega} := \max_{n=1,\dots,K} \|g^n\|_\omega.$$

For a matrix $A \in \mathbb{R}^{N\times N}$, the norm $\|A\|$ is the usual norm subordinate to the maximum (vector) norm on $\mathbb{R}^N$.

2. Discretization

Define the second-order central difference approximation in space

$$L^n v := -\varepsilon^2 \left(\delta_x^2 v + \delta_y^2 v\right) + r^n v$$

with

$$[\delta_x^2 v]_{i,j} := \frac{2}{h_i + h_{i+1}} \left(\frac{v_{i+1,j} - v_{i,j}}{h_{i+1}} - \frac{v_{i,j} - v_{i-1,j}}{h_i} \right),$$
$$[\delta_y^2 v]_{i,j} := \frac{2}{k_j + k_{j+1}} \left(\frac{v_{i,j+1} - v_{i,j}}{k_{j+1}} - \frac{v_{i,j} - v_{i,j-1}}{k_j} \right),$$

and the first-order backward difference

$$\delta_t v^n := \frac{v^n - v^{n-1}}{\tau_n}.$$

A possible discretization of (1) is: Find $U_{i,j}^n \approx u_{ij}^n$ as the solution to

$$[\delta_t U^n + L^n U^n]_{i,j} = f_{i,j}^n, \quad i,j = 1,\dots,N-1, \quad n = 1,\dots,K,$$

supplemented with the obvious discretizations of the initial and boundary conditions. This implicit Euler method may be formulated as

$$(I + \tau_n L^n)\, U^n = U^{n-1} + \tau_n f^n, \quad n = 1,\dots,K. \tag{2}$$

However we shall discretize using an alternating directions approach. Split $r = r_1 + r_2$ and $f = f_1 + f_2$ with $r_1, r_2 \geq 0$. Set

$$L_x^n v := -\varepsilon^2 \delta_x^2 v + r_1^n v \quad \text{and} \quad L_y^n v := -\varepsilon^2 \delta_y^2 v + r_2^n v.$$

Then an approximation can be sought as

$$\left.\begin{aligned} (I + \tau_n L_x^n)\, U^{n-1/2} &= U^{n-1} + \tau_n f_1^n, \\ \left(I + \tau_n L_y^n\right) U^n &= U^{n-1/2} + \tau_n f_2^n \end{aligned}\right\}, \quad n = 1,\dots,K. \tag{3}$$

In each half-step of the algorithm, $N-1$ linear tridiagonal systems must be solved. This can be done with optimal complexity by means of the Thomas algorithm. Moreover these systems are independent of one-another, and therefore allow for easy parallelization.

Remark 1. *Note that* (3) *is equivalent to*

$$(I + \tau_n L^n)\, U^n = U^{n-1} + \tau_n f^n + \tau_n^2 L_x^n \left(f_2^n - L_y^n U^n\right).$$

Thus the schemes (2) *and* (3) *differ by an order of* τ_n^2*. Taking error accumulation into account, both schemes will give the same first-order accuracy in time.*

Remark 2. *Second-order ADI schemes have also been constructed, for example the method by Peaceman-Rachford* [15]*. It is closely related to the Crank-Nicolson method* [6]*. The maximum-norm stability analysis for these higher-order methods is very intricate, in particular when applied to singularly perturbed problems.*

3. Analysis of the method

In this section nodal error estimates are derived of the alternating directions method (3). First, we state derivative bounds for the solution of (1) which are essential in the construction of layer-adapted meshes and in the convergence analysis.

We then give a shout outline of the analysis from [5], before presenting an alternative approach that simplifies the analysis somewhat, and generalises more easily to fitted meshes other than the piecewise uniform mesh of Shishkin.

3.1. Properties of the exact solution. In the course of the analysis we require the solution to be sufficiently smooth: $u \in C^{4+\alpha}(\bar{Q})$, with $\alpha \in (0,1)$ arbitrary. Essentially this means that the derivatives $\partial_x^k \partial_y^m \partial_t^\ell u$ are Hölder continuous with an arbitrary Hölder-exponent for $k, m, \ell \geq 0$ and $k + m + 2\ell \leq 4$. For a full discussion of these function spaces the reader is referred to [7, 10].

If $u \in C^{2+\alpha}(\bar{Q})$ then (1) implies

$$f(\cdot, 0) = 0 \quad \text{on} \quad \partial\Omega. \tag{4a}$$

Furthermore, let $u \in C^{4+\alpha}(\bar{Q})$ and $r, f \in C^{2+\alpha}(\bar{Q})$. Then differentiate (1a) with respect to time, use (1b) and (4a) to get

$$-\varepsilon^2 \Delta f(\cdot, 0) = \partial_t f(\cdot, 0) \quad \text{on} \quad \partial\Omega. \tag{4b}$$

In [5] derivative bounds in terms of ε are derived under the assumption that $u \in C^{4+\alpha}(\bar{Q})$:

$$\left|\partial_x^\ell u(x,y,t)\right| \leq C \left\{ \varepsilon^{\min\{0, 2-\ell\}} + \varepsilon^{-\ell} \left(\mathrm{e}^{-\varrho x/\varepsilon} + \mathrm{e}^{-\varrho(1-x)/\varepsilon} \right) \right\}, \quad \ell = 0, \ldots, 4, \tag{5a}$$

$$\left|\partial_y^\ell u(x,y,t)\right| \leq C \left\{ \varepsilon^{\min\{0, 2-\ell\}} + \varepsilon^{-\ell} \left(\mathrm{e}^{-\varrho y/\varepsilon} + \mathrm{e}^{-\varrho(1-y)/\varepsilon} \right) \right\}, \quad \ell = 0, \ldots, 4, \tag{5b}$$

$$\left|\partial_x^\ell \partial_y^m u(x,y,t)\right| \leq C \varepsilon^{-(\ell+m)}, \quad \ell + m \leq 4, \tag{5c}$$

$$\left|\partial_t^\ell u(x,y,t)\right| \leq C, \quad \ell = 0, 1, 2, \tag{5d}$$

for all $(x, y, t) \in \bar{Q}$.

