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NUMERICAL IDENTIFICATION OF MAGNETIC PERMEABILITY

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Abstract. This work is concerned with the analysis on a numerical reconstruction of the magnetic permeability. The ill-posed problem is solved through a stabilized nonlinear minimization system by an appropriately selected Tikhonov regularization. The existence and stability of the optimization system are demonstrated. The nonlinear optimization problem is approximated by an edge element method, whose convergence is established.

Key Words. Numerical identification, Maxwell system, permeability, edge element method, stability, convergence.

1. Introduction

In this work we are interested in the numerical reconstruction of the distribution of the magnetic permeability in the following Maxwell system:

$$\varepsilon(\mathbf{x})\frac{\partial \mathbf{E}}{\partial t} - \nabla\times\mathbf{H} = \mathbf{J} \text{ in } \Omega\,, \tag{1.1}$$

$$\nu(\mathbf{x})\frac{\partial \mathbf{H}}{\partial t} + \nabla\times\mathbf{E} = 0 \text{ in } \Omega\,, \tag{1.2}$$

where $\mathbf{E}$ and $\mathbf{H}$ represent the electric and magnetic fields of the physical medium which occupies a domain Ω in $\mathbf{R}^3$. The Maxwell system (1.1)-(1.2) is formed by the Ampere's law (1.1), and Faraday's law (1.2), and plays an important role in most applications that involve electromagnetism. The coefficients $\varepsilon(\mathbf{x})$ and $\nu(\mathbf{x})$ in (1.1) and (1.2) are the electric permittivity and magnetic permeability of the medium in Ω, while $\mathbf{J}$ is the applied electric current density. When the physical properties of the medium involved are known, i.e. $\varepsilon(\mathbf{x})$ and $\nu(\mathbf{x})$ are given, one can solve the system (1.1)-(1.2) to find the behaviors of the electric and magnetic field $\mathbf{E}$ and $\mathbf{H}$ in Ω. This is usually called a direct Maxwell problem. While in many applications, the inverse Maxwell problem may be more interesting and practically important, where the electric or magnetic property of the physical medium occupied by Ω is unknown. But knowing them is indispensable to some research investigations in Ω or to a good understanding of the physical medium Ω and how the fields $\mathbf{E}$ and $\mathbf{H}$ behave in Ω. In this work we shall consider the case when the electric permittivity of the physical medium occupied by Ω is available, but the magnetic permeability of the medium is unknown. In order to recover the magnetic permeability of the medium, we need to have some extra measurement data from the electric field $\mathbf{E}$

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or magnetic field $\mathbf{H}$. We shall assume the measurement data of $\mathbf{E}$ is available in some small subregion inside Ω. So the inverse problem to be considered can be formulated as follows:

Inverse Problem I. Let ω be a subregion in Ω. Given the measurement data

$$\mathbf{E}^{\delta}(\mathbf{x},t) \approx \mathbf{E}(\mathbf{x},t)\,, \quad (\mathbf{x},t) \in \omega \times (0,T)\,, \tag{1.3}$$

we will reconstruct the distribution profile of the magnetic permeability $\nu(\mathbf{x})$ in the entire domain Ω.

Noting that only the measurement data of the electric field $\mathbf{E}$ is available in Inverse Problem I, but the Maxwell system (1.1)-(1.2) involves both the electric and magnetic fields. So it is more natural to deal with a system that involves only the electric field. To do so, taking the time derivative on both sides of equation (1.1) and applying equation (1.2) gives the following electric field equation

$$\varepsilon(\mathbf{x})\frac{\partial^2 \mathbf{E}}{\partial t^2} + \nabla \times (\nu^{-1}(\mathbf{x})\nabla \times \mathbf{E}) = \frac{\partial \mathbf{J}}{\partial t} \quad \text{in} \quad \Omega\,. \tag{1.4}$$

Complementary to this electric field equation we shall consider the boundary condition

$$\mathbf{E} \times \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \tag{1.5}$$

and the initial conditions

$$\mathbf{E}(\mathbf{x},0) = \mathbf{E}_0(\mathbf{x})\,, \quad \mathbf{E}_t(\mathbf{x},0) = \mathbf{E}_1(\mathbf{x}) \quad \text{in} \quad \Omega\,. \tag{1.6}$$

For the known electric permittivity $\varepsilon(\mathbf{x})$, we know physically that it should be always bounded below and above. Hence we will assume that

$$\varepsilon_0 \le \varepsilon(\mathbf{x}) \le \varepsilon_1 \quad a.e. \quad \text{in} \quad \Omega\,, \tag{1.7}$$

where ε_0 and ε_1 are two positive constants.

Inverse problems of parameter identifications have attracted a great attention in the recent two decades due to their practically important applications in engineering and scientific computing; see, e.g. [1] [5] and the references therein. The mathematical and numerical analysis of identifications of parameters in many partial differential equations were available in the literature, see [1] [2] for the elliptic system; and [5] [6] [8] [11] for parabolic systems. But very little has been done for the analysis of numerical reconstruction of the parameters in the electromagnetic Maxwell system. This motivates the central topic of this current investigation.

2. Problem formulation and existence of solutions

In this section we shall formulate the ill-posed Inverse Problem I stated in Section 1 as a stabilized minimization system and establish the existence of the solutions and stability of the minimization formulation. For the sake of convenience, we shall rewrite the electric system (1.4) as

$$\varepsilon(\mathbf{x})\frac{\partial^2 \mathbf{E}}{\partial t^2} + \nabla \times (\mu(\mathbf{x})\nabla \times \mathbf{E}) = \mathbf{j} \quad \text{in} \quad \Omega \tag{2.1}$$

where $\mathbf{j} = \frac{\partial \mathbf{J}}{\partial t}$, and $\mu(\mathbf{x}) = \nu^{-1}(\mathbf{x})$ is the magnetic susceptibility. If $\mu(\mathbf{x})$ is known, then the magnetic permeability $\nu(\mathbf{x})$ targeted in Inverse Problem I can be obtained by taking the simple inverse of $\mu(\mathbf{x})$. So in the subsequent sections, we shall address the reconstruction of the magnetic susceptibility $\mu(\mathbf{x})$ under the measurement data $\mathbf{E}^\delta$ provided in (1.3). For the purpose, we will need the following Sobolev spaces:

$$\begin{aligned} H(\mathbf{curl};\Omega) &= \Big\{\mathbf{v}\in L^2(\Omega)^3;\quad \mathbf{curl}\mathbf{v}\in L^2(\Omega)^3\Big\}, \\ H_0(\mathbf{curl};\Omega) &= \Big\{\mathbf{v}\in L^2(\Omega)^3;\quad \mathbf{curl}\mathbf{v}\in L^2(\Omega)^3,\ \mathbf{v}\times\mathbf{n}=0 \text{ on } \partial\Omega\Big\}, \\ H^\alpha(\mathbf{curl};\Omega) &= \Big\{\mathbf{v}\in H^\alpha(\Omega)^3;\quad \mathbf{curl}\mathbf{v}\in H^\alpha(\Omega)^3\Big\}, \end{aligned}$$

for $0\le\alpha\le 1$, with the norms

$$\|\mathbf{v}\|_{0,\mathbf{curl}} = \Big\{\|\mathbf{v}\|^2+\|\nabla\times\mathbf{v}\|^2\Big\}^{\frac12},\quad \|\mathbf{v}\|_{\alpha,\mathbf{curl}} = \Big\{\|\mathbf{v}\|_\alpha^2+\|\nabla\times\mathbf{v}\|_\alpha^2\Big\}^{\frac12}.$$

Here and in the sequel, $\|\cdot\|$ will always mean the $L^2(\Omega)^3$-norm (or $L^2(\Omega)$-norm if only scalar functions are involved). And as usual, we will use $\|\cdot\|_\alpha$ to denote the norm of the Sobolev space $H^\alpha(\Omega)^3$ (or $H^\alpha(\Omega)$ if only scalar functions are involved).

In general, Inverse Problem I is not solvable. Instead we shall transform it into a mathematically solvable minimization system. For this we introduce the admissible set for the recovering parameter $\mu(\mathbf{x})$:

$$K=\Big\{\mu\in H^1(\Omega);\ \mu_0\le\mu(\mathbf{x})\le\mu_1\ a.e. \text{ in } \Omega\Big\} \tag{2.2}$$

where μ_0 and μ_1 are two a priori lower and upper bounds of the parameter $\mu(\mathbf{x})$. Then we formulate the Inverse Problem I as the following minimization process:

$$\min_{\mu\in K}\ G(\mu)=\int_0^T\int_\omega |\mathbf{E}(\mu)-\mathbf{E}^\delta|^2 d\mathbf{x}dt+\frac{\beta}{2}\|\nabla\mu\|^2_{L^2(\Omega)} \tag{2.3}$$

where β is a positive constant, called *the regularization parameter*, and $\mathbf{E}=\mathbf{E}(\mu)\in H_0(\mathbf{curl};\Omega)$ satisfies the initial conditions

$$\mathbf{E}(\mathbf{x},0)=\mathbf{E}_0(\mathbf{x})\quad\text{and}\quad \mathbf{E}_t(\mathbf{x},0)=\mathbf{E}_1(\mathbf{x})\quad\text{in}\quad\Omega \tag{2.4}$$

and the variational system associated with the electric field equation (2.1):

$$\int_\Omega \varepsilon(\mathbf{x})\mathbf{E}_{tt}\cdot\mathbf{v}d\mathbf{x}+\int_\Omega \mu(\mathbf{x})\mathbf{curl}\mathbf{E}\cdot\mathbf{curl}\mathbf{v}d\mathbf{x}=\int_\Omega \mathbf{j}\cdot\mathbf{v}d\mathbf{x},\ \forall\,\mathbf{v}\in H_0(\mathbf{curl};\Omega) \tag{2.5}$$

for $a.e.\ t\in(0,T)$.

In the rest of this section, we shall justify the formulation (2.3)-(2.5), and demonstrate that the nonlinear optimization system always has solutions and its solutions are stable with respect to the change in the error of the observation data $\mathbf{E}^\delta$. The next theorem establishes the existence of solutions.

Theorem 2.1. *There exists at least one minimizer to the optimization problem (2.3)-(2.5).*

Proof. As $G(\mu)\ge 0$, we know that inf $G(\mu)$ is finite over K. Thus there exists a minimizing sequence $\Big\{\mu^n\Big\}$ in K such that

$$\lim_{n\to\infty} G(\mu^n)=\inf_{\mu\in K} G(\mu).$$

Then by the definition of $G(\mu)$, there exists some constant C such that $\beta\|\nabla\mu^n\|^2_{L^2(\Omega)}\le C$. This, combining with the fact that $\mu^n\in K$, shows the boundedness of μ^n in $H^1(\Omega)$, thus there exists a subsequence, still denoted as $\Big\{\mu^n\Big\}$, and some μ^* in $H^1(\Omega)$ such that

$$\mu^n\rightharpoonup\mu^* \text{ in } H^1(\Omega)\quad\text{and}\quad \mu^n\to\mu^* \text{ in } L^2(\Omega) \text{ as } n\to\infty.$$

As K is a closed convex subset of $H^1(\Omega)$, hence K is weakly-closed and we have $\mu^* \in K$. Next, we will prove that μ^* is a global minimizer of (2.3)–(2.5). As it is rather technical, we will divide the whole proof into four steps.

