INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 7, Number 3, Pages 428–443

SECOND ORDER UNIFORM APPROXIMATIONS FOR THE SOLUTION OF TIME DEPENDENT SINGULARLY PERTURBED REACTION-DIFFUSION SYSTEMS

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This paper is dedicated to Grisha Shishkin, on the occasion of his 70th birthday

Abstract. In this work we consider a parabolic system of two linear singularly perturbed equations of reaction-diffusion type coupled in the reaction terms. To obtain an efficient approximation of the exact solution we propose a numerical method combining the Crank-Nicolson method used in conjunction with the central finite difference scheme defined on a piecewise uniform Shishkin mesh. The method gives uniform numerical approximations of second order in time and almost second order in space. Some numerical experiments are given to support the theoretical results.

Key Words. reaction-diffusion problems, uniform convergence, coupled system, Shishkin mesh, second order.

1. Introduction

We consider the parabolic singularly perturbed problem

$$
\text{(1)} \qquad \begin{cases} L_\varepsilon \mathbf{u} \equiv \dfrac{\partial \mathbf{u}}{\partial t} + L_{x,\varepsilon}\mathbf{u} = \mathbf{f}, & (x,t) \in Q = \Omega \times (0,T] = (0,1) \times (0,T], \\ \mathbf{u}(0,t) = \mathbf{0},\ \mathbf{u}(1,t) = \mathbf{0}, & \forall t \in [0,T], \\ \mathbf{u}(x,0) = \mathbf{0}, \quad \forall x \in \bar{\Omega}, & \end{cases}
$$

where the spatial differential operator is defined by

$$
\text{(2)} \qquad L_{x,\varepsilon} \equiv \begin{pmatrix} -\varepsilon_1 \frac{\partial^2}{\partial x^2} & \\ & -\varepsilon_2 \frac{\partial^2}{\partial x^2} \end{pmatrix} + A, \qquad A = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}.
$$

We denote by $\Gamma_0 = \{(x,0) \,|\, x \in \Omega\}$, $\Gamma_1 = \{(x,t) \,|\, x = 0,1,\ t \in [0,T]\}$, $\Gamma = \Gamma_0 \cup \Gamma_1$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)^T$, with $0 < \varepsilon_1 \le \varepsilon_2 \ll 1$, the vectorial singular perturbation parameter. The components of the right hand side function $\mathbf{f}(x,t) = (f_1(x,t), f_2(x,t))^T$ and the reaction matrix A are assumed to be sufficiently smooth. Also we suppose that the following positivity condition on the matrix reaction A is satisfied:

$$
\text{(3)} \qquad a_{i,1} + a_{i,2} \ge \alpha > 0,\ a_{ii} > 0, \quad i = 1,2,
$$

$$
\text{(4)} \qquad a_{ij} \le 0 \ \text{ if } \ i \ne j.
$$

If (3) is not satisfied, we could consider the transformation $\mathbf{v}(x,t) = \mathbf{u}(x,t)e^{-\alpha_0 t}$ with $\alpha_0 > 0$ sufficiently large, and therefore in the new problem (3) holds. Finally we assume that sufficient compatibility conditions among the data of the differential

Received by the editors March 18, 2009 and, in revised form, September 14, 2009.
2000 *Mathematics Subject Classification.* 65M06, 65N06, 65N12.
This research was partially supported by the project MEC/FEDER MTM2007-63204 and the Diputación General de Aragón.

equation hold in order that the exact solution $\mathbf{u} \in C^{4,3}(\bar{Q})$. In particular, for the later posterior analysis we will assume the following compatibility conditions

$$\frac{\partial^{k+k_0}\mathbf{f}}{\partial x^k \partial t^{k_0}}(0,0) = \frac{\partial^{k+k_0}\mathbf{f}}{\partial x^k \partial t^{k_0}}(1,0) = \mathbf{0}, \quad 0 \leq k + 2k_0 \leq 4. \tag{5}$$

Nevertheless, these hypothesis can be weakened in practice.

Linear coupled systems of type (1) appear in the modelization of the flow in fractured porous media, concretely in the double diffusion model of Barenblatt (see [2]). Other process involving similar problems are the model for turbulent interactions of waves and currents (see [15, 20]) or the diffusion process in electroanalytic chemistry (see [19]). It is well known (see [19]) that the exact solution of problem (1) has a multiscale character. Then, to find good approximations of the solution for any value of the diffusion parameters ε_1 and ε_2, it is necessary to use uniformly convergent methods (see [10, 12, 13, 14, 16]). In [10] a decomposition of the exact solution of problem (1) into its regular and singular components was given for any ratio between ε_1 and ε_2, proving bounds for their derivatives. In that work, also a first order in time and almost second order in space uniformly convergent method was obtained.

In practice it is important to use high order convergent schemes to find accurate numerical solutions with a low computational cost. In the context of singularly perturbed problems some papers follow this direction; for instance in [5, 9] a high order numerical method is defined to solve a 2D elliptic reaction-diffusion problem, in [4] the Crank-Nicolson and a HODIE scheme is used for a 1D parabolic convection-diffusion problem, in [3] a method combining the Peaceman-Rachford scheme with a HODIE scheme is used for a 2D parabolic reaction-diffusion problem, and in [11] the defect correction method is used to increase the order of convergence of the Euler and central differences schemes used for a 1D parabolic convection-diffusion problem. So far, we do not know of any paper proving uniform order of convergence bigger than one in both time and space for a method used to solve (1). Here to increase the order of convergence in time we use the Crank-Nicolson method; note that the totally discrete scheme obtained by using the Crank-Nicolson method and the central finite difference scheme, does not satisfy the discrete maximum principle except if the restrictive and unpractical restriction $\Delta t \leq C(N^{-1}\ln N)^2$ is imposed. In this paper we follow [4] to avoid this difficulty.

The paper is structured as follows. In Section 2 we establish the asymptotic behaviour of the solution of (1) and its partial derivatives. We note that this asymptotic analysis cannot be straightforwardly extended to the case of systems with an arbitrary number of parabolic equations. In Section 3 the analysis of the convergence is done by defining some specific auxiliary problems, which allows us to prove appropriate bounds for the local error of the Crank-Nicolson scheme. We also give the asymptotic behaviour of the exact solution of the semidiscrete problems resulting after the time discretization process. In Section 4 we construct the central finite difference scheme, defined on an appropriate piecewise uniform Shihskin mesh, to discretize in space and using a recursive argument and the uniform stability of the totally discrete operator, we deduce almost second order uniform approximation for the totally discrete method. Finally, in Section 5 we display some numerical experiments showing clearly the improvement in the order of uniform convergence of the numerical method.

We denote by $\mathbf{v} \leq \mathbf{w}$ if $v_i \leq w_i$, $i = 1, 2$, $|\mathbf{v}| = (|v_1|, |v_2|)^T$, $\|f\|_H$ is the maximum norm of f on the closed set H and $\|\mathbf{f}\|_H = \max\{\|f_1\|_H, \|f_2\|_H\}$. Henceforth, C denotes a generic positive constant independent of the diffusion parameters ε_1

and ε_2, and also of the discretization parameters N and Δt; sometimes we use a subscripted C with the same purpose. We use $\mathbf{v} \le C$ meaning that $v_1 \le C,\ v_2 \le C$.

2. Asymptotic behaviour of the solution

We extend the analysis given in [10], showing the asymptotic behaviour of the exact solution of (1). The proofs are based on the continuous maximum principle (see [18]).

Theorem 1 (Maximum principle)**.** *If $\boldsymbol{\psi} \ge \mathbf{0}$ on Γ and $L_\varepsilon \boldsymbol{\psi} \ge \mathbf{0}$ in Q, then $\boldsymbol{\psi} \ge \mathbf{0}$ for all $(x,t) \in \bar{Q}$.*

Corollary 2 (Comparison principle)**.** *If $|\boldsymbol{\psi}| \le \boldsymbol{\varphi}$ on Γ and $|L_\varepsilon \boldsymbol{\psi}| \le L_\varepsilon \boldsymbol{\varphi}$ in Q, then $|\boldsymbol{\psi}| \le \boldsymbol{\varphi}$ for all $(x,t) \in \bar{Q}$.*

Lemma 3. *The solution of problem (1) satisfies* $\left\| \dfrac{\partial^k \mathbf{u}}{\partial t^k} \right\|_{\bar{Q}} \le C, \quad 0 \le k \le 3.$

To obtain bounds for the spatial partial derivatives of $\mathbf{u}$, we consider a decomposition of the exact solution $\mathbf{u} = \mathbf{v} + \mathbf{w}$, where the regular component $\mathbf{v}$ is the solution of

$$L_\varepsilon \mathbf{v} = \mathbf{f}, \quad \text{in } Q, \quad \mathbf{v}(x,0) = \mathbf{0}, \quad \text{on } \Gamma_0, \quad \mathbf{v} = \mathbf{z}, \quad \text{on } \Gamma_1, \tag{6}$$

where $\mathbf{z}$ satisfies the initial value problem

$$\mathbf{z}_t + A\mathbf{z} = \mathbf{f}, \quad (x,t) \in \{0,1\} \times (0,T], \quad \mathbf{z}(x,0) = \mathbf{0} \quad x \in \{0,1\}, \tag{7}$$

and the singular component $\mathbf{w}$ is the solution of

$$L_\varepsilon \mathbf{w} = \mathbf{0}, \quad \text{in } Q, \quad \mathbf{w} = \mathbf{u} - \mathbf{v}, \quad \text{on } \Gamma. \tag{8}$$

Note that the right hand side of problem (6) satisfies the conditions (5) and also that $\mathbf{z}(x,0) = \mathbf{z}_t(x,0) = \mathbf{z}_{tt}(x,0) = \mathbf{z}_{ttt}(x,0) = 0,\ x = 0,1$. Then, we have that $\mathbf{v} \in C^{4,3}(\bar{Q})$ and therefore $\mathbf{w} \in C^{4,3}(\bar{Q})$.