3.2. The analysis in [5]**.** Introducing the continuous operator splitting $\mathcal{L}^n = \mathcal{L}_x^n + \mathcal{L}_y^n$ by

$$\mathcal{L}_x^n v := -\varepsilon^2 \partial_x^2 v + r_1^n v \quad \text{and} \quad \mathcal{L}_y^n v := -\varepsilon^2 \partial_y^2 v + r_2^n v,$$

the authors of [5] study the time semidiscretization (with constant temporal step size τ): Find functions $\bar{u}^{n-1/2}$ and $\bar{u}^n$, $n = 1, \ldots, K$ with

$$\left.\begin{aligned} \left(I + \tau \mathcal{L}_x^n\right) \bar{u}^{n-1/2} &= \bar{u}^{n-1} + \tau f_1^n, \\ \left(I + \tau \mathcal{L}_y^n\right) \bar{u}^{n} &= \bar{u}^{n-1/2} + \tau f_2^n \end{aligned}\right\}, \quad n = 1, \ldots, K \tag{6a}$$

subject to boundary and initial conditions

$$\bar{u}^{n-1/2}(0,\cdot) = \bar{u}^{n-1/2}(1,\cdot) = 0, \quad \bar{u}^{n-1}(\cdot,0) = \bar{u}^{n-1}(\cdot,1) = 0 \quad \text{and} \quad \bar{u}^0 \equiv 0. \tag{6b}$$

In [5] the local error of this semidiscretization is shown to be of order τ^2.

Next, in order to complete the analysis of the local error, the error of the central-difference approximation on a Shishkin mesh applied to (6) is studied. For this purpose—in addition to (5)—derivative bounds for two auxiliary functions $\hat{u}^{n-1/2}$ and $\hat{u}^n$ are required. These are defined as solutions of

$$\left.\begin{aligned} \left(I + \tau \mathcal{L}_x^n\right) \hat{u}^{n-1/2} &= u^{n-1} + \tau f_1^n, \\ \left(I + \tau \mathcal{L}_y^n\right) \hat{u}^{n} &= \hat{u}^{n-1/2} + \tau f_2^n \end{aligned}\right\}, \quad n = 1, \ldots, K \tag{7a}$$

with

$$\hat{u}^{n-1/2}(0,\cdot) = \hat{u}^{n-1/2}(1,\cdot) = 0, \quad \hat{u}^{n-1}(\cdot,0) = \hat{u}^{n-1}(\cdot,1) = 0, \tag{7b}$$

i.e., with the exact solution u at time level t_{n-1} as starting value for a single time step. With these derivative bounds it can be established that the local error is bounded by $C\tau\left(\tau + N^{-1}\right)$. Now, as shown in [3], error accumulation yields the uniform error bound

$$\|u - U\|_{\omega_t \times \omega} \leq C\left(\tau + N^{-1}\right). \tag{8}$$

Remark 3. *A detailed inspection of the analysis in* [5] *reveals, that by slightly shifting the transition point in the Shishkin mesh, the local error can be bounded by* $C\tau\left(\tau + N^{-2}\ln^2 N + \varepsilon N^{-1}\right)$. *This in turn gives the convergence result*

$$\|u - U\|_{\omega_t \times \omega} \leq C\left(\tau + N^{-2}\ln^2 N + \varepsilon N^{-1}\right).$$

Taking the maximum over $\varepsilon \in (0,1]$, *we recover the uniform result* (8).

Remark 4. *The analysis of the asymptotic behaviour to the semidiscrete problem* (7) *forms the central part of the convergence analysis in* [5]. *This is an interesting problem in its own right; for example it can be used when studying other spatial discretisations.*

In considering just the central differencing approximation, our approach avoids the use of the auxiliary functions $\hat{u}^{n-1/2}$ *and* $\hat{u}^n$, *thereby giving an analysis that is more succinct.*

3.3. Stability. Given the mesh ω_x, consider the one-dimensional difference operator

$$[\Lambda v]_i := -\mu\left[\delta_x^2 v\right]_i + c_i v_i, \quad i = 1,\ldots,N-1,$$

with a constant $\mu > 0$ and a function c.

Lemma 1. *Suppose* $c_i \geq \gamma > 0$, $i = 0,\ldots,N$. *Then the operator* Λ *satisfies a comparison principle. That is, for any two mesh functions* $v, w \in \mathbb{R}^{N+1}$

$$\left.\begin{array}{r} v_0 \leq w_0, \\ [\Lambda v]_i \leq [\Lambda w]_i, \quad i = 1,\ldots,N-1, \\ v_N \leq w_N \end{array}\right\} \quad \Longrightarrow \quad v_i \leq w_i, \quad i = 0,\ldots,N.$$

Furthermore

$$\max_{i=0,\ldots,N} |v_i| \leq \frac{1}{\gamma} \max_{i=1,\ldots,N-1} |[\Lambda v]_i| \quad \forall\ v \in \mathbb{R}_0^{N+1} := \left\{v \in \mathbb{R}^{N+1} : v_0 = v_N = 0\right\},$$

or $\|\Lambda^{-1}\| \leq \gamma^{-1}$ *for short.*

Proof. The matrix associated with Λ has non-positive off-diagonal entries and row sum $c_i \geq \gamma > 0$. Therefore, the M-criterion [17, §II.2.1] applies and yields the desired results. □

