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ON A NONLINEAR 4-POINT TERNARY AND INTERPOLATORY MULTIRESOLUTION SCHEME ELIMINATING THE GIBBS PHENOMENON

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Abstract. A nonlinear ternary 4-point interpolatory subdivision scheme is presented. It is based on a nonlinear perturbation of the ternary subdivision scheme studied in Hassan M.F., Ivrissimtzis I.P., Dodgson N.A. and Sabin M.A. (2002): "An interpolating 4-point ternary stationary subdivision scheme", Comput. Aided Geom. Design, **19**, 1-18. The convergence of the scheme and the regularity of the limit function are analyzed. It is shown that the Gibbs phenomenon, classical in linear schemes, is eliminated. The stability of the associated nonlinear multiresolution scheme is established. Up to our knowledge, this is the first interpolatory scheme of regularity larger than one, avoiding Gibbs oscillations and for which the stability of the associated multiresolution analysis is established. All these properties are very important for real applications.

Key Words. Nonlinear ternary subdivision scheme, regularity, nonlinear multiresolution, stability, Gibbs phenomenon, signal processing

1. Introduction

As a generalization of the binary subdivision schemes [13, 12, 10], ternary schemes have been proposed in the last years [16, 20, 27, 28, 7]. A general increasing interest for investigating higher arities has emerged since Hassan et al. [16] showed that one can achieve higher smoothness and smaller support for the so-called interpolating 4-point stationary scheme, by going from binary [12] to ternary. In [7], a non-stationary 4-point ternary interpolatory subdivision scheme has been presented. It provides the user with a tension parameter that, when increased within its range of definition, can generate at convergence C^2-continuous limit curves showing considerable variations of shape.

All these approaches deal with linear subdivision schemes and in particular the Gibbs phenomenon oscillations appear in the presence of discontinuities in the data.

On the other hand, multiresolution representations of data are useful tools in signal processing applications. Given f^L a set of data where L stands for a resolution level, a multiresolution representation of f^L is any sequence of type $\{f^0, d^0, d^1, \ldots, d^{L-1}\}$ where f^k is an approximation of f^L at resolution $k < L$ and d^k stands for the details required to recover f^{k+1} from f^k. The couple $\{f^k, d^k\}$ contains the same information as f^{k+1} and therefore the same is true for $\{f^0, d^0, d^1, \ldots, d^{L-1}\}$ and f^L.

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Again, due to the Gibbs phenomenon, it turns out to be that the efficiency of linear multiresolution decompositions for instance for signal compression is generally limited by the presence of discontinuities.

Moreover, in signal processing applications, the multi-scale representation $(f^0, d^0, d^1, \dots, d^{L-1})$ is usually processed obtaining $(\hat{f}^0, \hat{d}^0, \hat{d}^1, \dots, \hat{d}^{L-1})$ that are *close to but different from* the original one. Decoding recovers the discrete set $\hat{f}^L$ from the processed representation. The stability property deals with the ability to control the error between f^L and $\hat{f}^L$ by the difference between $(f^0, d^0, d^1, \dots, d^{L-1})$ and $(\hat{f}^0, \hat{d}^0, \hat{d}^1, \dots, \hat{d}^{L-1})$.

Recently, various attempts to improve the classical linear subdivision schemes and their associated multiresolution algorithms have led to various nonlinear multiresolution schemes. In such frameworks, only few results for convergence and stability are available [1, 3, 5, 8, 9, 11, 21, 23, 25].

The aim of this paper is to introduce a new nonlinear ternary subdivision scheme. We successively analyze the properties of the scheme and of the associated nonlinear multiresolution transform. Convergence, regularity of the limit functions and stability of the multiresolution transform are established as well as the elimination of Gibbs oscillations in presence of discontinuities.

Up to our knowledge, this is the first interpolatory scheme of regularity larger than one, avoiding Gibbs oscillations and for which the stability of the associated multiresolution analysis is established.

The paper is organized as follows. In section 2 we introduce the basic notations and some necessary results that we use in the rest of the paper. In section 3 we present a new nonlinear ternary subdivision scheme based on the scheme studied in [16]. Writing the scheme as a perturbation of a linear scheme and establishing a contractivity property of this perturbation, we deduce the convergence of the subdivision scheme and the stability of the associated multiresolution algorithm, that due to the nonlinear nature of the scheme is not a consequence of the convergence. The elimination of the Gibbs phenomenon in presence of discontinuities is studied rigorously. Section 4 is devoted to numerical examples.

2. The basic framework

The multiresolution framework studied in this paper can be considered as a particular example of the Harten interpolatory multiresolution setting [17, 6] transposed to ternary refinement.

2.1. The Harten interpolatory multiresolution setting. One considers a set of nested bi-infinite regular grids:

$$X^j = \{x_n^j\}_{n\in\mathbb{Z}}, \quad x_n^j = n3^{-j},$$

where j is called a scale parameter. The point-value discretization (sampling) operators are defined by

$$\mathcal{D}_j : f \in C(\mathbb{R}) \mapsto f^j = (f_n^j)_{n\in\mathbb{Z}} := (f(x_n^j))_{n\in\mathbb{Z}} \in V^j,$$

where V^j is the space of real sequences and $C(\mathbb{R})$ the set of continuous functions on $\mathbb{R}$. A reconstruction operator $\mathcal{R}_j$ associated to this discretization is any right inverse of $\mathcal{D}_j$ on V^j which means that

$$(\mathcal{R}_j f^j)(x_n^j) = f_n^j = f(x_n^j).$$

The operator defined by $\mathcal{D}_{j+1}\mathcal{R}_j$ acts between the coarse level (j) and the fine level $(j+1)$ and is called a prediction operator.

Since $\mathcal{D}_{j+1}$ is the sampling operator on the grid X^{j+1} that contains the grid X^j, the prediction operator identifies with an interpolating subdivision scheme [6, 10].

Moreover, since for most function f, $\mathcal{D}_{j+1}\mathcal{R}_j f^j \neq f^{j+1}$, details, called d^j, should be added to $\mathcal{D}_{j+1}\mathcal{R}_j f^j$ to recover f^{j+1}. The multiresolution transforms (see [6] for more details) make the connection between f^L and the sequence $\{f^0, d^0, \ldots, d^{L-1}\}$.

2.2. Two General Results. In [3], a general study of convergence and stability of dyadic nonlinear schemes associated to particular perturbations of linear subdivision schemes was presented.

Adapted to the triadic subdivision, this approach deals with nonlinear subdivision schemes, $S_{NL}f$, that can be defined by

$$\left\{ \begin{array}{l} (S_{NL}f)_{3n}=f_n, \\ (S_{NL}f)_{3n+1}=(Sf)_{3n+1}+F(\delta f)_{3n+1}, \\ (S_{NL}f)_{3n+2}=(Sf)_{3n+2}+F(\delta f)_{3n+2}, \end{array} \right. \tag{1}$$

with F a nonlinear operator on $l^\infty(\mathbb{Z})$, δ a continuous linear operator on $l^\infty(\mathbb{Z})$ and S an interpolatory and convergent linear subdivision scheme.

Recalling the following definitions:

Definition 1. *A triadic subdivision scheme S is said to be convergent if*

$$\forall f \in l^\infty(\mathbb{Z}),\ \exists g \in C^0(\mathbb{R}) \text{ such that } \lim_{j\to+\infty} sup_{n\in\mathbb{Z}}|(S^j f)_n - g(n3^{-j})| = 0.$$

We note $g = S^\infty f$.

Definition 2. *A convergent subdivision scheme is stable if*

$$\exists C < +\infty \text{ such that } \forall f^0, g^0 \in l^\infty(\mathbb{Z}) \quad ||S^\infty f - S^\infty g||_{L^\infty} \leq C||f^0 - g^0||_{l^\infty}.$$

The following results can be proved using the same track followed in [3]:

Theorem 1. *(Convergence)*
If S is a linear convergent subdivision scheme and if S_{NL}, F and δ verify

$$\exists M > 0 \quad \text{such that} \quad \forall d \in l^\infty(\mathbb{Z}) \quad ||F(d)||_\infty \leq M||d||_\infty\,, \tag{2}$$

$$\exists L > 0,\ \exists c < 1 \text{ such that } \quad \forall f \in l^\infty(\mathbb{Z}) \quad ||\delta S^L_{NL}(f)||_\infty \leq c||\delta f||_\infty, \tag{3}$$

then the subdivision scheme S_{NL} defined by (1) converges.