Lemma 4. *The regular component $\mathbf{v} = (v_1, v_2)^T$ satisfies*

$$\begin{gathered}
\left\| \frac{\partial^k \mathbf{v}}{\partial t^k} \right\|_{\bar{Q}} \le C,\ 0 \le k \le 3, \quad \left\| \frac{\partial^k \mathbf{v}}{\partial x^k} \right\|_{\bar{Q}} \le C,\ k = 0,1,2, \\
\left\| \frac{\partial^k v_1}{\partial x^k} \right\|_{\bar{Q}} \le C(1 + \varepsilon_1^{1-k/2}), \quad \left\| \frac{\partial^k v_2}{\partial x^k} \right\|_{\bar{Q}} \le C(1 + \varepsilon_2^{1-k/2}),\ k = 3,4, \\
\left\| \frac{\partial^2 \mathbf{v}}{\partial t \partial x} \right\|_{\bar{Q}} \le C, \quad \left\| \frac{\partial^3 \mathbf{v}}{\partial t \partial x^2} \right\|_{\bar{Q}} \le C, \quad \left\| \frac{\partial^3 \mathbf{v}}{\partial t^2 \partial x} \right\|_{\bar{Q}} \le C, \quad \left\| \frac{\partial^4 \mathbf{v}}{\partial t^2 \partial x^2} \right\|_{\bar{Q}} \le C.
\end{gathered} \tag{9}$$

Proof. We only give the main ideas of the proof for the crossed derivatives $\mathbf{v}_{ttx}$ and $\mathbf{v}_{ttxx}$. From (6) and (7) we have that $\mathbf{v}_{xx} = \mathbf{0}$ on Γ_1; hence $\mathbf{v}_{xxtt} = \mathbf{0}$ on Γ_1. Using that $\mathbf{v}(x,0) = \mathbf{0}$ on Γ_0 and differentiating (6) twice w.r.t. x we have that $\mathbf{v}_{xxt} = f_{xx}$ on Γ_0. Then, differentiating now (6) twice w.r.t. x and once w.r.t. t, it follows that

$$\|\mathbf{v}_{xxtt}(x,0)\|_{\bar{\Omega}} = \| \left(-L_{x,\boldsymbol{\varepsilon}} \mathbf{f}_{xx} + \mathbf{f}_{xxt} - 2A_x \mathbf{v}_{xt} - A_{xx} \mathbf{v}_t\right)(x,0)\|_{\bar{\Omega}} \le C,$$

where $A_x = (a'_{ij})$ and $A_{xx} = (a''_{ij})$. Differentiating (6) twice w.r.t. x and twice w.r.t. t, we can obtain $|L_\varepsilon \mathbf{v}_{xxtt}| = |(\mathbf{f}_{xxtt} - 2A_x \mathbf{v}_{xtt} - A_{xx} \mathbf{v}_{tt})| \le C + C_1 \|\mathbf{v}_{xtt}\|_{\bar{Q}}$, where $C_1 = 2\max\{|a'_{ij}|\}$. The comparison principle applied on the barrier function $\boldsymbol{\psi} = (1+t)(\mathbf{C} + \mathbf{C_1} \|\mathbf{v}_{xtt}\|_{\bar{Q}})$ proves

$$\|\mathbf{v}_{xxtt}\|_{\bar{Q}} \le (1+T)(C + C_1 \|\mathbf{v}_{xtt}\|_{\bar{Q}}). \tag{10}$$

Similarly to [[14], Lemma 3], we can apply the mean value theorem on the interval $[a, a+C_2] \subseteq [0,1]$, where $C_2 = \min\{1, 1/(2C_1(1+T))\}$ and $a \geq 0$, obtaining

$$\|\mathbf{v}_{xtt}\|_{\bar{Q}} \leq C + \|\mathbf{v}_{xxtt}\|_{\bar{Q}} /(2C_1). \tag{11}$$

Then, from (10) and (11) the result follows. □

Below we use the auxiliary function $B_\gamma(x) = e^{-x\sqrt{\alpha/\gamma}} + e^{-(1-x)\sqrt{\alpha/\gamma}}$, where γ is an arbitrary positive constant and α was defined in (3).

Lemma 5. *For all $(x,t) \in \bar{Q}$, the singular component $\mathbf{w} = (w_1, w_2)^T$ satisfies*

$$\left|\frac{\partial^k \mathbf{w}}{\partial t^k}\right| \leq C B_{\varepsilon_2}(x), \quad 0 \leq k \leq 3, \tag{12}$$

$$\left|\frac{\partial w_1}{\partial x}\right| \leq C(\varepsilon_1^{-1/2} B_{\varepsilon_1}(x) + \varepsilon_2^{-1/2} B_{\varepsilon_2}(x)), \quad \left|\frac{\partial w_2}{\partial x}\right| \leq C\varepsilon_2^{-1/2} B_{\varepsilon_2}(x), \tag{13}$$

$$\left|\frac{\partial^2 w_1}{\partial x^2}\right| \leq C(\varepsilon_1^{-1} B_{\varepsilon_1}(x) + \varepsilon_2^{-1} B_{\varepsilon_2}(x)), \quad \left|\frac{\partial^2 w_2}{\partial x^2}\right| \leq C\varepsilon_2^{-1} B_{\varepsilon_2}(x), \tag{14}$$

$$\left|\frac{\partial^k w_1}{\partial x^k}\right| \leq C(\varepsilon_1^{-k/2} B_{\varepsilon_1}(x) + \varepsilon_2^{-k/2} B_{\varepsilon_2}(x)), \quad k = 3, 4, \tag{15}$$

$$\left|\frac{\partial^k w_2}{\partial x^k}\right| \leq C\varepsilon_2^{-1}(\varepsilon_1^{(2-k)/2} B_{\varepsilon_1}(x) + \varepsilon_2^{(2-k)/2} B_{\varepsilon_2}(x)), \quad k = 3, 4. \tag{16}$$

Lemma 6. *For all $(x,t) \in \bar{Q}$, the singular component $\mathbf{w} = (w_1, w_2)^T$ satisfies*

$$\left|\frac{\partial^{k+1} w_i}{\partial x^k \partial t}\right| \leq C(\varepsilon_1^{-k/2} B_{\varepsilon_1}(x) + \varepsilon_2^{-k/2} B_{\varepsilon_2}(x)), \quad i = 1, 2, \quad k = 1, 2, \tag{17}$$

$$\left|\frac{\partial^4 w_1}{\partial x^3 \partial t}\right| \leq C(\varepsilon_1^{-3/2} B_{\varepsilon_1}(x) + \varepsilon_2^{-3/2} B_{\varepsilon_2}(x)), \tag{18}$$

$$\left|\frac{\partial^4 w_2}{\partial x^3 \partial t}\right| \leq C\varepsilon_2^{-1}(\varepsilon_1^{-1/2} B_{\varepsilon_1}(x) + \varepsilon_2^{-1/2} B_{\varepsilon_2}(x)). \tag{19}$$

Proof. Bounds (17) are obtained by differentiating (8) once and twice w.r.t. x respectively. Differentiating (8) twice w.r.t. t, we deduce

$$\left|\frac{\partial^4 w_i}{\partial x^2 \partial t^2}\right| \leq C\varepsilon_i^{-1} B_{\varepsilon_2}(x), \quad i = 1, 2,$$

and hence, using the mean value theorem it can be proved that

$$\left|\frac{\partial^3 w_i}{\partial x \partial t^2}\right| \leq C\varepsilon_i^{-1/2} B_{\varepsilon_2}(x), \quad i = 1, 2.$$

Differentiating (8) once w.r.t. x and once w.r.t. t, we can obtain (19), but for the first component we only prove the crude bound

$$\left|\frac{\partial^4 w_1}{\partial x^3 \partial t}\right| \leq C\varepsilon_1^{-3/2}. \tag{20}$$

Although (20) is not the required bound, it will by useful in the boundary points $(0,t), (1,t)$ with $t \in [0,1]$. To improve bound (20), we define the auxiliary problem

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\dfrac{\partial^4 w_1}{\partial x^3 \partial t}\right) - \varepsilon_1 \dfrac{\partial^2}{\partial x^2}\left(\dfrac{\partial^4 w_1}{\partial x^3 \partial t}\right) + a_{11}\dfrac{\partial^4 w_1}{\partial x^3 \partial t} = g(x,t), \quad (x,t) \in Q, \\ \dfrac{\partial^4 w_1}{\partial x^3 \partial t} \quad \text{given on } \Gamma, \end{cases}$$

where

$$g(x,t) = -\frac{\partial^4(a_{12}w_2)}{\partial x^3 \partial t} - \frac{\partial^3 a_{11}}{\partial x^3}\frac{\partial w_1}{\partial t} - 3\frac{\partial^2 a_{11}}{\partial x^2}\frac{\partial^2 w_1}{\partial x \partial t} - 3\frac{\partial a_{11}}{\partial x}\frac{\partial^3 w_1}{\partial x^2 \partial t}.$$