Set

$$M_n := \left(I + \tau_n L_x^n\right)\left(I + \tau_n L_y^n\right).$$

Theorem 1. *Assume that* $r = r_1 + r_2$ *with* $r_1 \geq 0$ *and* $r_2 \geq 0$ *on* $\bar{\Omega}$. *Then the operator* M^n *satisfies a comparison principle. In particular for any two mesh functions* $v, w \in \left(\mathbb{R}_0^{N+1}\right)^2$

$$[M^n v]_{i,j} \leq [M^n w]_{i,j}, \ i,j = 1,\ldots,N-1, \quad \Longrightarrow \quad v_{i,j} \leq w_{i,j}, \ i,j = 0,\ldots,N.$$

Furthermore

$$\max_{i,j=0,\dots,N} |v_{i,j}| \leq \max_{i,j=1,\dots,N} \left|[M^n v]_{i,j}\right| \quad \textit{for all } v \in \left(\mathbb{R}_0^{N+1}\right)^2.$$

Proof. Note that

$$(I + \tau_n L_x^n)\, v = -\tau_n \varepsilon^2 \delta_x^2 v + (1 + \tau_n r_1^n) v$$

and

$$\left(I + \tau_n L_y^n\right) v = -\tau_n \varepsilon^2 \delta_y^2 v + (1 + \tau_n r_2^n) v.$$

Lemma 1 can be applied with $\mu = \tau_n \varepsilon^2$ and $c = 1 + \tau_n r_k^n \geq 1$, $k = 1, 2$. Thus both operators $I + \tau_n L_x^n$ and $I + \tau_n L_y^n$ are inverse monotone. Consequently their product M^n is also inverse monotone and satisfies a comparison principle.

Lemma 1 also implies that

$$\left\|(I + \tau_n L_x^n)^{-1}\right\| \leq 1 \quad \text{and} \quad \left\|\left(I + \tau_n L_y^n\right)^{-1}\right\| \leq 1.$$

Thus $\left\|(M^n)^{-1}\right\| \leq 1$ which is equivalent to the last inequality of the theorem. □

3.4. Error analysis. Let $\eta^n := U^n - u^n$ denote the error of the scheme at time t_n. Then

$$M^n \eta^n = \eta^{n-1} + \tau_n \left(\mathcal{L}^n - L^n\right) u^n + \tau_n \left(\partial_t u^n - \delta_t u^n\right) + \tau_n^2 L_x^n \left(f_2^n - L_y^n u^n\right).$$

Note that $\eta^n \in \left(\mathbb{R}_0^{N+1}\right)^2$. Theorem 1 and a triangle inequality yield

$$\|\eta^n\|_\omega \leq \left\|\eta^{n-1}\right\|_\omega + \tau_n \|\psi^n\|_\omega + \tau_n \|\varphi^n\|_\omega + \tau_n^2 \|\chi^n\|_\omega,$$

where $\psi, \varphi, \chi \in \left(\mathbb{R}_0^{N+1}\right)^2$ solve

$$M^n \psi^n = \left(\mathcal{L}^n - L^n\right) u^n, \quad M^n \varphi^n = \left(\partial_t - \delta_t\right) u^n \quad \text{and} \quad M^n \chi^n = L_x^n \left(f_2^n - L_y^n u^n\right)$$

for $n = 1, \dots, K$. Induction for n yields

$$\max_{n=1,\dots,K} \|\eta^n\|_\omega \leq \max_{n=1,\dots,K} \|\psi^n\|_\omega + \max_{n=1,\dots,K} \|\varphi^n\|_\omega + \tau \max_{n=1,\dots,K} \|\chi^n\|_\omega, \tag{9}$$

since $\eta^0 \equiv 0$. The three error components will be studied separately.

For the truncation error in time, a Taylor expansion and (5d) give

$$\left|[M^n \varphi]_{i,j}^n\right| = \left|\left(\partial_t - \delta_t\right) u_{i,j}^n\right| \leq \frac{\tau_n}{2} \left\|\partial_t^2 u\right\|_{\omega_t \times \omega} \leq C\tau, \quad i, j = 1, \dots, N-1.$$

Thus $\|\varphi^n\| \leq C\tau$, by Theorem 1.

We continue with χ^n.

$$L_x^n \left(f_2^n - L_y^n u^n\right) = r_1^n \left(f_2^n - r_2^n u^n\right) + \varepsilon^2 r_1^n \delta_y^2 u^n - \varepsilon^2 \delta_x^2 \left(f_2^n - r_2^n u^n\right) - \varepsilon^4 \delta_x^2 \delta_y^2 u^n.$$

Clearly

$$\left\|r_1^n \left(f_2^n - r_2^n u^n\right)\right\|_\omega \leq C.$$

There exist $\xi \in (x_{i-1}, x_{i+1})$ and $\zeta \in (y_{j-1}, y_{j+1})$ such that

$$\left|(\delta_y^2 u)_{i,j}^n\right| = \left|\partial_y^2 u(x_i, \zeta, t_n)\right| \leq C\varepsilon^{-2} \quad \text{by (5a)}$$

and

$$\left|\left(\delta_x^2 \left(f_2 - r_2 u\right)\right)_{i,j}^n\right| = \left|\partial_x^2 \left(f_2 - r_2 u\right)(\xi, y_j, t_n)\right| \leq C\varepsilon^{-2} \quad \text{by (5b)}.$$