Step 1. Show the convergence of the sequence $\{\mathbf{E}(\mu^n)\}$.

We shall write $\mathbf{E}^n = \mathbf{E}(\mu^n)$. Using the equation (2.5), we know $\mathbf{E}^n \in H_0(\mathbf{curl};\Omega)$ satisfies

$$\int_\Omega \varepsilon(\mathbf{x})\mathbf{E}^n_{tt}\cdot\mathbf{v}d\mathbf{x}+\int_\Omega \mu^n(\mathbf{x})\mathbf{curl}\mathbf{E}^n\cdot\mathbf{curl}\mathbf{v}d\mathbf{x} = \int_\Omega \mathbf{j}\cdot\mathbf{v}d\mathbf{x}\,, \forall\,\mathbf{v}\in H_0(\mathbf{curl};\Omega)\,. \tag{2.6}$$

Taking $\mathbf{v} = \mathbf{E}^n_t$ in (2.6), we obtain

$$\frac{1}{2}\int_\Omega \varepsilon(\mathbf{x})\frac{d}{dt}|\mathbf{E}^n_t|^2 d\mathbf{x} + \frac{1}{2}\int_\Omega \mu^n(\mathbf{x})\frac{d}{dt}|\mathbf{curl}\mathbf{E}^n|^2 d\mathbf{x} = \int_\Omega \mathbf{j}\cdot\mathbf{E}^n_t d\mathbf{x}\,.$$

Integrating over [0,t], we have

$$\begin{aligned} & \frac{1}{2}\int_\Omega \varepsilon(\mathbf{x})(|\mathbf{E}^n_t(\mathbf{x},t)|^2 - |\mathbf{E}_1(\mathbf{x})|^2)d\mathbf{x} \\ + & \frac{1}{2}\int_\Omega \mu^n(\mathbf{x})(|\mathbf{curl}\mathbf{E}^n(\mathbf{x},t)|^2 - |\mathbf{curl}\mathbf{E}_0(\mathbf{x})|^2)d\mathbf{x} \\ = & \int_0^t\int_\Omega \mathbf{j}\cdot\mathbf{E}^n_t d\mathbf{x}dt. \end{aligned}$$

By the Cauchy-Schwarz inequality and the bounds of ε and K, we further derive

$$\begin{aligned} & \varepsilon_0\|\mathbf{E}^n_t(t)\|^2_{L^2(\Omega)^3} + \mu_0\|\mathbf{curl}\mathbf{E}^n(t)\|^2_{L^2(\Omega)^3} \\ \le & \varepsilon_1\int_\Omega |\mathbf{E}_1|^2 d\mathbf{x} + \mu_1\|\mathbf{curl}\mathbf{E}_0\|^2_{L^2(\Omega)^3} + \|\mathbf{j}\|^2_{L^2(0,T;L^2(\Omega)^3)} + \int_0^t \|\mathbf{E}^n_t\|^2_{L^2(\Omega)^3}dt, \end{aligned}$$

applying the Gronwall's inequality yields for all $t\in(0,T)$ that

$$\begin{aligned} & \varepsilon_0 \sup_{0\le t\le T}\|\mathbf{E}^n_t(t)\|^2_{L^2(\Omega)^3} + \mu_0 \sup_{0\le t\le T}\|\mathbf{curl}\mathbf{E}^n(t)\|^2_{L^2(\Omega)^3} \\ \le & C\Big(\|\mathbf{E}_1\|^2_{L^2(\Omega)^3} + \|\mathbf{curl}\mathbf{E}_0\|^2_{L^2(\Omega)^3} + \|\mathbf{j}\|^2_{L^2(0,T;L^2(\Omega)^3)}\Big)\,. \end{aligned} \tag{2.7}$$

This, along with the relation

$$\mathbf{E}^n(\mathbf{x},t) = E_0(\mathbf{x}) + \int_0^t \mathbf{E}^n_t(\mathbf{x},\tau)d\tau\,, \forall\,t\in[0,T]\,,$$

gives

$$\|\mathbf{E}^n\|_{L^2(0,T;L^2(\Omega)^3)} \le C\Big(\|\mathbf{E}_1\|_{L^2(\Omega)^3} + \|\mathbf{E}_0\|_{0,\mathbf{curl}} + \|\mathbf{j}\|_{L(0,T;L^2(\Omega)^3)}\Big)\,. \tag{2.8}$$

On the other hand, it follows from (2.5) that for any $\mathbf{v}\in H_0(\mathbf{curl};\Omega)$,

$$\begin{aligned} |(\varepsilon(\mathbf{x})\mathbf{E}^n_{tt},\mathbf{v})| & \le |\int_\Omega \mathbf{j}\cdot\mathbf{v}d\mathbf{x}| + |\int_\Omega \mu^n(\mathbf{x})\mathbf{curl}\mathbf{E}^n\cdot\mathbf{curl}\mathbf{v}d\mathbf{x}| \\ & \le (\|\mathbf{j}\|_{L^2(\Omega)^3} + \mu_1\|\mathbf{curl}\mathbf{E}^n\|_{L^2(\Omega)^3})\|\mathbf{v}\|_{0,\mathbf{curl}}\,, \end{aligned}$$

hence we obtain from (2.7) that

$$\|\varepsilon(\mathbf{x})\mathbf{E}^n_{tt}\|_{L^2(0,T;(H_0(\mathbf{curl};\Omega))')} \le C\,, \tag{2.9}$$

where, and throughout the rest of the work, C will always stand for a generic constant which depends only on some given data such as $\mathbf{E}_0$, $\mathbf{E}_1$, $\mathbf{j}$ and the bounds ε_0, ε_1, μ_0 and μ_1 in (1.7) and (2.2).

One observes from (2.7) and (2.8) that $\mathbf{curl}\mathbf{E}^n$, $\mu^n\mathbf{curl}\mathbf{E}^n$, $\mathbf{E}^n$ and $\mathbf{E}^n_t$ are all bounded in $L^2(0,T;L^2(\Omega)^3)$, while $\varepsilon(\mathbf{x})\mathbf{E}^n_{tt}$ is bounded in $L^2(0,T;(H_0(\mathbf{curl};\Omega))')$.

This implies the existence of a subsequence of $\Big\{\mathbf{E}^n\Big\}$, still denoted as $\Big\{\mathbf{E}^n\Big\}$, and some $\mathbf{P}$, $\mathbf{W}$, $\hat{\mathbf{E}}$, $\hat{\hat{\mathbf{E}}} \in L^2(0,T;L^2(\Omega)^3)$ and $\mathbf{U} \in L^2(0,T;(H_0(\mathbf{curl};\Omega))')$ such that

$$(2.10)\qquad \begin{cases} \mathbf{curl}\,\mathbf{E}^n & \rightharpoonup & \mathbf{P} \text{ in } L^2(0,T;L^2(\Omega)^3); \\ \mu^n\mathbf{curl}\,\mathbf{E}^n & \rightharpoonup & \mathbf{W} \text{ in } L^2(0,T;L^2(\Omega)^3); \\ \mathbf{E}^n & \rightharpoonup & \hat{\mathbf{E}} \text{ in } L^2(0,T;L^2(\Omega)^3); \\ \mathbf{E}^n_t & \rightharpoonup & \hat{\hat{\mathbf{E}}} \text{ in } L^2(0,T;L^2(\Omega)^3); \\ \varepsilon(\mathbf{x})\mathbf{E}^n_{tt} & \rightharpoonup & \mathbf{U} \text{ in } L^2(0,T;(H_0(\mathbf{curl};\Omega))'). \end{cases}$$

Step 2. For the limits $\mathbf{P}$, $\mathbf{W}$, $\hat{\mathbf{E}}$, $\hat{\hat{\mathbf{E}}}$ and $\mathbf{U}$ in (2.10), we prove that

$$\hat{\mathbf{E}} = \mathbf{E}(\mu^*), \quad \mathbf{P} = \mathbf{curl}\,\hat{\mathbf{E}}, \quad \mathbf{W} = \mu^*\mathbf{curl}\,\hat{\mathbf{E}}, \quad \hat{\hat{\mathbf{E}}} = \hat{\mathbf{E}}_t, \quad \mathbf{U} = \varepsilon(\mathbf{x})\hat{\mathbf{E}}_{tt}\,.$$

Firstly, for any $\varphi \in L^2(0,T;H_0(\mathbf{curl};\Omega))$, we have by the integration by parts formula that

$$\int_0^T\int_\Omega \mathbf{curl}\mathbf{E}^n\cdot\varphi d\mathbf{x}dt = \int_0^T\int_\Omega \mathbf{E}^n\cdot\mathbf{curl}\,\varphi d\mathbf{x}dt\,.$$

Letting $n\to\infty$ in the above equation and using (2.10), we obtain

$$(2.11)\int_0^T\int_\Omega \mathbf{P}\cdot\varphi d\mathbf{x}dt = \int_0^T\int_\Omega \hat{\mathbf{E}}\cdot\mathbf{curl}\,\varphi d\mathbf{x}dt = \int_0^T\int_\Omega \mathbf{curl}\,\hat{\mathbf{E}}\cdot\varphi d\mathbf{x}dt\,.$$

Then by the density of $L^2(0,T;H_0(\mathbf{curl};\Omega))$ in $L^2(0,T;L^2(\Omega)^3)$, we obtain $\mathbf{P} = \mathbf{curl}\hat{\mathbf{E}}$.