Using that

$$\begin{aligned}
&\frac{\partial^4 w_1}{\partial x^3 \partial t}(x,0) = 0,\ x \in [0,1], \quad \left|\frac{\partial^4 w_1}{\partial x^3 \partial t}(x,t)\right| \le C\varepsilon_1^{-3/2}, \quad x = 0,1,\\
&|g(x,t)| \le C(\varepsilon_1^{-1/2}(\varepsilon_1^{-1/2} + \varepsilon_2^{-1})B_{\varepsilon_1}(x) + \varepsilon_2^{-3/2}B_{\varepsilon_2}(x)) \le\\
&\le C(\varepsilon_1^{-3/2}B_{\varepsilon_1}(x) + \varepsilon_2^{-3/2}B_{\varepsilon_2}(x)),
\end{aligned}$$

the maximum principle proves the result. □

3. The time semidiscretization: the Crank-Nicolson scheme

In $[0,T]$ we consider a uniform mesh $\bar{\omega}^M = \{k\Delta t,\ 0 \le k \le M,\ \Delta t = T/M\}$. On this mesh the Crank–Nicolson scheme is given by

$$\begin{aligned}
&\mathbf{u}^0 = \mathbf{u}(x,0) = \mathbf{0},\\
&\left\{\begin{array}{l} (I + (\Delta t/2)L_{x,\varepsilon})\,\mathbf{u}^{n+1} = (\Delta t/2)(\mathbf{f}^n + \mathbf{f}^{n+1}) + (I - (\Delta t/2)L_{x,\varepsilon})\,\mathbf{u}^n,\\ \mathbf{u}^{n+1}(0) = \mathbf{0},\ \mathbf{u}^{n+1}(1) = \mathbf{0},\ n = 0,1,\cdots,M-1, \end{array}\right.
\end{aligned} \tag{21}$$

where $\mathbf{f}^n = \mathbf{f}(x,t_n), n = 0,1,\cdots,M-1$. To study the local error of this method, we consider the following auxiliary problem

$$\left\{\begin{array}{l} (I + (\Delta t/2)L_{x,\varepsilon})\,\widehat{\mathbf{u}}^{n+1} = (\Delta t/2)(\mathbf{f}^n + \mathbf{f}^{n+1}) + (I - (\Delta t/2)L_{x,\varepsilon})\,\mathbf{u}(x,t_n),\\ \widehat{\mathbf{u}}^{n+1}(0) = \mathbf{0}, \quad \widehat{\mathbf{u}}^{n+1}(1) = \mathbf{0}. \end{array}\right. \tag{22}$$

Lemma 7. *(See [4]). The local error associated to (21), defined as* $\mathbf{e}^{n+1}(x) = \mathbf{u}(x,t_{n+1}) - \widehat{\mathbf{u}}^{n+1}(x)$*, satisfies* $|\mathbf{e}^{n+1}(x)| \le C(\Delta t)^3,\ x \in \bar{\Omega}$.

For the later analysis of the spatial discretization we need a more precise information about the asymptotic behaviour of the exact solution of the semidiscrete problems (22) and their derivatives with respect to the variable x. For that, we decompose $\widehat{\mathbf{u}}^{n+1} = \widehat{\mathbf{v}}^{n+1} + \widehat{\mathbf{w}}^{n+1}$, where $= \widehat{\mathbf{v}}^{n+1}$ is the solution of

$$\left\{\begin{array}{l} (I + (\Delta t/2)L_{x,\varepsilon})\,\widehat{\mathbf{v}}^{n+1}(x) = (\Delta t/2)(\mathbf{f}^n + \mathbf{f}^{n+1}) + (I - (\Delta t/2)L_{x,\varepsilon})\,\mathbf{v}(x,t_n),\ x \in (0,1),\\ (I + (\Delta t/2)A)\,\widehat{\mathbf{v}}^{n+1}(x) = (\Delta t/2)(\mathbf{f}^n + \mathbf{f}^{n+1}) + (I - (\Delta t/2)A)\,\mathbf{v}(x,t_n),\ x = 0,1, \end{array}\right. \tag{23}$$

and $\widehat{\mathbf{w}}^{n+1}$ is the solution of

$$\left\{\begin{array}{l} (I + (\Delta t/2)L_{x,\varepsilon})\,\widehat{\mathbf{w}}^{n+1} = (I - (\Delta t/2)L_{x,\varepsilon})\,\mathbf{w}(x,t_n),\ x \in (0,1),\\ \widehat{\mathbf{w}}^{n+1}(0) = \widehat{\mathbf{u}}^{n+1}(0) - \widehat{\mathbf{v}}^{n+1}(0), \quad \widehat{\mathbf{w}}^{n+1}(1) = \widehat{\mathbf{u}}^{n+1}(1) - \widehat{\mathbf{v}}^{n+1}(1), \end{array}\right. \tag{24}$$

whit $\mathbf{v}$ the solution of (6)-(7), and $\mathbf{w}$ the solution of (8).

Remark 8. *Note that from* $\mathbf{v}_{xx}(x,t) = \mathbf{0}$, $x = 0,1$*, it follows trivially that* $\widehat{\mathbf{v}}_{xx}(x) = \mathbf{0}$, $x = 0,1$. *Moreover,* $\widehat{\mathbf{v}}^{n+1}(x)$, $x = 0,1$, *are the approximations given by the Crank-Nicolson method to solve the initial value problem*

$$\mathbf{v}_t + A\mathbf{v} = \mathbf{f}, \quad (x,t) \in \{0,1\} \times (t_n, t_{n+1}], \quad \mathbf{v}(x,t_n),\ x \in \{0,1\}\ \textit{known}. \tag{25}$$

Then, it follows that

$$|\mathbf{v}(x,t_{n+1}) - \widehat{\mathbf{v}}^{n+1}(x)| \le C(\Delta t)^3, \quad x \in \{0,1\}. \tag{26}$$

From (26), Lemma 7 and the triangular inequality we have

$$|\mathbf{w}(x,t_{n+1}) - \widehat{\mathbf{w}}^{n+1}(x)| \le C(\Delta t)^3, \quad x \in \{0,1\}.$$

Lemma 9. *The local errors associated to* $\widehat{\mathbf{v}}^{n+1}$ *and* $\widehat{\mathbf{w}}^{n+1}$ *satisfy*

$$|\mathbf{v}(x,t_{n+1})-\widehat{\mathbf{v}}^{n+1}(x)| \le C(\Delta t)^3, \quad |\mathbf{w}(x,t_{n+1})-\widehat{\mathbf{w}}^{n+1}(x)| \le C(\Delta t)^3 B_{\varepsilon_2}(x), \quad x \in \bar{\Omega}.$$

Proof. The proof is completely analogous to this one given in [4]. □

To find precise bounds of the derivatives of $\widehat{\mathbf{v}}^{n+1}$, we will use the following technical result.

Lemma 10. *It holds that*

$$\left|(\mathbf{v}_{xx}(x,t_{n+1}) - \widehat{\mathbf{v}}^{n+1}_{xx}(x))/\Delta t\right| \le C.$$

Proof. First, using that $\|\mathbf{v}_{xxtt}\|_{\bar{Q}} \le C$, we have

$$\begin{aligned} &(\mathbf{v}_{xx}(x,t_{n+1}) - \mathbf{v}_{xx}(x,t_n))/\Delta t = \mathbf{v}_{xxt}(x,t_n+\Delta t/2) + O(\Delta t) \\ &= -L_{x,\boldsymbol{\varepsilon}}\mathbf{v}_{xx}(x,t_n+\Delta t/2) + \mathbf{f}_{xx}(x,t_n+\Delta t/2) - 2A_x\mathbf{v}_x(x,t_n+\Delta t/2) \\ &\ -A_{xx}\mathbf{v}(x,t_n+\Delta t/2) + O(\Delta t). \end{aligned} \tag{27}$$

Secondly, differentiating the equation (6) twice w.r.t. x and once w.r.t. t, we obtain $|L_{x,\boldsymbol{\varepsilon}}\mathbf{v}_{txx}(x,t)| = |\mathbf{f}_{txx} - \mathbf{v}_{ttxx} - A_{xx}\mathbf{v}_t - 2A_x\mathbf{v}_{tx}| \le C$, and therefore

$$L_{x,\boldsymbol{\varepsilon}}\mathbf{v}_{xx}(x,t_n+\Delta t/2) = L_{x,\boldsymbol{\varepsilon}}((\mathbf{v}_{xx}(x,t_{n+1}) + \mathbf{v}_{xx}(x,t_n))/2) + O(\Delta t). \tag{28}$$

Then, defining the problem

$$(I+(\Delta t/2)L_{x,\boldsymbol{\varepsilon}})(\mathbf{v}_{xx}(x,t_{n+1})-\widehat{\mathbf{v}}^{n+1}_{xx}(x)) = \mathbf{g}(x),\ \mathbf{v}_{xx}(x,t_{n+1})-\widehat{\mathbf{v}}^{n+1}_{xx}(x) = \mathbf{0},\ x=0,1,$$

from (27) and (28) the right hand side can be bounded by

$$\begin{aligned} |\mathbf{g}(x)| = |&\Delta t(\mathbf{f}^{n+\Delta t/2}_{xx}(x) - (\mathbf{f}^{n}_{xx}(x) + \mathbf{f}^{n+1}_{xx}(x))/2) \\ &+(\Delta t/2)[2A_x(\widehat{\mathbf{v}}^{n+1}_x(x) + \mathbf{v}_x(x,t_n) - 2\mathbf{v}_x(x,t_n+\Delta t/2)) \\ &+A_{xx}(\widehat{\mathbf{v}}^{n+1}(x) + \mathbf{v}(x,t_n) - 2\mathbf{v}(x,t_n+\Delta t/2))] + O(\Delta t)| = O(\Delta t). \end{aligned}$$

Then, using the maximum principle for $(I + (\Delta t/2)L_{x,\boldsymbol{\varepsilon}})$ the result follows. □

Proposition 11. *The regular component* $\widehat{\mathbf{v}}^{n+1} = (\widehat{v}^{n+1}_1, \widehat{v}^{n+1}_2)^T$ *satisfies*

$$\left\|\frac{d^k\widehat{v}^{n+1}_i}{dx^k}\right\|_{\bar{\Omega}} \le C,\ 0 \le k \le 2,\ \left\|\frac{d^k\widehat{v}^{n+1}_i}{dx^k}\right\|_{\bar{\Omega}} \le C(1+\varepsilon_i^{1-k/2}),\ 3 \le k \le 4,\ i=1,2. \tag{29}$$

Proof. Using that $|\partial^k v_i/\partial x^k| \le C,\ 0 \le k \le 2,\ |\partial^k v_i/\partial x^k| \le C\varepsilon_i^{1-k/2},\ 3 \le k \le 4,\ i=1,2$, and that $\widehat{\mathbf{v}}^{n+1}_{xx}(x) = \mathbf{0}$, $x = 0,1$ (see Remark 8), we can reproduce the proof given in [14] for the regular component, obtaining

$$\left\|\frac{d^k\widehat{\mathbf{v}}^{n+1}}{dx^k}\right\|_{\bar{\Omega}} \le C,\ 0 \le k \le 2.$$

For higher derivatives differentiating (23) twice w.r.t. x, we obtain

$$\left\{\begin{array}{l} L_{x,\boldsymbol{\varepsilon}}\widehat{\mathbf{v}}^{n+1}_{xx}(x) = (\mathbf{f}^n_{xx} + \mathbf{f}^{n+1}_{xx}) + (2/\Delta t)(\mathbf{v}_{xx}(x,t_n) - \widehat{\mathbf{v}}^{n+1}_{xx}(x)) - \\ -L_{x,\boldsymbol{\varepsilon}}\mathbf{v}_{xx}(x,t_n) - 2A_x(\widehat{\mathbf{v}}^{n+1}_x(x) + \mathbf{v}_x(x,t_n)) - A_{xx}(\widehat{\mathbf{v}}^{n+1}(x) + \mathbf{v}(x,t_n)),\ x \in (0,1), \\ \widehat{\mathbf{v}}^{n+1}_{xx}(0) = \widehat{\mathbf{v}}^{n+1}_{xx}(1) = \mathbf{0}. \end{array}\right. \tag{30}$$

From Lemmas 4 and 10 it follows that the right hand side of problem (30) is parameter uniform bounded and therefore the result can be proved using a similar argument to this one of [14]. □

As in the case of the regular component, before finding appropriate bounds for the derivatives of $\widehat{\mathbf{w}}^{n+1}$, we prove the following technical result.