Theorem 2. *(Regularity)*
Let S and S_{NL} satisfying hypotheses of Theorem 1. If S is C^{α^-} convergent [1] *then, for all sequence $f \in l^\infty(\mathbb{Z})$, $S^\infty_{NL}(f) \in C^{\beta^-}$ with $\beta = \min\left(\alpha, -\frac{log_3(c)}{L}\right)$.*

Theorem 3. *(Stability)*
If S_{NL}, F and δ verify

$$\exists M > 0, \text{ such that, } \forall d_1, d_2 \in l^\infty(\mathbb{Z})$$
$$||F(d_1) - F(d_2)||_\infty \leq M||d_1 - d_2||_\infty, \tag{4}$$
$$\exists c < 1, \text{ such that, } \forall f, g \in l^\infty(\mathbb{Z})$$
$$||\delta(S_{NL}(f) - S_{NL}(g))||_\infty \leq c||\delta(f - g)||_\infty, \tag{5}$$

then the multiresolution transform associated to the nonlinear scheme S_{NL} is stable.

[1] For $0 < \alpha < 1$, $f \in C^\alpha(\mathbb{R})$ if and only if f is bounded and $\exists C > 0$ such that $\forall x, y \in \mathbb{R}, |f(x) - f(y)| \leq C|x - y|^\alpha$
For $\alpha > 1$, $f \in C^\alpha(\mathbb{R})$ if and only if $f^{([\alpha])}$ is bounded and $f^{([\alpha])} \in C^{(\alpha-[\alpha])}$ where $[\alpha]$ is the integer part of α

3. A nonlinear ternary subdivision scheme

In this section we introduce a nonlinear ternary subdivision scheme as a perturbation of the linear scheme introduced in [16]. Other ternary nonlinear schemes can be found in [7, 11, 25].

In [16], Hassan et al. show that one can achieve higher smoothness and smaller support for the so-called interpolating 4-point stationary scheme, by going from binary to ternary. Given a real w that will be called a tension parameter, the following ternary subdivision scheme is analyzed:

$$\begin{aligned}(S_w f)_{3n} &= f_n,\\ (S_w f)_{3n+1} &= a_0 f_{n-1} + a_1 f_n + a_2 f_{n+1} + a_3 f_{n+2},\\ (S_w f)_{3n+2} &= a_3 f_{n-1} + a_2 f_n + a_1 f_{n+1} + a_0 f_{n+2},\end{aligned} \tag{6}$$

with

$$a_0 = -\frac{1}{18} - \frac{1}{6}w,\ a_1 = \frac{13}{18} + \frac{1}{2}w,\ a_2 = -\frac{7}{18} - \frac{1}{2}w \text{ and } a_3 = -\frac{1}{18} + \frac{1}{6}w.$$

The value $w = \frac{1}{27}$ corresponds to the ternary Lagrange scheme, therefore denoted $S_{\frac{1}{27}}$.

In [16], it is proved that *the scheme (6) converges for* $0 \leq w < \frac{1}{2}$ *to a limit function with regularity* C^2 *for* $\frac{1}{15} < w < \frac{1}{9}$, $C^{(1+\beta)-}$ *with* $\beta = -log_3(1-2w)$ *for* $w \in]0, \frac{1}{15}] \cup [\frac{1}{9}, \frac{1}{2}[$ *and* C^{1-} *for* $w = 0$.

A generalization of the scheme using a non-stationary tension parameter (i.e. a value for w different for every level j) can be found in [7].
An important weakness of such linear interpolatory schemes is the occurrence of oscillations (Gibbs phenomenon) in the limit function when the scheme starts from the sampling of a discontinuous function (see Figure 1-Left).

In order to avoid these oscillations, a nonlinear modification of the scheme is required. The nonlinear scheme we are going to present takes its roots from the schemes introduced by Harten, Osher, Engquist and Chakravarthy in [18, 19] for the interpolation of fluxes in the numerical solution of hyperbolic conservation laws. In all these works, over/undershoots around discontinuities are controlled using so-called flux limiters that are based on the substitution of an arithmetic mean $AMEAN(x,y) = \frac{x+y}{2}$ by other means.

The initial scheme of [18, 19], namely the ENO scheme (for essentially non oscillatory scheme) suffered from several drawbacks from which we mention:

- loss of accuracy on smooth regions with specific input data,
- smearing of certain discontinuities,
- smoothing up of corners,
- too wide stencil.

Several remedies were proposed, among which one find the work of Marquina [24], Amat, Busquier and Candela [2] and Serna and Marquina[26].

The nonlinear scheme we are going to analyze in this paper follows [4, 5], and is based on the substitution of the arithmetic mean by the PPH mean defined as:

$$(x,y) \in \mathbb{R}^2 \mapsto \text{PPH}(x,y) := \frac{xy}{x+y}(\text{sgn}(xy)+1),$$

with $\text{sgn}(x) = 1$ if $x \geq 0$ and $\text{sgn}(x) = -1$ if $x < 0$.

We first propose two new formulations of the scheme (6):

$$\begin{aligned}
(S_w f)_{3n} &= f_n,\\
(S_w f)_{3n+1} &= (\frac{2}{3}+\frac{w}{3})f_n + (\frac{1}{3}-\frac{2w}{3})f_{n+1} + \frac{w}{3}f_{n+2}\\
&\quad -(\frac{1}{9}+\frac{w}{3})\text{AMEAN}(d^2f_n, d^2f_{n+1}),\\
(S_w f)_{3n+2} &= (\frac{1}{3}-\frac{w}{3})f_n + (\frac{2}{3}+\frac{2w}{3})f_{n+1} - \frac{w}{3}f_{n+2}\\
&\quad -(\frac{1}{9}-\frac{w}{3})\text{AMEAN}(d^2f_n, d^2f_{n+1}),
\end{aligned}$$

or

$$\begin{aligned}
(S_w f)_{3n} &= f_n,\\
(S_w f)_{3n+1} &= -\frac{w}{3}f_{n-1} + (\frac{2}{3}+\frac{2w}{3})f_n + (\frac{1}{3}-\frac{w}{3})f_{n+1}\\
&\quad -(\frac{1}{9}-\frac{w}{3})\text{AMEAN}(d^2f_n, d^2f_{n+1}),\\
(S_w f)_{3n+2} &= \frac{w}{3}f_{n-1} + (\frac{1}{3}-\frac{w}{3})f_n + (\frac{2}{3}+\frac{w}{3})f_{n+1}\\
&\quad -(\frac{1}{9}+\frac{w}{3})\text{AMEAN}(d^2f_n, d^2f_{n+1}).
\end{aligned}$$

The nonlinear scheme S_{PPH} that we propose, removing w in the notation, is then given by

$$(S_{\text{PPH}}f)_{3n} = f_n,$$

and, if $|d^2f_n| \geq |d^2f_{n+1}|$,

$$\begin{aligned}
(S_{\text{PPH}} f)_{3n+1} &= (\frac{2}{3}+\frac{w}{3})f_n + (\frac{1}{3}-\frac{2w}{3})f_{n+1} + \frac{w}{3}f_{n+2}\\
&\quad -(\frac{1}{9}+\frac{w}{3})\text{PPH}(d^2f_n, d^2f_{n+1}),\\
(S_{\text{PPH}} f)_{3n+2} &= (\frac{1}{3}-\frac{w}{3})f_n + (\frac{2}{3}+\frac{2w}{3})f_{n+1} - \frac{w}{3}f_{n+2}\\
&\quad -(\frac{1}{9}-\frac{w}{3})\text{PPH}(d^2f_n, d^2f_{n+1}),
\end{aligned} \tag{7}$$

or, if $|d^2f_n| < |d^2f_{n+1}|$,

$$\begin{aligned}
(S_{\text{PPH}} f)_{3n+1} &= -\frac{w}{3}f_{n-1} + (\frac{2}{3}+\frac{2w}{3})f_n + (\frac{1}{3}-\frac{w}{3})f_{n+1}\\
&\quad -(\frac{1}{9}-\frac{w}{3})\text{PPH}(d^2f_n, d^2f_{n+1}),\\
(S_{\text{PPH}} f)_{3n+2} &= \frac{w}{3}f_{n-1} + (\frac{1}{3}-\frac{w}{3})f_n + (\frac{2}{3}+\frac{w}{3})f_{n+1}\\
&\quad -(\frac{1}{9}+\frac{w}{3})\text{PPH}(d^2f_n, d^2f_{n+1}).
\end{aligned} \tag{8}$$

Before analyzing in details the properties of the new scheme S_{PPH} we summarize the most important properties of the harmonic mean in the following proposition (see [5] for more details).

