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POINTWISE APPROXIMATION OF CORNER SINGULARITIES FOR SINGULARLY PERTURBED ELLIPTIC PROBLEMS WITH CHARACTERISTIC LAYERS

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This paper is dedicated to 70th anniversary of Grigorii I. Shishkin

Abstract. A Dirichlet problem for a singularly perturbed steady-state convection-diffusion equation with constant coefficients on the unit square is considered. In the equation under consideration the convection term is represented by only a single derivative with respect to one coordinate axis. This problem is discretized by the classical five-point upwind difference scheme on a rectangular piecewise uniform mesh that is refined in the neighborhood of the regular and the characteristic boundary layers. It is proved that, for sufficiently smooth right-hand side of the equation and the restrictions of the continuous boundary function to the sides of the square, without additional compatibility conditions at the corners, the error of the discrete solution is $O(N^{-1}\ln^2 N)$ uniformly with respect to the small parameter, in the discrete maximum norm, where N is the number of mesh points in each coordinate direction.

Key Words. parabolic boundary layers, elliptic equation, piecewise uniform mesh, corner singularities.

1. Introduction

In the unit square $\Omega = (0,1)^2$ with the boundary $\partial\Omega$ the following problem is considered

$$Lu := -\varepsilon\Delta u + a\frac{\partial u}{\partial x} + qu = f(x,y), \quad (x,y)\in\Omega, \quad u\,\Big|_{\partial\Omega} = g, \tag{1}$$

where

$$a = \text{const} > 0, \quad q = \text{const} > 0, \tag{2}$$

and $\varepsilon \in (0,1]$ is a small parameter.

Let Γ_k be the sides of the square Ω enumerated counter-clockwise, beginning with $\Gamma_1 = \big\{(x,y)\in\partial\Omega \;\big|\; x=0\big\}$ and let $a_k = (x_k, y_k)$ be its vertices, enumerated in the same way with $a_1 = (0,0)$. Let also $g_k := g\,\Big|_{\Gamma_k}$ denote a restriction of the boundary function g to the side Γ_k of the square Ω.

Numerous investigations (see [11] and the references) show that the solution to problem (1)-(2) has a complicated structure, involving a regular boundary layer in the neighborhood of the right boundary Γ_3, two characteristic layers in the neighborhood of the bottom and the top boundaries Γ_2 and Γ_4, corner layers with corner singularities in the neighborhood of the vertices a_2 and a_3, and corner singularities in the neighborhood of the inflow vertices a_1 and a_4; see Figure 1.

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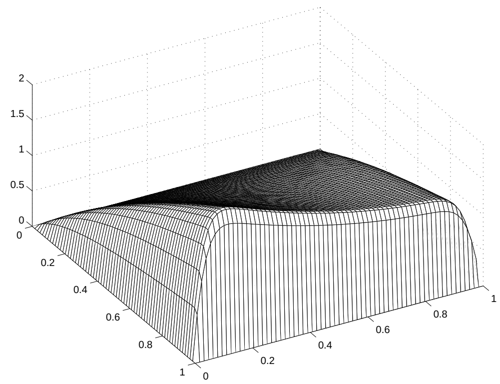

FIGURE 1. Solution of model problem (1), (30); $\varepsilon = 2^{-9}$.

In the recent work [5], which was improved in [6], a detailed analysis is given of the solution to problem (1)-(2) with ε-explicit estimates of all its derivatives in general case $g \notin C(\partial\Omega)$. A few years earlier, the solution of this problem, under assumptions that the compatibility conditions of the first order are satisfied, was analyzed in [10]. In [9] equation (1) with variable convection coefficient a was investigated, but under very severe compatibility conditions at the corner points excluding appearance of corner singularities both in the solution itself and in its derivatives up to desired order for equation (1) and for the reduced equation as well. Under the same assumptions, in [9] the convergence of classical five-point upwind difference scheme is analyzed for which, on a Shishkin mesh, the convergence estimate of $O(N^{-1}\ln^2 N)$ is obtained, where N is the number of mesh points in each coordinate direction. Earlier in [10], a comment was made that one might get the error estimate of $O(N^{-1}\ln N)$ for this scheme (using the obtained solution decomposition). In spite of heaviness of compatibility conditions, in most works dealing with analysis of numerical methods for singularly perturbed equations in a rectangle, such assumptions are made in order to provide smoothness to the solution being approximated. The book of Shishkin [12] is an exception. In this book for some problems certain compatibility conditions are posed, while those are not posed for other problems. But for the cases when the compatibility conditions are not posed, estimates, obtained in [12] (for a much more general problem than (1), (2)), give low orders of convergence. For example, for the finite-difference scheme (9) applied to problem (1), (2), only the error bound $O((N^{-1}\ln^2 N)^{1/14})$ is given in [12].

In recent years for some singularly perturbed problems the author of this paper has succeeded in carrying a more thorough analysis of the convergence rate of difference schemes, when the problem data at the corner points have minimal compatibility (only the continuity is required). Thus, in [1], [2] for a singularly perturbed reaction-diffusion equation on a unit square, with the Dirichlet and the Dirichlet-Neumann boundary conditions, the error of the classical difference scheme on a Shishkin mesh is proved to be $O(N^{-2}\ln^2 N)$. However, in the case of Dirichlet-Neumann boundary conditions, it was necessary to use an additional power refinement in the neighborhood of those corner points where boundary conditions of different types were imposed at the adjacent sides. A similar situation occurs in [4] too, where the reaction-diffusion equation in an L-shaped domain is investigated. In [3] the convection-diffusion equation in a rectangle with a regular boundary layer is considered. For this problem, singularities at different corner points are of very different nature (see, for example, [7]). But now it does not suffice to use only continuity of the boundary data at the corner points for tensor-product meshes. In [3], provided that the stringent compatibility conditions at the inflow corner (as in [9]) and continuity at the others are satisfied, the uniform convergence with the rate of $O(N^{-1}\ln^2 N)$ is proved.