Lemma 12. *It holds that*

$$\begin{array}{l}\left|(\widehat{\mathbf{w}}_x^{n+1}(x)-\mathbf{w}_x(x,t_n))/\Delta t\right| \le CB_{\varepsilon_2}(x)(\varepsilon_1^{-1/2},\varepsilon_2^{-1/2})^T,\\ \left|(\widehat{\mathbf{w}}_{xx}^{n+1}(x)-\mathbf{w}_{xx}(x,t_n))/\Delta t\right| \le CB_{\varepsilon_2}(x)(\varepsilon_1^{-1},\varepsilon_2^{-1})^T.\end{array}$$

Proof. We consider the function

$$\boldsymbol{\varphi}_1^{n+1}(x) = (\widehat{\mathbf{w}}^{n+1}(x)-\mathbf{w}(x,t_n))/\Delta t. \tag{31}$$

From Lemmas 5 and 9 it follows

$$|\boldsymbol{\varphi}_1^{n+1}(x)| = \left|(\widehat{\mathbf{w}}^{n+1}(x)-\mathbf{w}(x,t_{n+1}))/\Delta t + (\mathbf{w}(x,t_{n+1})-\mathbf{w}(x,t_n))/\Delta t\right| \le CB_{\varepsilon_2}(x). \tag{32}$$

The function $\boldsymbol{\varphi}_1^{n+1}(x)$ is the solution of the following boundary value problem:

$$L_{x,\boldsymbol{\varepsilon}}\boldsymbol{\varphi}_1^{n+1} = \boldsymbol{\varphi}_2^{n+1}, \quad \boldsymbol{\varphi}_1^{n+1}(0),\ \boldsymbol{\varphi}_1^{n+1}(1) \ \text{ given}, \tag{33}$$

where $\boldsymbol{\varphi}_2^{n+1}(x)$ is appropriately chosen. Then, from (24) we can deduce that

$$|(I+(\Delta t/2)L_{x,\boldsymbol{\varepsilon}})\boldsymbol{\varphi}_2^{n+1}(x)| = |-L_{x,\boldsymbol{\varepsilon}}^2\mathbf{w}(x,t_n)| = |L_{x,\boldsymbol{\varepsilon}}\mathbf{w}_t(x,t_n)| = |-\mathbf{w}_{tt}(x,t_n)| \le CB_{\varepsilon_2}(x),$$

for the interior mesh points. On the boundary, using (24), (33), a continuity argument and Remark 8, we have

$$\begin{aligned}\boldsymbol{\varphi}_2^{n+1}(x) &= -(2/\Delta t)(\boldsymbol{\varphi}_1^{n+1}(x)+L_{x,\boldsymbol{\varepsilon}}\mathbf{w}(x,t_n))\\ &= (2/\Delta t)((\mathbf{w}(x,t_n)-\widehat{\mathbf{w}}^{n+1}(x))/\Delta t+\mathbf{w}_t(x,t_n))\\ &= (2/\Delta t)\left((\mathbf{w}(x,t_n)-\mathbf{w}(x,t_{n+1}))/\Delta t+(\mathbf{w}(x,t_{n+1})-\widehat{\mathbf{w}}^{n+1}(x))/\Delta t+\mathbf{w}_t(x,t_n)\right)\\ &= O(\Delta t)+(2/\Delta t)\left((\mathbf{w}(x,t_n)-\mathbf{w}(x,t_{n+1}))/\Delta t+\mathbf{w}_t(x,t_n)\right), \quad x=0,1.\end{aligned}$$

Hence $|\boldsymbol{\varphi}_2^{n+1}(x)| \le C$ for $x=0,1$ and from the maximum principle for $(I+(\Delta t/2)L_{x,\boldsymbol{\varepsilon}})$ it follows that $|\boldsymbol{\varphi}_2^{n+1}(x)| \le CB_{\varepsilon_2}(x)$, $x\in\bar\Omega$. Using this bound in (32) and (33), we deduce that

$$|[d^2\boldsymbol{\varphi}_1^{n+1}/dx^2]_1| \le C\varepsilon_1^{-1}B_{\varepsilon_2}(x),\ |[d^2\boldsymbol{\varphi}_1^{n+1}/dx^2]_2| \le C\varepsilon_2^{-1}B_{\varepsilon_2}(x),\ x\in\bar\Omega, \tag{34}$$

where we denote $\boldsymbol{\varphi}_1^{n+1}(x) = ([\boldsymbol{\varphi}_1^{n+1}(x)]_1,[\boldsymbol{\varphi}_1^{n+1}(x)]_2)^T$. From (34) and using the mean value theorem (see [14]), we can obtain

$$|[d\boldsymbol{\varphi}_1^{n+1}/dx]_1| \le C\varepsilon_1^{-1/2}B_{\varepsilon_2}(x),\ |[d\boldsymbol{\varphi}_1^{n+1}/dx]_2| \le C\varepsilon_2^{-1/2}B_{\varepsilon_2}(x),\ x\in\bar\Omega. \tag{35}$$

□

Proposition 13. *The singular component* $\widehat{\mathbf{w}}^{n+1} = (\widehat{w}_1^{n+1},\widehat{w}_2^{n+1})^T$ *satisfies*

$$|\widehat{w}_1^{n+1}(x)| \le CB_{\varepsilon_2}(x), \quad |\widehat{w}_2^{n+1}(x)| \le CB_{\varepsilon_2}(x), \tag{36}$$

$$\left|\frac{d\widehat{w}_1^{n+1}}{dx}\right| \le C(\varepsilon_1^{-1/2}B_{\varepsilon_1}(x)+\varepsilon_2^{-1/2}B_{\varepsilon_2}(x)),\ \left|\frac{d\widehat{w}_2^{n+1}}{dx}\right| \le C\varepsilon_2^{-1/2}B_{\varepsilon_2}(x), \tag{37}$$

$$\left|\frac{d^2\widehat{w}_1^{n+1}}{dx^2}\right| \le C(\varepsilon_1^{-1}B_{\varepsilon_1}(x)+\varepsilon_2^{-1}B_{\varepsilon_2}(x)), \quad \left|\frac{d^2\widehat{w}_2^{n+1}}{dx^2}\right| \le C\varepsilon_2^{-1}B_{\varepsilon_2}(x). \tag{38}$$

Proof. From Lemma 5 we have $|\mathbf{w}(x,t_n)| \le CB_{\varepsilon_2}(x)$ and also

$$\varepsilon_1\left|\frac{\partial^2 w_1}{\partial x^2}\right| \le C(B_{\varepsilon_1}(x)+(\varepsilon_1/\varepsilon_2)B_{\varepsilon_2}(x)) \le CB_{\varepsilon_2}(x),\ \varepsilon_2\left|\frac{\partial^2 w_2}{\partial x^2}\right| \le CB_{\varepsilon_2}(x).$$

Then, it is straightforward to prove that

$$|(I+(\Delta t/2)L_{x,\boldsymbol{\varepsilon}})\widehat{\mathbf{w}}^{n+1}(x)| = |(I-(\Delta t/2)L_{x,\boldsymbol{\varepsilon}})\mathbf{w}(x,t_n)| \le CB_{\varepsilon_2}(x).$$

Defining the barrier function $\boldsymbol{\psi} = B_{\varepsilon_2}(x)\mathbf{C}$, it holds

$$(I+(\Delta t/2)L_{x,\boldsymbol{\varepsilon}})\boldsymbol{\psi} = \boldsymbol{\psi}+CB_{\varepsilon_2}(x)(\Delta t/2)\left(-(\alpha/\varepsilon_2)\boldsymbol{\varepsilon}+A\cdot\mathbf{1}\right) \ge \boldsymbol{\psi},$$

and using again the maximum principle for $(I+(\Delta t/2)L_{x,\boldsymbol{\varepsilon}})$, we obtain $|\widehat{\mathbf{w}}^{n+1}(x)| \le \boldsymbol{\psi}(x)$. For higher order derivatives, we define the auxiliary problem

$$\begin{cases} (I+(\Delta t/2)L_{x,\boldsymbol{\varepsilon}})\boldsymbol{\varphi}_1^{n+1} = -L_{x,\boldsymbol{\varepsilon}}\mathbf{w}(x,t_n), \\ \boldsymbol{\varphi}_1(0) = (\widehat{\mathbf{w}}^{n+1}(0)-\mathbf{w}(0,t_n))/\Delta t, \quad \boldsymbol{\varphi}_1(1) = (\widehat{\mathbf{w}}^{n+1}(1)-\mathbf{w}(1,t_n))/\Delta t, \end{cases}$$

whose solution is given in (31). Using that $|L_{x,\boldsymbol{\varepsilon}}\mathbf{w}(x,t_n)| = |\mathbf{w}_t(x,t_n)| \le CB_{\varepsilon_2}$ and $|\boldsymbol{\varphi}_1^{n+1}(0)| \le C,\ |\boldsymbol{\varphi}_1^{n+1}(1)| \le C$, the maximum principle for $(I+(\Delta t/2)L_{x,\boldsymbol{\varepsilon}})$ proves that $|\boldsymbol{\varphi}_1^{n+1}(x)| \le CB_{\varepsilon_2}(x)$.