Proposition 1. *For all* $(x, y) \in \mathbb{R}^2$, *the harmonic mean* $\text{PPH}(x, y)$ *satisfies*

- $\text{PPH}(x,y) = \text{PPH}(y,x)$.
- $\text{PPH}(x,y) = 0 \quad if\ xy \le 0$.
- $\text{PPH}(-x,-y) = -\text{PPH}(x,y)$.
- $\text{PPH}(x,y) = \frac{sign(x)+sign(y)}{2} min(|x|,|y|)\left[1+\left|\frac{x-y}{x+y}\right|\right]$.
- $|\text{PPH}(x,y)| \le \max(|x|,|y|)$.
- $|\text{PPH}(x,y)| \le 2\min(|x|,|y|)$.
- *For* $x,y>0$, $\min(x,y) \le \text{PPH}(x,y) \le \frac{x+y}{2}$.
- *If* $x=O(1)$, $y=O(1)$, $|y-x|=O(h)$ *and* $xy>0$ *then*
$$|\frac{x+y}{2} - \text{PPH}(x,y)| = O(h^2).$$
- $|\text{PPH}(x_1,y_1) - \text{PPH}(x_2,y_2)| \le 2\max(|x_1-x_2|,|y_1-y_2|)$.

3.1. Convergence. First we rewrite the scheme S_{PPH} (7, 8) as a particular perturbation of the linear ternary scheme S_w, which is a convergent scheme with limit functions in the spaces C^1 for $0 \le w < \frac{1}{2}$.

For all $f \in l^\infty(\mathbb{R})$, we have

$$S_{\text{PPH}} f = S_w f + F(d^2 f),$$

with F the function defined by

$$F(d^2 f)_{3n} \quad = \quad 0,$$

and, $if |d^2 f_n| \ge |d^2 f_{n+1}|$,

$$\begin{aligned} F(d^2f)_{3n+1} &= (\frac{1}{9}+\frac{w}{3})(\text{AMEAN}(d^2f_n,d^2f_{n+1}) - \text{PPH}(d^2f_n,d^2f_{n+1})) \\ F(d^2f)_{3n+2} &= (\frac{1}{9}-\frac{w}{3})(\text{AMEAN}(d^2f_n,d^2f_{n+1}) - \text{PPH}(d^2f_n,d^2f_{n+1})) \end{aligned}$$

or, $if |d^2 f_n| < |d^2 f_{n+1}|$,

$$\begin{aligned} F(d^2f)_{3n+1} &= (\frac{1}{9}-\frac{w}{3})(\text{AMEAN}(d^2f_n,d^2f_{n+1}) - \text{PPH}(d^2f_n,d^2f_{n+1})) \\ F(d^2f)_{3n+2} &= (\frac{1}{9}+\frac{w}{3})(\text{AMEAN}(d^2f_n,d^2f_{n+1}) - \text{PPH}(d^2f_n,d^2f_{n+1})) \end{aligned}$$

We can use the general result presented in Theorem 1 to derive the convergence of the nonlinear scheme (7, 8) as well as the regularity of the limit functions.

Theorem 4. *(Convergence)*
The contraction constant of equation (3) written for S_{PPH} *is* $c = \max\left(\frac{2+3w}{9}, \frac{1}{9}+w, \frac{1-w}{3}\right)$. *Therefore, for* $0 \le w < \frac{8}{9}$, *the scheme* S_{PPH} *defined by (7, 8) is uniformly convergent.*

Theorem 5. *(Regularity)*
For all sequence $f \in l^\infty(\mathbb{Z})$, *the limit function* $S^\infty_{\text{PPH}} f$ *belongs to* $C^{\beta-}$ *with* $\beta = \min(1-log_3(1-2w), -log_3(c)) > 1$ *for* $0 < w \le \frac{2}{9}$, $\beta = \mathbf{1}$ *for* $w=0$ *and* $\beta = -log_3(c) < \mathbf{1}$ *for* $\frac{2}{9} \le w < \frac{8}{9}$.

(Both theorems are proved in the Annex, section 6).

Remark 1. *We can note that the theoretical regularity of Theorem 5 is close to the numerical regularity that can be evaluated following* [21] *(see table 1).*

w	$\frac{1}{27}$	$\frac{1}{11}$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{3}{4}$
theoretical regularity	1.035	1.0868	1.1072	0.738	0.136
numerical regularity	1.02	1.065	1.17	0.8775	0.26

Table 1. Comparison of the regularity constant estimated theoretically (theorem 5) and numerically [21].

3.2. Stability of the associated nonlinear interpolatory multiresolution scheme. Using Theorem 3 we get

Theorem 6. *(Stability)*
The nonlinear multiresolution algorithm associated to S_{PPH} *is stable for* $0 \leq w < \frac{2}{15}$.

(This theorem is proved in the Annex, section 6).

3.3. Properties of the limit functions. In this section, we analyze rigorously the properties of the limit functions and particularly the behavior of the scheme in presence of discontinuities.

Even if the goal we have here is different from the one that triggered the initial propositions of Harten et al [18, 19], we come back first to some drawbacks of the ENO scheme. Since the stencil of our scheme has four points, the wideness of the ENO stencil, mentionned as a drawback in section 3 is cured. Moreover, we have to add another drawback that is the non stability of the ENO subdivision scheme (see for instance [8]).

Concerning the order of approximation of our scheme, the following proposition holds.

Proposition 2. *For all function* $g \in C^4([0,1])$ *and* $h > 0$, *if*

$$f = g((nh))_{n\in\mathbb{Z}},$$

then

if $d^2 f_n d^2 f_{n+1} > 0$ *for all* $n \in \mathbb{Z}$, *then*

$$||(S_{\text{PPH}} f)_n - g(\frac{hn}{3})||_\infty = O(h^4),$$

otherwise

$$||(S_{\text{PPH}} f)_n - g(\frac{hn}{3})||_\infty = O(h^3).$$

Proof: According to Proposition 1, we have that if $d^2 f_n d^2 f_{n+1} > 0$ for all $n \in \mathbb{N}$ then

$$|\text{PPH}(d^2 f_n, d^2 f_{n+1}) - \text{AMEAN}(d^2 f_n, d^2 f_{n+1})| = O(h^4).$$

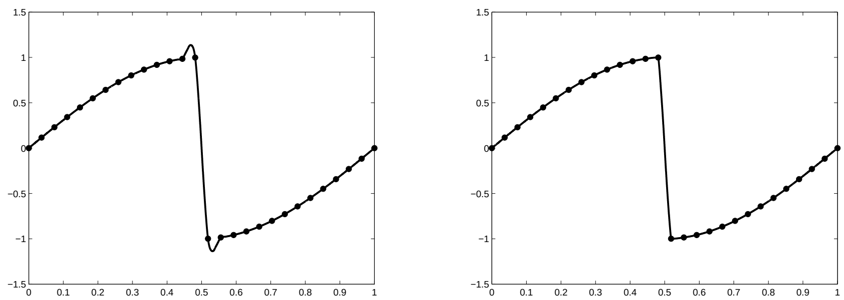
Therefore, according to the definition of the S_{PPH},

$$||S_{\text{PPH}} f - S_w f||_\infty = O(h^4).$$

Since the scheme S_w is of order of approximation 4 we get the result when $d^2 f_n d^2 f_{n+1} > 0$.
If not, the reproduction of polynomials of the linear parts in the definition of S_{PPH} (7, 8) leads to