In this work, it is assumed that the boundary function g from (1) is continuous

$$g \in C(\partial\Omega), \tag{3}$$

and is sufficiently smooth on $\Gamma_k = \overline{\Gamma}_k$, $k = 1, \ldots, 4$. Any other compatibility conditions at the corner points a_k, $k = 1, \ldots, 4$, are not assumed.

The aim of the present paper is to show that our weak compatibility conditions, complemented by sufficient smoothness of the right-hand side $f(x, y)$ of equation (1), are sufficient for the error estimate from [9], where additional compatibility conditions were assumed; see theorem 2 below. Note that to prove this theorem, we essentially use results from [9] and [5], [6].

The present paper is organized as follows. In § 2, we present a solution decomposition and give a sharper version of well-known estimates for the derivatives of the decomposition components. In § 3, we discretize problem (1) and then prove the error estimate of theorem 2, which is our main result. In § 4, some numerical results are presented that support this theorem.

In the course of the paper c, c_1 and so on, denote different positive constants that do not depend on ε and N.

2. Decomposition of the solution and a priori estimates

As we have already mentioned, to prove the main theorem of this work, we essentially use results from [5], [6], where a decomposition of the solution to (1) was constructed and a priori estimates of its components were obtained. We formulate a version of some results from [5], [6]. Note that certain assumptions on smoothness of f and g_k made in [5], [6] do not seem necessary. Instead, we simply assume that f and g_k are sufficiently smooth. Moreover, the results from [5, 6] are rather general. Those include both the case of a discontinuity of a boundary function at the corner points of the domain, and all possible cases when the compatibility conditions of any order are satisfied. Our interest is only in one case of the boundary function being continuous at the corner points without additional compatibility conditions; this corresponds to $\nu = 0$ in [5, 6].

Unfortunately, the estimates of the solution derivatives in [5] are too rough for our purposes in the neighborhood of the corners for $m + n = 2$ (the case $\gamma = 0$) from [5, theorem 5.1] because of a logarithmic factor. This factor is necessary to estimate mixed derivatives. As for us, we need only pure derivatives with respect to x and to y. As it will be proved below, the logarithmic factor can be omitted for these derivatives. With this correction, an abridged variant of [5, theorem 5.1], which was improved in [6], will be formulated.

Introduce some notations before the theorem to be formulated.

Set

$$D^m := D_x^m := \partial^m/\partial x^m, \quad D^n := D_y^n := \partial^n/\partial y^n, \quad D^{m,n} := D^m D^n.$$

By $r_k = \sqrt{(x - x_k)^2 + (y - y_k)^2}$ denote the distance from the point (x, y) to the vertex a_k and by $\rho_k = \rho((x, y), \Gamma_k)$ denote the distance from the same point to the straight line on which the segment Γ_k lies.

Theorem 1. *Let $u(x, y)$ be a solution to problem (1)-(3). Let $f(x, y)$ for $(x, y) \in \overline{\Omega} = \Omega \cup \partial\Omega$ and $g_k(s)$, $k = 1, \ldots, 4$ for $s \in \Gamma_k = \overline{\Gamma}_k$ be sufficiently smooth. Then* *the following decomposition takes place*

$$u(x,y) = v(x,y) + w(x,y) + \sum_{k=1}^{4} z_k(x,y), \tag{4}$$

where

$$Lv = f(x,y), \quad Lw = Lz_k = 0, \quad k = 1,\dots,4, \quad (x,y) \in \Omega,$$

and also for $m + n \leqslant 3$

$$|D^{m,n} v(x,y)| \leqslant c, \tag{5}$$

$$|D^{m,n} w(x,y)| \leqslant c\,\varepsilon^{-m} e^{-a(1-x)/\varepsilon}, \tag{6}$$

$$|D^{m,n} z_k(x,y)| \leqslant$$
$$\leqslant c \begin{cases} (\varepsilon^{-n/2} + \varepsilon^{1-m-n}) & \textit{for} \quad m+n \leqslant 1 \textit{ and } \quad r_k < \varepsilon, \\ \varepsilon^{-1} \quad \textit{for} \quad m+n = 2, mn = 0 \textit{ and } \quad r_k < \varepsilon, \\ (\varepsilon^{-n/2} + \varepsilon^{-1} r_k^{-1}) \quad \textit{for} \quad m+n = 3 \quad \textit{and} \quad r_k < \varepsilon, \\ \varepsilon^{-n/2}[1 + r_k^{1-m-n/2}] e^{-\sqrt{q}\,\rho_{2(k+2)/3}/(2\sqrt{\varepsilon})} \quad \textit{for} \quad r_k \geqslant \varepsilon, \end{cases} \qquad k = 1, 4 \tag{7}$$

and

$$|D^{m,n} z_k(x,y)| \leqslant$$
$$\leqslant c \begin{cases} (\varepsilon^{-m-n/2} + \varepsilon^{1-m-n}) \quad \textit{for} \quad m+n \leqslant 1 \textit{ and } \quad r_k < \varepsilon, \\ \varepsilon^{-m-n/2} \quad \textit{for} \quad m+n = 2,\ mn = 0 \quad \textit{and} \quad r_k < \varepsilon, \\ (\varepsilon^{-m-n/2} + \varepsilon^{-1} r_k^{-1}) \quad \textit{for} \quad m+n = 3 \quad \textit{and} \quad r_k < \varepsilon, \\ \varepsilon^{-m-n/2}[1 + r_k^{1-n/2}] e^{-\sqrt{q}\,\rho_{2(k-1)}/(2\sqrt{\varepsilon}) - a(1-x)/\varepsilon} \quad \textit{for} \quad r_k \geqslant \varepsilon, \end{cases} \qquad k = 2, 3. \tag{8}$$

