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NOVEL FINITE DIFFERENCE SCHEME FOR THE NUMERICAL SOLUTION OF TWO-DIMENSIONAL INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. In the present article, a new methodology has been developed to solve two-dimensional (2D) Navier-Stokes equations (NSEs) in new form proposed by Pukhnachev (J. Appl. Mech. Tech. Phys., 45:2 (2004), 167–171) who introduces a new unknown function that is related to the pressure and the stream function. The important distinguish of this formulation from vorticity-stream function form of NSEs is that stream function satisfies to the transport equation and the new unknown function satisfies to the elliptic equation. The scheme and algorithm treat the equations as a coupled system which allows one to satisfy two conditions for stream function with no condition on the new function. The numerical algorithm is applied to the lid-driven cavity flow as the benchmark problem. The characteristics of this flow are adequately represented by the new numerical model.

Key Words. Navier-Stokes equations, incompressible viscous flow, finite-difference scheme.

1. Introduction

There are many numerical schemes for the solution of the Navier-Stokes Equations (NSEs) representing incompressible viscous flows. Some of these are schemes utilize primitive variables (velocity-pressure), vorticity-stream function, stream function (biharmonic equation), and vorticity-velocity formulation. The primary difficulty in obtaining numerical solutions with primitive variable formulation is that there is no evolution equation for the pressure variable. To avoid the troubles associated with primitive variable approach, stream function vorticity and vorticity-velocity formulations of the NSEs are widely used. Unfortunately, the correct boundary values of vorticity are not always easy to get. In the present study we use a new form of the NSEs. Aristov and Pukhnachev [1] and Pukhnachev [7] proposed new form of the NSEs for the case of axisymmetric and 2D flows, respectively. The 2D NSEs in the terms of new unknown functions contain one transport equation for the stream function and one elliptic equation for the new unknown function. This system only resembles the vorticity and stream function's form but the physical meaning of the coupling function is different from the vorticity. We have constructed finite-difference scheme for the NSEs in the new form. Our algorithm treats the equations as a coupled system which allows us to satisfy two conditions for stream function with no condition on the new unknown function. The proposed

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scheme can easily be extended to solving axisymmetric NSEs. The performance of the proposed method is investigated by considering well-known benchmark problem. A test case involves simulating a 2D lid-driven cavity flow at Reynolds number $Re \leqslant 1000$, when the motion is strictly laminar and steady. Several numerical investigations have been reported the characteristics of this cavity flow. Moreover, viscous fluid flow inside a driven cavity has been a common experiment approach used to check or improve numerical techniques (see for example, [2, 3, 4, 5, 6, 8]).

The content of this paper is organized as follows. In the next section, we derive new formulation of the NSEs with no-slip boundary conditions. Section 3 briefly describes the problem used for the test case and detailed description of numerical algorithm. The results of validation of the finite-difference scheme are presented in Section 4, where we make a detailed comparison with available numerical and experimental data.

2. The New Formulation of Navier-Stokes equations

To make paper self completed we first represent the transformation of viscous incompressible NSEs in 2D to a new form. The viscous incompressible flow is governed by the NSEs in a Cartesian coordinate system (x, y),

$$u_t + uu_x + vu_y = -\frac{1}{\rho}p_x + \nu\big(u_{xx} + u_{yy}\big), \tag{1}$$

$$v_t + uv_x + vv_y = -\frac{1}{\rho}p_y + \nu\big(v_{xx} + v_{yy}\big), \tag{2}$$

$$u_x + v_y = 0, \tag{3}$$

where u and v are the velocity components in $x-$ and $y-$ directions, respectively; p is the pressure, ρ is the fluid density, and ν is the kinematic viscosity. The fluid is subjected to potential external forces. In 2D, the constrain of incompressibility $\nabla \cdot \mathbf{v} = 0$ can be satisfied exactly by expressing velocity vector in terms of stream function ψ according to