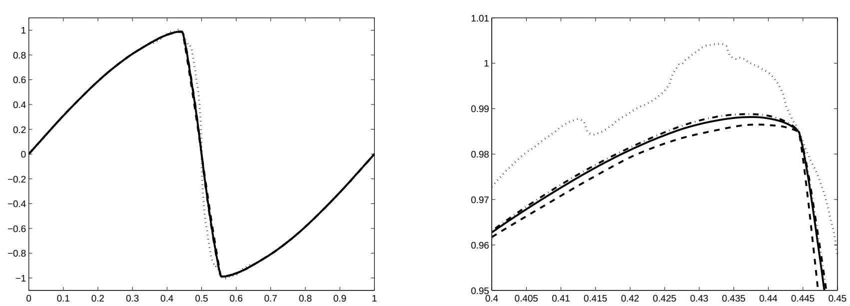
$$||(S_{\text{PPH}} f)_n - g(\frac{hn}{3})||_\infty = O(h^3).$$

□



(a) The linear scheme (6). (b) The nonlinear scheme S_{PPH}.

FIGURE 1. Comparison of the limit functions obtained, starting from (●), for a tension parameter $w = \frac{1}{11}$.



(a) The nonlinear scheme S_{PPH}. (b) The nonlinear scheme S_{PPH}. Zoom on $(0.45, 0.5)$

FIGURE 2. Nonlinear scheme S_{PPH} for different tension parameters w. $w = \frac{1}{27}$ − −, $w = \frac{1}{11}$ −, $w = \frac{1}{9}$ - . -, $w = \frac{1}{2}$ ....

From a numerical point of view, the scheme S_{PPH} does not suffer from oscillations even if the original data come from the sampling of a discontinuous function (see Figure 1).

According to D. Gottlieb and C.W. Shu [15], given a punctually discontinuous function f and its sampling f^h defined by $f^h_n = f(nh)$, the Gibbs phenomenon deals with the properties of $S^\infty f^h$. It can be characterized by two features ([15]):

- Away from the discontinuity the convergence of $S^\infty(f^h)$ towards f is rather slow and for any point x,
$$|f(x) - (S^\infty f^h)(x)| = O(h).$$
- There are over/undershoots, close to the discontinuity, that do not diminish with reducing h; thus
$$||f - (S^\infty f^h)||_\infty \text{ does not tend to zero with } h.$$

When linear operator are involved in the construction of $S^\infty f^h$ these two features relate to specific properties of elementary functions (sometimes bi orthogonal bases) such as compact/non compact support that influences feature 1) and oscillations that influence feature 2). In our non linear context such functions do not exist.

We are however going to prove that the nonlinear schemes S_{PPH} does not suffer from the Gibbs phenomenon oscillations, as it can be guessed from Figure 1. We have indeed the following

Proposition 3. *Given $0 \leq \xi \leq h$, for any function f defined by:*
$$\forall x \leq \xi,\ f(x) = f_-(x) \quad with \quad f_- \in C^\infty(]-\infty, \xi]\,,$$
$$\forall x > \xi,\ f(x) = f_+(x) \quad with \quad f_+ \in C^\infty([\xi, +\infty[\,,$$
and discontinuous in ξ, we have, supposing that $f_-(\xi) > f_+(\xi)$:
- *if $|x| \geq \frac{5}{2}h,\ |f(x) - (S^\infty_{\text{PPH}} f^h)(x)| = O(h^3)$,*
- *if $|x| \leq \frac{5}{2}h,\ f_+(0) + O(h) \leq (S^\infty_{\text{PPH}} f^h)(x) \leq f_-(h) + O(h)$.*

Proof Without loss of generality, we focus on $[0, +\infty[$.

First we rewrite the scheme S_{PPH} (7, 8) as a particular perturbation of the linear ternary scheme $S_{1,1}$ defined by

(9)
$$\begin{aligned}
(S_{1,1}f)_{3n} &= f_n,\\
(S_{1,1}f)_{3n+1} &= \frac{2}{3}f_n + \frac{1}{3}f_{n+1},\\
(S_{1,1}f)_{3n+2} &= \frac{1}{3}f_n + \frac{2}{3}f_{n+1}.
\end{aligned}$$

For all $f \in l^\infty(\mathbb{R})$, we have
$$S_{\text{PPH}}f = S_{1,1}f + F(d^2 f),$$
with F the function defined by
$$F(d^2 f)_{3n} = 0,$$
and, $if|d^2 f_n| \geq |d^2 f_{n+1}|$,
$$\begin{aligned}
F(d^2 f)_{3n+1} &= \frac{w}{3}d^2 f_{n+1} - (\frac{1}{9} + \frac{w}{3})\text{PPH}(d^2 f_n, d^2 f_{n+1})\\
F(d^2 f)_{3n+2} &= -\frac{w}{3}d^2 f_{n+1} - (\frac{1}{9} - \frac{w}{3})\text{PPH}(d^2 f_n, d^2 f_{n+1})
\end{aligned}$$
or, $if|d^2 f_n| < |d^2 f_{n+1}|$,
$$\begin{aligned}
F(d^2 f)_{3n+1} &= -\frac{w}{3}d^2 f_n - (\frac{1}{9} - \frac{w}{3})\text{PPH}(d^2 f_n, d^2 f_{n+1})\\
F(d^2 f)_{3n+2} &= \frac{w}{3}d^2 f_n - (\frac{1}{9} + \frac{w}{3})\text{PPH}(d^2 f_n, d^2 f_{n+1})
\end{aligned}$$

We first consider a single application of S_{PPH}. Using Proposition 2 we get:

- for $n \geq 2$ and $n_1 \in \{3n, 3n+1\}$, $|(S_{\text{PPH}} f^h)_{n_1} - f_+(\frac{hn}{3})| = O(h^3)$
- for $n = 1$ since f is discontinuous in ξ, $d^2 f_n = O(1)$ and $d^2 f_{n+1} = O(h^2)$. Then from Proposition 1, $\text{PPH}(d^2 f_n, d^2 f_{n+1}) = O(h^2)$.
  Moreover, since the linear scheme $S_{1,1}$ (9) is a second order scheme, we get that $|(S_{\text{PPH}} f^h)_{n_1} - f_+(\frac{hn_1}{3})| = O(h^2)$ for $n_1 \in \{3n, 3n+1\}$.
- for $n = 0$, $d^2 f_n d^2 f_{n+1} \leq 0$ and therefore, according to the Proposition 1, $\text{PPH}(d^2 f_n, d^2 f_{n+1}) = 0$. It is then easy to check, from the definition of S_{PPH} that $f_+(h) \leq (S_{\text{PPH}} f)_{3n} \leq (S_{\text{PPH}} f)_{3n+1} \leq f_-(0)$.

Iterating, according to the stability of S_{PPH} we get:

- for $x \geq \frac{5}{2}h$, $|(S^{\infty}_{\text{PPH}} f^h)(x) - f_+(x)| = O(h^3)$.
- for $0 \leq x \leq \frac{5}{2}h$, the contraction of the second order differences (see Annex 6) and the fact that the linear scheme S in (9), does not produce Gibbs oscillations allows to conclude.

□

4. Numerical tests

We now present some comparisons between the multiresolution schemes associated to the linear ternary $S_{\frac{1}{27}}$ subdivision scheme and to the nonlinear ternary S_{PPH} subdivision scheme for different values of w.

The initial data are obtained through a regular sampling on 2049 points of $[0, 1]$ of a function with two corners localized at $x = \frac{1}{6}$ and $x = \frac{2}{3}$ (Figure 3).