All propositions of theorem 1, except those for $m + n = 2$, are contained in theorem 1 from [6] and are proved in [5, 6]. Proposition (7) on the estimate of the second non-mixed derivatives follows from [5] as well. To be precise, the proof of theorem 5.1 from [5] for this case involves a number of other results from the same paper. In particular, this proof uses theorem 4.2 which, in its own turn, refers to theorem 4.1, and the latter refers to lemma 4.6, where the bound of the even order derivatives with respect to y is proved. This estimate implies (7) for $m = 0$, $n = 2$. As by virtue of (7), $D_x z_k = O(1)$, then it follows from the equation $Lz_k = 0$ that $D_x^2 z_k = O(\varepsilon^{-1})$. Thus we have established the desired estimate from (7).

Estimates (8) for the second pure derivatives of the functions Z_2 and Z_3 are proved using a similar argument.

Remark 1. In [8], a mixed problem for the equation (1), subject to Neumann boundary condition on the outflow boundary Γ_3, is considered, and it is proved that for the solution to this problem, there is no logarithmic factor in the estimates of the pure derivatives with respect to y near a_2 and a_3.

3. The discrete problem and the error analysis

We present error bounds for the approximation of (1) by the standard five-point upwind difference scheme

$$L^N U = -\varepsilon(\delta_x^2 + \delta_y^2)U + aD_x^- U + qU = f, \quad (x_i, y_j) \in \Omega^N, \quad U\,\big|_{\partial\Omega^N} = g \tag{9}$$

on a mesh $\Omega^N = \omega_x \times \omega_y$, which is a tensor product of two piecewise uniform one-dimensional Shishkin meshes ω_x, ω_y. The finite difference operators D_x^- and δ_x^2 are the standard first order backward difference and the second order centered difference on a non-uniform mesh. To be exact, if $h_i = x_i - x_{i-1}$ is the local mesh size of the mesh ω_x and $\hbar_i = (h_i + h_{i+1})/2$, then, for example,

$$D_x^- v_{ij} := \frac{v_{ij} - v_{i-1,j}}{h_i}, \quad \delta_x^2 v_{ij} = \left(D_x^- v_{i+1,j} - D_x^- v_{ij}\right)/\hbar_i.$$

Here the mesh ω_x places $N/2$ mesh intervals into both $[0, 1-\sigma_x]$ and $[1-\sigma_x, 1]$ and the mesh ω_y places $N/4$ mesh intervals into intervals $[0, \sigma_y]$ and $[1-\sigma_y, 1]$ and $N/2$ intervals into the region $[\sigma_y, 1-\sigma_y]$. The transition parameters are taken to be

$$\sigma_x = \min\left\{\frac{1}{2}, \frac{\varepsilon}{a}\ln N\right\} \text{ and } \sigma_y = \min\left\{\frac{1}{4}, \frac{2}{\sqrt{q}}\sqrt{\varepsilon}\ln N\right\}.$$

Let

$$h = 2\sigma_x/N \quad \text{and} \quad \mathfrak{h} = 4\sigma_y/N$$

be the "fine" mesh widths of ω_x and ω_y, respectively, and let

$$H = 2(1-\sigma_x)/N \quad \text{and} \quad \mathfrak{H} = 2(1-2\sigma_y)/N$$

be their "coarse" mesh widths.

One can easily prove that the matrix, generated by the operator L^N, is an M-matrix. Therefore the discrete operator L^N satisfies the discrete maximum/comparison principle.

The discrete solution U is decomposed similarly to the continuous solution (4)

$$U = V + W + \sum_{k=1}^{4} Z_k, \tag{10}$$

where

$$L^N V = f, \quad L^N W = L^N Z_k = 0, \quad k = 1, \ldots, 4, \quad (x_i, y_j) \in \Omega^N,$$

and

$$V = v, \quad W = w, \quad Z_k = z_k, \quad k = 1, \ldots, 4, \quad (x_i, y_j) \in \partial\Omega^N.$$

The functions V and W from (10) are similar to the corresponding functions V and W_R from [9]. (Compared to [9], we use a slightly different method of extension to construct $v(x,y)$ and $w(x,y)$ in (4).) Therefore, by virtue of the estimates (5), (6), from theorem 1 and results from [9], it follows that

$$|v - V| + |w - W| = O(N^{-1}\ln^2 N). \tag{11}$$

The case of the functions Z_k is more complicated. Thus, the function Z_1 is the analogue to the function W_B from [9] (That is missed in the decomposition of U on p. 1770 there.) This function is an approximation to the boundary layer function of the lower characteristic layer. But, in contrast to W_B, the function Z_1 includes description of the corner singularity too, arising from the lack of the appropriate compatibility condition at the vertex a_1. The functions Z_4 and W_T are associated in a similar manner. The functions W_{BR} and W_{TR} from [9] characterize approximations of the corner layers near the vertices a_2 and a_3, respectively. Our functions Z_2 and Z_3, aside from these, include corresponding corner singularities. In spite of the pointed out difference in the functions Z_k and W from [9], the behavior of these functions outside of the neighborhoods to the corresponding sides and vertices of Ω, is similar. Therefore, in virtue of bounds (7) and (8), from theorem 1 and results from [9], the following estimates are valid