Since $\mu^n\mathbf{curl}\,\mathbf{E}^n \rightharpoonup \mathbf{W}$ in $L^2(0,T;L^2(\Omega)^3)$, we know

$$\lim_{n\to\infty}\int_0^T\int_\Omega \mu^n\mathbf{curl}\mathbf{E}^n\cdot\varphi d\mathbf{x}dt = \int_0^T\int_\Omega \mathbf{W}\cdot\varphi d\mathbf{x}dt,\ \forall\varphi\in L^2(0,T;L^2(\Omega)^3).$$

Now we show $\mathbf{W} = \mu^*\mathbf{curl}\,\hat{\mathbf{E}}$, namely it holds for any $\varphi \in L^2(0,T;L^2(\Omega)^3)$ that

$$(2.12)\qquad \lim_{n\to\infty}\int_0^T\int_\Omega \mu^n\mathbf{curl}\mathbf{E}^n\cdot\varphi d\mathbf{x}dt = \int_0^T\int_\Omega \mu^*\mathbf{curl}\hat{\mathbf{E}}\cdot\varphi d\mathbf{x}dt\,.$$

To see this, we derive by the Cauchy-Schwarz inequality, (2.7) and (2.2) that

$$\begin{aligned} &\Big|\int_0^T\int_\Omega \mu^n\mathbf{curl}\mathbf{E}^n\cdot\varphi d\mathbf{x}dt - \int_0^T\int_\Omega \mu^*\mathbf{curl}\hat{\mathbf{E}}\cdot\varphi d\mathbf{x}dt\Big| \\ \le\ &\Big|\int_0^T\int_\Omega (\mu^n-\mu^*)\mathbf{curl}\mathbf{E}^n\cdot\varphi d\mathbf{x}dt\Big| + \Big|\int_0^T\int_\Omega \mu^*(\mathbf{curl}\mathbf{E}^n - \mathbf{curl}\hat{\mathbf{E}})\cdot\varphi d\mathbf{x}dt\Big| \\ \le\ &\Big(\int_0^T\int_\Omega |\mu^n-\mu^*||\varphi|^2 d\mathbf{x}dt\Big)^{1/2}\Big(\int_0^T\int_\Omega |\mu^n-\mu^*||\mathbf{curl}\mathbf{E}^n|^2 d\mathbf{x}dt\Big)^{1/2} \\ &+\Big|\int_0^T\int_\Omega \mu^*(\mathbf{curl}\mathbf{E}^n - \mathbf{curl}\hat{\mathbf{E}})\cdot\varphi d\mathbf{x}dt\Big| \\ \le\ &C\Big(\int_0^T\int_\Omega |\mu^n-\mu^*||\varphi|^2 d\mathbf{x}dt\Big)^{1/2} + \Big|\int_0^T\int_\Omega \mu^*(\mathbf{curl}\mathbf{E}^n - \mathbf{curl}\hat{\mathbf{E}})\cdot\varphi d\mathbf{x}dt\Big|, \end{aligned}$$

which converges to zero as $n\to\infty$ by using the Lebesgue dorminated convergence theorem, and the weak convergence of $\mathbf{curl}\mathbf{E}^n$ to $\mathbf{P}$ in (2.10) and $\mathbf{P} = \mathbf{curl}\hat{\mathbf{E}}$.

Next, taking $\psi(t) \in C_0^\infty[0,T]$, we have for any $\mathbf{v}\in L^2(\Omega)^3$,

$$\int_0^T\int_\Omega \psi(t)\mathbf{E}^n_t\cdot\mathbf{v}dxdt = -\int_0^T\int_\Omega \psi_t(t)\mathbf{E}^n\cdot\mathbf{v}d\mathbf{x}dt\,.$$

Letting $n \to \infty$ in the above equation and using (2.10), we obtain

$$\int_0^T \int_\Omega \psi(t)\hat{\hat{\mathbf{E}}} \cdot \mathbf{v} d\mathbf{x} dt = -\int_0^T \int_\Omega \psi_t(t)\hat{\mathbf{E}} \cdot \mathbf{v} d\mathbf{x} dt = \int_0^T \int_\Omega \psi(t)\hat{\mathbf{E}}_t \cdot \mathbf{v} d\mathbf{x} dt,$$

that implies $\hat{\hat{\mathbf{E}}} = \hat{\mathbf{E}}_t$.

Now taking $\psi(t) \in C^1[0,T]$ such that $\psi(T) = 0$, we obtain by integration by parts for any $\mathbf{v} \in L^2(\Omega)^3$ that

$$\int_0^T \int_\Omega \psi(t)\mathbf{E}_t^n \cdot \mathbf{v} d\mathbf{x} dt = -\int_0^T \int_\Omega \psi_t(t)\mathbf{E}^n \cdot \mathbf{v} d\mathbf{x} dt - \int_\Omega \psi(0)\mathbf{E}_0 \cdot \mathbf{v} d\mathbf{x}.$$

Letting $n \to \infty$ in the above equation and using (2.10) and $\hat{\hat{\mathbf{E}}} = \hat{\mathbf{E}}_t$, we obtain

$$\int_0^T \int_\Omega \psi(t)\hat{\mathbf{E}}_t \cdot \mathbf{v} d\mathbf{x} dt = -\int_0^T \int_\Omega \psi_t(t)\hat{\mathbf{E}} \cdot \mathbf{v} d\mathbf{x} dt - \int_\Omega \psi(0)\mathbf{E}_0 \cdot \mathbf{v} d\mathbf{x},$$

namely

$$\int_0^T \int_\Omega (\hat{\mathbf{E}}\psi(t))_t \cdot \mathbf{v} d\mathbf{x} dt = -\int_\Omega \psi(0)\mathbf{E}_0 \cdot \mathbf{v} d\mathbf{x}.$$

This gives that $\int_\Omega \psi(0)\hat{\mathbf{E}}(\mathbf{x},0) \cdot \mathbf{v} d\mathbf{x} = \int_\Omega \psi(0)\mathbf{E}_0(\mathbf{x}) \cdot \mathbf{v} d\mathbf{x}$, and we know

$$\hat{\mathbf{E}}(\mathbf{x},0) = \mathbf{E}_0(\mathbf{x}).$$

Next by taking $\psi(t) \in C_0^\infty[0,T]$, we derive for any $\mathbf{v} \in H_0(\mathbf{curl};\Omega)$ that

$$\int_0^T \int_\Omega \varepsilon(\mathbf{x})\psi(t)\mathbf{E}_{tt}^n \cdot \mathbf{v} d\mathbf{x} dt = -\int_0^T \int_\Omega \varepsilon(\mathbf{x})\psi_t(t)\mathbf{E}_t^n \cdot \mathbf{v} d\mathbf{x} dt,$$

Letting $n \to \infty$ in the above equation, we obtain

$$\int_0^T \int_\Omega \psi(t)\mathbf{U} \cdot \mathbf{v} d\mathbf{x} dt = -\int_0^T \int_\Omega \psi_t(t)\varepsilon(\mathbf{x})\hat{\hat{\mathbf{E}}} \cdot \mathbf{v} d\mathbf{x} dt,$$

which implies

$$\mathbf{U} = \varepsilon(\mathbf{x})\hat{\hat{\mathbf{E}}}_{\mathbf{t}} = \varepsilon(\mathbf{x})\hat{\mathbf{E}}_{\mathbf{tt}}.$$

Furthermore, we take a $\psi(t) \in C^1[0,T]$ such that $\psi(T) = 0$ to deduce for any $\mathbf{v} \in H_0(\mathbf{curl};\Omega)$ that

$$\int_0^T \int_\Omega \psi(t)\varepsilon(\mathbf{x})\mathbf{E}_{tt}^n \cdot \mathbf{v} d\mathbf{x} dt = -\int_0^T \int_\Omega \psi_t(t)\varepsilon(\mathbf{x})\mathbf{E}_t^n \cdot \mathbf{v} d\mathbf{x} dt - \int_\Omega \psi(0)\varepsilon(\mathbf{x})\mathbf{E}_1(\mathbf{x}) \cdot \mathbf{v} d\mathbf{x} dt$$

Letting $n \to \infty$ in the above equation and using $\mathbf{U} = \varepsilon(\mathbf{x})\hat{\hat{\mathbf{E}}}_{\mathbf{t}}$, we obtain

$$\int_0^T \int_\Omega \varepsilon(\mathbf{x})(\hat{\hat{\mathbf{E}}}\psi(t))_t \cdot \mathbf{v} d\mathbf{x} dt = -\int_\Omega \psi(0)\varepsilon(\mathbf{x})\mathbf{E}_1(\mathbf{x}) \cdot \mathbf{v} d\mathbf{x}.$$

Hence we have

$$\hat{\hat{\mathbf{E}}}(\mathbf{x},0) = \hat{\mathbf{E}}_t(\mathbf{x},0) = \mathbf{E}_1(\mathbf{x}).$$

Finally, we multiply both sides of (2.5) by a function $\psi(t) \in C_0^\infty[0,T]$, then integrate with respect to t to obtain

$$\int_0^T \int_\Omega \varepsilon(\mathbf{x})\psi(t)\mathbf{E}_{tt}^n \cdot \mathbf{v} d\mathbf{x} dt + \int_0^T \int_\Omega \mu^n(\mathbf{x})\psi(t)\mathbf{curl}\mathbf{E}^n \cdot \mathbf{curl}\mathbf{v} d\mathbf{x} dt = \int_0^T \int_\Omega \psi(t)\mathbf{j} \cdot \mathbf{v} d\mathbf{x} dt.$$

Letting $n \to \infty$ in the above equation, we obtain

(2.13)
$$\int_0^T \int_\Omega \varepsilon(\mathbf{x})\psi(t)\hat{\mathbf{E}}_{tt} \cdot \mathbf{v} d\mathbf{x} dt + \int_0^T \int_\Omega \mu^*\psi(t)\mathbf{curl}\hat{\mathbf{E}} \cdot \mathbf{curl}\mathbf{v} d\mathbf{x} dt = \int_0^T \int_\Omega \psi(t)\mathbf{j} \cdot \mathbf{v} d\mathbf{x} dt\,,$$

comparing with (2.5) and using the fact that $\hat{\mathbf{E}}(\mathbf{x},0)=\mathbf{E}_0(\mathbf{x})$ and $\hat{\mathbf{E}}_t(\mathbf{x},0)=\mathbf{E}_1(\mathbf{x})$ and the definition of $\mathbf{E}(\mu^*)$, we derive

$$\hat{\mathbf{E}}=\mathbf{E}(\mu^*)\equiv\mathbf{E}^*.$$

Step 3. We prove

$$\lim_{n\to\infty}\int_0^T\int_\omega|\mathbf{E}(\mu^n)-\mathbf{E}^\delta|^2d\mathbf{x}dt=\int_0^T\int_\omega|\mathbf{E}(\mu^*)-\mathbf{E}^\delta|^2d\mathbf{x}dt. \tag{2.14}$$

It suffices to show that

$$\lim_{n\to\infty}\int_0^T\|\mathbf{E}(\mu^n)-\mathbf{E}(\mu^*)\|^2_{L^2(\Omega)^3}dt=0. \tag{2.15}$$

Indeed, if (2.15) holds, then we obtain by using the Cauchy-Schwarz inequality and the bounds (2.7)-(2.8) that

$$\begin{aligned}
&\Big|\int_0^T\|\mathbf{E}^n-\mathbf{E}^\delta\|^2_{L^2(\omega)^3}dt-\int_0^T\|\mathbf{E}^*-\mathbf{E}^\delta\|^2_{L^2(\omega)^3}dt\Big|\\
=\ &\Big|\int_0^T\int_\omega(\mathbf{E}^n-\mathbf{E}^*)\cdot(\mathbf{E}^n+\mathbf{E}^*-2\mathbf{E}^\delta)d\mathbf{x}dt\Big|\\
\le\ &(\int_0^T\int_\omega|\mathbf{E}^n-\mathbf{E}^*|^2d\mathbf{x}dt)^{\frac12}(\int_0^T\int_\omega|\mathbf{E}^n+\mathbf{E}^*-2\mathbf{E}^\delta|^2d\mathbf{x}dt)^{\frac12}\\
\le\ &C(\int_0^T\|\mathbf{E}(\mu^n)-\mathbf{E}(\mu^*)\|^2_{L^2(\Omega)^3}dt)\to0\quad\text{as}\quad n\to\infty,
\end{aligned}$$

so (2.14) is verified. Next, we prove (2.15). Taking $\mathbf{v}=\mathbf{E}^n_t$ in (2.5), we obtain

$$\frac12\int_\Omega\varepsilon(\mathbf{x})\frac{d}{dt}|\mathbf{E}^n_t|^2d\mathbf{x}+\frac12\int_\Omega\mu^n(\mathbf{x})\frac{d}{dt}|\mathbf{curl}\mathbf{E}^n|^2d\mathbf{x}=\int_\Omega\mathbf{j}\cdot\mathbf{E}^n_td\mathbf{x}. \tag{2.16}$$