Given an integer $0 < L$ and a real ϵ, we consider the truncation operator $\mathbf{tr}^{\epsilon}_L$ defined as

$$\mathbf{tr}^{\epsilon}_L(\{f^0, d^0, \ldots, d^{L-1}\}) = (\{f^0, \hat{d}^0, \ldots, \hat{d}^{L-1}\}),$$

with

$$\hat{d}^k_j = \begin{cases} 0 & |d^k_j| \leq \epsilon, \\ d^k_j & \textit{otherwise}. \end{cases}$$

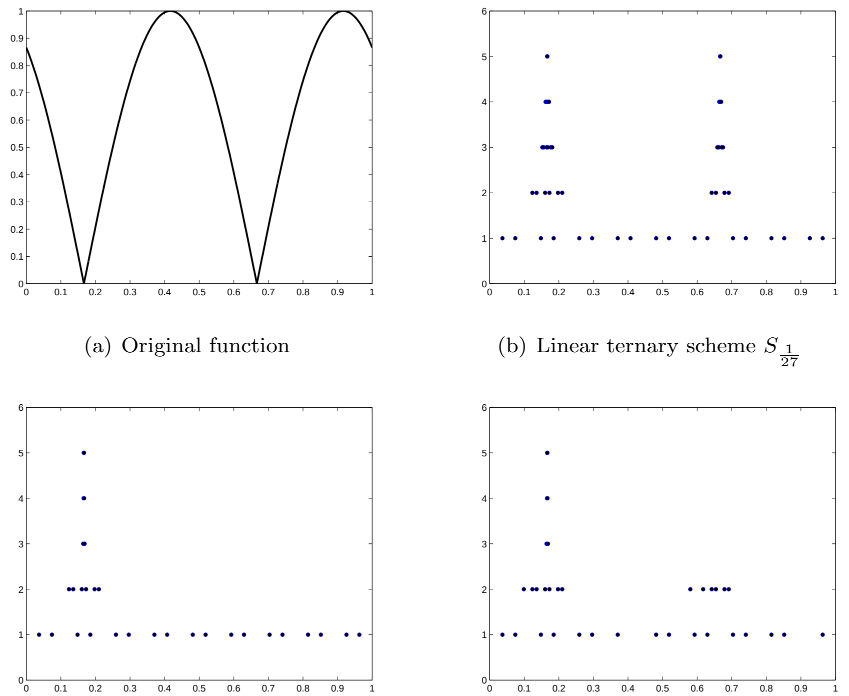
For $L = 5$ and $\epsilon = 10^{-3}$, the non-zero remaining details after truncation are plotted on Figure 3. The number nnz of non-zero remaining details after truncation and the error after recovering $\hat{f}^L$ are evaluated in Table 2.

	E_∞	E_1	E_2	nnz
Linear ternary $S_{\frac{1}{27}}$	8.51×10^{-4}	1.92×10^{-5}	9.18×10^{-7}	67
S_{PPH}, $w = \frac{1}{27}$	$\mathbf{8.00 \times 10^{-4}}$	$\mathbf{8.44 \times 10^{-5}}$	$\mathbf{3.37 \times 10^{-6}}$	45
S_{PPH}, $w = \frac{1}{11}$	9.37×10^{-4}	1.16×10^{-4}	5.17×10^{-6}	**43**
S_{PPH}, $w = \frac{1}{2}$	1.20×10^{-3}	3.35×10^{-4}	8.06×10^{-6}	51

Table 2. Comparison of the truncated multiresolution decompositions for $L = 5$ and $\epsilon = 10^{-3}$ for the function of Figure 3 top-left.

>From Table 2 and Figure 3, it appears that all the algorithms perform similarly with respect to accuracy in the approximation to the original signal. However the number of nonzero detail coefficients in the nonlinear scheme is significantly smaller.

Our last test deals with a numerical comparison of S_{PPH} and S_{ENO} for the generation of bidimensionnal curves in presence of corners. From Figure 4 it appears that



(a) Original function

(b) Linear ternary scheme $S_{\frac{1}{27}}$

(c) Nonlinear ternary scheme S_{PPH}, $w = \frac{1}{11}$

(d) Nonlinear ternary scheme S_{PPH}, $w = \frac{1}{2}$

FIGURE 3. Comparison of the non-zero remaining details after truncation of parameters $L = 5$ and $\epsilon = 10^{-3}$ for the associated multiresolution. For each non zero coefficient d_n^j, a point is plotted at the position $(n3^{-j}, j+1)$.

S_{PPH} provides significantly better results with no spurious over/undershoots close to the corners.

5. Conclusions

The new nonlinear subdivision schemes presented in this paper have the following properties: they are ternary interpolatory schemes, that converge towards functions of regularity larger than one; if the initial data come from the sampling of a discontinuous function, the limit functions does not oscillate as it classically happens when using linear interpolatory schemes; the associated multiresolution transforms are stable.

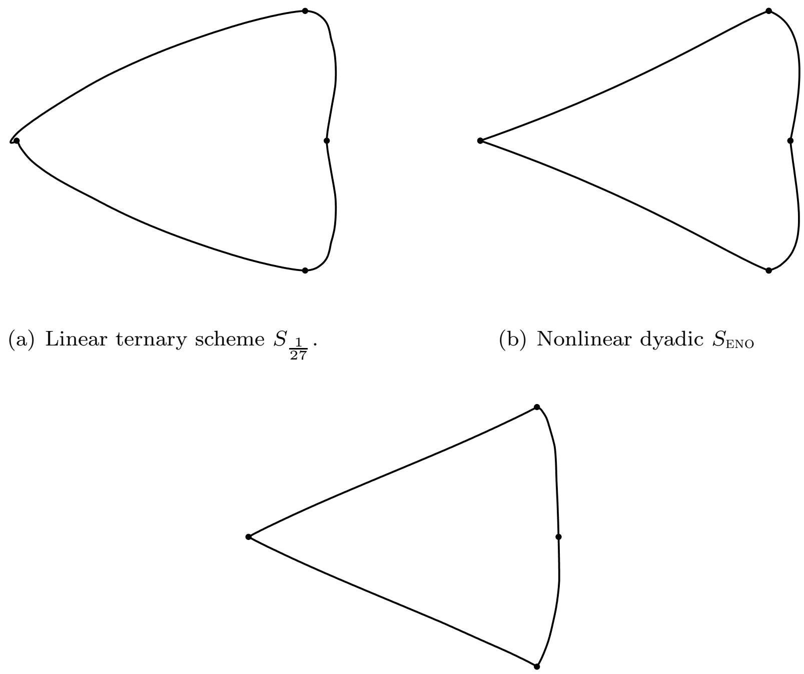
To our knowledge, it is the first family of schemes that share simultaneously such desirable properties.

6. ANNEX

We start with the following technical lemma that will be used for the forthcoming results.

Lemma 1. *If* $|x_1| > |y_1|$ *and* $|x_2| < |y_2|$, *then*

$$|x_1 - x_2 - (\text{PPH}(x_1, y_1) - \text{PPH}(x_2, y_2))| \leq 2||(x_1, y_1) - (x_2, y_2)||_\infty$$



(a) Linear ternary scheme $S_{\frac{1}{27}}$.

(b) Nonlinear dyadic S_{ENO}

(c) Nonlinear ternary scheme S_{PPH}, $w = \frac{1}{27}$

FIGURE 4. Comparison of curve generation from the initial points (•).

and

$$|x_1 + y_2 - 2\text{PPH}(y_1, y_2)| \leq 3||(x_1, y_1) - (x_2, y_2)||_\infty.$$

Proof

• If $x_1 y_1 > 0$, $x_2 y_2 < 0$ and $x_1 x_2 < 0$,

$$\begin{aligned} |x_1 - x_2 - (\text{PPH}(x_1, y_1) - \text{PPH}(x_2, y_2))| &\leq |x_1 - x_2 - \text{PPH}(x_1, y_1)| \\ &\leq \max(|x - x_1|, \text{PPH}(x_1, y_2). \end{aligned}$$

or, $|\text{PPH}(x_1, y_1)| \leq 2 \min(|x_1|, |y_1|)$.