$$
(12)\qquad
\begin{aligned}
&|z_1 - Z_1| \leqslant c\,N^{-1}, \quad 0 \leqslant x_i \leqslant 1, \quad \sigma_y \leqslant y_j \leqslant 1,\\
&|z_4 - Z_4| \leqslant c\,N^{-1}, \quad 0 \leqslant x_i \leqslant 1, \quad 0 \leqslant y_j \leqslant 1-\sigma_y,\\
&|z_2 - Z_2| \leqslant c\,N^{-1}, \quad 0 \leqslant x_i \leqslant 1-\sigma_x, \quad \text{or} \quad \sigma_y \leqslant y_j \leqslant 1,\\
&|z_3 - Z_3| \leqslant c\,N^{-1}, \quad 0 \leqslant x_i \leqslant 1-\sigma_x, \quad \text{or} \quad 0 \leqslant y_j \leqslant 1-\sigma_y.
\end{aligned}
$$

Thus, it remains to estimate $z_k - Z_k$ on their "own" fine meshes. Since the behavior of Z_1 and Z_4 are identical, as well as of Z_2 and Z_3, it is sufficient to analyze (z_1-Z_1) for $0 \leqslant x_i \leqslant 1$, $0 \leqslant y_j \leqslant \sigma_y$ and $(z_2 - Z_2)$ for $1-\sigma_x \leqslant x_i \leqslant 1$, $0 \leqslant y_j \leqslant \sigma_y$. Let us begin with $(z_1 - Z_1)$. Using Taylor series expansions, we get

$$
(13)\quad |L^N(z_1-Z_1)| \leqslant c\,N^{-1}\left[\varepsilon\left|\frac{\partial^3 z_1}{\partial x^3}(\xi_i, y_j)\right| + \left|\frac{\partial^2 z_1}{\partial x^2}(\xi_i^*, y_j)\right| + \varepsilon\sigma_y\left|\frac{\partial^3 z_1}{\partial y^3}(x_i, \eta_j)\right|\right],
$$

where ξ_i, ξ_i^* and η_j are some points in (x_{i-1}, x_{i+1}), (x_{i-1}, x_i) and (y_{j-1}, y_{j+1}), respectively, that are different in different situations. Now it is necessary to use (7) to obtain estimate (13) in terms of r_1. But first, it should be clarified when we have

$$
\sqrt{\xi_i^2 + y_j^2} \geqslant c\sqrt{x_i^2 + y_j^2}, \quad \sqrt{x_i^2 + \eta_j^2} \geqslant c\sqrt{x_i^2 + y_j^2}
$$

with some constant c not depending on N and ε. By virtue of the definition of ξ_i, the inequality $\xi_i^2 + y_j^2 \geqslant x_{i-1}^2 + y_j^2$ is always correct and it is sufficient to estimate the constant c from the inequality

$$
(x_i - H)^2 + y_j^2 \geqslant c^2\,(x_i^2 + y_j^2),
$$

which holds true only for $c < 1$. This inequality is equivalent to the inequality

$$
(14)\qquad \left(x_i - \frac{H}{1-c^2}\right)^2 + y_j^2 \geqslant \frac{c^2 H^2}{(1-c^2)^2},
$$

which is true for the nodes of Ω^N disposed outside of a disk of the radius $cH/(1-c^2)$ with the center at the point $(H/(1-c^2), 0)$. If $i = 1$, then inequality (14) is correct only for the nodes whose coordinates $y_j \geqslant cH/(\sqrt{1-c^2})$. In order that inequality (14), for $\mathfrak{h} \leqslant H$, to be valid for all nodes of Ω^N with $i \geqslant 2$, it is necessary that the constant c to be not greater than $1/2$.

So,

$$
(15)\qquad \sqrt{\xi_i^2 + y_j^2} \geqslant r_1/2 \quad \text{for} \quad x_i \geqslant 2H \quad \text{or} \quad y_j \geqslant H/\sqrt{3}.
$$

Since $H \geqslant N^{-1}$ and $\mathfrak{h} \leqslant N^{-1}$, then the inequality

$$
(16)\qquad \sqrt{x_i^2 + \eta_j^2} \geqslant r_1/2
$$

holds true at all points of Ω^N.

Let us estimate (13) using (7) and then take into account (15), (16). Since estimates (7) for $r_k \leqslant \varepsilon$ are the same as for $r_k > \varepsilon$ if $r_k = O(\varepsilon)$, we have

(17)

$$
\begin{aligned}
&|L^N(z_1 - Z_1)_{ij}| \leqslant\\
&\leqslant c\,N^{-1}\begin{cases}
\varepsilon\left(1 + \dfrac{1}{\varepsilon r_1}\right) + \left(1 + \dfrac{1}{\varepsilon}\right) + \varepsilon^{3/2}\ln N\left(\dfrac{1}{\varepsilon^{3/2}} + \dfrac{1}{\varepsilon r_1}\right), & r_1 < \varepsilon\\
\varepsilon\left(1 + \dfrac{1}{r_1^2}\right) + \left(1 + \dfrac{1}{r_1}\right) + \varepsilon^{3/2}\ln N\left(1 + \dfrac{1}{r_1^{1/2}}\right)\varepsilon^{-3/2}, & r_1 \geqslant \varepsilon
\end{cases} \leqslant\\
&\leqslant c\,r_1^{-1}N^{-1}\ln N, \quad x_i \geqslant 2H \quad \text{or} \quad y_j \geqslant H/\sqrt{3}.
\end{aligned}
$$

For other nodes of Ω^N, that is for nodes (x_1, y_j) when $y_j \leqslant H/\sqrt{3}$, estimate (17) does not follow from (13). So one must operate in another way.