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{4}$$

New form of NSEs is based on the following observation. The substitution of Eq. (4) into Eq. (1) yields

$$\frac{\partial}{\partial y}\left(\psi_t - \psi_x\psi_y - \nu\Delta\psi\right) + \frac{\partial}{\partial x}\left(\frac{1}{\rho}p + \psi_y^2\right) = 0, \tag{5}$$

where

$$\Delta \stackrel{\text{def}}{=} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Therefore, there is a function Φ satisfies the relations

$$\frac{1}{\rho}p = -\psi_y^2 + \Phi_y, \tag{6}$$

and

$$\psi_t - \psi_x\psi_y + \Phi_x = \nu\Delta\psi. \tag{7}$$

Differentiating Eq. (6) and Eq. (7) with respect to y and x, respectively, and substituting the resulting expressions into Eq. (2), where u and v are expressed in terms of ψ, we obtain

$$\Delta\Phi = 2\psi_y\Delta\psi. \tag{8}$$

We will consider only the case where the no-slip conditions are satisfied at the boundary of the flow domain. In term of the stream function ψ only, boundary conditions are

$$\psi = 0,\ \frac{\partial \psi}{\partial n} = b(x, y), \tag{9}$$

where $\frac{\partial \psi}{\partial n}$ means derivative in the direction of normal vector to the boundary. To complete the formulation of the problem it is necessary to specify the initial conditions

$$\psi = \psi_0(x, y),\ t = 0. \tag{10}$$

The main goal of this work is to develop and validate a finite-difference scheme for solving the system (7)–(10).

3. Numerical Technique

The standard benchmark problem for testing 2D NSEs is the lid driven cavity flow as shown in Fig. 1. The fluid contained inside a squared cavity is set into motion by the top wall which is sliding at constant velocity from left to right. Let L be the characteristic length scale associated with the cavity geometry and U be the characteristic velocity scale associated with the moving boundary. The

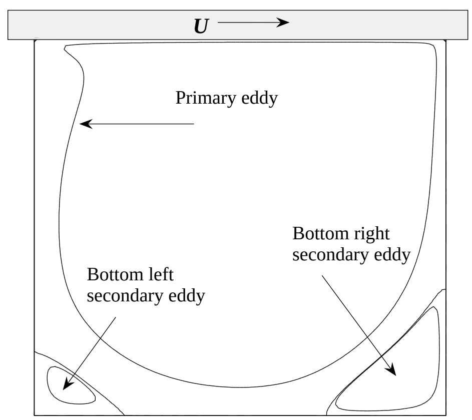

FIGURE 1. Lid driven square cavity flow configuration.

non-dimensional parameter of the problem is

$$\mathrm{Re} = \frac{LU}{\nu}. \tag{11}$$

The system of equations (1)–(3) is rendered dimensionless as follows

$$x = \frac{x^*}{L},\quad y = \frac{y^*}{L},\quad t = \frac{t^*\nu}{L^2},\quad u = \frac{u^*}{U},\quad v = \frac{v^*}{U}. \tag{12}$$

We cover the domain $Q = \{0 \leqslant x \leqslant 1,\ \ 0 \leqslant y \leqslant 1\}$, with a uniform grid

$$Q_h = \{(x_i, y_j)|x_i = (i - 1.5)h_x,\ \ y_j = (j - 1.5)h_y, i = 1, \dots, N_x,\ j = 1, \dots, N_y\}$$

with spacings

$$h_x = \frac{1}{N_x - 2},\quad h_y = \frac{1}{N_y - 2},$$

in the $x-$ and $y-$ directions, respectively. Such grid allows one to use central differences to approximate boundary conditions with second-order on two point stencils.