Next, the representation

$$\left(\delta_y^2 g\right)_{i,j} = \frac{2}{k_j + k_{j+1}} \left(\frac{1}{k_{j+1}} \int_{y_j}^{y_{j+1}} \int_{y_j}^{s} \partial_y^2 g(x_i, \tau) d\tau ds + \frac{1}{k_j} \int_{y_j}^{y_{j-1}} \int_{y_j}^{s} \partial_y^2 g(x_i, \tau) d\tau ds \right) \tag{10}$$

for any function $g \in C^2(\bar{\Omega})$ yields

$$\left| \left(\delta_y^2 g\right)_{i,j} \right| \le \left\| \partial_y^2 g(x_i, \cdot) \right\|_{[y_{j-1}, y_{j+1}]}.$$

Moreover, for any $g \in C^4(\bar{\Omega})$

$$\left(\delta_x^2 \delta_y^2 g\right)_{i,j} = \frac{2}{k_j + k_{j+1}} \left(\frac{1}{k_{j+1}} \int_{y_j}^{y_{j+1}} \int_{y_j}^{s} \delta_x^2 \partial_y^2 g(x_i, \tau) d\tau ds + \frac{1}{k_j} \int_{y_j}^{y_{j-1}} \int_{y_j}^{s} \delta_x^2 \partial_y^2 g(x_i, \tau) d\tau ds \right).$$

Use an integral representation for δ_x^2 similar to (10) to get

$$\left| \left(\delta_y^2 \delta_x^2 g\right)_{i,j} \right| \le \left\| \partial_x^2 \partial_y^2 g \right\|_{[x_{i-1}, x_{i+1}] \times [y_{j-1}, y_{j+1}]}.$$

This yields

$$\left| \left(\delta_x^2 \delta_y^2 u\right)_{i,j}^n \right| \le C \left\| \partial_x^2 \partial_y^2 u \right\| \le C\varepsilon^{-4}, \quad \text{by (5c)}.$$

Collecting the above bounds, we obtain

$$\left\| L_x^n \left(f_2^n - L_y^n u^n \right) \right\|_\omega \le C.$$

Finally, Theorem 1 yields $\|\chi\|_\omega \le C$.

Note, the error components φ and χ have been bounded without using any property of the spatial mesh. Thus these bounds hold for arbitrary meshes. Recalling (9), we arrive at our main general error bound.

Theorem 2. *Let the assumptions of Theorem 1 hold true. Then the error of the alternating-direction method* (3) *applied to* (1) *satisfies*

$$\max_{n=1,\dots,K} \|U^n - u^n\|_\omega \le C\tau + \max_{n=1,\dots,K} \|\psi^n\|_\omega,$$

where $\psi^n \in \left(\mathbb{R}_0^{N+1}\right)^2$ *solves* $M^n \psi^n = \left(\mathcal{L}^n - L^n\right) u^n$ *on* ω.

3.5. Layer-adapted meshes.

Bakhvalov meshes [1]. These meshes are based on user-chosen mesh parameters $\sigma > 0$ and $q \in (0, 1/2)$, where the grid points are $x_i = i/N$ if $\sigma\varepsilon \ge \varrho q$, while when $\sigma\varepsilon < \varrho q$ one sets

$$x_i = \begin{cases} \vartheta(i/N) & \text{for } i \le N/2, \\ 1 - \vartheta(1 - i/N) & \text{for } i > N/2, \end{cases}$$

with a mesh generating function ϑ defined by

$$\vartheta(\xi) = \begin{cases} \chi(\xi) := -\dfrac{\sigma\varepsilon}{\varrho} \ln\left(1 - \dfrac{\xi}{q}\right) & \text{for} \quad \xi \in [0, \tau], \\ \pi(\xi) := \chi(\tau) + \chi'(\tau)(\xi - \tau) & \text{for} \quad \xi \in [\tau, 1/2]. \end{cases}$$

The transition point τ is determined by the equation $(1-2\tau)\chi'(\tau) = 1 - 2\chi(\tau)$. Thus the tangent to the graph of χ at $\left(\tau, \chi(\tau)\right)$ passes through $(1/2, 1/2)$. This defines the

mesh on $[0,1/2]$ and it is extended to $[0,1]$ by reflection about $(x,\xi)=(1/2,1/2)$. The resulting mesh generating function φ lies in $C^1[0,1]$.

An easy adaptation of the truncation-error analysis in [1] (see also the appendix in [8]) together with the derivative bounds (5a) and (5b) gives

$$\left|(\mathcal{L}^n - L^n)u^n_{i,j}\right| = \varepsilon^2 \left|\left(\delta_x^2 - \partial_x^2 + \delta_y^2 - \partial_y^2\right) u^n_{i,j}\right| \leq CN^{-2}.$$

Theorem 1 yields $\|\psi^n\|_\omega \leq CN^{-2}$.

Thus in view of Theorem 2 the error of our discretization on Bakhvalov meshes satisfies

$$\|U-u\|_{\omega_t\times\omega} \leq C\left(\tau + N^{-2}\right). \tag{11}$$

Shishkin meshes [20]. These meshes are constructed as follows. Choose mesh parameters $\sigma > 0$ and $q \in (0,1/2)$. Define the transition point

$$\lambda := \min\left\{q, \frac{\sigma\varepsilon}{\varrho}\ln N\right\}.$$

Assuming that qN is an integer, we divide each of the two intervals $[0,\lambda]$ and $[1-\lambda,1]$ uniformly into qN subintervals and $[\lambda,1-\lambda]$ into $(1-2q)N$ subintervals of equal length. A typical choice would be $q=1/4$ and N divisible by 4. The mesh generating function for a Shishkin mesh is piecewise linear and continuous.