At first, note that $L^N(z_1 - Z_1) = L^N z_1$. Using Lagrange mean value theorem, we get

$$D_x^- g(x_1, y_j) = \frac{\partial g}{\partial x}(\xi_1^*, y_j).$$

A similar formula takes place for the second difference quotient. Namely, we have

$$\delta_x^2 g(x_1, y_j) = \frac{\partial^2 g}{\partial x^2}(\xi_1, y_j).$$

On the uniform mesh with the mesh width H this formula, for example, follows from the representation

$$\delta_x^2 g(x_1, y_j) = H^{-2} \int_0^{2H} (H - |x - H|) \frac{\partial^2 g}{\partial x^2}(x, y_j) d\, x$$

and the general mean value formula. It follows, from what has been said, that

$$L^N z_1(x_1, y_j) = -\varepsilon \left[\frac{\partial^2 z_1}{\partial x^2}(\xi_1, y_j) + \frac{\partial^2 z_1}{\partial y^2}(x_1, \eta_j) \right] + a \frac{\partial z_1}{\partial x}(\xi_1^*, y_j) + q z_1(x_1, y_j). \tag{18}$$

Taking into account (7) and bearing in mind that $y_j \leqslant c\, H$, we obtain

$$|L^N(z_1 - Z_1)(x_1, y_j)| \leqslant c \leqslant c\, N^{-1}(H^2 + y_j^2)^{-1/2}, \quad y_j \leqslant c\, H. \tag{19}$$

This estimate is not worse than (17) for the other nodes. So it is possible, when it is necessary, to assume that (17) combines the estimate (17) for $i > 1$ and (19). That is, (17) is valid for all nodes (x_i, y_j) for $0 < y_j < \sigma_y$, we are interested in.

Now we turn to the function $(z_2 - Z_2)$. As for (13), we find, for $1 - \sigma_x < x_i < 1$, $0 < y_j < \sigma_y$, that

$$\begin{aligned} &|L^N(z_2 - Z_2)(x_i, y_j)| \leqslant \\ &\leqslant c\, N^{-1} \left\{ \varepsilon \sigma_x \left| \frac{\partial^3 z_2}{\partial x^3}(\xi_i, y_j) \right| + \sigma_x \left| \frac{\partial^2 z_2}{\partial x^2}(\xi_i^*, y_j) \right| + \varepsilon \sigma_y \left| \frac{\partial^3 z_2}{\partial y^3}(x_i, \eta_j) \right| \right\}. \end{aligned}$$

Now $\sqrt{(1 - \xi_i)^2 + y_j^2} \geqslant r_2/2$ for all mesh points of Ω^N and $\sqrt{(1 - x_i)^2 + \eta_j^2} \geqslant r_2/2$ for $y_j \geqslant 2\mathfrak{h}$ or $x_i \leqslant 1 - \mathfrak{h}/\sqrt{3}$. Therefore, with regard to (8), we get

$$\begin{aligned} &|L^N(z_2 - Z_2)(x_i, y_j)| \leqslant c\, N^{-1} \ln N \left(\varepsilon^{-1} + \begin{cases} \sqrt{\varepsilon}/r_2, & r_2 < \varepsilon, \\ 1/\sqrt{r_2}, & r_2 \geqslant \varepsilon \end{cases} \right), \\ &\qquad\qquad y_j \geqslant 2\mathfrak{h} \quad \text{or} \quad x_i \leqslant 1 - \mathfrak{h}/\sqrt{3}. \end{aligned} \tag{20}$$

In just the same way as for z_1 from (18), we find that

$$\begin{aligned} L^N z_2(x_i, y_1) = &- \varepsilon \left[\frac{\partial^2 z_2}{\partial x^2}(\xi_i, y_1) + \frac{\partial^2 z_2}{\partial y^2}(x_i, \eta_1) \right] + a \frac{\partial z_2}{\partial x}(\xi_i^*, y_1) + \\ &+ q z_2(x_i, y_1), \quad 1 - x_i < \mathfrak{h}/\sqrt{3}. \end{aligned} \tag{21}$$

But now from (8) it follows immediately only the bound

$$|L^N z_2(x_i, y_j)| \leqslant c/\varepsilon, \tag{22}$$

which is much worse than (20) and is too rough for using in further estimates. To sharpen (22), we write the equation for z_2 at the mesh point (ξ_i, y_1) and subtract this, equaled zero, expression from (21). Using Lagrange mean value theorem, we have

$$\begin{aligned} L^N z_2(x_i, y_1) &= L^N z_2(x_i, y_1) - L z_2(\xi_i, y_1) = \\ &= -\varepsilon \left[\frac{\partial^2 z_2}{\partial y^2}(x_i, \eta_1) - \frac{\partial^2 z_2}{\partial y^2}(\xi_i, y_1) \right] + a \left[\frac{\partial z_2}{\partial x}(\xi_i^*, y_1) - \frac{\partial z_2}{\partial x}(\xi_i, y_1) \right] + \\ &\quad + q[z_2(x_i, y_1) - z_2(\xi_i, y_1)] = \\ &= -\varepsilon \left[\frac{\partial^2 z_2}{\partial y^2}(x_i, \eta_1) - \frac{\partial^2 z_2}{\partial y^2}(\xi_i, y_1) \right] + a(\xi_i^* - \xi_i) \frac{\partial^2 z_2}{\partial x^2}(\xi_i^{**}, y_1) + \\ &\quad + q[z_2(x_i, y_1) - z_2(\xi_i, y_1)], \end{aligned}$$

where $\xi_i^{**} \in (\xi_i^*, \xi_i)$. Now, using (8), we obtain

$$|L^N(z_2 - Z_2)(x_i, y_1)| \leqslant c \left(1 + \frac{\ln N}{\varepsilon N}\right), \quad x_i \geqslant 1 - \mathfrak{h}/\sqrt{3}, \tag{23}$$

which is not worse than (20). So it is possible to assume that the estimate (20) is correct for all nodes satisfying $1 - \sigma_x < x_i < 1$, $0 < y_j < \sigma_y$. Thus, the following lemma is proved.