The essential element of the proposed here algorithm is that Eqs. (7) and (8) for ψ and Φ are considered as a coupled system. This formulation is based on the idea of regarding the two boundary conditions for ψ as actual conditions for the $\psi - \Phi$ system. Note that ψ and Φ are evaluated on the full-time steps. We employ second-order central-difference approximations for the operators in Eqs. (7) and (8). The system of difference equations is

$$\begin{aligned}(13)\quad & \frac{\psi_{i,j}^{n+1} - \psi_{i,j}^{n}}{\tau} - \\ & Re\frac{(\psi_{i+1,j}^{n} - \psi_{i-1,j}^{n})(\psi_{i,j+1}^{n+1} - \psi_{i,j-1}^{n+1}) + (\psi_{i+1,j}^{n+1} - \psi_{i-1,j}^{n+1})(\psi_{i,j+1}^{n} - \psi_{i,j-1}^{n})}{8h_x h_y} + \\ & Re\frac{\left(\Phi_{i+1,1}^{n+1} - \Phi_{i-1,j}^{n+1}\right)}{2h_x} = \frac{1}{2}\left(\triangle\psi_{i,j}^{n+1} + \triangle\psi_{i,j}^{n}\right),\end{aligned}$$

$$\begin{aligned}(14)\quad & \triangle\Phi_{i,j}^{n+1} = \frac{1}{2h_y}\left[(\psi_{i,j+1}^{n} - \psi_{i,j-1}^{n})\triangle\psi_{i,j}^{n+1} + (\psi_{i+1,j}^{n+1} - \psi_{i-1,j}^{n+1})\triangle\psi_{i,j}^{n}\right], \\ & i = 2,\dots,N_x - 1, \quad j = 1,\dots,N_y - 1.\end{aligned}$$

The boundary conditions are written in the following form

$$(15)\quad \begin{array}{lll} \dfrac{\psi_{2,j}^{n+1} + \psi_{1,j}^{n+1}}{2} = 0, & \dfrac{\psi_{2,j}^{n+1} - \psi_{1,j}^{n+1}}{h_x} = 0, & \\ \dfrac{\psi_{N_x,j}^{n+1} + \psi_{N_x-1,j}^{n+1}}{2} = 0, & \dfrac{\psi_{N_x,j}^{n+1} - \psi_{N_x-1,j}^{n+1}}{h_x} = 0, & j = 2,\dots,N_y - 1, \\ \dfrac{\psi_{i,2}^{n+1} + \psi_{i,1}^{n+1}}{2} = 0, & \dfrac{\psi_{i,2}^{n+1} - \psi_{i,1}^{n+1}}{h_y} = 0, & \\ \dfrac{\psi_{i,N_y}^{n+1} + \psi_{i,N_y-1}^{n+1}}{2} = 0, & \dfrac{\psi_{i,N_y}^{n+1} - \psi_{i,N_y-1}^{n+1}}{h_y} = 1, & i = 2,\dots,N_x - 1. \end{array}$$

To combine Eqs. (13)–(15) as a *single* linear system with banded matrix we introduce new system of indices as follows

$$\begin{aligned} k_{(i,j)} &= 2(j-1)N_x + 2i - 1, & i = 1,\dots,N_x, \\ m_{(i,j)} &= 2(j-1)N_x + 2i = k_{(i,j)} + 1, & j = 1,\dots,N_y. \end{aligned}$$

Each node (i,j) of grid Q_h associates with two indices $k_{(i,j)}$ and $m_{(i,j)}$. Index $k_{(i,j)}$ is an odd number and index $m_{(i,j)}$ is an even number. It is easy to see that

$$(16a)\quad \begin{aligned} k_{(i+1,j)} = k_{(i,j)} + 2, \quad k_{(i,j+1)} = k_{(i,j)} + 2N_x, \\ k_{(i-1,j)} = k_{(i,j)} - 2, \quad k_{(i,j-1)} = k_{(i,j)} - 2N_x, \end{aligned}$$

$$(16b)\quad \begin{aligned} m_{(i+1,j)} = m_{(i,j)} + 2, \quad m_{(i,j+1)} = m_{(i,j)} + 2N_x, \\ m_{(i-1,j)} = m_{(i,j)} - 2, \quad m_{(i,j-1)} = m_{(i,j)} - 2N_x. \end{aligned}$$