• If $x_1 y_1 > 0$, $x_2 y_2 < 0$ and $x_1 x_2 > 0$,

$$|x_1 - x_2 - (\text{PPH}(x_1, y_1) - \text{PPH}(x_2, y_2))| \leq |x_1 - x_2 - \text{PPH}(x_1, y_1)|.$$

If $x_1 - x_2 - \text{PPH}(x_1, y_1) > 0$,

$$|x_1 - x_2 - \text{PPH}(x_1, y_1)| \leq |x_1 - x_2|.$$

If $x_1 - x_2 - \text{PPH}(x_1, y_1) < 0$, from the hypothesis we have $|x_1| > |y_1|$, thus

$$\begin{aligned} |x_1 - x_2 - \text{PPH}(x_1, y_1)| &\leq |x_1 - x_2 - x_1| \\ &\leq |y_2| \\ &\leq |y_1 - y_2|. \end{aligned}$$

• If $x_1y_1 > 0$, $x_2y_2 > 0$ and $x_1x_2 < 0$, from the equation

$$|c_1d^2f_{n+1} - c_2\text{PPH}(d^2f_n, d^2f_{n+1})| \leq \max(c_1, c_2)||d^2f||_\infty, \tag{10}$$

we have

$$\begin{aligned} |x_1 - x_2 - (\text{PPH}(x_1, y_1) - \text{PPH}(x_2, y_2))| &\leq \max(|x_1|, |y_1|) + \max(|x_2|, |y_2|) \\ &\leq 2\max(|x_1 - x_2|, |y_1 - y_2|). \end{aligned}$$

• If $x_1y_1 > 0$, $x_2y_2 > 0$ and $x_1x_2 > 0$, assuming for instance that $x_1 > 0$.

$$\begin{aligned} &x_1 - x_2 - (\text{PPH}(x_1, y_1) - \text{PPH}(x_2, y_2)) = x_1 - x_2 - \frac{2x_1y_1}{x_1 + y_1} + .\frac{2x_2y_2}{x_2 + y_2} \\ = &\frac{x_1^2x_2 - x_1x_2^2 - y_1x_2^2 + x_1^2y_2 - x_1y_1x_2 - x_1y_1y_2 + y_1x_2y_2 + y_1x_2y_2}{(x_1 + y_1)(x_2 + y_2)} \\ = &\frac{x_1x_2x_1y_2 + y_1x_2 - y_1y_2}{(x_1 + y_1)(x_2 + y_2)}(x_1 - x_2) + \frac{2x_1x_2}{(x_1 + y_1)(x_2 + y_2)}(y_1 - y_2) \\ = &\left(1 - \frac{2y_1y_2}{(x_1 + y_1)(x_2 + y_2)}\right)(x_1 - x_2) + \frac{2x_1x_2}{(x_1 + y_1)(x_2 + y_2)}(y_1 - y_2). \end{aligned}$$

>From the hypothesis, $|x_2| < |y_2|$ and $x_1x_2 > 0$, thus $2x_1x_2 \leq |x_1x_2 + x_1y_2|$.
The second inequality is obtained using a similar strategy.
□

6.1. Convergence and regularity of S_{pph}.
Proof of Theorem 4 and of Theorem 5

>From the expression of F in section 3.1, the hypothesis (2) is verified. Indeed, for all $d \in l^\infty(\mathbb{Z})$,

$$||F(d)||_\infty \leq 2\left(\frac{1}{9} + \frac{w}{3}\right)||df||_\infty. \tag{11}$$

In order to obtain the contraction property (3), we have to consider different cases

Case 1: k=3n+1,: study of $f^1_{3n+2} - 2f^1_{3n+1} + f^1_{3n} = d^2f^1_{3n+1}$

case 1A: $|d^2f_n| \geq |d^2f_{n+1}|$,
case 1B: $|d^2f_n| < |d^2f_{n+1}|$,

Case 2: k=3n+2,: study of $f^1_{3n+3} - 2f^1_{3n+2} + f^1_{3n+1} = d^2f^1_{3n+2}$

case 2A: $|d^2f_n| \geq |d^2f_{n+1}|$,
case 2B: $|d^2f_n| < |d^2f_{n+1}|$,

Case 3: k=3n,: study of $f^1_{3n+1} - 2f^1_{3n} + f^1_{3n-1} = d^2f^1_{3n}$

case 3A: $|d^2f_n| \geq |d^2f_{n+1}|$ and $|d^2f_{n-1}| \geq |d^2f_n|$,
case 3B: $|d^2f_n| < |d^2f_{n+1}|$ and $|d^2f_{n-1}| < |d^2f_n|$,
case 3C: $|d^2f_n| \geq |d^2f_{n+1}|$ and $|d^2f_{n-1}| < |d^2f_n|$,
case 3D: $|d^2f_n| < |d^2f_{n+1}|$ and $|d^2f_{n-1}| \geq |d^2f_n|$.

Moreover, for the linear triadic scheme $S_{1,1}$ (9), we have

$$(d^2(Sf))_{3n+1} = (d^2(Sf))_{3n+2} = 0 \qquad \text{and} \qquad (d^2(Sf))_{3n} = \frac{1}{3}d^2f_n.$$

We analyze the different cases.

• Case 1: for the case 1A, we have

$$d^2 f^1_{3n+1} = -w d^2 f_{n+1} + \left(\frac{1}{9} + w\right) \mathrm{PPH}(d^2 f_n, d^2 f_{n+1}).$$

In order to obtain a better bound we use the property (10) of the harmonic mean, we then have

$$|d^2 f^1_{3n+1}| \leq \left(\frac{1}{9} + w\right) ||d^2 f||_\infty. \tag{12}$$

Similarly, the case 1B gives

$$d^2 f^1_{3n+1} = w d^2 f_n + \left(\frac{1}{9} - w\right) \mathrm{PPH}(d^2 f_n, d^2 f_{n+1}), \tag{13}$$

and

$$|d^2 f^1_{3n+1}| \leq \max\left(w, \frac{1}{9}\right) ||d^2 f||_\infty. \tag{14}$$

• Case 2: for the case 2A, we obtain

$$d^2 f^1_{3n+2} = w d^2 f_{n+1} + \left(\frac{1}{9} - w\right) \mathrm{PPH}(d^2 f_n, d^2 f_{n+1}), \tag{15}$$

and, for the case 2B,

$$d^2 f^1_{3n+2} = -w d^2 f_n + \left(\frac{1}{9} + w\right) \mathrm{PPH}(d^2 f_n, d^2 f_{n+1}). \tag{16}$$

• Case 3: for the case 3A, we obtain

$$\begin{aligned} d^2 f^1_{3n} = {} & \frac{1-w}{3} d^2 f_n + \frac{w}{3} d^2 f_{n+1} - \left(\frac{1}{9} - \frac{w}{3}\right) \mathrm{PPH}(d^2 f_{n-1}, d^2 f_n) \\ & - \left(\frac{1}{9} + \frac{w}{3}\right) \mathrm{PPH}(d^2 f_n, d^2 f_{n+1}). \end{aligned} \tag{17}$$

>From the equation (10) and considering different cases, we have

$$|d^2 f^1_{3n}| \leq \max\left(\frac{2+3w}{9}, \frac{1-w}{3}\right) ||d^2 f||_\infty. \tag{18}$$

The case 3B gives

$$\begin{aligned} d^2 f^1_{3n} = {} & \frac{1}{3} d^2 f_n + \frac{w}{3} d^2 f_{n+1} + \frac{w}{3} d^2 f_{n-1} \\ & - \left(\frac{1}{9} + \frac{w}{3}\right) \left(\mathrm{PPH}(d^2 f_{n-1}, d^2 f_n) + \mathrm{PPH}(d^2 f_n, d^2 f_{n+1})\right). \end{aligned} \tag{19}$$

Similarly, we obtain

$$|d^2 f^1_{3n}| \leq \max\left(\frac{2+3w}{9}, \frac{1-w}{3}\right) ||d^2 f||_\infty.$$

For the case 3C, we find

$$d^2 f^1_{3n} = \frac{1-2w}{3} d^2 f_n \tag{20}$$

$$- \left(\frac{1}{9} - \frac{w}{3}\right) \left(\mathrm{PPH}(d^2 f_{n-1}, d^2 f_n) + \mathrm{PPH}(d^2 f_n, d^2 f_{n+1})\right), \tag{21}$$

and then

$$|d^2 f^1_{3n}| \quad \leq \quad \max\left(\frac{1}{9}, \frac{|1-2w|}{3}, \frac{w}{3}\right) ||d^2 f||_\infty. \tag{22}$$

Finally, for the case 3D

$$\begin{aligned} d^2 f^1_{3n} \quad = \quad & \frac{1-w}{3} d^2 f_n + \frac{w}{3} d^2 f_{n-1} - \left(\frac{1}{9} + \frac{w}{3}\right) \text{PPH}(d^2 f_{n-1}, d^2 f_n) \\ & - \left(\frac{1}{9} - \frac{w}{3}\right) \text{PPH}(d^2 f_n, d^2 f_{n+1}). \end{aligned} \tag{23}$$

We obtain the same inequality of the case 3A.