Similarly, taking $\mathbf{v}=\mathbf{E}^*_t$ in (2.13), we obtain

$$\frac12\int_\Omega\varepsilon(\mathbf{x})\frac{d}{dt}|\mathbf{E}^*_t|^2d\mathbf{x}+\frac12\int_\Omega\mu^*(\mathbf{x})\frac{d}{dt}|\mathbf{curl}\mathbf{E}^*|^2d\mathbf{x}=\int_\Omega\mathbf{j}\cdot\mathbf{E}^*_td\mathbf{x}. \tag{2.17}$$

Subtracting (2.17) from (2.16), we have

$$\begin{aligned}
&\frac12\int_\Omega\varepsilon(\mathbf{x})\frac{d}{dt}(|\mathbf{E}^n_t|^2-|\mathbf{E}^*_t|^2)d\mathbf{x}+\frac12\int_\Omega\mu^n(\mathbf{x})\frac{d}{dt}(|\mathbf{curl}\mathbf{E}^n|^2-|\mathbf{curl}\mathbf{E}^*|^2)d\mathbf{x}\\
=&\int_\Omega\mathbf{j}\cdot(\mathbf{E}^n_t-\mathbf{E}^*_t)d\mathbf{x}+\frac12\int_\Omega(\mu^*(\mathbf{x})-\mu^n(\mathbf{x}))\frac{d}{dt}|\mathbf{curl}\mathbf{E}^*|^2d\mathbf{x}\,.
\end{aligned} \tag{2.18}$$

One can rewrite (2.18) as

$$\begin{aligned}
&\frac12\int_\Omega\varepsilon(\mathbf{x})\frac{d}{dt}|\mathbf{E}^n_t-\mathbf{E}^*_t|^2d\mathbf{x}+\frac12\int_\Omega\mu^n(\mathbf{x})\frac{d}{dt}|\mathbf{curl}\mathbf{E}^n-\mathbf{curl}\mathbf{E}^*|^2d\mathbf{x}\\
=\ &\int_\Omega\mathbf{j}\cdot(\mathbf{E}^n_t-\mathbf{E}^*_t)d\mathbf{x}+\frac12\int_\Omega(\mu^*(\mathbf{x})-\mu^n(\mathbf{x}))\frac{d}{dt}|\mathbf{curl}\mathbf{E}^*|^2d\mathbf{x}\\
&+\int_\Omega\varepsilon(\mathbf{x})\frac{d}{dt}(\mathbf{E}^*_t\cdot(\mathbf{E}^*_t-\mathbf{E}^n_t))d\mathbf{x}\\
&+\int_\Omega\mu^n(\mathbf{x})\frac{d}{dt}(\mathbf{curl}\mathbf{E}^*\cdot(\mathbf{curl}\mathbf{E}^*-\mathbf{curl}\mathbf{E}^n))d\mathbf{x}\,.
\end{aligned} \tag{2.19}$$

By integrating both sides of (2.19) with respect to t we obtain

$$
\begin{aligned}
& \frac{1}{2}\int_\Omega \varepsilon(\mathbf{x})|\mathbf{E}_t^n(\mathbf{x},t)-\mathbf{E}_t^*(\mathbf{x},t)|^2 d\mathbf{x} \\
& +\frac{1}{2}\int_\Omega \mu^n(\mathbf{x})|\mathbf{curlE}^n(\mathbf{x},t)-\mathbf{curlE}^*(\mathbf{x},t)|^2 d\mathbf{x} \\
= & \int_0^t\int_\Omega \mathbf{j}\cdot(\mathbf{E}_t^n-\mathbf{E}_t^*)d\mathbf{x}dt \\
& +\frac{1}{2}\int_\Omega (\mu^*(\mathbf{x})-\mu^n(\mathbf{x}))(|\mathbf{curlE}^*(\mathbf{x},t)|^2-|\mathbf{curlE}_0(\mathbf{x})|^2)d\mathbf{x} \\
& +\int_\Omega \varepsilon(\mathbf{x})\mathbf{E}_t^*(\mathbf{x},t)\cdot(\mathbf{E}_t^*(\mathbf{x},t)-\mathbf{E}_t^n(\mathbf{x},t))d\mathbf{x} \\
& +\int_\Omega \mu^n(\mathbf{x})\mathbf{curlE}^*(\mathbf{x},t)\cdot(\mathbf{curlE}^*(\mathbf{x},t)-\mathbf{curlE}^n(\mathbf{x},t))d\mathbf{x}\,, \qquad (2.20)
\end{aligned}
$$

then integrating both sides of (2.20) over $t\in(0,T)$, we come to the following relations

$$
\begin{aligned}
& \frac{1}{2}\int_0^T\int_\Omega \varepsilon(\mathbf{x})|\mathbf{E}_t^n(\mathbf{x},t)-\mathbf{E}_t^*(\mathbf{x},t)|^2 d\mathbf{x}dt \\
& +\frac{1}{2}\int_0^T\int_\Omega \mu^n(\mathbf{x})|\mathbf{curlE}^n(\mathbf{x},t)-\mathbf{curlE}^*(\mathbf{x},t)|^2 d\mathbf{x}dt \\
= & \int_0^T\int_0^t\int_\Omega \mathbf{j}\cdot(\mathbf{E}_\tau^n(\mathbf{x},\tau)-\mathbf{E}_\tau^*(\mathbf{x},\tau))d\mathbf{x}d\tau dt \\
& +\frac{1}{2}\int_0^T\int_\Omega (\mu^*(\mathbf{x})-\mu^n(\mathbf{x}))(|\mathbf{curlE}^*(\mathbf{x},t)|^2-|\mathbf{curlE}_0(\mathbf{x})|^2)d\mathbf{x}dt \\
& +\int_0^T\int_\Omega \varepsilon(\mathbf{x})\mathbf{E}_t^*(\mathbf{x},t)\cdot(\mathbf{E}_t^*(\mathbf{x},t)-\mathbf{E}_t^n(\mathbf{x},t))d\mathbf{x}dt \\
& +\int_0^T\int_\Omega \mu^n(\mathbf{x})\mathbf{curlE}^*(\mathbf{x},t)\cdot(\mathbf{curlE}^*(\mathbf{x},t)-\mathbf{curlE}^n(\mathbf{x},t))d\mathbf{x}dt \\
\equiv & \sum_{i=1}^4 (I)_i\,. \qquad (2.21)
\end{aligned}
$$

Next we estimate all the terms $(I)_i$'s in (2.21) one by one.

First for the estimation of $(I)_1$, by the Canchy-Schwarz inequality and (2.10) we know

$$\int_0^t\int_\Omega \mathbf{j}\cdot(\mathbf{E}_\tau^n(\mathbf{x},\tau)-\mathbf{E}_\tau^*(\mathbf{x},\tau))d\mathbf{x}d\tau$$

is bounded independent of n and t, and tends to zero as $n\to\infty$ for all $t\in(0,T)$. Hence, by using the Lebesgue dorminated convergence theorem we have $(I)_1\to 0$ as n $\to\infty$.

For $(I)_2$, we know that $\mu^n\to\mu^*$ in $\mathrm{L}^2(\Omega)$ from the fact that $\mu^n\rightharpoonup\mu^*$ in $\mathrm{H}^1(\Omega)$. Then the convergence of $(I)_2$ follows directly from the Lebesgue dorminated convergence theorem. The convergence of $(I)_3$ follows directly from the previously proved fact that $\mathbf{E}_t^n\rightharpoonup\mathbf{E}_t^*$ in $L^2(0,T;L^2(\Omega)^3)$.

Finally we come to analyse term $(I)_4$. Following the proof of (2.12), we can deduce

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^T\int_\Omega \mu^n \mathbf{curl}\mathbf{E}^n(\mathbf{x},t)\cdot\mathbf{curl}\mathbf{E}^*(\mathbf{x},t)d\mathbf{x}dt \\
&= \int_0^T\int_\Omega \mu^* \mathbf{curl}\mathbf{E}^*(\mathbf{x},t)\cdot\mathbf{curl}\mathbf{E}^*(\mathbf{x},t)d\mathbf{x}dt.
\end{aligned}\tag{2.22}$$

Moreover, by using the Lebesgue dorminated convergence theorem we know

$$\int_0^T\int_\Omega (\mu^n(\mathbf{x})-\mu^*(\mathbf{x}))|\mathbf{curl}\mathbf{E}^*(\mathbf{x},t)|^2 d\mathbf{x}dt \to 0 \quad as \quad n\to\infty,$$

which, along with (2.22) and the triangle inequality, yields

$$\begin{aligned}
(I)_4 \quad \leq \quad & |\int_0^T\int_\Omega \mu^n \mathbf{curl}\mathbf{E}^n(\mathbf{x},t)\cdot\mathbf{curl}\mathbf{E}^*(\mathbf{x},t)d\mathbf{x}dt \\
& -\int_0^T\int_\Omega \mu^* \mathbf{curl}\mathbf{E}^*(\mathbf{x},t)\cdot\mathbf{curl}\mathbf{E}^*(\mathbf{x},t)d\mathbf{x}dt| \\
& +|\int_0^T\int_\Omega (\mu^n(\mathbf{x})-\mu^*(\mathbf{x}))\mathbf{curl}\mathbf{E}^*(\mathbf{x},t)\cdot\mathbf{curl}\mathbf{E}^*(\mathbf{x},t)d\mathbf{x}dt| \to 0
\end{aligned}$$

as $n\to\infty$. By the previously established convergence for $(I)_1$ up to $(I)_4$, we know from (2.21) that

$$\|\mathbf{E}_t^n-\mathbf{E}_t^*\|_{L^2(0,T;L^2(\Omega)^3)} \to 0 \quad as \quad n\to\infty\,.$$

Hence we have

$$\begin{aligned}
& \int_0^T\int_\Omega |\mathbf{E}^n(\mathbf{x},t)-\mathbf{E}^*(\mathbf{x},t)|^2 d\mathbf{x}dt \\
= \quad & \int_0^T\int_\Omega |\int_0^t (\mathbf{E}_\tau^n(\mathbf{x},\tau)-\mathbf{E}_\tau^*(\mathbf{x},\tau))d\tau|^2 d\mathbf{x}dt \\
\leq \quad & T^2\int_\Omega\int_0^T |\mathbf{E}_t^n(\mathbf{x},t)-\mathbf{E}_t^*(\mathbf{x},t)|^2 dtd\mathbf{x} \to 0 \quad as \quad n\to\infty,
\end{aligned}$$

which leads to (2.15).