Next, we write the problem (24) as follows:

$$L_{x,\boldsymbol{\varepsilon}}\widehat{\mathbf{w}}^{n+1}(x) = -2\boldsymbol{\varphi}_1^{n+1}(x) - L_{x,\boldsymbol{\varepsilon}}\mathbf{w}(x,t_n), \quad \widehat{\mathbf{w}}^{n+1}(0),\ \widehat{\mathbf{w}}^{n+1}(1) \ \text{ given.}$$

From $|\boldsymbol{\varphi}_1^{n+1}(x)| \le CB_{\varepsilon_2}(x)$, $|L_{x,\boldsymbol{\varepsilon}}\mathbf{w}(x,t_n)| \le CB_{\varepsilon_2}(x)$ and $|\widehat{\mathbf{w}}^{n+1}(x)| \le CB_{\varepsilon_2}(x)$, it is straightforward that

$$\left|\frac{d^2\widehat{w}_1^{n+1}(x)}{dx^2}\right| \le C\varepsilon_1^{-1}B_{\varepsilon_2}(x), \quad \left|\frac{d^2\widehat{w}_2^{n+1}(x)}{dx^2}\right| \le C\varepsilon_2^{-1}B_{\varepsilon_2}(x), \tag{39}$$

and therefore, following similar techniques to the ones used in [14], we can deduce

$$\left|\frac{d\widehat{w}_1^{n+1}(x)}{dx}\right| \le C\varepsilon_1^{-1/2}B_{\varepsilon_2}(x), \quad \left|\frac{d\widehat{w}_2^{n+1}(x)}{dx}\right| \le C\varepsilon_2^{-1/2}B_{\varepsilon_2}(x). \tag{40}$$

From (39) and (40) it follows the required result for the first and the second derivative of w_2. Nevertheless, for w_1 we need to improve the previous bounds. Then, differentiating the first equation of (24) w.r.t. x, we obtain

$$\begin{cases} (I+(\Delta t/2)L^1_{x,\varepsilon_1})\varphi_3^{n+1}(x) \\ \qquad = -L^1_{x,\varepsilon_1}\dfrac{dw_1}{dx}(x,t_n) - \dfrac{1}{2}\dfrac{d}{dx}\left[a_{12}(x)(w_2(x,t_n)+\widehat{w}_2^{n+1}(x))\right] \\ \qquad -\dfrac{1}{2}a'_{11}(x)(w_1(x,t_n)+\widehat{w}_1^{n+1}(x)), \\ \varphi_3^{n+1}(0) = (\dfrac{d\widehat{w}_1^{n+1}}{dx}(0) - \dfrac{dw_1}{dx}(0,t_n))/\Delta t, \\ \varphi_3^{n+1}(1) = (\dfrac{d\widehat{w}_1^{n+1}}{dx}(1) - \dfrac{dw_1}{dx}(1,t_n))/\Delta t, \end{cases}$$

where $L^1_{x,\varepsilon_1}z \equiv -\varepsilon_1 z'' + a_{11}z$, whose solution is

$$\varphi_3^{n+1}(x) = (\frac{d\widehat{w}_1^{n+1}}{dx}(x) - \frac{dw_1}{dx}(x,t_n))/\Delta t. \tag{41}$$

From Lemma 5, the bounds given in Lemma 12 applied on the boundary, the estimates (36)-(38) and the maximum principle, we obtain $|\varphi_3^{n+1}(x)| \le C(\varepsilon_1^{-1/2}B_{\varepsilon_1}(x) + \varepsilon_2^{-1/2}B_{\varepsilon_2}(x))$. Now, we define the problem

$$\begin{cases} L_{x,\varepsilon_1}\dfrac{d\widehat{w}_1^{n+1}}{dx}(x) = h(x) \equiv -2\varphi_3^{n+1}(x) - L_{x,\varepsilon_1}\dfrac{dw_1}{dx}(x,t_n) \\ -\dfrac{d}{dx}\left[a_{12}(x)(w_2(x,t_n)+\widehat{w}_2^{n+1}(x))\right] \\ \quad -a'_{11}(x)(w_1(x,t_n)+\widehat{w}_1^{n+1}(x)), \\ \dfrac{d\widehat{w}_1^{n+1}}{dx}(0),\ \dfrac{d\widehat{w}_1^{n+1}}{dx}(1) \ \text{ given.} \end{cases}$$

It is straightforward to prove that the right hand side $|h(x)| \leq C(\varepsilon_1^{-1/2}B_{\varepsilon_1}(x) + \varepsilon_2^{-1/2}B_{\varepsilon_2}(x))$, which together with the crude bounds (40) for the boundary conditions, permit us to deduce

$$\left|\frac{d\widehat{w}_1^{n+1}}{dx}(x)\right| \leq C(\varepsilon_1^{-1/2}B_{\varepsilon_1}(x) + \varepsilon_2^{-1/2}B_{\varepsilon_2}(x)).$$

For the second derivative the proof follows similar ideas (see [6] for a full proof). $\square$

Proposition 14. *The singular component* $\widehat{\mathbf{w}}^{n+1} = (\widehat{w}_1^{n+1}, \widehat{w}_2^{n+1})^T$ *satisfies*

$$\left|\frac{d^3\widehat{w}_1^{n+1}}{dx^3}\right| \leq C(\varepsilon_1^{-3/2}B_{\varepsilon_1}(x) + {\varepsilon_2}^{-3/2}B_{\varepsilon_2}(x)), \tag{42}$$

$$\left|\frac{d^4\widehat{w}_1^{n+1}}{dx^4}\right| \leq C\varepsilon_1^{-1}(\varepsilon_1^{-1}B_{\varepsilon_1}(x) + \varepsilon_2^{-1}B_{\varepsilon_2}(x)). \tag{43}$$

$$\left|\frac{d^k\widehat{w}_2^{n+1}}{dx^k}\right| \leq C\varepsilon_2^{-1}(\varepsilon_1^{(2-k)/2}B_{\varepsilon_1}(x) + \varepsilon_2^{(2-k)/2}B_{\varepsilon_2}(x)),\ \ k = 3, 4 \tag{44}$$

Proof. From the differential equations satisfied by $\dfrac{d^2\widehat{w}_2^{n+1}}{dx^2}$ and $\dfrac{d^2\widehat{w}_1^{n+1}}{dx^2}$ respectively (see [6]), (44) for $k = 4$ and also (43) are immediate. Using the argument given in [14], we deduce (44) for $k = 3$ and

$$\left|\frac{d^3\widehat{w}_1^{n+1}}{dx^3}\right| \leq C\varepsilon_1^{-1}(\varepsilon_1^{-1/2}B_{\varepsilon_1}(x) + \varepsilon_2^{-1/2}B_{\varepsilon_2}(x)), \tag{45}$$

which is not the required bound (42). It will be obtained by setting the problem

$$L^1_{x,\varepsilon_1}\varphi_3^{n+1}(x) = p(x) \quad x \in (0,1), \quad \varphi_3^{n+1}(0),\, \varphi_3^{n+1}(1) \ \ \text{given}. \tag{46}$$

First, we obtain appropriate bounds for p using the maximum principle. With this purpose, we note that

$$\begin{aligned}&(I + (\Delta t/2)L^1_{x,\varepsilon_1})p = L^1_{x,\varepsilon_1}(I + (\Delta t/2)L^1_{x,\varepsilon_1})\varphi_3^{n+1}(x) = \\ &= L^1_{x,\varepsilon_1}\left(\frac{\partial^2 w_1}{\partial x\partial t} + \frac{1}{2}\left(a'_{11}(w_1 - \widehat{w}_1^{n+1}) + \frac{\partial}{\partial x}(a_{12}(w_2 - \widehat{w}_2^{n+1}))\right)\right).\end{aligned}$$

All functions appearing in the last term have already been bounded and therefore we can easily conclude $|(I + (\Delta t/2)L^1_{x,\varepsilon_1})p| \leq C\varepsilon_1^{-1/2}$. Using a continuity argument, the function p on the boundary satisfies

$$\begin{aligned}&p(x) = L^1_{x,\varepsilon_1}\varphi_3^{n+1}(x) = (2/\Delta t)\left(\frac{\partial^2 w_1}{\partial x\partial t} + (\frac{\partial w_1}{\partial x}(x,t_n) - \frac{\partial w_1}{\partial x}(x,t_{n+1}))/\Delta t + \right.\\ &\left. + (\frac{\partial w_1}{\partial x}(x,t_{n+1}) - \frac{d\widehat{w}_1^{n+1}}{dx}(x))/\Delta t\right) + a'_{12}(w_2(x,t_n) - \widehat{w}_2^{n+1}(x))/\Delta t + \\ &+ a_{12}(\frac{\partial w_2}{\partial x}(x,t_n) - \frac{d\widehat{w}_2^{n+1}}{dx}(x))/\Delta t + a'_{11}(w_1(x,t_n) - \widehat{w}_1^{n+1}(x))/\Delta t.\end{aligned}$$

Lemmas 6, 9 and 12 prove $|p(x)| \leq C\varepsilon_1^{-1/2}$, $x = 0, 1$. Then, from the maximum principle it follows $|p(x)| \leq C\varepsilon_1^{-1/2}$ for $x \in [0,1]$, and therefore, from (46), we deduce

$$\left|\frac{\partial^2\varphi_3^{n+1}}{\partial x^2}\right| \leq C\varepsilon_1^{-3/2}.$$