Lemma 1. *Under the conditions of theorem 1, the following estimates are valid*

$$\begin{gathered} |L^N(z_1 - Z_1)_{ij}| \leqslant c\, r_1^{-1} N^{-1} \ln N, \quad (x_i, y_j) \in (0,1) \times (0, \sigma_y), \\ |L^N(z_2 - Z_2)_{ij}| \leqslant c\, N^{-1} \ln N \left(\varepsilon^{-1} + \begin{cases} \sqrt{\varepsilon}/r_2, & r_2 < \varepsilon, \\ 1/\sqrt{r_2}, & r_2 \geqslant \varepsilon \end{cases} \right), \\ (x_i, y_j) \in (1 - \sigma_x, 1) \times (0, \sigma_y). \end{gathered}$$

It remains to estimate the functions $(z_1 - Z_1)$ and $(z_2 - Z_2)$ themselves on a proper subset of the nodes of Ω^N. These estimates are crucial in our analysis.

As it was noted above, the operator L^N of problem (9) satisfies the comparison principle. So it is sufficient to construct appropriate barriers to estimate $(z_k - Z_k)$.

Lemma 2. *If a positive constant b is sufficiently large, then the function*

$$\begin{aligned} \mathcal{B}(x, y) &= \ln \frac{\sqrt{(x + bH)^2 + y^2}}{H} + \\ &+ \left(\frac{\pi}{2} - \arctan \frac{y}{x + bH} \right) \left(1 + \frac{\pi}{2} + \arctan \frac{y}{x + bH} \right) + 1 \geqslant 1, \quad (x, y) \in \overline{\Omega}, \end{aligned}$$

is such that there exists sufficiently small positive number C, for which

$$L^N \mathcal{B}_{ij} \geqslant C\, (x_i^2 + y_j^2)^{-1/2}, \quad (x_i, y_j) \in \Omega^N.$$

Proof. Assume $x' = x + b\,H$, $y' = y$. Let now r', φ' be the polar coordinates in new axes. It is obvious that for $(x, y) \in \overline{\Omega}$ the variables r' and φ' are subjected to constraints $0 \leqslant \varphi' < \pi/2$ and $r' \geqslant bH$. In variables r' and φ', the function $\mathcal{B}(x, y)$ takes the form

$$\mathcal{B}(x, y) = \ln(r'/H) + (\pi/2 - \varphi')(\pi/2 + 1 + \varphi') + 1 \geqslant 1, \quad (x, y) \in \overline{\Omega}. \tag{24}$$

Since

$$L v \equiv L' v := -\varepsilon \Delta' v + a\, \partial v / \partial x' + q v,$$

then, going over to polar coordinates, we get

(25)
$$L\mathcal{B}(x,y) \equiv -\varepsilon\left[\frac{1}{r'}\frac{\partial}{\partial r'}\left(r'\frac{\partial\mathcal{B}}{\partial r'}\right)+\frac{1}{r'^2}\frac{\partial^2\mathcal{B}}{\partial\varphi'^2}\right]+a\left[\cos\varphi'\frac{\partial\mathcal{B}}{\partial r'}-\frac{\sin\varphi'}{r'}\frac{\partial\mathcal{B}}{\partial\varphi'}\right]+q\mathcal{B}\geqslant$$
$$\geqslant\frac{2\varepsilon}{r'^2}+\frac{a}{r'},\quad (x,y)\in\Omega.$$

Applying the operator L^N to $\mathcal{B}$, we obtain

(26)
$$L^N\mathcal{B}_{ij}=(L\mathcal{B})_{ij}-\frac{\varepsilon}{6}\left[H\frac{\partial^3\mathcal{B}}{\partial x'^3}(x_i+\theta_1H,y_j)-H\frac{\partial^3\mathcal{B}}{\partial x'^3}(x_i-\theta_2H,y_j)+\right.$$
$$\left.+\,\mathfrak{h}\frac{\partial^3\mathcal{B}}{\partial y^3}(x_i,y_j+\theta_3\mathfrak{h})-\mathfrak{h}\frac{\partial^3\mathcal{B}}{\partial y^3}(x_i,y_j-\theta_4\mathfrak{h})\right]+\frac{a}{2}H\frac{\partial^2\mathcal{B}}{\partial x'^2}(\xi_i^*,y_j),\quad \theta_l\in[0,1).$$

It follows from (24) that
$$\left|\frac{\partial^l\mathcal{B}}{\partial x'^l}\right|+\left|\frac{\partial^l\mathcal{B}}{\partial y^l}\right|\leqslant\frac{c}{r'^l},\quad l=2,3.$$

Bearing this in mind, with regard to (25), we obtain from (26) that
$$L^N\mathcal{B}_{ij}\geqslant\frac{2\varepsilon}{r'^2}+\frac{a}{r'}-$$
$$-\,c\,\varepsilon\left[\frac{H}{[(x_i'-H)^2+y_j^2]^{3/2}}+\frac{\mathfrak{h}}{[{x_i'}^2+(y_j-\mathfrak{h})^2]^{3/2}}\right]-\frac{c\,H}{(x_i'-H)^2+y_j^2}.$$