Now we introduce a new grid function $\sigma = \{\sigma_k,\, k = 1, ..., 2N_xN_y\}$ which is defined on the composite grid. Let components of the grid function σ_k with even indices represent $\psi_{i,j}$ and components with odd indices $\sigma_m (= \sigma_{k+1})$ represent $\Phi_{i,j}$. Substituting σ_k instead $\psi_{i,j}$ and substituting σ_m instead $\Phi_{i,j}$ into Eqs. (13)–(14), we recast algebraic system as follows

(17a)
$$\frac{\sigma_k^{n+1}-\sigma_k^n}{\tau}+\frac{Re}{8h_xh_y}\Big[\left(\sigma_{k+2}^n-\sigma_{k-2}^n\right)\left(\sigma_{k+2N_x}^{n+1}-\sigma_{k-2N_x}^{n+1}\right)+\left(\sigma_{k+2}^{n+1}-\sigma_{k-2}^{n+1}\right)$$
$$\left(\sigma_{k+2N_x}^n-\sigma_{k-2N_x}^n\right)\Big]-\frac{Re}{2h_x}\left(\sigma_{k+1}^{n+1}-\sigma_{k-3}^{n+1}\right)=\frac{1}{2}\left(\Delta\sigma_k^{n+1}+\Delta\sigma_k^n\right),$$

(17b)
$$\Delta\sigma_m^{n+1}=\frac{\left(\sigma_{m+2N_x-1}^n-\sigma_{m-2N_x-1}^n\right)}{2h_y}\Delta\sigma_{m-1}^{n+1}+\frac{\left(\sigma_{m+2N_x-1}^{n+1}-\sigma_{m-2N_x-1}^{n+1}\right)}{2h_y}\Delta\sigma_{m-1}^n,$$

where
$$\Delta\sigma_k=\frac{\left(\sigma_{k+2}-2\sigma_k+\sigma_{k-2}\right)}{h_x^2}+\frac{\left(\sigma_{k+2N_x}-2\sigma_k+\sigma_{k-2N_x}\right)}{h_y^2}.$$

The straightforward implementation of the algorithm leads to a problem with a numerically singular matrix. There are different ways to regularize the problem. We found that adding a small term at the boundary gives the best results. According to this idea, Eq. (15) can be rewritten for function σ as follows

(18a) $$\frac{\sigma_k^{n+1}+\sigma_{k+2}^{n+1}}{2}=0,\quad \frac{\sigma_{m+1}^{n+1}-\sigma_{m-1}^{n+1}}{h_x}=\varepsilon\sigma_m^{n+1},\quad i=1,\ j=1,\dots,N_y,$$

(18b) $$\frac{\sigma_k^{n+1}+\sigma_{k-2}^{n+1}}{2}=0,\quad \frac{\sigma_{m-1}^{n+1}-\sigma_{m-3}^{n+1}}{h_x}=\varepsilon\sigma_m^{n+1},\quad i=N_x,\ j=1,\dots,N_y,$$

(18c) $$\frac{\sigma_k^{n+1}+\sigma_{k+2N_x}^{n+1}}{2}=0,\quad \frac{\sigma_{m+2N_x+1}^{n+1}-\sigma_{m-1}^{n+1}}{h_y}=0,\quad j=1,\ i=1,\dots,N_x,$$

(18d) $$\frac{\sigma_k^{n+1}+\sigma_{k-2N_x}^{n+1}}{2}=0,\quad \frac{\sigma_{m-1}^{n+1}-\sigma_{m-2N_x-1}^{n+1}}{h_y}=1,\quad j=N_y,\ i=1,\dots,N_x,$$

where ε is a small number. If the steady flow is needed then the algorithm can be considered as an iterative procedure and iterations are terminated at certain time $n=N$ when the following criterion is satisfied
$$\frac{\max_{i,j}\left|\sigma_{i,j}^{N+1}-\sigma_{i,j}^N\right|}{\max_{i,j}\left|\sigma_{i,j}^{N+1}\right|}\leqslant 10^{-8}$$

Note that the linear system for the coupled formulation of the $\psi-\Phi$ problem can be written as the following multi-diagonal system for the composite grid function σ