Differentiating equations (8) and (24) three times w.r.t. x, we obtain

$$\begin{cases} \big(I + (\Delta t/2) L^1_{x,\varepsilon_1}\big) \dfrac{\partial^3}{\partial x^3} \big((\widehat{w}_1^{n+1} - w_1)/\Delta t)\big) \\ = -L^1_{x,\varepsilon_1} \dfrac{d^3 w_1}{dx^3} - \dfrac{1}{2} \dfrac{\partial^3}{\partial x^3} \big(a_{12}(\widehat{w}_2^{n+1} + w_2)\big) \\ -\dfrac{1}{2} \left(a'''_{11}(\widehat{w}_1^{n+1} + w_1) + 3a''_1 \dfrac{\partial}{\partial x}(\widehat{w}_1^{n+1} + w_1) + 3a'_1 \dfrac{\partial^2}{\partial x^2}(\widehat{w}_1^{n+1} + w_1) \right), \quad x \in (0,1), \\ \dfrac{\partial^3}{\partial x^3} \big((\widehat{w}_1^{n+1} - w_1)/\Delta t\big), \ \dfrac{\partial^3}{\partial x^3} \big((\widehat{w}_1^{n+1} - w_1)/\Delta t\big), \quad \text{given in } x = 0, 1. \end{cases}$$

Taking into account that

$$\left| L^1_{x,\varepsilon_1} \frac{\partial^3 w_1}{\partial x^3} \right| = \left| \frac{\partial^4 w_1}{\partial^3 x \partial t} + a'''_{11} w_1 + 3a''_{11} \frac{\partial w_1}{\partial x} + 3a'_{11} \frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^3}{\partial x^3}(a_{12} w_2) \right| \leq$$
$$\leq C(\varepsilon_1^{-3/2} B_{\varepsilon_1}(x) + \varepsilon_2^{-3/2} B_{\varepsilon_2}(x)),$$

the maximum principle proves

$$\left| \frac{\partial^3}{\partial x^3} \big((\widehat{w}_1^{n+1} - w_1)/\Delta t\big) \right| \leq C(\varepsilon_1^{-3/2} B_{\varepsilon_1}(x) + \varepsilon_2^{-3/2} B_{\varepsilon_2}(x)), \quad x \in [0,1].$$

Finally, we consider the problem

$$\begin{cases} L^1_{x,\varepsilon_1} \dfrac{d^3 \widehat{w}_1}{dx^3} = -2 \dfrac{\partial^3}{\partial x^3} \big((\widehat{w}_1^{n+1} - w_1)/\Delta t\big) - L^1_{x,\varepsilon_1} \dfrac{\partial^3 w_1}{\partial x^3} - \dfrac{\partial^3}{\partial x^3} \big(a_{12}(\widehat{w}_2^{n+1} + w_2)\big) \\ - \left(a'''_{11}(\widehat{w}_1^{n+1} + w_1) + 3a''_1 \dfrac{\partial}{\partial x}(\widehat{w}_1^{n+1} + w_1) + 3a'_1 \dfrac{\partial^2}{\partial x^2}(\widehat{w}_1^{n+1} + w_1) \right), \quad x \in (0,1), \\ \dfrac{d^3 \widehat{w}_1}{dx^3}(0), \ \dfrac{\partial^3 \widehat{w}_1}{\partial x^3}(1) \text{ given.} \end{cases}$$

Using in the boundary the crude bounds (45), again from the maximum principle we obtain (42).

□

Proposition 15. *Suppose that $\varepsilon_1 < \varepsilon_2$. Then, the singular component can be decomposed as*

$$\widehat{w}_1^{n+1} = \widehat{w}_{1,\varepsilon_1}^{n+1} + \widehat{w}_{1,\varepsilon_2}^{n+1}, \quad \widehat{w}_2^{n+1} = \widehat{w}_{2,\varepsilon_1}^{n+1} + \widehat{w}_{2,\varepsilon_2}^{n+1},$$

where

$$\left| \frac{d^2 \widehat{w}_{1,\varepsilon_1}^{n+1}}{dx^2} \right| \leq C\varepsilon_1^{-1} B_{\varepsilon_1}(x), \qquad \left| \frac{d^3 \widehat{w}_{1,\varepsilon_2}^{n+1}}{dx^3} \right| \leq C\varepsilon_2^{-3/2} B_{\varepsilon_2}(x), \tag{47}$$

$$\left| \frac{d^2 \widehat{w}_{2,\varepsilon_1}^{n+1}}{dx^2} \right| \leq C\varepsilon_2^{-1} B_{\varepsilon_1}(x), \qquad \left| \frac{d^3 \widehat{w}_{2,\varepsilon_2}^{n+1}}{dx^3} \right| \leq C\varepsilon_2^{-3/2} B_{\varepsilon_2}(x), \tag{48}$$

and also it can be decomposed as

$$\widehat{w}_1^{n+1} = \widehat{z}_{1,\varepsilon_1}^{n+1} + \widehat{z}_{1,\varepsilon_2}^{n+1}, \quad \widehat{w}_2^{n+1} = \widehat{z}_{2,\varepsilon_1}^{n+1} + \widehat{z}_{2,\varepsilon_2}^{n+1},$$

where

$$\left| \frac{d^2 \widehat{z}_{1,\varepsilon_1}^{n+1}}{dx^2} \right| \leq C\varepsilon_1^{-1} B_{\varepsilon_1}(x), \qquad \left| \frac{d^4 \widehat{z}_{1,\varepsilon_2}^{n+1}}{dx^4} \right| \leq C\varepsilon_1^{-1}\varepsilon_2^{-1} B_{\varepsilon_2}(x), \tag{49}$$

$$\left| \frac{d^2 \widehat{z}_{2,\varepsilon_1}^{n+1}}{dx^2} \right| \leq C\varepsilon_2^{-1} B_{\varepsilon_1}(x), \qquad \left| \frac{d^4 \widehat{z}_{2,\varepsilon_2}^{n+1}}{dx^4} \right| \leq C\varepsilon_2^{-2} B_{\varepsilon_2}(x). \tag{50}$$

Proof. It is completely similar to the proofs given in [13, 14]. □

4. The totally discrete scheme

We discretize (21) by the central difference scheme defined on a piecewise uniform mesh $\bar{\Omega}^N$ of Shishkin type (see [8]). From Proposition 13 we know that there are two overlapping boundary layers at $x=0$ and $x=1$. Then, to define the Shishkin mesh we use two transition parameters given by

$$\tau_{\varepsilon_2} = \min\left\{1/4, 2\sqrt{\varepsilon_2/\alpha}\ln N\right\}, \quad \tau_{\varepsilon_1} = \min\left\{\tau_{\varepsilon_2}/2, 2\sqrt{\varepsilon_1/\alpha}\ln N\right\},$$

where α is given in (3). In the subintervals $[0,\tau_{\varepsilon_1}]$, $[\tau_{\varepsilon_1},\tau_{\varepsilon_2}]$, $[\tau_{\varepsilon_2},1-\tau_{\varepsilon_2}]$, $[1-\tau_{\varepsilon_2},1-\tau_{\varepsilon_1}]$ and $[1-\tau_{\varepsilon_1},1]$ we distribute uniformly $N/8+1$, $N/8+1$, $N/2+1$, $N/8+1$ and $N/8+1$ mesh points respectively. So, the mesh points are

$$x_j = \begin{cases} jh_{\varepsilon_1}, & j=0,\dots,N/8, \\ x_{N/8}+(j-N/8)h_{\varepsilon_2}, & j=N/8+1,\dots,N/4, \\ x_{N/4}+(j-N/4)H, & j=N/4+1,\dots,3N/4, \\ x_{3N/4}+(j-3N/4)h_{\varepsilon_2}, & j=3N/4+1,\dots,7N/8, \\ x_{7N/8}+(j-7N/8)h_{\varepsilon_1}, & j=7N/8+1,\dots,N, \end{cases}$$

where $h_{\varepsilon_1} = 8\tau_{\varepsilon_1}/N,\ h_{\varepsilon_2} = 8(\tau_{\varepsilon_2}-\tau_{\varepsilon_1})/N,\ H = 2(1-2\tau_{\varepsilon_2})/N$. Here we only consider the most interesting case when $\sqrt{\varepsilon_2} \le CN^{-1}$ and therefore $\tau_{\varepsilon_2} = 2\sqrt{\varepsilon_2/\alpha}\ln N$ $\tau_{\varepsilon_1} = 2\sqrt{\varepsilon_1/\alpha}\ln N$. Below we denote the local step sizes by $h_j = x_j - x_{j-1},\ j = 1,\dots,N$. On $\bar{\Omega}^N$ the central finite difference scheme is given by

$$\tag{51} \begin{array}{l} \mathbf{U}_j^0 = \mathbf{0}, \qquad 0 \le j \le N, \\ \begin{cases} \text{For } n=0,\dots,M-1, \\ \left(I+(\Delta t/2)L_{x,\varepsilon}^N\right)\mathbf{U}_j^{n+1} = \left(I-(\Delta t/2)L_{x,\varepsilon}^N\right)\mathbf{U}_j^n + (\Delta t/2)(\mathbf{f}_j^{n+1}+\mathbf{f}_j^n), \\ \mathbf{U}_0^{n+1} = \mathbf{U}_N^{n+1} = \mathbf{0}, \end{cases} \end{array}$$

where

$$L_{x,\varepsilon}^N \equiv \begin{pmatrix} -\varepsilon_1 & \\ & -\varepsilon_2 \end{pmatrix}\delta^2 + AI, \quad \delta^2 Z_j = \frac{2}{h_j+h_{j+1}}\left(\frac{Z_{j+1}-Z_j}{h_{j+1}} - \frac{Z_j - Z_{j-1}}{h_j}\right).$$

Lemma 16. *For each value of Δt, the discrete operator $\left(I+(\Delta t/2)L_{x,\varepsilon}^N\right)$ is uniformly stable and it satisfies a discrete maximum principle.*

Proof. It trivially follows using that $\left(I+(\Delta t/2)L_{x,\varepsilon}^N\right)$ is an M-matrix . □