Step 4. We prove the limit μ^* is the minimizer to the system (2.3)–(2.5).

Using the results of Step 3 and the lower semi-continuity of a norm, we have

$$\begin{aligned}
G(\mu^*) \quad = \quad & \int_0^T \|\mathbf{E}(\mu^*)-\mathbf{E}^\delta\|^2_{L^2(\omega)^3}dt + \frac{\beta}{2}\|\nabla\mu^*\|^2_{L^2(\Omega)} \\
= \quad & \lim_{n\to\infty}\int_0^T \|\mathbf{E}(\mu^n)-\mathbf{E}^\delta\|^2_{L^2(\omega)^3}dt + \frac{\beta}{2}\|\nabla\mu^*\|^2_{L^2(\Omega)} \\
\leq \quad & \lim_{n\to\infty}\int_0^T \|\mathbf{E}(\mu^n)-\mathbf{E}^\delta\|^2_{L^2(\omega)^3}dt + \frac{\beta}{2}\liminf_{n\to\infty}\|\nabla\mu^n\|^2_{L^2(\Omega)} \\
\leq \quad & \liminf_{n\to\infty} G(\mu^n) = \inf_{\mu\in K} G(\mu),
\end{aligned}$$

so μ^* is indeed a minimizer to the system (2.3)-(2.5). This completes the proof of Theorem 2.1. ♯

The next theorem demonstrates that the minimization system (2.3)-(2.5) is indeed a stabilization of the ill-posed Inverse Problem I with respect to the changes of the observation errors, so justifies the regularizing effect of the formulation (2.3)-(2.5).

Theorem 2.2. *Let* $\left\{\mathbf{E}_n^\delta\right\}$ *be a sequence such that*

$$\mathbf{E}_n^\delta \to \mathbf{E}^\delta \quad in \quad L^2(0,T;L^2(\omega)^3) \quad as \quad n\to\infty\,, \tag{2.23}$$

and let $\left\{\mu^{(n)}\right\}$ *be the sequence of the minimizers of problem (2.3)-(2.5) with* $\mathbf{E}^\delta$ *replaced by* $\mathbf{E}_n^\delta$*. Then the whole sequence* $\left\{\mu^{(n)}\right\}$ *converges strongly in* $H^1(\Omega)$ *to a minimizer of (2.3)–(2.5).*

Proof. By the definition of $\left\{\mu^{(n)}\right\}$, for any $\mu\in K$ we have

$$\int_0^T\int_\omega |\mathbf{E}(\mu^{(n)})-\mathbf{E}_n^\delta|^2 d\mathbf{x}dt+\frac{\beta}{2}\|\nabla\mu^{(n)}\|_{L^2(\Omega)}^2 \le \int_0^T\int_\omega |\mathbf{E}(\mu)-\mathbf{E}_n^\delta|^2 d\mathbf{x}dt+\frac{\beta}{2}\|\nabla\mu\|_{L^2(\Omega)}^2.$$

This implies $\{\|\nabla\mu^{(n)}\|\}$ is bounded. Furthermore, $\{\mu^{(n)}\}$ is also bounded in $L^2(\Omega)$ by noting that $\mu^{(n)}\in K$, so we know $\{\mu^{(n)}\}$ is bounded in $H^1(\Omega)$. Therefore there exists a subsequence, still denoted by $\mu^{(n)}$, and some $\mu^*\in K$, such that $\mu^{(n)}\rightharpoonup\mu^*$ in $H^1(\Omega)$ as $n\to\infty$.

Now applying the Cauchy-Schwarz inequality, we can derive

$$\begin{aligned}
&\Big|\int_0^T\int_\omega |\mathbf{E}(\mu^{(n)})-\mathbf{E}_n^\delta|^2 d\mathbf{x}dt-\int_0^T\int_\omega |\mathbf{E}(\mu^*)-\mathbf{E}^\delta|^2 d\mathbf{x}dt\Big|\\
=\;&\Big|\int_0^T\int_\omega (\mathbf{E}^\delta-\mathbf{E}_n^\delta)\cdot(2\mathbf{E}(\mu^{(n)})-\mathbf{E}_n^\delta-\mathbf{E}^\delta)d\mathbf{x}dt\\
&+\int_0^T\int_\omega (\mathbf{E}(\mu^{(n)})-\mathbf{E}(\mu^*))\cdot(\mathbf{E}(\mu^{(n)})+\mathbf{E}(\mu^*)-2\mathbf{E}^\delta)d\mathbf{x}dt\Big|\\
\le\;&(\int_0^T\int_\omega |\mathbf{E}^\delta-\mathbf{E}_n^\delta|^2 d\mathbf{x}dt)^{\frac12}(\int_0^T\int_\omega |2\mathbf{E}(\mu^{(n)})-\mathbf{E}_n^\delta-\mathbf{E}^\delta|^2 d\mathbf{x}dt)^{\frac12}\\
&+(\int_0^T\int_\omega |\mathbf{E}(\mu^{(n)})-\mathbf{E}(\mu^*)|^2 d\mathbf{x}dt)^{\frac12}(\int_0^T\int_\omega |\mathbf{E}(\mu^{(n)})+\mathbf{E}(\mu^*)-2\mathbf{E}^\delta|^2 d\mathbf{x}dt)^{\frac12}\\
\equiv\;& R_1+R_2\,.
\end{aligned}$$

We know that $\int_0^T\int_\omega|\mathbf{E}^\delta-\mathbf{E}_n^\delta|^2d\mathbf{x}dt\to 0\; as\; n\to\infty$ by using (2.23), and $\int_0^T\int_\omega|2\mathbf{E}(\mu^{(n)})-\mathbf{E}_n^\delta-\mathbf{E}^\delta|^2d\mathbf{x}dt$ is bounded from the proof of Theorem 2.1. Hence we have $R_1\to 0\; as\; n\to\infty$. Similarly to the proof of Step 3 in Theorem 2.1, we can show that $\int_0^T\int_\omega|\mathbf{E}(\mu^{(n)})-\mathbf{E}(\mu^*)|^2d\mathbf{x}dt\to 0$, thus $R_2\to 0\; as\; n\to\infty$. This proves

$$\lim_{n\to\infty}\int_0^T\int_\omega |\mathbf{E}(\mu^{(n)})-\mathbf{E}_n^\delta|^2 d\mathbf{x}dt=\int_0^T\int_\omega |\mathbf{E}(\mu^*)-\mathbf{E}^\delta|^2 d\mathbf{x}dt. \tag{2.24}$$

Now we can show that μ^* is a minimizer of (2.3)-(2.5). To see this, for any $\mu \in K$ we derive from (2.23), (2.24) and the lower semi-continuity of a norm that

$$\begin{aligned}
G(\mu^*) &= \int_0^T \|\mathbf{E}(\mu^*) - \mathbf{E}^\delta\|^2_{L^2(\omega)^3} dt + \frac{\beta}{2}\|\nabla\mu^*\|^2_{L^2(\Omega)} \\
&= \lim_{n\to\infty} \int_0^T \|\mathbf{E}(\mu^{(n)}) - \mathbf{E}^\delta_n\|^2_{L^2(\omega)^3} dt + \frac{\beta}{2}\|\nabla\mu^*\|^2_{L^2(\Omega)} \\
&\le \lim_{n\to\infty} \int_0^T \|\mathbf{E}(\mu^{(n)}) - \mathbf{E}^\delta_n\|^2_{L^2(\omega)^3} dt + \frac{\beta}{2}\liminf_{n\to\infty}\|\nabla\mu^{(n)}\|^2_{L^2(\Omega)} \\
&\le \liminf_{n\to\infty}[\int_0^T \|\mathbf{E}(\mu^{(n)}) - \mathbf{E}^\delta_n\|^2_{L^2(\omega)^3} dt + \frac{\beta}{2}\|\nabla\mu^{(n)}\|^2_{L^2(\Omega)}] \\
&\le \liminf_{n\to\infty}[\int_0^T \|\mathbf{E}(\mu) - \mathbf{E}^\delta_n\|^2_{L^2(\omega)^3} dt + \frac{\beta}{2}\|\nabla\mu\|^2_{L^2(\Omega)}] \\
&= \int_0^T \|\mathbf{E}(\mu) - \mathbf{E}^\delta\|^2_{L^2(\omega)^3} dt + \frac{\beta}{2}\|\nabla\mu\|^2_{L^2(\Omega)} \\
&= G(\mu)\,, \qquad (2.25)
\end{aligned}$$

which verifies that μ^* is a minimizer of (2.3)-(2.5).

It remains to prove that $\mu^{(n)} \to \mu^*$ in $H^1(\Omega)$ as $n \to \infty$. First it follows from the weak convergence of $\mu^{(n)}$ to μ^* in $H^1(\Omega)$ that $\mu^{(n)} \to \mu^*$ in $\mathrm{L}^2(\Omega)$. Next, we get from (2.25) that

$$G(\mu^*) = \min_{\mu\in K} G(\mu) = \lim_{n\to\infty}[\int_0^T \|\mathbf{E}(\mu^{(n)}) - \mathbf{E}^\delta_n\|^2_{L^2(\omega)^3} dt + \frac{\beta}{2}\|\nabla\mu^{(n)}\|^2_{L^2(\Omega)}],$$

which, along with (2.24), gives

$$\lim_{n\to\infty}\|\nabla\mu^{(n)}\|^2_{L^2(\Omega)} = \|\nabla\mu^*\|^2_{L^2(\Omega)}.$$

Hence we derive

$$\begin{aligned}
\|\nabla\mu^{(n)} - \nabla\mu^*\|^2_{L^2(\Omega)} &= \|\nabla\mu^{(n)}\|^2_{L^2(\Omega)} + \|\nabla\mu^*\|^2_{L^2(\Omega)} - 2(\nabla\mu^{(n)}, \nabla\mu^*) \\
&\to 2\|\nabla\mu^*\|^2_{L^2(\Omega)} - 2(\nabla\mu^*, \nabla\mu^*) = 0
\end{aligned}$$

as $n \to \infty$, leading to the desired strong convergence of $\mu^{(n)}$ to μ^* in $H^1(\Omega)$. ♯

3. Finite element approximation

We now propose a finite element method for solving the continuous nonlinear minimization problem (2.3)–(2.5). For the purpose, we first triangulate the space domain Ω and assume that $\mathcal{T}^h$ is a shape regular triangulation of Ω with a mesh size h, consisting of tetrahedral elements. We will approximate the electric field $\mathbf{E}$ by the edge element space of second family (cf. [9]):

$$\mathbf{V}_h = \Big\{\mathbf{v}_h \in H(\mathbf{curl};\Omega);\ \mathbf{v}_h|_A \in (P_1(A))^3,\ \forall A \in \mathcal{T}^h\Big\},$$

where $P_1(A)$ is the space of linear polynomials on A. It was proved in [10] that any function in $\mathbf{V}_h$ can be uniquely determined by its degrees of freedom in the moment set $M_h(\mathbf{v})$ given by

$$M_h(\mathbf{v}) = \Big\{\int_e (\mathbf{v}\cdot\tau) q ds;\ \forall\, q \in P_1(e) \text{ on any edge } e \text{ of } A \in \mathcal{T}^h\Big\}.$$