It follows from reasoning, leading to (15), that
$$\sqrt{(x_i'-H)^2+y_j^2}\geqslant r'/2\quad\text{for}\quad (x_i,y_j)\in\Omega^N,\quad\text{if}\quad b\geqslant 1.$$

This inequality together with (16) makes it possible to conclude that
$$L^N\mathcal{B}_{ij}\geqslant\frac{2\varepsilon}{r'^2}+\frac{a}{r'}-\frac{c\,\varepsilon}{r'^3}(H+\mathfrak{h})-\frac{c\,H}{r'^2}\quad\text{for}\quad b\geqslant 1.$$

If, moreover,

(27)
$$r'\geqslant\max\{c\,H,c\,\mathfrak{h},2c\,H/a\},$$

then
$$L^N\mathcal{B}_{ij}\geqslant\frac{a}{2r'}.$$

In order that all nodes of Ω^N lie outside of the disk from(27), it is sufficient that its radius is not greater than $(b+1)H$. As $H\geqslant\mathfrak{h}$, then this condition is satisfied for
$$c\,H\,\max(1,2/a)\equiv c_1\,H\leqslant(b+1)H.$$

It remains to obtain an estimate of the constant c from the inequality
$$r\geqslant c\,r',\quad (x_i,y_j)\in\Omega^N.$$

Taking into account (14), we conclude that this inequality is equivalent to the inequality
$$\left(x_i'-\frac{bH}{1-c^2}\right)^2+y_j^2\geqslant\left(\frac{cbH}{1-c^2}\right)^2,\quad (x_i,y_j)\in\Omega^N,$$

or, that is the same,
$$\left(x_i-\frac{c^2bH}{1-c^2}\right)^2+y_j^2\geqslant\left(\frac{cbH}{1-c^2}\right)^2,\quad (x_i,y_j)\in\Omega^N.$$

This inequality will be correct, if we require that the node $(H, \mathfrak{h})$ will not be inside of the disk. For this, it is sufficient that

$$c \leqslant 1/(1+b).$$

The lemma is proved with $C = ac/2$, where $c \leqslant (1+b)^{-1} \leqslant c_1^{-1}$. □

Lemma 3. *There exists positive constants b and C such that*

$$\mathcal{B}_2(x,y) = \ln \frac{\sqrt{(1-x)^2 + (y+b\mathfrak{h})^2}}{\mathfrak{h}} + \arctan \frac{y+b\mathfrak{h}}{1-x} - $$
$$- \arctan^2 \frac{y+b\mathfrak{h}}{1-x} + \frac{\pi^2}{4} + 1 \geqslant 1, \quad (x,y) \in \Omega,$$

and

$$L^N \mathcal{B}_2(x_i, y_j) \geqslant C \left((1-x_i)^2 + y_j^2\right)^{-1/2} = C r_2^{-1}, \quad (x_i, y_j) \in \Omega^N.$$

Proof. Make a change of variables $x^* = 1-x$, $y^* = y$. In new variables the operator L takes the form

$$L^* u^* = -\varepsilon \Delta^* u^* - a \frac{\partial u^*}{\partial x^*} + q u^*.$$

Further reasoning repeats arguments used in the proof of lemma 2, with a difference that now we have y instead of x and vice versa. Thus, there will be $\mathfrak{h}$ instead of H and h instead of $\mathfrak{h}$. □

Now we have everything to obtain missing estimates of $(z_1 - Z_1)$ and $(z_2 - Z_2)$. Denote $e = z_1 - Z_1$. By virtue of (17), (19), with regard to (12), and taking account the construction of z_1, we get

$$|L^N e| \leqslant c\,(N^{-1} \ln N) r_1^{-1} \quad \text{for} \quad 0 < x_i < 1, \quad 0 < y_j < \sigma_y,$$
$$e = O(N^{-1}) \quad \text{for} \quad y_j = \sigma_y, \quad \text{and} \quad e = 0 \quad \text{for} \quad (x_i, y_j) \in \partial\Omega^N.$$

On Ω^N, because of lemma 2, $r_1^{-1} \leqslant C^{-1} L^N \mathcal{B}$, and, by virtue of (24), $\mathcal{B} \geqslant 1$. Therefore, for appropriate c, the function $c\,(N^{-1} \ln N)\mathcal{B}(x_i, y_j)$ is a barrier for e. Since $\mathcal{B} \leqslant c \ln N$ on Ω^N, then

$$|e| = |z_1 - Z_1| \leqslant c\, N^{-1} \ln^2 N, \quad 0 < x_i < 1, \quad 0 < y_j < \sigma_y. \tag{28}$$

Now denote $e = z_2 - Z_2$. By virtue of (20), (23) and (12),

$$|L^N e| \leqslant c\,(N^{-1} \ln N)(\varepsilon^{-1} + r_2^{-1}) \quad \text{for} \quad 1 - \sigma_x < x_i < 1, \quad 0 < y_j < \sigma_y,$$
$$e = O(N^{-1}) \quad \text{for} \quad x_i = 1 - \sigma_x, \quad y_j = \sigma_y, \quad \text{and} \quad e = 0 \quad \text{on} \quad \partial\Omega^N.$$

Assume that $e = \mathring{e} + \bar{e}$, where

$$|L^N \mathring{e}| \leqslant c\,(N^{-1} \ln N)\varepsilon^{-1}, \quad \text{and} \quad |L^N \bar{e}| \leqslant c\,(N^{-1} \ln N) r_2^{-1}.$$

The estimate for $\mathring{e}$ is obtained in [9] and it has the form

$$|\mathring{e}| = O((N^{-1} \ln^2 N)).$$

For $\bar{e}$ from lemma 3, using the function $\mathcal{B}_2$, we find, as for the case of $e = z_1 - Z_1$, that

$$|\bar{e}| \leqslant c\,(N^{-1} \ln N)\mathcal{B}_2 \quad \text{for} \quad 1 - \sigma_x < x_i < 1, \quad 0 < y_j < \sigma_y.$$