With the equations (12), (14), (22) and (18), the contraction hypothesis (3) is verified with $c = \max\left(\frac{1}{9} + w, \frac{2+3w}{9}, \frac{1-w}{3}\right)$.

For $0 \leq w < \frac{2}{9}$, we can then use Theorem 1 obtaining the desired convergence and regularity, since the linear ternary schemes S_w are convergent schemes with limit functions in the spaces $C^{(1+\beta)-}$ when $0 \leq w \leq \frac{1}{2}$ and C^2 when $\frac{1}{15} < w < \frac{1}{9}$.

For $\frac{2}{9} \leq w < \frac{8}{9}$, we write (proposition 3)

$$S_{\text{PPH}} f = S_{1,1} f + F(d^2 f),$$

with $S_{1,1}$ the two point scheme (9) which have a regularity C^{1-}.
By using Theorem 1 and the constant c, we can conclude.
□

6.2. Stability of $S_{\mathbf{pph}}$.
Proof of Theorem 6
Using the expressions of the perturbation F in section 3.3, and the definition of the harmonic mean, we have for all $d_1, d_2 \in l^\infty(\mathbb{Z})$, that

$$||F(d_1) - F(d_2)||_\infty \quad \leq \quad \left(\frac{2}{9} + w\right) ||d_1 - d_2||_\infty.$$

Thus, the hypothesis (4) for F is verified.

For the contraction property (5), we analyze $(d^2 f^1 - d^2 g^1)_k$ for $k = 3n+1$ (case 1), $k = 3n+2$ (case 2) and $k = 3n$ (case 3).

We consider different cases according to the sub cases verified by f and g (see the proof of Theorem 4.

For k=3n+1, we have to study 2 cases .

• If f and g verify the case $1A$, from the equation (12), we have

$$|d^2 f^1_{3n+1} - d^2 g^1_{3n+1}| \quad \leq \quad \left(\frac{2}{9} + 3w\right) ||d^2 f - d^2 g||_\infty. \tag{24}$$

For $0 \leq w < \frac{2}{15}$, we have $\frac{2}{9} + 3w < 1$.
Similarly, if f and g verify the case $1B$.

• If f verifies the case $1A$ and g the case $1B$, from equations (12) and (13), we obtain

$$\begin{aligned} d^2f^1_{3n+1} - d^2g^1_{3n+1} &= -w(d^2f_{n+1} + d^2g_n) + \left(\frac{1}{9} + w\right) \mathrm{PPH}(d^2f_n, d^2f_{n+1}) \\ &\quad - \left(\frac{1}{9} - w\right) \mathrm{PPH}(d^2g_n, d^2g_{n+1}) \\ &= -w(d^2f_{n+1} + d^2g_n - 2\mathrm{PPH}(d^2g_n, d^2g_{n+1})) \\ &\quad + \left(\frac{1}{9} + w\right) \left(\mathrm{PPH}(d^2f_n, d^2f_{n+1}) - \mathrm{PPH}(d^2g_n, d^2g_{n+1})\right), \end{aligned}$$

where $|d^2f_n| \geq |d^2f_{n+1}|$ and $|d^2g_n| < |d^2g_{n+1}|$, and applying Lemma 1

$$\begin{aligned} |d^2f^1_{3n+1} - d^2g^1_{3n+1}| &\leq \left(\frac{2}{9} + 5w\right) ||d^2f - d^2g||_\infty \\ &\leq \frac{11}{9} ||d^2f - d^2g||_\infty. \end{aligned}$$

For k=3n+2, we have again to study 2 cases.

• If f and g verify the case $2A$, from the equation (15), we have

$$|d^2f^1_{3n+2} - d^2g^1_{3n+2}| \leq \left(\frac{2}{9} - w\right) ||d^2f - d^2g||_\infty. \tag{25}$$

Similarly, if f and g verify the case $2B$.

• If f verifies the case $2A$ and g the case $2B$, from the equations (15) and (16), we obtain

$$\begin{aligned} d^2f^1_{3n+2} - d^2g^1_{3n+2} &= w(d^2f_{n+1} + d^2g_n) + \left(\frac{1}{9} - w\right) \mathrm{PPH}(d^2f_n, d^2f_{n+1}) \\ &\quad - \left(\frac{1}{9} + w\right) \mathrm{PPH}(d^2g_n, d^2g_{n+1}). \end{aligned}$$

Using the same strategy than the case $k = 3n + 1$, we have

$$\begin{aligned} |d^2f^1_{3n+2} - d^2g^1_{3n+2}| &\leq \left(\frac{2}{9} + 5w\right) ||d^2f - d^2g||_\infty \\ &\leq \frac{11}{9} ||d^2f - d^2g||_\infty. \end{aligned}$$

For k=3n, we have to study the seven cases that we can see in Table 3, the other cases are obtained by symmetry.

f	g	notation	f	g	notation	f	g	notation
Case A	Case A	$af - ag$	Case B	Case B	$bf - bg$	Case C	Case C	$cf - cg$
	Case B	$af - bg$		Case C	$bf - cg$		Case D	$cf - dg$
	Case C	$af - cg$		Case D	$bf - dg$			
	Case D	$af - dg$						

TABLE 3. Cases to consider for $k = 3n$ in the stability proof of S_{PPH}.

• Case $af - ag$, from the equation (17), we obtain

$$\begin{aligned} |d^2 f^1_{3n} - d^2 g^1_{3n}| &\leq \left(\frac{1-w}{3} + \frac{w}{3} + \frac{2}{9} - \frac{2w}{3} + \frac{2}{9} + \frac{2w}{3}\right) ||d^2 f - d^2 g||_\infty \\ &\leq \frac{7}{9} ||d^2 f - d^2 g||_\infty. \end{aligned}$$

• Case $bf - bg$, from the equation (19), we obtain

$$\begin{aligned} |d^2 f^1_{3n} - d^2 g^1_{3n}| &\leq \left(\frac{1+2w}{3} + \frac{4}{9} + \frac{4w}{3}\right) ||d^2 f - d^2 g||_\infty \\ &< ||d^2 f - d^2 g||_\infty. \end{aligned}$$

For the cases $cf - cg$ and $df - dg$, the strategy is identical.

• Case $af - bg$, from the equations (17) and (19), we obtain

$$\begin{aligned} d^2 f^1_{3n} - d^2 g^1_{3n} &= \frac{1}{3}(d^2 f_n - d^2 g_n) \\ &+ \frac{w}{3}(d^2 f_{n+1} - d^2 g_{n+1}) \\ &- \left(\frac{1}{9} + \frac{w}{3}\right) \big(\text{PPH}(d^2 f_n, d^2 f_{n+1}) - \text{PPH}(d^2 g_n, d^2 g_{n+1})\big) \\ &- \left(\frac{1}{9} - \frac{w}{3}\right) \big(\text{PPH}(d^2 f_{n-1}, d^2 f_n) - \text{PPH}(d^2 g_{n-1}, d^2 g_n)\big) \\ &- \frac{w}{3}(d^2 f_n + d^2 g_{n-1} - 2\text{PPH}(d^2 g_{n-1}, d^2 g_n)). \end{aligned}$$

Using Lemma 1, we have

$$\begin{aligned} |d^2 f^1_{3n} - d^2 g^1_{3n}| &\leq \left(\frac{1}{3} + \frac{w}{3} + \frac{2}{9} + \frac{2w}{3} + \frac{2}{9} - \frac{2w}{3} + w\right) ||d^2 f - d^2 g||_\infty \\ &\leq \left(\frac{7}{9} + \frac{4w}{3}\right) ||d^2 f - d^2 g||_\infty, \end{aligned}$$

with $\frac{7}{9} + \frac{4w}{3} < \frac{41}{45}$.