For these meshes we have

$$\left|(\mathcal{L}^n - L^n)u^n_{i,j}\right| = \varepsilon^2 \left|\left(\delta_x^2 - \partial_x^2 + \delta_y^2 - \partial_y^2\right) u^n_{i,j}\right| \leq C\left(N^{-2}\ln^2 N + \varepsilon N^{-1}\right);$$

see, e.g., [14, 8]. Thus,

$$\|\psi^n\|_\omega \leq C\left(N^{-2}\ln^2 N + \varepsilon N^{-1}\right). \tag{12}$$

Application of Theorem 1 gives the error bound

$$\|U-u\|_{\omega_t\times\omega} \leq C\left(\tau + N^{-2}\ln^2 N + \varepsilon N^{-1}\right). \tag{13}$$

Remark 5. *In view of results for stationary problems* [2, 8, 14]*, one might expect instead of* (12) *that the following sharper bound would hold:*

$$\|\psi^n\|_\omega \leq CN^{-2}\ln^2 N.$$

However, we have not succeeded in obtaining such a result for this class of problem by using the usual mathematical techniques for stationary problems; neither the piecewise linear barrier-function technique from [14] *nor the strong stability technique from* [19]*. The reason for this is that the maximum norm of the Green's function for* $I + \tau_n L_x^N$ *is not bounded by* $C\varepsilon^{-1}$*, but by* $C\tau_n^{-1/2}\varepsilon^{-1}$*.*

Alternatively other layer-adapted meshes may be used, for example those of van Veldhuizen [22] or the generalized Shishkin meshes of Roos and Linß [16].

4. Numerical results

We present the results of numerical experiments for two example problems, with the goal of demonstrating the veracity of Theorem 2, and its application to the meshes described in §3.5. Furthermore, we consider issues of a parallel implementation when the algorithm is applied on computers with different architectures.

4.1. Example 1. For our first test case we construct a problem based on an example from [9], that exhibits boundary and corner layers, and for which the solution is known:

$$u_t(x,y,t) - \varepsilon^2 \Delta u(x,y,t) + u(x,y,t) = f(x,y,t) \quad (x,y,t) \in \Omega \times (0,1], \tag{14a}$$

with boundary and initial conditions

$$u(x,y,0) = 0 \quad \text{on} \quad \partial\Omega \times (0,T], \quad u(\cdot,0) = 0 \quad \text{on} \quad \bar{\Omega}. \tag{14b}$$

We choose f so that the solution is

$$u(x,y,t) = (1-e^{-t})\Big(\cos\frac{\pi x}{2} - \frac{e^{-x/\varepsilon} - e^{-1/\varepsilon}}{1-e^{-1/\varepsilon}}\Big)\Big(1 - y - \frac{e^{-y/\varepsilon} - e^{-1/\varepsilon}}{1-e^{-1/\varepsilon}}\Big).$$

This will satisfy the compatibility conditions (4). We choose to use the splitting

$$r_1 = r, \quad r_2 \equiv 0, \quad f_1 = f \quad \text{and} \quad f_2 \equiv 0.$$

Note that in general, it is computationally advantageous to take either r_1 or r_2 to be identically zero so that the same factorization of the associated linear system can be used at each time step.

To indicate the dependence of the numerical solution on the perturbation parameter ε, the number K of mesh intervals in time and the number N of mesh intervals in each space directions, we write $U = U_\varepsilon^{K,N}$.

In this section we concentrate on the Bakhvalov mesh described in §3.5, and in particular in verifying separately the rates of convergence with respect to τ and N in (11) as was done, for example, in [11]. To that end, we first apply the ADI scheme to problem (14) for $\varepsilon = 1, 10^{-1}, 10^{-2}, \ldots, 10^{-8}$ on a mesh with K equally sized time steps and $N = 2K$ intervals in the Bakhvalov mesh. This allows us to verify that the error is independent of the parameter ε. Furthermore, because the scheme is second-order in space, and first-order in time, we expect the temporal component of the error to dominate. The results are summarized in Table 1, though to save space we show only the cases $\varepsilon = 1, 10^{-2}, 10^{-4}, 10^{-6}$ and 10^{-8}.

The computed error and rate of convergence are denoted

$$\mathcal{E}_\varepsilon^N = \left\| u_\varepsilon - U_\varepsilon^{K(N),N} \right\|_{\omega_t \times \omega}, \qquad \varrho_\varepsilon^N = \log_2\left(\mathcal{E}_\varepsilon^N / \mathcal{E}_\varepsilon^{2N}\right).$$

It is clear that, for sufficiently small ε, the error is essentially independent of this parameter. Furthermore, the rates of convergence are as predicted.

	$\varepsilon = 1$		$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-6}$		$\varepsilon = 10^{-8}$	
N	$\mathcal{E}_\varepsilon^N$	ϱ_ε^N	$\mathcal{E}_\varepsilon^N$	ϱ_ε^N	$\mathcal{E}_\varepsilon^N$	ϱ_ε^N	$\mathcal{E}_\varepsilon^N$	ϱ_ε^N	$\mathcal{E}_\varepsilon^N$	ϱ_ε^N
32	6.15e-3	0.82	1.15e-2	1.00	1.18e-2	1.01	1.18e-2	1.01	1.18e-2	1.01
64	3.49e-3	0.90	5.74e-3	1.01	5.89e-3	1.00	5.89e-3	1.00	5.89e-3	1.00
128	1.87e-3	0.95	2.86e-3	1.00	2.93e-3	1.00	2.93e-3	1.00	2.93e-3	1.00
256	9.70e-4	0.97	1.42e-3	1.00	1.46e-3	1.00	1.46e-3	1.00	1.46e-3	1.00
512	4.94e-4	0.99	7.11e-4	1.00	7.31e-4	1.00	7.31e-4	1.00	7.31e-4	1.00
1024	2.49e-4	0.99	3.55e-4	1.00	3.65e-4	1.00	3.65e-4	1.00	3.65e-4	1.00
2048	1.25e-4	1.00	1.77e-4	1.00	1.83e-4	1.00	1.83e-4	1.00	1.83e-4	1.00
4096	6.28e-5	—	8.87e-5	—	9.13e-5	—	9.13e-5	—	9.13e-5	—

TABLE 1. Errors in the ADI scheme on a Bakhvalov mesh with $K(N) = N/2$ intervals, applied to (14)

For Table 2 we take $K = N^2$. Now the spatial error should dominate. Indeed this is the case and we observe that, not only is the method robust with respect to ε, the scheme is second-order convergent with respect to N.