To enforce the boundary condition (1.5), we introduce the following subspace of $\mathbf{V}_h$ and $H_0(\mathbf{curl};\Omega)$:

$$\mathbf{V}_h^0 = \Big\{\mathbf{v}_h \in \mathbf{V}_h;\ \mathbf{v}_h \times \mathbf{n} = 0 \text{ on } \partial\Omega\Big\}.$$

We will approximate the parameter function $\mu(\mathbf{x})$ to be recovered by the standard nodal finite element space of piecewise linear functions (cf. [3]):

$$S_h = \Big\{v_h \in H^1(\Omega);\ v_h|_A \in P_1(A),\ \forall A \in \mathcal{T}^h\Big\}.$$

Let $\{\mathbf{x}_i\}_{i=1}^N$ be the set of all the nodal points of the triangulation $\mathcal{T}^h$, then the constrained set K in (2.2) can be approximated by

$$K_h = \Big\{v_h \in S_h;\ \mu_0 \le v_h(\mathbf{x}_i) \le \mu_1 \ for\ i = 1, 2, \cdots, N\Big\},$$

To fully discretize the system (2.3)-(2.5), we also need the time discretization. To do so, we divide the time interval $(0, T)$ into M equally-spaced subintervals by using the nodal points

$$\triangle_h:\ 0 = t^0 < t^1 < \cdots < t^M = T$$

with $t^m = m\tau, \tau = M/T$. We will denote the m-th subinterval by $I^m = (t^{m-1}, t^m]$. For a given sequence $\{\mathbf{E}^m\}_{m=0}^M \subset L^2(\Omega)^3$, we introduce its first and second order backward finite differences:

$$\partial_\tau \mathbf{E}^m = \frac{\mathbf{E}^m - \mathbf{E}^{m-1}}{\tau}, \quad \partial_\tau^2 \mathbf{E}^m = \frac{\partial_\tau \mathbf{E}^m - \partial_\tau \mathbf{E}^{m-1}}{\tau}.$$

The differences above may also be applied to a continuous function $\mathbf{v}(\mathbf{x}, t)$, in which case we will write $\mathbf{v}^m = \mathbf{v}(\cdot, t^m)$ for $0 \le m \le M$. For a vector-valued function $\mathbf{v}$ with some appropriate smoothness, we introduce its edge element interpolation $\Pi_h \mathbf{v}$ such that $\Pi_h \mathbf{v} \in \mathbf{V}_h$, and $\Pi_h \mathbf{v}$ and $\mathbf{v}$ have the same moments in $M_h(\mathbf{v})$. Similarly, for a scalar function v that is continuous, we can introduce its nodal element interpolation $\pi_h v$ such that $\pi_h v \in S_h$, and $\pi_h v$ and v have the same values at all the nodal points.

With the above preparations, we are now ready to formulate the finite element approximation of the nonlinear optimization system (2.3)-(2.5):

$$\min_{\mu_h \in K_h} G_{h,\tau}(\mu_h) = \tau \sum_{m=0}^{M} \alpha_m \int_\omega |\mathbf{E}_h^m - \mathbf{E}^\delta|^2 d\mathbf{x} + \frac{\beta}{2} \|\nabla \mu_h\|_{L^2(\Omega)}^2 \tag{3.1}$$

where $\mathbf{E}_h^m = \mathbf{E}_h^m(\mu_h) \in \mathbf{V}_h^0$ satisfies

$$\mathbf{E}_h^0 = \Pi_h \mathbf{E}_0, \quad \mathbf{E}_h^0 - \mathbf{E}_h^{-1} = \tau\, \Pi_h \mathbf{E}_1 \tag{3.2}$$

and

$$\int_\Omega \varepsilon(\mathbf{x}) \partial_\tau^2 \mathbf{E}_h^m \cdot \mathbf{v}_h d\mathbf{x} + \int_\Omega \mu_h(\mathbf{x}) \mathbf{curl} \mathbf{E}_h^m \cdot \mathbf{curl} \mathbf{v}_h d\mathbf{x} = \int_\Omega \partial_\tau \mathbf{J}^m \cdot \mathbf{v}_h d\mathbf{x} \tag{3.3}$$

for all $\mathbf{v}_h \in \mathbf{V}_h^0$, and $m = 1, 2, \cdots, M$. Here $\Big\{\alpha_m\Big\}$ are the coefficients of the composite trapezoidal rule for the time integration over $[0, T]$, i.e. $\alpha_0 = \alpha_M = \frac{1}{2}$ and $\alpha_m = 1$ for all $m \ne 0, M$.

The next theorem shows the existence of the solutions to the discrete system (3.1)-(3.3).

Theorem 3.1. *For each fixed $\tau > 0$ and $h > 0$, there exists at least a minimizer to the discrete optimization problem (3.1)-(3.3).*

Proof. We emphasize that the parameters $\tau > 0$ and $h > 0$ and the integer M can be viewed to be all fixed constants in this proof. Clearly, as $G_{h,\tau}(\mu_h) \geq 0$ we know $\inf G_{h,\tau}(\mu_h)$ is finite. Thus there exists a minimizing sequence $\left\{\mu_h^k\right\}$ in K_h such that

$$\lim_{k\to\infty} G_{h,\tau}(\mu_h^k) = \inf_{\mu_h \in K_h} G_{h,\tau}(\mu_h).$$

Then it is easy to see the boundedness of $\left\{\mu_h^k\right\}$ in $H^1(\Omega)$, hence the existence of a subsequence, still denoted as $\left\{\mu_h^k\right\}$, and some $\mu_h^* \in K_h$ such that $\mu_h^k \rightharpoonup \mu_h^*$ in $H^1(\Omega)$ *as* $k \to \infty$.

We now prove that μ_h^* is a minimizer of (3.1)-(3.3). We first show that for $m = 0, 1, \cdots, M$,

$$\mathbf{E}_h^m(\mu_h^k) \to \mathbf{E}_h^m(\mu_h^*) \quad \text{in} \quad L^2(\Omega)^3 \quad as \quad k \to \infty\,.$$

To see this, it follows from the definition of $\mathbf{E}_h^m$ that for any $\mathbf{v}_h \in \mathbf{V}_h^0$,

$$\int_\Omega \varepsilon(\mathbf{x})\partial_\tau^2 \mathbf{E}_h^m(\mu_h^k)\cdot\mathbf{v}_h d\mathbf{x} + \int_\Omega \mu_h^k(\mathbf{x})\mathbf{curl}\mathbf{E}_h^m(\mu_h^k)\cdot\mathbf{curl}\mathbf{v}_h d\mathbf{x} = \int_\Omega \partial_\tau \mathbf{J}^m\cdot\mathbf{v}_h d\mathbf{x}\,. \tag{3.4}$$

$$\int_\Omega \varepsilon(\mathbf{x})\partial_\tau^2 \mathbf{E}_h^m(\mu_h^*)\cdot\mathbf{v}_h d\mathbf{x} + \int_\Omega \mu_h^*(\mathbf{x})\mathbf{curl}\mathbf{E}_h^m(\mu_h^*)\cdot\mathbf{curl}\mathbf{v}_h d\mathbf{x} = \int_\Omega \partial_\tau \mathbf{J}^m\cdot\mathbf{v}_h d\mathbf{x}\,. \tag{3.5}$$

Taking $\mathbf{v}_h = \tau\partial_\tau \mathbf{E}_h^m(\mu_h^k)$ in (3.4) and using the fact that $(a-b)a \geq \frac{1}{2}(a^2-b^2), \forall a, b \in$ R, one gets

$$\begin{aligned}
& \int_\Omega \varepsilon(\mathbf{x})(|\partial_\tau \mathbf{E}_h^m(\mu_h^k)|^2 - |\partial_\tau \mathbf{E}_h^{m-1}(\mu_h^k)|^2)d\mathbf{x} \\
& + \int_\Omega \mu_h^k(\mathbf{x})(|\mathbf{curl}\mathbf{E}_h^m(\mu_h^k)|^2 - |\mathbf{curl}\mathbf{E}_h^{m-1}(\mu_h^k)|^2)d\mathbf{x} \\
\leq \quad & \tau \int_\Omega |\partial_\tau \mathbf{J}^m|^2 d\mathbf{x} + \tau \int_\Omega |\partial_\tau \mathbf{E}_h^m(\mu_h^k)|^2 d\mathbf{x}\,.
\end{aligned}$$

Summing up the inequality with respect to m, we derive for $m = 1, 2, \cdots, M$ that

$$\begin{aligned}
& \varepsilon_0\|\partial_\tau \mathbf{E}_h^m(\mu_h^k)\|^2_{L^2(\Omega)^3} + \mu_0\|\mathbf{curl}\mathbf{E}_h^m(\mu_h^k)\|^2_{L^2(\Omega)^3} \\
\leq \quad & \varepsilon_1\|\Pi_h \mathbf{E}_1\|^2_{L^2(\Omega)^3} + \mu_1\|\mathbf{curl}\Pi_h \mathbf{E}_0\|^2_{L^2(\Omega)^3} + \tau\sum_{n=1}^m \|\partial_\tau \mathbf{J}^n\|^2_{L^2(\Omega)^3} \\
& + \tau\sum_{n=1}^m \|\partial_\tau \mathbf{E}_h^n(\mu_h^k)\|^2_{L^2(\Omega)^3}.
\end{aligned}$$

Then by the discrete Gronwall's inequality, we deduce that

$$\max_{1\leq m\leq M} \|\partial_\tau \mathbf{E}_h^m(\mu_h^k)\|^2_{L^2(\Omega)^3} \leq C\,, \quad \max_{1\leq m\leq M} \|\mathbf{curl}\mathbf{E}_h^m(\mu_h^k)\|^2_{L^2(\Omega)^3} \leq C \tag{3.6}$$

for some constant C independent of k, h and τ.