Since

$$\max_{\substack{1-\sigma_x < x < 1 \\ 0 < y < \sigma_y}} \mathcal{B}_2(x,y) \leqslant c \ln \frac{\sqrt{\sigma_x^2 + \sigma_y^2}}{\mathfrak{h}},$$

and $\sigma_x \leqslant 2\sigma_y$ (at least, for $\varepsilon < 4a^2$), then

$$\ln \frac{\sqrt{\sigma_x^2 + \sigma_y^2}}{\mathfrak{h}} \leqslant \ln \frac{\sqrt{2}\sigma_y}{\mathfrak{h}} \leqslant c \ln N.$$

Therefore,

$$|\bar{e}| \leqslant c\, N^{-1} \ln^2 N \quad \text{for} \quad 1 - \sigma_x < x_i < 1, \quad 0 < y_j < \sigma_y,$$

and consequently,

$$|z_2 - Z_2| \leqslant c\, N^{-1} \ln^2 N \quad \text{for} \quad 1 - \sigma_x < x_i < 1, \quad 0 < y_j < \sigma_y. \tag{29}$$

Collecting together estimates (11), (12), (28), (29), we conclude that the following theorem is valid.

Theorem 2. *Let u be the solution to problem* (1) - (3), *which satisfies the conditions of theorem 1, and U be the solution to problem* (9) *on the piece-wise Shishkin mesh Ω^N. Then*

$$|U_{ij} - u(x_i, y_j)| \leqslant c\, N^{-1} \ln^2 N.$$

Remark 2. The arguments used to prove theorem 2, remain valid for the case of variable coefficients a and q of equation (1). Therefore, the following conditional proposition holds: if theorem 1 is valid for the case of variable coefficients a and q in equation (1), then theorem 2 holds true as well.

4. Numerical results

Our test problem is (1) with

$$q = 0, \quad a = 1, \quad f = 1 + (1-x)^2 + (1-y)^2, \quad g = 0; \tag{30}$$

see Figure 1. It is easy to see that for such definition of coefficients and right-hand sides, compatibility conditions even of the first order are not satisfied at vertices.

Table 1. Maximum nodal values of errors e^N and its products with N, $N/\ln N$ and $N/\ln^2 N$.

N	$\varepsilon = 1$	$\varepsilon = 2^{-3}$	$\varepsilon = 2^{-6}$	$\varepsilon = 2^{-9}$	$\varepsilon = 2^{-12}$	$\varepsilon = 2^{-15}$
	5.35e-4	1.98e-2	5.28e-2	6.01e-2	6.06e-2	6.11e-2
	1.71e-2	6.34e-1	1.69e 0	1.92e 0	1.94e 0	1.95e 0
32	4.94e-3	1.83e-1	4.87e-1	5.55e-1	5.59e-1	5.64e-1
	1.43e-3	5.28e-2	1.41e-1	1.60e-1	1.61e-1	1.63e-1
	2.48e-4	1.13e-2	2.93e-2	3.47e-2	3.48e-2	3.51e-2
	1.58e-2	7.24e-1	1.88e 0	2.22e 0	2.23e 0	2.25e 0
64	3.81e-3	1.74e-1	4.51e-1	5.34e-1	5.36e-1	5.40e-1
	9.16e-4	4.19e-2	1.08e-1	1.29e-1	1.29e-1	1.30e-1
	1.19e-4	5.84e-3	1.59e-2	1.97e-2	1.95e-2	1.96e-2
	1.52e-2	7.47e-1	2.04e 0	2.52e 0	2.50e 0	2.51e 0
128	3.13e-3	1.54e-1	4.20e-1	5.19e-1	5.16e-1	5.17e-1
	6.45e-4	3.17e-2	8.65e-2	1.07e-1	1.06e-1	1.07e-1
	5.81e-5	2.97e-3	8.60e-3	1.09e-2	1.09e-2	1.09e-2
	1.49e-2	7.59e-1	2.20e 0	2.79e 0	2.79e 0	2.78e 0
256	2.68e-3	1.37e-1	3.97e-1	5.04e-1	5.04e-1	5.01e-1
	4.84e-4	2.47e-2	7.16e-2	9.08e-2	9.08e-2	9.04e-2

Table 1 presents numerical results for problem (9). Since the exact solution is unknown, to investigate the rates of convergence, for each N and ε, we compute the double mesh errors $e^N := \max_{ij} |U_{ij}^N - \bar{U}_{2i,2j}^{2N}|$. Here $U^N := U$ is the computed solution on the mesh described in § 3 with $N-1$ interior mesh nodes in each direction, while $\bar{U}^{2N}$ is the computed solution on a similar mesh that uses the same transition parameters σ_x and σ_y, but the (almost) doubled number $2N-1$ of interior mesh nodes in each direction.

In each block of four numbers in Table 1 values of e^N, Ne^N, $Ne^N/\ln N$ and $Ne^N/\ln^2 N$ are sequentially placed.

An analysis of table data in rows shows that the double mesh errors e^N stabilize as $\varepsilon \to 0$. An analysis of the same data in columns shows that $e^N = O(N^{-1})$ on uniform meshes ($\varepsilon = 1$, $\varepsilon = 2^{-3}$) and that $e^N = O(N^{-1}\ln N)$ for $\varepsilon \ll 1$. These results do not contradict theorem 2 proved above, but suggest that estimates in theorem 2 and in [9] are somewhat rough.

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