	$\varepsilon = 1$		$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-6}$		$\varepsilon = 10^{-8}$	
N	$\mathcal{E}_\varepsilon^N$	ϱ_ε^N	$\mathcal{E}_\varepsilon^N$	ϱ_ε^N	$\mathcal{E}_\varepsilon^N$	ϱ_ε^N	$\mathcal{E}_\varepsilon^N$	ϱ_ε^N	$\mathcal{E}_\varepsilon^N$	ϱ_ε^N
16	4.90e-4	1.97	3.61e-3	1.83	3.65e-3	1.83	3.65e-3	1.83	3.65e-3	1.83
32	1.25e-4	1.99	1.01e-3	1.94	1.03e-3	1.94	1.03e-3	1.94	1.03e-3	1.94
64	3.14e-5	2.00	2.64e-4	1.99	2.67e-4	1.99	2.67e-4	1.99	2.67e-4	1.99
128	7.86e-6	2.00	6.65e-5	2.00	6.72e-5	2.00	6.72e-5	2.00	6.72e-5	2.00
256	1.96e-6	2.00	1.67e-5	2.00	1.68e-5	2.00	1.68e-5	2.00	1.68e-5	2.00
512	4.91e-7	—	4.17e-6	—	4.21e-6	—	4.21e-6	—	4.21e-6	—

TABLE 2. Errors in the ADI scheme on a Bakhvalov mesh with $K(N) = N^2$ intervals, applied to (14)

Finally, we wish to make a direct comparison between the errors in the computed solutions to (14) using the ADI scheme (3) and the standard two-dimensional scheme (2). For this experiment (and all subsequent ones) we will balance the temporal and spatial errors by taking N to be $\mathcal{O}(\sqrt{K})$. More precisely, to ensure that N is divisible by 4, we take $N = 4\lfloor\sqrt{K}\rfloor$.

The results of Table 3 show the errors when the solution is computed using the ADI scheme (3), the standard Euler scheme (2), and the difference between the two. Let $\tilde{U}_\varepsilon^{K,N}$ be the solution computed using the Euler scheme, and

$$\bar{\mathcal{E}}_\varepsilon^K = \left\| u_\varepsilon - U_\varepsilon^{K,4\lfloor\sqrt{K}\rfloor} \right\|_{\omega_t \times \omega}, \quad \tilde{\mathcal{E}}_\varepsilon^K = \left\| u_\varepsilon - \tilde{U}_\varepsilon^{K,4\lfloor\sqrt{K}\rfloor} \right\|_{\omega_t \times \omega},$$
$$\tilde{\mathcal{D}}_\varepsilon^K = \left\| U_\varepsilon^{K,4\lfloor\sqrt{K}\rfloor} - \tilde{U}_\varepsilon^{K,4\lfloor\sqrt{K}\rfloor} \right\|_{\omega_t \times \omega}, \quad \tilde{\varrho}_\varepsilon^K = \log_2\left(\tilde{\mathcal{D}}_\varepsilon^K / \tilde{\mathcal{D}}_\varepsilon^{2K}\right).$$

¿From Table 3 we see that for small ε the accuracy of the two methods is very similar, and furthermore that the difference in time is first-order (see Remark 1).

		$\varepsilon = 1$				$\varepsilon = 10^{-4}, 10^{-5}, \ldots, 10^{-8}$			
K	N	$\bar{\mathcal{E}}_\varepsilon^K$	$\tilde{\mathcal{E}}_\varepsilon^K$	$\tilde{\mathcal{D}}_\varepsilon^K$	$\tilde{\varrho}_\varepsilon^K$	$\bar{\mathcal{E}}_\varepsilon^{K,N}$	$\tilde{\mathcal{E}}_\varepsilon^{K,N}$	$\tilde{\mathcal{D}}_\varepsilon^{K,N}$	$\tilde{\varrho}_\varepsilon^{K,N}$
16	16	6.13e-3	2.46e-5	6.11e-3	0.82	1.30e-2	1.15e-2	3.32e-3	0.71
32	20	3.47e-3	1.20e-5	3.46e-3	0.90	6.88e-3	6.00e-3	2.03e-3	0.84
64	32	1.87e-3	6.03e-6	1.86e-3	0.95	3.47e-3	3.00e-3	1.14e-3	0.98
128	44	9.69e-4	3.00e-6	9.66e-4	0.97	1.78e-3	1.52e-3	5.76e-4	0.96
256	64	4.94e-4	1.50e-6	4.93e-4	0.99	8.83e-4	7.56e-4	2.96e-4	0.97
512	88	2.49e-4	7.49e-7	2.49e-4	0.99	4.49e-4	3.82e-4	1.51e-4	0.99
1024	128	1.25e-4	3.75e-7	1.25e-4	1.00	2.22e-4	1.90e-4	7.57e-5	1.00
2048	180	6.28e-5	1.87e-7	6.26e-5	1.00	1.11e-4	9.50e-5	3.79e-5	1.00
4096	256	3.14e-5	9.37e-8	3.14e-5	—	5.56e-5	4.75e-5	1.90e-5	—

TABLE 3. Comparing the Euler (2) and ADI (3) discritizations on a Bakhvalov mesh for Example (14)

4.2. Example 2. For our second example, we study a more typical problem, but for which the true solution is unknown:

$$u_t(x,y,t) - \varepsilon^2 \Delta u(x,y,t) + (1+y)u(x,y,t) = x\left(1-\sqrt{x}\right)\sin\left(\pi y^2\right) + t \qquad (x,y,t) \in \Omega \times (0,1], \tag{15a}$$

with boundary and initial conditions

$$u(x,y,0) = 0 \quad \text{on} \quad \partial\Omega \times (0,T], \quad u(\cdot,0) = 0 \quad \text{on} \quad \bar{\Omega}. \tag{15b}$$

Again, the data satisfies the compatibility condition (4a), but not (4b).