Similarly, taking $\mathbf{v}_h = \tau\partial_\tau \mathbf{E}_h^m(\mu_h^*)$ in (3.5), we can deduce that

$$\max_{1\leq m\leq M} \|\partial_\tau \mathbf{E}_h^m(\mu_h^*)\|^2_{L^2(\Omega)^3} \leq C\,, \quad \max_{1\leq m\leq M} \|\mathbf{curl}\mathbf{E}_h^m(\mu_h^*)\|^2_{L^2(\Omega)^3} \leq C\,. \tag{3.7}$$

Now we set $\omega_h^m(k) = \mathbf{E}_h^m(\mu_h^k) - \mathbf{E}_h^m(\mu_h^*)$. Clearly $\omega_h^0(k) = 0$. By subtracting (3.5) from (3.4), we derive

$$\begin{aligned} &\int_\Omega \varepsilon(\mathbf{x})\partial_\tau^2\omega_h^m(k)\cdot\mathbf{v}_h d\mathbf{x} + \int_\Omega \mu_h^k(\mathbf{x})\mathbf{curl}\omega_h^m(k)\cdot\mathbf{curl}\mathbf{v}_h d\mathbf{x} \\ = &\int_\Omega (\mu_h^*(\mathbf{x}) - \mu_h^k(\mathbf{x}))\mathbf{curl}\mathbf{E}_h^m(\mu_h^*)\cdot\mathbf{curl}\mathbf{v}_h d\mathbf{x}\,, \forall\,\mathbf{v}_h\in\mathbf{V}_h^0\,. \end{aligned} \tag{3.8}$$

Taking $\mathbf{v}_h = \tau\partial_\tau\omega_h^m(k)$ in (3.8), we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega \varepsilon(\mathbf{x})[|\partial_\tau\omega_h^m(k)|^2 - |\partial_\tau\omega_h^{m-1}(k)|^2]d\mathbf{x} \\ &+\frac{1}{2}\int_\Omega \mu_h^k(\mathbf{x})[|\mathbf{curl}\omega_h^m(k)|^2 - |\mathbf{curl}\omega_h^{m-1}(k)|^2]d\mathbf{x} \\ \le &\int_\Omega (\mu_h^*(\mathbf{x}) - \mu_h^k(\mathbf{x}))\mathbf{curl}\mathbf{E}_h^m(\mu_h^*)\cdot\mathbf{curl}(\omega_h^m(k) - \omega_h^{m-1}(k))d\mathbf{x}. \end{aligned}$$

Summing up both sides over m, and noting that $\omega_h^0(k) = 0$, we derive

$$\begin{aligned} &\frac{1}{2}\varepsilon_0\|\partial_\tau\omega_h^m(k)\|^2_{L^2(\Omega)^3} + \frac{1}{2}\mu_0\|\mathbf{curl}\omega_h^m(k)\|^2_{L^2(\Omega)^3} \\ \le &\int_\Omega |\mu_h^*(\mathbf{x}) - \mu_h^k(\mathbf{x})|\sum_{n=1}^m |\mathbf{curl}\mathbf{E}_h^n(\mu_h^*)|\Big\{|\mathbf{curl}\mathbf{E}_h^n(\mu_h^*) - \mathbf{curl}\mathbf{E}_h^{n-1}(\mu_h^*)| \\ &+|\mathbf{curl}\mathbf{E}_h^n(\mu_h^k) - \mathbf{curl}\mathbf{E}_h^{n-1}(\mu_h^k)|\Big\}d\mathbf{x} \\ \le &\int_\Omega |\mu_h^*(\mathbf{x}) - \mu_h^k(\mathbf{x})|\sum_{n=1}^m [C_n(C_n + C_{n-1}) + C_n(|\mathbf{curl}\mathbf{E}_h^n| + |\mathbf{curl}\mathbf{E}_h^{n-1}|)]d\mathbf{x}, \end{aligned}$$

with $C_n = \max_{A\in\mathcal{T}^h}|\mathbf{curl}\mathbf{E}_h^n(\mu_h^*)|$. Letting $\hat{C} = \max\Big\{C_0, C_1, \cdots, C_M\Big\}$, then we deduce

$$\begin{aligned} &\frac{1}{2}\varepsilon_0\|\partial_\tau\omega_h^m(k)\|^2_{L^2(\Omega)^3} + \frac{1}{2}\mu_0\|\mathbf{curl}\omega_h^m(k)\|^2_{L^2(\Omega)^3} \\ \le\ & 2m\hat{C}^2\int_\Omega |\mu_h^*(\mathbf{x}) - \mu_h^k(\mathbf{x})|d\mathbf{x} + 2\hat{C}\sum_{n=0}^m\int_\Omega |\mu_h^*(\mathbf{x}) - \mu_h^k(\mathbf{x})||\mathbf{curl}\mathbf{E}_h^n(\mu_h^k)|d\mathbf{x} \\ \le\ & 2m\hat{C}^2|\Omega|^{\frac{1}{2}}\|\mu_h^*(\mathbf{x}) - \mu_h^k(\mathbf{x})\|_{L^2(\Omega)} \\ &+2\hat{C}\sum_{n=0}^m \|\mu_h^*(\mathbf{x}) - \mu_h^k(\mathbf{x})\|_{L^2(\Omega)}\|\mathbf{curl}\mathbf{E}_h^n(\mu_h^k)\|_{L^2(\Omega)^3}\,. \end{aligned}$$

Noting that $\mu_h^k \to \mu_h^*$ in $L^2(\Omega)$ as $k\to\infty$ by means of the convergence of μ_h^k to μ_h^* in $H^1(\Omega)$, and the boundedness of $\|\mathbf{curl}\mathbf{E}_h^n(\mu_h^k)\|_{L^2(\Omega)^3}$ for all $n = 1, 2, \cdots, M$ implied by (3.6), we have

$$\varepsilon_0\|\partial_\tau\omega_h^m(k)\|^2_{L^2(\Omega)^3} + \mu_0\|\mathbf{curl}\omega_h^m(k)\|^2_{L^2(\Omega)^3} \to 0 \quad as \quad k\to\infty. \tag{3.9}$$

Using (3.9) and the fact that $\omega_h^0(k) = 0$ we deduce

$$\|\tau^{-1}\omega_h^m(k)\|_{L^2(\Omega)^3} = \|\sum_{n=1}^m \partial_\tau\omega_h^n(k)\|_{L^2(\Omega)^3} \le \sum_{n=1}^m \|\partial_\tau\omega_h^n(k)\|_{L^2(\Omega)^3} \to 0$$

as $k\to\infty$, or equivalently we have for $m = 1, 2, \cdots, M$ that

$$\|\mathbf{E}_h^m(\mu_h^k) - \mathbf{E}_h^m(\mu_h^*)\|_{L^2(\Omega)^3} \to 0 \quad as \quad k\to\infty\,. \tag{3.10}$$

Next we show that

$$\lim_{k\to\infty} \tau \sum_{m=0}^{M} \alpha_m \int_\omega |\mathbf{E}_h^m(\mu_h^k) - \mathbf{E}^\delta|^2 d\mathbf{x} = \tau \sum_{m=0}^{M} \alpha_m \int_\omega |\mathbf{E}_h^m(\mu_h^*) - \mathbf{E}^\delta|^2 d\mathbf{x}. \tag{3.11}$$

In fact, the strong convergence in (3.11) follows directly from (3.10) and the following derivations:

$$\begin{aligned}
&\Big|\tau \sum_{m=0}^{M} \alpha_m \int_\omega |\mathbf{E}_h^m(\mu_h^k) - \mathbf{E}^\delta|^2 d\mathbf{x} - \tau \sum_{m=0}^{M} \alpha_m \int_\omega |\mathbf{E}_h^m(\mu_h^*) - \mathbf{E}^\delta|^2 d\mathbf{x}\Big| \\
= \ &\Big|\tau \sum_{m=0}^{M} \alpha_m \int_\omega (\mathbf{E}_h^m(\mu_h^k) - \mathbf{E}_h^m(\mu_h^*)) \cdot (\mathbf{E}_h^m(\mu_h^k) + \mathbf{E}_h^m(\mu_h^*) - 2\mathbf{E}^\delta) d\mathbf{x}\Big| \\
\leq \ &(\tau \sum_{m=0}^{M} \|\mathbf{E}_h^m(\mu_h^k) - \mathbf{E}_h^m(\mu_h^*)\|^2_{L^2(\omega)^3})^{\frac{1}{2}} (\tau \sum_{m=0}^{M} \|\mathbf{E}_h^m(\mu_h^k) + \mathbf{E}_h^m(\mu_h^*) - 2\mathbf{E}^\delta\|^2)^{\frac{1}{2}} \\
\leq \ &(\tau \sum_{m=0}^{M} \|\mathbf{E}_h^m(\mu_h^k) - \mathbf{E}_h^m(\mu_h^*)\|^2_{L^2(\Omega)^3})^{\frac{1}{2}} (\tau \sum_{m=0}^{M} \|\mathbf{E}_h^m(\mu_h^k) + \mathbf{E}_h^m(\mu_h^*) - 2\mathbf{E}^\delta\|^2)^{\frac{1}{2}} \\
\leq \ &C(\tau \sum_{m=0}^{M} \|\mathbf{E}_h^m(\mu_h^k) - \mathbf{E}_h^m(\mu_h^*)\|^2_{L^2(\Omega)^3})^{\frac{1}{2}} \to 0 \quad as \quad k \to \infty\,.
\end{aligned}$$

Finally we prove that μ_h^* is a minimizer to the optimization problem (3.1)-(3.3). Using (3.11) and the lower semi-continuity of a norm, we obtain

$$\begin{aligned}
G_{h,\tau}(\mu_h^*) &= \tau \sum_{m=0}^{M} \alpha_m \int_\omega |\mathbf{E}_h^m(\mu_h^*) - \mathbf{E}^\delta|^2 d\mathbf{x} + \frac{\beta}{2}\|\nabla \mu_h^*\|^2_{L^2(\Omega)} \\
&= \lim_{k\to\infty} \tau \sum_{m=0}^{M} \alpha_m \int_\omega |\mathbf{E}_h^m(\mu_h^k) - \mathbf{E}^\delta|^2 d\mathbf{x} + \frac{\beta}{2}\|\nabla \mu_h^*\|^2_{L^2(\Omega)} \\
&\leq \lim_{k\to\infty} \tau \sum_{m=0}^{M} \alpha_m \int_\omega |\mathbf{E}_h^m(\mu_h^k) - \mathbf{E}^\delta|^2 d\mathbf{x} + \frac{\beta}{2}\liminf_{k\to\infty}\|\nabla \mu_h^k\|^2_{L^2(\Omega)} \\
&\leq \liminf_{k\to\infty} G_{h,\tau}(\mu_h^k) = \inf_{\mu_h\in K_h} G_{h,\tau}(\mu_h).
\end{aligned}$$

So μ_h^* is indeed a minimizer of $G_{h,\tau}(\mu_h)$ over K_h. ♯

4. Convergence of the finite element approximation

In this section we are going to show that the finite element approximation (3.1)-(3.3) converges to the continuous minimization problem (2.3)-(2.5). For the purpose, we first introduce some approximation properties of the edge element interpolation Π_h (cf. [4] [9] [10]):

Lemma 4.1. *The interpolation Π_h has the following approximation properties for $\frac{1}{2} < \alpha \leq 1$:*

$$\|\mathbf{curl}(\mathbf{v} - \Pi_h\mathbf{v})\|_0 \leq Ch\|\mathbf{curl}\mathbf{v}\|_1 \ \forall \mathbf{v} \in H^1(\mathbf{curl};\Omega)\,, \tag{4.12}$$

$$\|\mathbf{v} - \Pi_h\mathbf{v}\|_0 \leq Ch^\alpha\|\mathbf{v}\|_{\alpha,\mathbf{curl}} \ \forall \mathbf{v} \in H^\alpha(\mathbf{curl};\Omega). \tag{4.13}$$