(19) $$B_{l-2N_x-1}\sigma_{l-2N_x-1}^{n+1}+B_{l-2N_x}\sigma_{l-2N_x}^{n+1}+B_{l-3}\sigma_{l-3}^{n+1}+B_{l-2}\sigma_{l-2}^{n+1}$$
$$+B_{l-1}\sigma_{l-1}^{n+1}+B_l\sigma_l^{n+1}+B_{l+1}\sigma_{l+1}^{n+1}+B_{l+2}\sigma_{l+2}^{n+1}$$
$$+B_{l+2N_x-1}\sigma_{l+2N_x-1}^{n+1}+B_{l+2N_x}\sigma_{l+2N_x}^{n+1}+B_{l+2N_x+1}\sigma_{l+2N_x+1}^{n+1}=F_l,$$

where $l=1,\dots,2N_xN_y$. The matrix of the linear system (19) is banded with $2N_x+1$ lower and upper bandwidths. We used standard routings DGBSV and DGBSVX of the LAPACK to compute the solution of Eq. (19).

4. Results and Discussions

A validation test involves a 2D cavity flow at Reynolds number up to 1000, wherein the flow is laminar and steady. Computations were performed for the lid-driven cavity problem on grids from 32×32 up to 102×102. The impact of ε on the results is judiciously evaluated by numerical experiments and shown that for $\varepsilon \in [10^{-10}, 10^{-4}]$ the approximate solution agreed with know test case.

In order to validate the scheme, we compare the extremal values and space location of stream function with [2, 3, 4, 5, 6]. Table 1 reports the characteristics of the primary and right bottom secondary eddies at $Re = 100$ and $Re = 1000$, respectively. This table shows the extreme values of ψ and the space location of the extremal values of ψ. Top rows present our quantities obtained from simulation. Then bottom rows display the quantities obtained by the other authors. For the primary vortex in cases $Re = 100$ and 1000, our results agree within 5% with those obtained by the other authors. For the secondary vortex in the case $Re = 1000$ with the grids 52×52, our results agree within 5% with Christov et al. [5] but differ significantly up to 10% with those obtained by Botella et al. [2]. Our data obtained using 102×102 grid exhibit perfect match with other results from [2, 3, 4, 6]. The geometrical structures of the flow are displayed in Figs. 2 and 3. The grid 102×102 is used. Note that the values of u (velocity along $x-$ direction), v (velocity along $y-$direction), and vorticity ω were computed from ψ after the iteration converge. The values of u, v, and ω inside the domain were approximated using central-difference while the values of u, v, and ω on the boundary were approximated using one side first-order difference.

In Fig. 4, we compare the centerline $u-$ and $v-$ velocity profiles with data of Ghia et al. [6]. The computational has been done for Reynolds number $Re = 1000$ with the grid 102×102. The velocity profiles are similar to the data of Ghia et al. [6].

Table 1. Strength and location of the primary and bottom right secondary vortices at $Re = 100$ and $Re = 1000$.

Ref.	Grid	Primary eddy		Bottom right second. eddy	
		ψ_{min}	(x_{min}, y_{min})	ψ_{max}	(x_{max}, y_{max})
			$Re = 100$		
Present	32×32	-0.103615	$(0.600, 0.733)$	3.34039×10^{-6}	$(0.967, 0.067)$
	52×52	-0.103569	$(0.620, 0.740)$	9.42029×10^{-6}	$(0.940, 0.060)$
	102×102	-0.103510	$(0.620, 0.740)$	1.18920×10^{-5}	$(0.940, 0.060)$
[2]	48	-0.10008	–	–	–
[4]	162×162	-0.10397	$(0.6198, 0.7369)$	–	–
[6]	129×129	-0.103	$(0.5844, 0.7400)$	1.25×10^{-5}	$(0.9401, 0.0599)$
			$Re = 1000$		
Present	52×52	-0.119825	$(0.540, 0.560)$	1.56192×10^{-3}	$(0.860, 0.100)$
	102×102	-0.119280	$(0.530, 0.560)$	1.68817×10^{-3}	$(0.870, 0.110)$
[2]	160	-0.118937	$(0.5308, 0.5652)$	1.72972×10^{-3}	$(0.8640, 0.1118)$
[3]	1024×1024	-0.11892	$(0.5312, 0.5654)$	1.7292×10^{-3}	$(0.8643, 0.1123)$
[4]	80×80	-0.118710	$(0.5346, 0.5645)$	–	–
[5]	512×512	-0.116269	$(0.5316, 0.5660)$	1.640×10^{-3}	$(0.8651, 0.1118)$
[6]	128	-0.117929	$(0.5313, 0.5625)$	1.751×10^{-3}	$(0.8594, 0.1094)$