The exact solution to the test problem is not available, so we estimate the accuracy of the numerical solution by comparing it to the numerical solution of the Richardson extrapolation method, which is of higher order: Let $U_\varepsilon^{K,N}$ be the solution of the difference scheme on the original mesh and $\tilde{U}_\varepsilon^{4K,2N}$ that on the mesh obtained by uniformly bisecting the spatial mesh and using 4 times as many time steps. Asymptotically $\tilde{U}_\varepsilon^{4K,2N}$ is 4 times as accurate as $U_\varepsilon^{K,N}$. Therefore the extrapolated solution is

$$\mathring{U}_\varepsilon^{K,N} := \frac{4\tilde{U}_\varepsilon^{4K,2N} - U_\varepsilon^{K,N}}{3}.$$

We estimate the error for fixed K, N and ε

$$\left\|u_\varepsilon - U_\varepsilon^{K,N}\right\|_\omega \approx \eta_\varepsilon^{K,N} := \left\|U_\varepsilon^{K,N} - \mathring{U}_\varepsilon^{K,N}\right\|_{\omega_t\times\omega} = \frac{4}{3}\left\|U_\varepsilon^{K,N} - \tilde{U}_\varepsilon^{4K,2N}\right\|_{\omega_t\times\omega}.$$

Rather than presenting results for various values of ε, as we done above, we present the *ε-uniform errors*, estimated by

$$\eta^{K,N} := \max_{\mu=0,-1,\ldots,-8} \eta_{10^\mu}^{K,N}.$$

In these experiments we again use the splitting $r_1 = r$, $r_2 \equiv 0$, $f_1 = f$ and $f_2 \equiv 0$, which satisfies the assumptions of our analysis. Following the observations in §4.1, we balance spatial and temporal accuracy by taking $N = 4\lfloor\sqrt{K}\rfloor$. The resulting method is formally of first order in K. The numerical rates of convergence are computed using the standard formula

$$\varrho^K = \log_2\left(\tilde{\eta}^K/\tilde{\eta}^{2K}\right), \quad \text{where} \quad \tilde{\eta}^K := \eta^{K,4\lfloor\sqrt{K}\rfloor}.$$

In Table 4 we present the results for both Shishkin and Bakhvalov meshes, verifying both (11) and (13).

		Shishkin mesh		Bakhvalov mesh	
K	N	error	rate	error	rate
16	16	2.480e-2	0.03	1.490e-2	0.77
32	20	2.424e-2	0.45	8.733e-3	0.87
64	32	1.775e-2	0.50	4.764e-3	0.93
128	44	1.251e-2	0.73	2.502e-3	0.96
256	64	7.550e-3	0.66	1.284e-3	0.98
512	88	4.785e-3	0.75	6.513e-4	0.99
1024	128	2.854e-3	0.77	3.278e-4	0.99
2048	180	1.669e-3	0.79	1.645e-4	1.00
4096	256	9.637e-4	—	8.241e-5	—

TABLE 4. The ADI scheme on layer-adapted meshes for test problem (15)

4.3. Speed up of parallel algorithm. In each half step of the algorithm $N-1$ decoupled tridiagonal linear system have to be solved. This can be done in parallel. In this section we evaluate the efficiency of a parallel implementation. To this end we consider (3) with $f_1 = 1 + ty$, $r_1 = 1 + (1+y)x$, $f_2 = r_2 = 0$ and homogeneous initial and boundary conditions. We give results for three different architectures:

- PC with 4 dual-core AMD Opteron 885 running at 2.6GHz,
- PC with 2 Intel Xeon quad-core E5430 running at 2.66GHz and
- SGI Altix 4700 with 128 Intel Itanium, at 1.6GHz, of which we use up to 64 CPUs.

Tables 5-7 display the results of our experiments. The first column in each table gives the number of processors (#p) used and is followed by the execution time (in sec) and the speed up compared to a single processor. The discretization parameters N and K are chosen such that the number of floating point operations is approximately constant. The execution times are essentially proportional to the number K of time steps. Therefore to measure the speed up K can be arbitrary. We have chosen K large enough to be able to neglect the time for initializations.