• Case $af - cg$, from the equations (17) and (20), we obtain

$$\begin{aligned} d^2 f^1_{3n} - d^2 g^1_{3n} &= \frac{1-w}{3}(d^2 f_n - d^2 g_n) \\ &- \left(\frac{1}{9} - \frac{w}{3}\right) \big(\text{PPH}(d^2 f_{n-1}, d^2 f_n) - \text{PPH}(d^2 g_{n-1}, d^2 g_n)\big) \\ &- \left(\frac{1}{9} + \frac{w}{3}\right) \big(\text{PPH}(d^2 f_n, d^2 f_{n+1}) - \text{PPH}(d^2 g_n, d^2 g_{n+1})\big) \\ &+ \frac{w}{3}(d^2 f_{n+1} + d^2 g_n - 2\text{PPH}(d^2 g_n, d^2 g_{n+1})). \end{aligned}$$

Using Lemma 1, we have

$$\begin{aligned} |d^2 f^1_{3n} - d^2 g^1_{3n}| &\leq \left(\frac{1-w}{3} + \frac{2}{9} - \frac{2w}{3} + \frac{2}{9} + \frac{2w}{3} + w\right) ||d^2 f - d^2 g||_\infty \\ &\leq \left(\frac{7}{9} + \frac{4w}{3}\right) ||d^2 f - d^2 g||_\infty. \end{aligned}$$

We find the same constant of the case $af - bg$.

• Case $af - dg$, from the equations (17) and (23), we obtain

$$\begin{aligned} d^2f^1_{3n} - d^2g^1_{3n} &= \frac{1-w}{3}(d^2f_n - d^2g_n) \\ &-\left(\frac{1}{9}+\frac{w}{3}\right)\big(\text{PPH}(d^2f_{n-1}, d^2f_n) - \text{PPH}(d^2g_{n-1}, d^2g_n)\big) \\ &-\frac{w}{3}(d^2f_n + d^2g_{n-1} - 2\text{PPH}(d^2g_{n-1}, d^2g_n)) \\ &-\left(\frac{1}{9}-\frac{w}{3}\right)\big(\text{PPH}(d^2f_n, d^2f_{n+1}) - \text{PPH}(d^2g_n, d^2g_{n+1})\big) \\ &+\frac{w}{3}(d^2f_{n+1} + d^2g_n - 2\text{PPH}(d^2g_n, d^2g_{n+1})). \end{aligned}$$

Using lemma 1, we have

$$\begin{aligned} |d^2f^1_{3n} - d^2g^1_{3n}| &\leq \left(\frac{1-w}{3}+\frac{2}{9}+\frac{2w}{3}+\frac{2}{9}-\frac{2w}{3}+2w\right)||d^2f - d^2g||_\infty \\ &\leq \left(\frac{7}{9}+\frac{5w}{3}\right)||d^2f - d^2g||_\infty, \end{aligned}$$

with $\frac{7}{9}+\frac{5w}{3} < 1$.

• Case $bf - cg$, from the equations (19) and (20), we obtain

$$\begin{aligned} d^2f^1_{3n} - d^2g^1_{3n} &= \frac{1}{3}(d^2f_n - d^2g_n) \\ &-\left(\frac{1}{9}+\frac{w}{3}\right)\big(\text{PPH}(d^2f_{n-1}, d^2f_n) - \text{PPH}(d^2g_{n-1}, d^2g_n)\big) \\ &-\frac{w}{3}(d^2f_{n-1} + d^2g_n - 2\text{PPH}(d^2g_{n-1}, d^2g_n)) \\ &-\left(\frac{1}{9}+\frac{w}{3}\right)\big(\text{PPH}(d^2f_n, d^2f_{n+1}) - \text{PPH}(d^2g_n, d^2g_{n+1})\big) \\ &-\frac{w}{3}(d^2f_{n+1} + d^2g_n - 2\text{PPH}(d^2g_n, d^2g_{n+1})), \end{aligned}$$

where $|d^2f_n| \geq \max(|d^2f_{n+1}|, |d^2f_{n-1}|)$ and $|d^2g_n| < \min(|d^2g_{n+1}|, |d^2g_{n-1}|)$, then using Lemma 1, we have

$$\begin{aligned} |d^2f^1_{3n} - d^2g^1_{3n}| &\leq \left(\frac{1}{9}+\frac{2}{9}+\frac{2}{9}+\frac{2w}{3}+\frac{2w}{3}+2w\right)||d^2f - d^2g||_\infty \\ &\leq \left(\frac{5}{9}+\frac{10w}{3}\right)||d^2f - d^2g||_\infty, \end{aligned}$$

with $\frac{5}{9}+\frac{10w}{3} < 1$.

• Case $bf-dg$, from the equations (19) and (23), we obtain

$$\begin{aligned} d^2f^1_{3n}-d^2g^1_{3n} &= \frac{1}{3}(d^2f_n-d^2g_n)+\frac{w}{3}(d^2f_{n-1}-d^2g_{n-1}) \\ &-\left(\frac{1}{9}-\frac{w}{3}\right)\big(\text{PPH}(d^2f_{n-1},d^2f_n)-\text{PPH}(d^2g_{n-1},d^2g_n)\big) \\ &-\left(\frac{1}{9}+\frac{w}{3}\right)\big(\text{PPH}(d^2f_n,d^2f_{n+1})-\text{PPH}(d^2g_n,d^2g_{n+1})\big) \\ &+\frac{w}{3}(d^2f_{n+1}+d^2g_n-2\text{PPH}(d^2g_n,d^2g_{n+1})). \end{aligned}$$

Using again Lemma 1, we have

$$\begin{aligned} |d^2f^1_{3n}-d^2g^1_{3n}| &\leq \left(\frac{1}{3}+\frac{w}{3}+\frac{2}{9}-\frac{2w}{3}+\frac{2}{9}+\frac{2w}{3}+w\right)||d^2f-d^2g||_\infty \\ &\leq \left(\frac{7}{9}+\frac{4w}{3}\right)||d^2f-d^2g||_\infty. \end{aligned}$$

We find the same constant than the cases $af-bg$ and $af-bd$.

• Case $cf-dg$, from the equations (20) and (23), we obtain

$$\begin{aligned} d^2f^1_{3n}-d^2g^1_{3n} &= \frac{1-w}{3}(d^2f_n-d^2g_n) \\ &-\left(\frac{1}{9}-\frac{w}{3}\right)\big(\text{PPH}(d^2f_n,d^2f_{n+1})-\text{PPH}(d^2g_n,d^2g_{n+1})\big) \\ &-\left(\frac{1}{9}-\frac{w}{3}\right)\big(\text{PPH}(d^2f_{n-1},d^2f_n)-\text{PPH}(d^2g_{n-1},d^2g_n)\big) \\ &-\frac{w}{3}(d^2f_n+d^2g_{n-1}-2\text{PPH}(d^2g_{n-1},d^2g_n)). \end{aligned}$$

>From Lemma 1, we have

$$\begin{aligned} |d^2f^1_{3n}-d^2g^1_{3n}| &\leq \left(\frac{1-w}{3}+\frac{2}{9}-\frac{2w}{3}+\frac{2}{9}-\frac{2w}{3}+w\right)||d^2f-d^2g||_\infty \\ &\leq \left(\frac{7}{9}-\frac{4w}{3}\right)||d^2f-d^2g||_\infty. \end{aligned}$$

Thus the contraction property (5) is verified in all the cases, for $0\leq w<\frac{2}{15}$, and we can use Theorem 3 to conclude.

□

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