We find it important to understand the behavior of function Φ for different Reynolds number. Fig. 5 shows the contour of Φ. Pattern of contour lines of Φ are drawn for several Reynolds numbers, $Re = 50$, 100, 200, 300, 500, and 1000. The set of figures is generated on the grid 52×52 with the parameter $\varepsilon = 10^{-6}$.

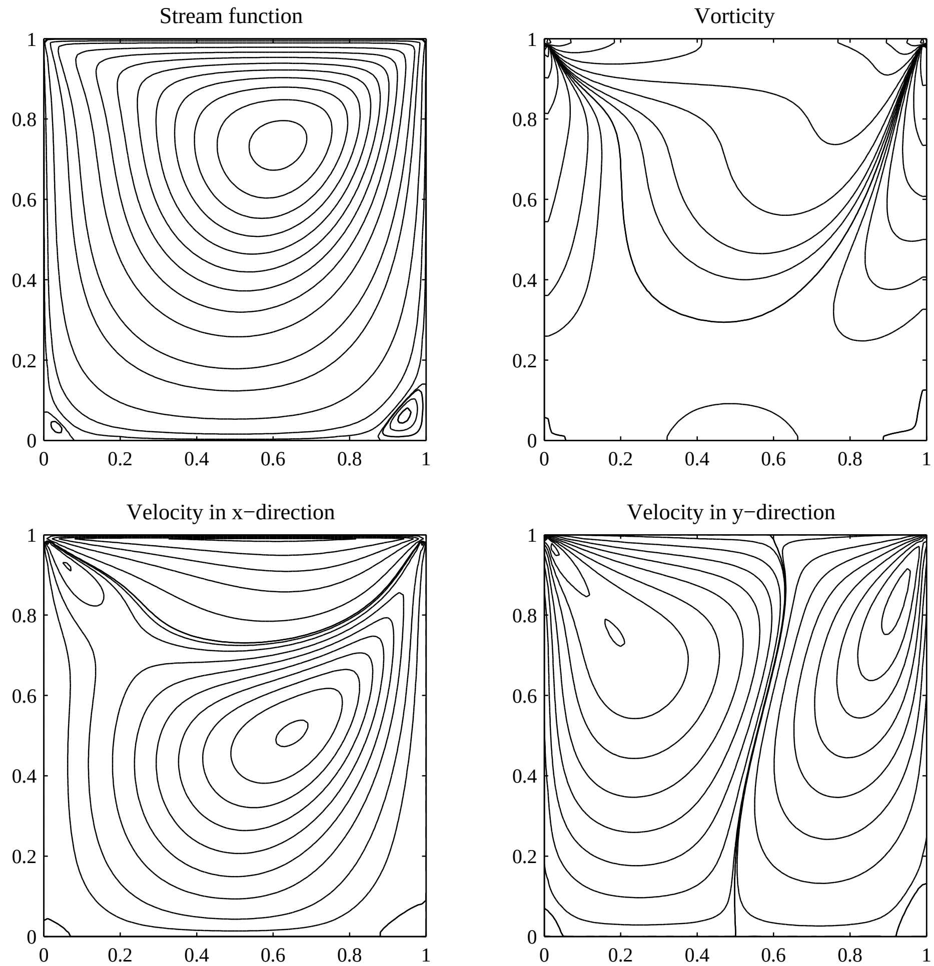

FIGURE 2. Stream function ψ, vorticity ω, and velocity components u and v contours for the lid-driven cavity flow at $Re = 100$.