	$N = 4096$		$N = 8192$		$N = 16384$	
#p	$K = 1024$		$K = 256$		$K = 64$	
1	2770.613	—	3804.458	—	4586.329	—
2	1435.968	1.93	1944.529	1.96	2441.954	1.88
4	744.734	3.72	1056.552	3.60	1316.491	3.48
8	441.500	6.28	612.680	6.21	801.864	5.72

TABLE 5. Four dual-core AMD Opteron 885, 2.6GHz

	$N = 4096$		$N = 8192$		$N = 16384$	
#p	$K = 1024$		$K = 256$		$K = 64$	
1	1148.302	—	1157.759	—	1242.505	—
2	575.995	1.99	580.589	1.99	624.769	1.99
4	298.216	3.85	299.897	3.86	318.920	3.90
8	163.358	7.03	169.671	6.82	235.230	5.28

TABLE 6. Two Intel Xeon quad-core E5430, 2.66GHz

	$N = 4096$		$N = 8192$		$N = 16384$	
#p	$K = 1024$		$K = 256$		$K = 64$	
1	2008.069	—	2196.300	—	2587.116	—
2	1006.552	2.00	1100.634	2.00	1240.626	2.09
4	512.595	3.92	552.724	3.97	616.088	4.20
8	260.792	7.70	279.759	7.85	316.437	8.18
16	139.389	14.41	142.038	15.46	162.278	15.94
32	83.056	24.18	85.385	26.72	97.808	26.45
64	66.566	30.31	56.585	38.81	69.013	37.49

TABLE 7. Altix 4700 (128 Intel Itanium, 1.6GHz)

The experiments demonstrate that a parallel implementation of the algorithm yields a signification speed up and reduction in execution time. However, limitations of the program used are apparent. On the two PCs (Tables 5 and 6) the maximum speed up is, depending on the discretization size, 5 to 7 compared to the optimal value of 8 given by the number of processors available.

For the Altix, a massively parallel main frame computer, the speed up for 8 processors is close to 8. The optimal number of processors to be used on this machine is about 32. For more than 64 processors (we have tested 96 and 128 processors) the execution time increases which is due to more time needed to synchronize the caches of all processors.

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References

[1] N. S. Bakhvalov. Towards optimization of methods for solving boundary value problems in the presence of boundary layers. *Zh. Vychisl. Mat. i Mat. Fiz.*, 9:841–859, 1969. in Russian.

[2] C. Clavero, J. L. Gracia, and E. O'Riordan. A parameter robust numerical method for a two dimensional reaction-diffusion problem. *Math. Comput.*, 74(252):1743–1758, 2005.

[3] C. Clavero, J. C. Jorge, and F. Lisbona. Uniformly convergent schemes for singular perturbation problems combining alternating directions and exponential fitting techniques. *Adv. Comput. Methods Bound. Inter. Layers* (J.J.H. Miller, ed.), Boole, Dublin 1993, pp.32–52.

[4] C. Clavero, J. C. Jorge, F. Lisbona, and G.I. Shishkin. A fractional step method on a special mesh for the resolution of multidimensional evolutionary convection-diffusion problems. *Appl. Numer. Math.*, 27(3):211-231, 1998.

[5] C. Clavero, J. C. Jorge, F. Lisbona, and G.I. Shishkin. An alternating direction scheme on a nonuniform mesh for reaction-diffusion parabolic problems. *IMA J. Numer. Anal.*, 20(2):263–280, 2000.

[6] J. Crank, P. Nicolson. A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type. *Proc. Camb. Philos. Soc.* 43:50-67, 1947.

[7] A. Friedman. *Partial differential equations of parabolic type.* Prentice-Hall, Englewood Cliffs, 1964.

[8] R. B. Kellogg, T. Linß, and M. Stynes. A finite difference method on layer-adapted meshes for an elliptic reaction-diffusion system in two dimensions. *Math. Comp.*, 77(264):2085–2096, 2008.

[9] N. Kopteva. Maximum norm a posteriori error estimate for a 2d singularly perturbed semilinear reaction-diffusion problem. *SIAM J. Numer. Anal.*, 46(3):1602–1608, 2001.

[10] O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'tseva. *Linear and quasi-linear equations of parabolic type.* American Mathematical Society, 1968.

[11] T. Linß and N. Madden. Parameter uniform approximations for time-dependent reaction-diffusion problems. *Numer. Methods Partial Differ. Equations*, 23(6):1290–1300, 2007.

[12] G. I. Marchuk, N. N. Yanenko. The solution of a multidimensional kinetic equation by the splitting method. *Sov. Math., Dokl.* 5:1121-1123, 1964.

[13] G. I. Marchuk. Splitting and alternating direction methods. In Handbook of Numerical Analysis, Vol 1, Ph. G. Ciarlet, J. L. Lions (eds.), North Holland, Amsterdam 1990, pp. 197-462.

[14] J. J. H. Miller, E. O'Riordan, G. I. Shishkin, and L. P. Shishkina. Fitted mesh methods for problems with parabolic boundary layers. *Math. Proc. R. Ir. Acad.*, 98A(2):173–190, 1998.

[15] D. W. Peaceman, H. H. Rachford Jr.: The numerical solution of parabolic and elliptic differential equations. *J. Soc. Ind. Appl. Math.* 3:28-41, 1955.

[16] H.-G. Roos and T. Linß. Sufficient conditions for uniform convergence on layer-adapted grids. *Computing*, 63(1):27–45, 1999.

[17] H.-G. Roos, M. Stynes, and L. Tobiska. *Robust Numerical Methods for Singularly Perturbed Differential Equations*, volume 24 of *Springer Series in Computational Mathematics.* Springer, Berlin, 2nd edition, 2008.

[18] A. A. Samarskii. *Theory of difference schemes.* Nauka, Moscow, 1989.

[19] I. A. Savin. On the rate of convergence, uniform with respect to a small parameter, of a difference scheme for an ordinary differential equation. *Comput. Math. Math. Phys.*, 35(11):1417-1422, 1995.

[20] G. I. Shishkin. *Grid Approximation of Singularly Perturbed Elliptic and Parabolic Equations.* Second doctorial thesis, Keldysh Institute, Moscow, 1990. In Russian.

[21] J. C. Strikwerda. *Finite Difference Schemes and Partial Differential Equations.* SIAM, Philadelphia, 2nd edition, 2004.

[22] M. van Veldhuizen. Higher order methods for a singularly perturbed problem. *Numer. Math.*, 30:267–279, 1978.

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