To prove the uniform convergence of (51), we split the global error at the time t_{n+1} in the form

$$\tag{52} \mathbf{u}(x_j,t_{n+1}) - \mathbf{U}_j^{n+1} = \left(\mathbf{u}(x_j,t_{n+1}) - \widehat{\mathbf{u}}^{n+1}(x_j)\right) + \left(\widehat{\mathbf{u}}^{n+1}(x_j) - \widehat{\mathbf{U}}_j^{n+1}\right) + \left(\widehat{\mathbf{U}}_j^{n+1} - \mathbf{U}_j^{n+1}\right),$$

where $\widehat{\mathbf{U}}_j^{n+1}$ is the solution of

$$\tag{53} \begin{cases} \left(I+(\Delta t/2)L_{x,\varepsilon}^N\right)\widehat{\mathbf{U}}_j^{n+1} = \left(I-(\Delta t/2)L_{x,\varepsilon}^N\right)\mathbf{u}(x_j,t_n) + (\Delta t/2)(\mathbf{f}_j^{n+1}+\mathbf{f}_j^n), \\ \widehat{\mathbf{U}}_0^{n+1} = \widehat{\mathbf{U}}_N^{n+1} = \mathbf{0},\ 0<j<N. \end{cases}$$

To bound appropriately $\left(\widehat{\mathbf{u}}^{n+1}(x_j) - \widehat{\mathbf{U}}_j^{n+1}\right)$ we consider a further decomposition $\widehat{\mathbf{U}}^{n+1} = \widehat{\mathbf{V}}^{n+1} + \widehat{\mathbf{W}}^{n+1},\ n=0,1,\cdots,M-1$, where

$$\begin{cases} \left(I+(\Delta t/2)L_{x,\varepsilon}^N\right)\widehat{\mathbf{V}}_j^{n+1} = \left(I-(\Delta t/2)L_{x,\varepsilon}^N\right)\mathbf{v}(x_j,t_n) + (\Delta t/2)(\mathbf{f}_j^{n+1}+\mathbf{f}_j^n), \\ \widehat{\mathbf{V}}_0^{n+1} = \widehat{\mathbf{v}}^{n+1}(0),\ \widehat{\mathbf{V}}_N^{n+1} = \widehat{\mathbf{v}}^{n+1}(1),\ 0<j<N, \end{cases}$$

and

$$\begin{cases} \left(I + (\Delta t/2)L^N_{x,\varepsilon}\right)\widehat{\mathbf{W}}^{n+1}_j = \left(I - (\Delta t/2)L^N_{x,\varepsilon}\right)\mathbf{w}(x_j, t_n),\ 0 < j < N, \\ \widehat{\mathbf{W}}^{n+1}_0 = \widehat{\mathbf{w}}^{n+1}(0),\ \widehat{\mathbf{W}}^{n+1}_N = \widehat{\mathbf{w}}^{n+1}(1). \end{cases}$$

Proposition 17. *Let $\widehat{\mathbf{u}}^{n+1}(x)$ be the solution of (22) and $\{\widehat{\mathbf{U}}^{n+1}_j\}$ the solution of (53). Then, if $\sqrt{\varepsilon_1} < \sqrt{\varepsilon_2} \le N^{-1}$, it holds*

$$\|\widehat{\mathbf{u}}^{n+1}(x_j) - \widehat{\mathbf{U}}^{n+1}_j\|_{\bar{\Omega}^N} \le C\Delta t (N^{-1}\ln N)^2. \tag{54}$$

Proof. The local error for the central difference scheme, at the mesh point x_j, satisfies

$$(I + (\Delta t/2)L^N_{x,\varepsilon})(\widehat{\mathbf{u}}^{n+1}(x_j) - \widehat{\mathbf{U}}^{n+1}_j) = (\Delta t/2)(L^N_{x,\varepsilon} - L_{x,\varepsilon})(\widehat{\mathbf{u}}^{n+1}(x_j) + \mathbf{u}(x_j, t_n)). \tag{55}$$

Using the decomposition into regular and singular components, appropriate Taylor expansions and the hypothesis $\sqrt{\varepsilon_1} < \sqrt{\varepsilon_2} \le N^{-1}$, we can obtain

$$\begin{array}{l} |(L^N_{x,\boldsymbol{\varepsilon}} - L_{x,\boldsymbol{\varepsilon}})(\widehat{\mathbf{v}}^{n+1}(x_j) + \mathbf{v}(x_j, t_n))| \le CN^{-1}(\varepsilon_1^{1/2}, \varepsilon_2^{1/2})^T \le CN^{-2}(1,1)^T, \\ |(L^N_{x,\boldsymbol{\varepsilon}} - L_{x,\boldsymbol{\varepsilon}})(\widehat{\mathbf{w}}^{n+1}(x_j) + \mathbf{w}(x_j, t_n))| \le C(N^{-1}\ln N)^2(1,1)^T, \end{array} \tag{56}$$

Using the uniform stability of the discrete operator $(I + (\Delta t/2)L^N_{x,\varepsilon})$, from (56) it follows (54).

∎

Theorem 18. *Let $\mathbf{u}(x,t)$ be the solution of (1) and $\{\mathbf{U}^{n+1}_j\}$ the solution of (51). Then, if $\sqrt{\varepsilon_1} < \sqrt{\varepsilon_2} \le N^{-1}$, and we assume that the powers of the transition operator $R_{N,\Delta t}$ associated to (51) given by*

$$R_{N,\Delta t} = (I + (\Delta t/2)L^N_{x,\varepsilon})^{-1}(I - (\Delta t/2)L^N_{x,\varepsilon}), \tag{57}$$

are uniformly bounded, then the error satisfies

$$\|\mathbf{u}(x_j, t_{n+1}) - \mathbf{U}^{n+1}_j\|_{\bar{Q}^N} \le C(N^{-2}\ln^2 N + (\Delta t)^2), \tag{58}$$

Proof. From (52) we must bound three different terms. First, from Lemma 7 we have

$$\|\mathbf{u}(x_j, t_{n+1}) - \widehat{\mathbf{u}}^{n+1}(x_j)\|_{\bar{\Omega}^N} \le C_{n+1}(\Delta t)^3.$$

In second place, from Proposition 17 it follows

$$\|\widehat{\mathbf{u}}^{n+1}(x_j) - \widehat{\mathbf{U}}^{n+1}_j\|_{\bar{\Omega}^N} \le C_{n+1}\Delta t (N^{-1}\ln N)^2.$$

Finally, to bound $\|\widehat{\mathbf{U}}^{n+1}_j - \mathbf{U}^{n+1}_j\|$, we take into account that

$$\widehat{\mathbf{U}}^{n+1}_j - \mathbf{U}^{n+1}_j = R_{N,\Delta t}\left(\mathbf{u}(x_j, t_n) - \mathbf{U}^n_j\right).$$

Similarly to previous bounds, we can obtain

$$\begin{array}{l} \mathbf{u}(x_j, t_n) - \mathbf{U}^n_j = (\mathbf{u}(x_j, t_n) - \widehat{\mathbf{u}}^n(x_j)) + \left(\widehat{\mathbf{u}}^n(x_j) - \widehat{\mathbf{U}}^n_j\right) + \left(\widehat{\mathbf{U}}^n_j - \mathbf{U}^n_j\right), \\ \|\mathbf{u}(x_j, t_n) - \widehat{\mathbf{u}}^n(x_j)\|_{\bar{\Omega}^N} \le C_n(\Delta t)^3, \\ \|\widehat{\mathbf{u}}^n(x_j) - \widehat{\mathbf{U}}^n_j\|_{\bar{\Omega}^N} \le C_n\Delta t(N^{-1}\ln N)^2), \\ \widehat{\mathbf{U}}^n_j - \mathbf{U}^n_j = R_{N,\Delta t}\left(\mathbf{u}(x_j, t_{n-1}) - \mathbf{U}^{n-1}_j\right). \end{array}$$

Then, using a recursive argument we deduce

$$\|\mathbf{u}(x_j,t_{n+1})-\mathbf{U}_j^{n+1}\|_{\bar{\Omega}^N}\leq C\sum_{i=1}^{n}\|R_{N,\Delta t}^{n-i}\|_{\bar{\Omega}^N}((\Delta t)^3+\Delta t(N^{-1}\ln N)^2)\leq$$
$$\leq C\Delta t\sum_{i=1}^{n}\|R_{N,\Delta t}^{n-i}\|_{\bar{\Omega}^N}((\Delta t)^2+N^{-2}\ln^2 N).$$

where $C=\max\{C_1,C_2,\cdots,C_{n+1}\}$. Then, using that the powers of the transition operators $R_{N,\Delta t}$ are uniformly bounded, the result follows. □

Remark 19. *From the results given in* [1], [7] *and* [17], *it is known the stability of the Crank-Nicolson method for a fixed value of the diffusion parameters, even on nonquasiuniform meshes. Nevertheless, so far, we do not have a theoretical proof giving the stability of this method uniformly with respect to the diffusion parameters on Shishkin meshes. On the other hand, the experiments performed in many cases (see next section for an example), give us a numerical evidence that the maximum norm of the powers of the transition operator $R_{N,\Delta t}$ are bounded independently of the parameters ε_1, and ε_2. The theoretical results in previously cited papers and the numerical experiments performed (see Tables 2, 3 and 4) allow us to conjecture that the uniform stability of our totally discrete method holds.*

Remark 20. *In the case that the the transition parameters are $\tau_{\varepsilon_1}=1/8$, $\tau_{\varepsilon_2}=1/4$, which corresponds to large values of ε_1 and ε_2 and therefore the mesh is uniform, a classical analysis proves the same bounds that in Theorem 18. Also, when $\varepsilon_1=\varepsilon_2\leq N^{-2}$ a sharp analysis (see* [6]*) gives the bound (58) for the error.*

In other case, i.e., when either $\varepsilon_1\ll\varepsilon_2=1$ or the hypothesis $\sqrt{\varepsilon_1}<\sqrt{\varepsilon_2}\leq N^{-1}$ assumed in Theorem 18 does not hold, it is possible to prove that (see [6] *for full details of the proof)*

$$\|\mathbf{u}(x_j,t_{n+1})-\mathbf{U}_j^{n+1}\|_{\bar{Q}^N}\leq C(N^{-2+q}\ln^2 N+(\Delta t)^2),$$

where $N^{-q}\leq C\Delta t$, with $0<q<1$, and therefore the order is reduced. Nevertheless, in practice this reduction is not observed.