5. Conclusion

A finite-difference scheme is developed and validated for the new formulation of the 2D Navier-Stokes equations proposed in [7]. The feature that qualitatively distinguishes this formulation from the vorticity-stream function formulation of NSEs is that *the stream function satisfies the evolution equation* (the elliptic equation in case of vorticity-stream function formulation) and *the new unknown function satisfies an elliptic equation* (vorticity satisfies the evolution equation in case of vorticity-stream function formulation). The scheme and algorithm treat the equations as a coupled system which allows one to satisfy two conditions for stream function with no condition on the new function. The new numerical algorithm demonstrated good accuracy and reasonable efficiency for the lid-driven cavity flow benchmark problem. The results obtained in all test cases are in excellent agreement with other established numerical results, underlining that the new formulation is a viable approach to the 2D Navier-Stokes and can serve as a basis for the efficient numerical models.

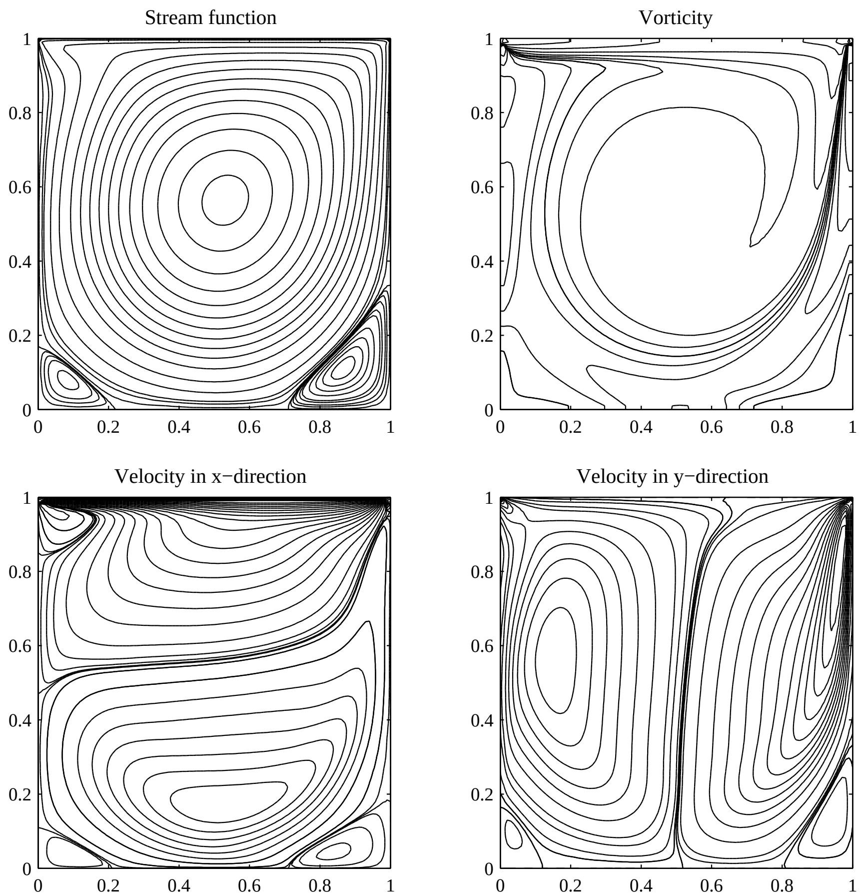

FIGURE 3. Stream function ψ, vorticity ω, and velocity components u and v contours for the lid-driven cavity flow at $Re = 1000$.

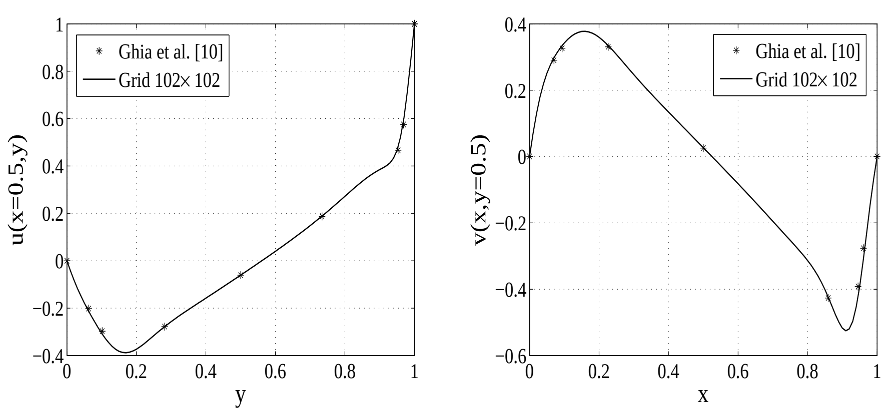

FIGURE 4. The vertical centerline $u-$profile and horizontal centerline $v-$profile for the lid-driven cavity flow at $Re = 1000$.

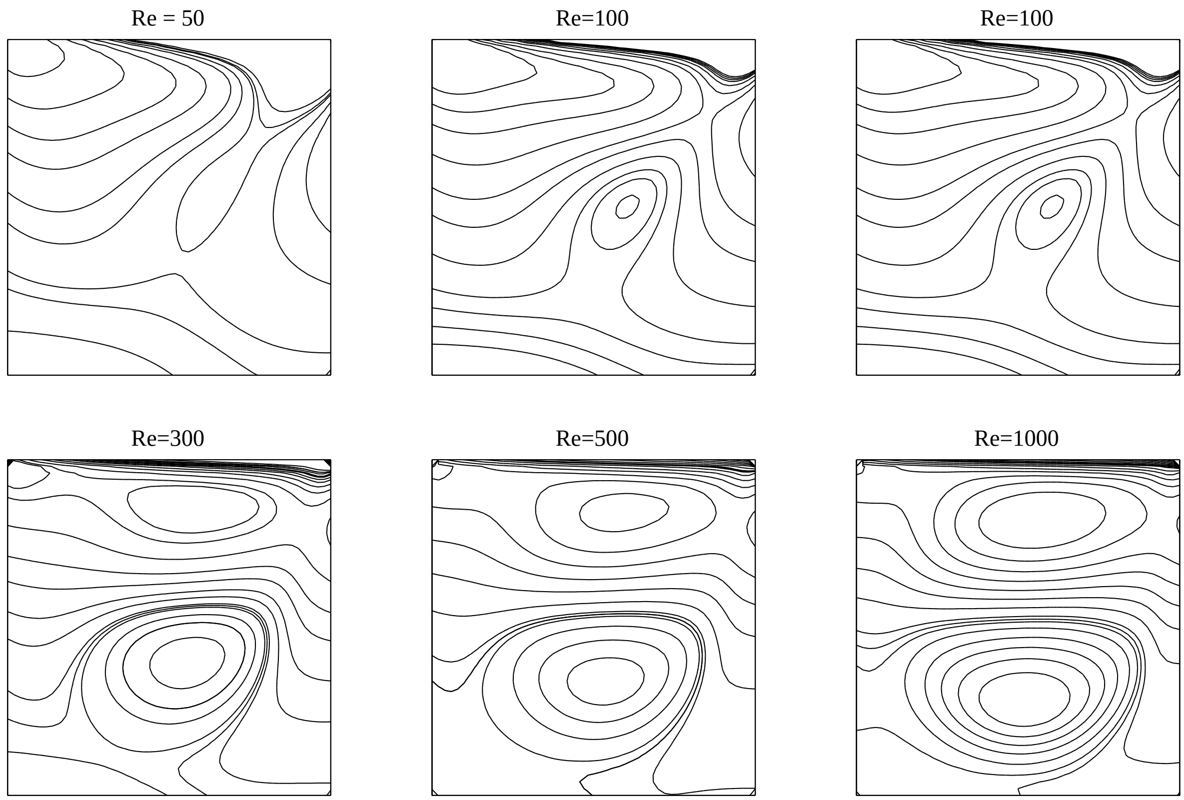

FIGURE 5. Contours of Φ for the lid-driven cavity flow.

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