5. Numerical results

In this section we only show the numerical results obtained in one example given by

$$\left.\begin{array}{l}\dfrac{\partial u_1}{\partial t}-\varepsilon_1\dfrac{\partial^2 u_1}{\partial x^2}+2(1+x)^2u_1-(1+x^3)u_2=2e^xt(1-t),\\ \dfrac{\partial u_2}{\partial t}-\varepsilon_2\dfrac{\partial^2 u_2}{\partial x^2}-2\cos(\pi x/4)u_1+4e^{1-x}u_2=(10x+1)t(1-t),\end{array}\right\}\ (x,t)\in\Omega_T$$
$$u(0,t)=u(1,t)=0,\quad t\in[0,1],\quad u(x,0)=0,\quad x\in(0,1].$$

where $\Omega_T=(0,1)\times(0,1]$. Note that the compatibility conditions (5) at the corners $(0,0)$ and $(1,0)$ are not satisfied in this example, but we obtain (see below) accurate results showing the second order uniform convergence of the numerical scheme. A more detailed analysis and more examples showing the influence of the compatibility conditions between data of the problem are given in [6].

To find an approximation to the pointwise errors $|\mathbf{U}_j^n-\mathbf{u}(x_j,t_n)|$, we use a variant of the double mesh principle. So, we calculate $\{\mathbf{Z}_j^n\}$ on the mesh $\{(\hat{x}_j,\hat{t}_n)\}$ that contains the mesh points of the original mesh and their midpoints, i.e., the mesh points are

$$\begin{array}{ll}\hat{x}_{2j}=x_j,\ \ j=0,\ldots,N, & \hat{x}_{2j+1}=(x_j+x_{j+1})/2,\ \ j=0,\ldots,N-1,\\ \hat{t}_{2n}=t_n,\ \ n=0,\ldots,M, & \hat{t}_{2n+1}=(t_n+t_{n+1})/2,\ \ n=0,\ldots,M-1.\end{array}$$

At the original mesh points (x_j, t_n), the maximum errors and the uniform errors are approximated by $\mathbf{d}_{\boldsymbol{\varepsilon},N,\Delta t} = \max_{0\le n\le M} \max_{0\le j\le N} |\mathbf{U}_j^n - \mathbf{Z}_{2j}^{2n}|$, $\mathbf{d}_{N,\Delta t} = \max_S d_{\boldsymbol{\varepsilon},N,\Delta t}$, where S is the set

$$S = \{(\varepsilon_1, \varepsilon_2) \mid \varepsilon_2 = 2^{-6}, \ldots, 2^{-30},\ \varepsilon_1 = \varepsilon_2, 2^{-2}\varepsilon_2, \ldots, 2^{-58}, 2^{-60}\}, \tag{59}$$

in order to permit that the maximum errors stabilize. From these values we obtain the corresponding orders of convergence and the uniform orders of convergence in a standard way, by using $\mathbf{p} = \log_2(\mathbf{d}_{\boldsymbol{\varepsilon},N,\Delta t}/\mathbf{d}_{\boldsymbol{\varepsilon},2N,\Delta t/2})$, $\mathbf{p}_{uni} = \log_2(\mathbf{d}_{N,\Delta t}/\mathbf{d}_{2N,\Delta t/2})$.

Table 1 displays the results obtained in this case. From it we see that the method gives almost second order of uniform convergence, in agreement with Theorem 18.

To illustrate the conjecture about the bounds of powers of operator $R_{N,\Delta t}$, we

TABLE 1. Maximum (in ε_1) errors and orders of convergence

ε_2	N=16 $\Delta t = 0.2$	N=32 $\Delta t = 0.2/2$	N=64 $\Delta t = 0.2/2^2$	N=128 $\Delta t = 0.2/2^3$	N=256 $\Delta t = 0.2/2^4$	N=512 $\Delta t = 0.2/2^5$	N=1024 $\Delta t = 0.2/2^6$
2^{-6}	1.933E-2 1.685	6.013E-3 0.285	4.935E-3 0.370	3.820E-3 1.152	1.719E-3 1.568	5.800E-4 1.657	1.839E-4
2^{-8}	4.895E-2 1.401	1.853E-2 1.807	5.294E-3 0.472	3.817E-3 1.152	1.718E-3 1.568	5.794E-4 1.657	1.837E-4
2^{-10}	8.561E-2 0.382	6.571E-2 1.735	1.974E-2 1.825	5.571E-3 1.697	1.718E-3 1.568	5.794E-4 1.657	1.837E-4
2^{-12}	8.555E-2 0.381	6.567E-2 0.799	3.775E-2 1.210	1.632E-2 1.499	5.775E-3 1.565	1.951E-3 1.649	6.222E-4
2^{-14}	8.551E-2 0.381	6.565E-2 0.799	3.775E-2 1.210	1.632E-2 1.498	5.775E-3 1.565	1.951E-3 1.649	6.223E-4
...	...	...	...	...	...	...	...
2^{-30}	8.549E-2 0.381	6.564E-2 0.799	3.774E-2 1.210	1.631E-2 1.498	5.776E-3 1.565	1.952E-3 1.649	6.223E-4
$\mathbf{d}_{N,\Delta t}$ $[\mathbf{p}_{uni}]$	8.561E-2 0.382	6.571E-2 0.799	3.775E-2 1.210	1.632E-2 1.499	5.776E-3 1.565	1.952E-3 1.649	6.223E-4

calculate the maximum norm for $R^p_{N,\Delta t}$, for some values of $\varepsilon_1, \varepsilon_2, N$ and Δt. Note that if we have $\|R^p_{N,\Delta t}\| < 1$ for some value of p, then for any $q > p$ with $q = p\cdot r + s$, $0 \le s < p$, it holds

$$\|R^q_{N,\Delta t}\| \le (\|R^p_{N,\Delta t}\|)^r (\|R_{N,\Delta t}\|)^s,$$

and then trivially it follows that $\|R^q_{N,\Delta t}\|$ is bounded. Tables 2 and 3 display the maximum norm of $R^p_{N,\Delta t}$; from them, we clearly observe that its value is bounded and the larger is p, the smaller is $\|R^p_{N,\Delta t}\|$ for N and Δt fixed. Same conclusions has been obtained for any value of the diffusion parameters considered in the experiments. Moreover, we also have observed that the maximum norm of the power of the discrete transition operator $R_{N,\Delta t}$ stabilize when the diffusion parameters are sufficiently small.

TABLE 2. Maximum norm for $R^p_{N,\Delta t}$, with $\varepsilon_1 = 10^{-6}, \varepsilon_2 = 10^{-3}$

	N=16 $\Delta t = 0.2$	N=32 $\Delta t = 0.2/2$	N=64 $\Delta t = 0.2/2^2$	N=128 $\Delta t = 0.2/2^3$	N=256 $\Delta t = 0.2/2^4$	N=512 $\Delta t = 0.2/2^5$	N=1024 $\Delta t = 0.2/2^6$
$\|R_{N,\Delta t}\|$	0.4464470	0.6916734	0.8418833	0.9208203	0.9608592	1.0542657	1.5341987
$\|R^3_{N,\Delta t}\|$	0.0646130	0.2973178	0.5677750	0.7655138	0.8809182	0.9410089	1.1171190
$\|R^5_{N,\Delta t}\|$	0.0097496	0.1282374	0.3792081	0.6292697	0.8030617	0.9009582	0.9510187

Finally we are interested in to see what happens when we consider a large value of the discretization parameter N and the time step Δt is not too small. Table 4 display the results obtained for different values of the diffusion parameters ε_1 and

TABLE 3. Maximum norm for $R^p_{N,\Delta t}$, with $\varepsilon_1 = 10^{-10}, \varepsilon_2 = 10^{-6}$

	N=16 $\Delta t = 0.2$	N=32 $\Delta t = 0.2/2$	N=64 $\Delta t = 0.2/2^2$	N=128 $\Delta t = 0.2/2^3$	N=256 $\Delta t = 0.2/2^4$	N=512 $\Delta t = 0.2/2^5$	N=1024 $\Delta t = 0.2/2^6$
$\|R_{N,\Delta t}\|$	0.4777628	0.7087096	0.8483049	0.9237030	0.9620024	0.9810943	0.9905794

ε_2. From it, clearly we again deduce that the powers of the transition operator are bounded in all cases, according with our conjecture.

TABLE 4. Maximum norm for $R^p_{N,\Delta t}$, with $N = 512, \Delta t = 0.1$

	$\varepsilon_1 = 10^{-2}$ $\varepsilon_2 = 1$	$\varepsilon_1 = 10^{-5}$ $\varepsilon_2 = 10^{-2}$	$\varepsilon_1 = 10^{-8}$ $\varepsilon_2 = 10^{-4}$	$\varepsilon_1 = 10^{-11}$ $\varepsilon_2 = 10^{-6}$
$\|R_{N,\Delta t}\|$	2.4634039	2.2992496	1.1744234	1.3164294
$\|R^3_{N,\Delta t}\|$	2.0801142	1.9027820	0.4448608	0.4548921
$\|R^5_{N,\Delta t}\|$	2.0851964	1.8157999	0.2321241	0.2353049
$\|R^7_{N,\Delta t}\|$	2.0809942	1.7585435	0.1188727	0.1196725
$\|R^9_{N,\Delta t}\|$	2.0786761	1.7103613	0.0614513	0.0619040
$\|R^{11}_{N,\Delta t}\|$	2.0752610	1.6677914	0.1186732	0.0325003
$\|R^{13}_{N,\Delta t}\|$	2.0717191	1.6307874	0.0321934	0.0169643
$\|R^{15}_{N,\Delta t}\|$	2.0691277	1.5943620	0.0087984	0.0088996

Acknowledgments

The authors are very grateful to the anonymous referees for their careful reading of the paper and their helpful suggestions to improve this work.

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