Also, we need to introduce an important projection operator P_h: $H_0(\mathbf{curl};\Omega) \to \mathbf{V}_h^0$, the energy-norm projection, defined by

$$a(P_h\mathbf{u}, \mathbf{v}) = a(\mathbf{u}, \mathbf{v}) \ \forall \mathbf{v} \in \mathbf{V}_h^0\,, \tag{4.14}$$

where the bilinear form

$$a(\mathbf{u},\mathbf{v}) = \int_{\mathbf{\Omega}} \mu(\mathbf{x})(\mathbf{curl}\mathbf{u}\cdot\mathbf{curl}\mathbf{v} + \mathbf{u}\cdot\mathbf{v})\mathbf{dx}\,.$$

It is easy to see by the definition of the projection P_h in (4.14) that

$$\begin{aligned} \|\mathbf{v}-P_h\mathbf{v}\|^2_{0,\mathbf{curl}} &\le \mu_0^{-1}\, a(\mathbf{v}-P_h\mathbf{v},\mathbf{v}-P_h\mathbf{v}) \\ &\le \mu_0^{-1}\, a(\mathbf{v}-\Pi_h\mathbf{v},\mathbf{v}-\Pi_h\mathbf{v}) \\ &\le \mu_1\mu_0^{-1}\, \|\mathbf{v}-\Pi_h\mathbf{v}\|^2_{0,\mathbf{curl}} \end{aligned} \tag{4.15}$$

for all $\mathbf{v}\in H_0(\mathbf{curl};\Omega)\cap H^\alpha(\mathbf{curl};\Omega)$ with $\alpha>\frac{1}{2}$.

In our subsequent anaylsis, we will frequently use the following estimates for $\mathbf{B}=H^1(\mathbf{curl};\Omega)$ or $H^\alpha(\Omega)^3$ with $\alpha\ge 0$,

$$\|\partial_\tau\mathbf{v}^m\|^2_{\mathbf{B}} \le \frac{1}{\tau}\int_{t^{m-1}}^{t^m}\|\mathbf{v}_t(t)\|^2_{\mathbf{B}}dt\ \forall\ \mathbf{v}\in H^1(0,T;\mathbf{B})\,, \tag{4.16}$$

$$\|\partial^2_\tau\mathbf{v}^m\|^2_{\mathbf{B}} \le \frac{1}{\tau}\int_{t^{m-2}}^{t^m}\|\mathbf{v}_{tt}(t)\|^2_{\mathbf{B}}dt\ \forall\ \mathbf{v}\in H^2(0,T;\mathbf{B})\,, \tag{4.17}$$

$$\|\partial_\tau\mathbf{v}^m_t-\partial^2_\tau\mathbf{v}^m\|^2_{\mathbf{B}} \le C\tau\int_{t^{m-2}}^{t^m}\|\mathbf{v}_{ttt}(t)\|^2_{\mathbf{B}}dt\ \forall\ \mathbf{v}\in H^3(0,T;\mathbf{B})\,, \tag{4.18}$$

and the following classical approximation result:

Lemma 4.2. *Let X be a Banach space. For a given function $f\in C([0,T];X)$, we define a step function approximation of f:*

$$S_\triangle f(\mathbf{x},t) = \sum_{m=1}^{M}\chi_m(t)f(\mathbf{x},t^m),$$

where $\chi_m(t)$ is the characteristic function on the interval I^m. Then we have

$$\lim_{\tau\to 0}\int_0^T \|S_\triangle f(\cdot,t)-f(\cdot,t)\|^2_X dt = 0.$$

As usual, it is necessary for the solution $\mathbf{E}$ to the forward Maxwell system (2.5) to have certain regularity in order to establish the convergence of the finite element approximation (3.1)-(3.3) to the continuous minimization problem (2.3)-(2.5). For this we will assume that for any $\gamma\in K$, the solution $\mathbf{E}(\gamma)$ to the system (2.5) has the following regularity

$$\mathbf{E}(\gamma)\in H^2(0,T;H^1(\mathbf{curl};\Omega))\cap H^3(0,T;L^2(\Omega)^3)\,. \tag{4.19}$$

Now we start with the following auxiliary estimates which will be needed for the subsequent convergence of the finite element approximation (3.1)-(3.3).

Lemma 4.3. *Let $\Big\{\mu_h\Big\}$ be a sequence such that $\mu_h\in K_h$ and μ_h converges to some $\mu\in K$ in $L^2(\Omega)$ as h tends to 0. Let $\{\mathbf{E}^m_h(\mu_h)\}$ be the solution of (3.2)-(3.3), and $\mathbf{E}(\mu)$ be the solution of (2.4)-(2.5), satisfying that $\mathbf{E}(\mu)\in H^1(0,T;H_0(\mathbf{curl};\Omega)\cap H^1(\mathbf{curl};\Omega))$. Then the following estimate holds for all sequences $\{\mathbf{v}^m_h\}^M_{m=0}\subset\mathbf{V}^0_h$*

and $1 \le n \le M$,

$$\begin{aligned}
& \tau \sum_{m=1}^{n} \int_\Omega (\mu(\mathbf{x}) - \mu_h(\mathbf{x}))\, \mathbf{curl} P_h \mathbf{E}^m(\mu) \cdot \mathbf{curl} \partial_\tau \mathbf{v}_h^m \, d\mathbf{x} \\
\le \quad & \mu_1 \|\mathbf{E}^1(\mu)\|_{0,\mathbf{curl}} \|\mathbf{curl}\mathbf{v}_h^0\| + C\|\mathbf{E}^n(\mu)\|_{1,\mathbf{curl}} (\|\mu(\mathbf{x}) - \mu_h(\mathbf{x})\|^{\frac{1}{2}} + h)\|\mathbf{curl}\mathbf{v}_h^n\| \\
& + C(h^2 + \|\mu(\mathbf{x}) - \mu_h(\mathbf{x})\|) \int_0^T \|\mathbf{E}_t(t)\|_{1,\mathbf{curl}}^2 dt + \frac{1}{2}\tau \sum_{m=1}^{n-1} \|\mathbf{curl}\mathbf{v}_h^m\|^2 .
\end{aligned} \tag{4.20}$$

Proof. We will write $\mathbf{E}^m(\mu) = \mathbf{E}(\mu)(\cdot, t^m)$ $(1 \le m \le M)$ for the exact solution $\mathbf{E}(\mu)(\mathbf{x}, t)$. First, using the discrete integration by parts formula for any sequences $\Big\{a^m\Big\}$ and $\Big\{b^m\Big\}$,

$$\sum_{m=1}^{n} a^m \, \partial_\tau b^m = -\sum_{m=2}^{n} \partial_\tau a^m \, b^{m-1} + \tau^{-1} a^n b^n - \tau^{-1} a^1 b^0 \,,$$

we obtain that

$$\begin{aligned}
& \tau \sum_{m=1}^{n} \int_\Omega (\mu(\mathbf{x}) - \mu_h(\mathbf{x}))\mathbf{curl} P_h \mathbf{E}^m(\mu) \cdot \mathbf{curl} \partial_\tau \mathbf{v}_h^m d\mathbf{x} \\
= \quad & -\tau \sum_{m=2}^{n} \int_\Omega (\mu(\mathbf{x}) - \mu_h(\mathbf{x}))\mathbf{curl} P_h \partial_\tau \mathbf{E}^m(\mu) \cdot \mathbf{curl}\mathbf{v}_h^{m-1} d\mathbf{x} \\
& + \int_\Omega (\mu(\mathbf{x}) - \mu_h(\mathbf{x}))\mathbf{curl} P_h \mathbf{E}^n(\mu) \cdot \mathbf{curl}\mathbf{v}_h^n d\mathbf{x} \\
& - \int_\Omega (\mu(\mathbf{x}) - \mu_h(\mathbf{x}))\mathbf{curl} P_h \mathbf{E}^1(\mu) \cdot \mathbf{curl}\mathbf{v}_h^0 d\mathbf{x} \\
\equiv \quad & -F_1 + F_2 - F_3 \,.
\end{aligned} \tag{4.21}$$

We next estimate F_1, F_2 and F_3 one by one. F_3 can be estimated directly by

$$\begin{aligned}
|F_3| \quad \le \quad & \|(\mu(\mathbf{x}) - \mu_h(\mathbf{x}))\mathbf{curl} P_h \mathbf{E}^1(\mu)\| \, \|\mathbf{curl}\mathbf{v}_h^0\| \\
\le \quad & 2\mu_1 \|\mathbf{curl} P_h \mathbf{E}^1(\mu)\| \, \|\mathbf{curl}\mathbf{v}_h^0\| \le 2\mu_1 \|\mathbf{E}^1(\mu)\|_{0,\mathbf{curl}} \|\mathbf{curl}\mathbf{v}_h^0\| \,.
\end{aligned}$$

For F_2, we can estimate by using the Hölder inequality, Sobolev embedding theorem, Lemma 4.1 and (4.15) as follows:

$$\begin{aligned}
|F_2| \quad \le \quad & |\int_\Omega (\mu(\mathbf{x}) - \mu_h(\mathbf{x}))\mathbf{curl}\mathbf{E}^n(\mu) \cdot \mathbf{curl}\mathbf{v}_h^n d\mathbf{x}| \\
& + |\int_\Omega (\mu(\mathbf{x}) - \mu_h(\mathbf{x}))(\mathbf{curl} P_h \mathbf{E}^n(\mu) - \mathbf{curl}\mathbf{E}^n(\mu)) \cdot \mathbf{curl}\mathbf{v}_h^n d\mathbf{x}| \\
\le \quad & \|\mu(\mathbf{x}) - \mu_h(\mathbf{x})\|_{L^4(\Omega)} \|\mathbf{curl}\mathbf{E}^n(\mu)\|_{L^4(\Omega)} \|\mathbf{curl}\mathbf{v}_h^n\| \\
& + 2\mu_1 \|P_h \mathbf{E}^n(\mu) - \mathbf{E}^n(\mu)\|_{0,\mathbf{curl}} \|\mathbf{curl}\mathbf{v}_h^n\| \\
\le \quad & (4\mu_1^2)^{\frac{1}{4}} \|\mu(\mathbf{x}) - \mu_h(\mathbf{x})\|_{L^2(\Omega)}^{\frac{1}{2}} \|\mathbf{curl}\mathbf{E}^n(\mu)\|_1 \|\mathbf{curl}\mathbf{v}_h^n\| \\
& + C\, h \, \|\mathbf{E}^n(\mu)\|_{1,\mathbf{curl}} \|\mathbf{curl}\mathbf{v}_h^n\| \\
\le \quad & C(\|\mu(\mathbf{x}) - \mu_h(\mathbf{x})\|_{L^2(\Omega)}^{\frac{1}{2}} + h)\|\mathbf{E}^n(\mu)\|_{1,\mathbf{curl}} \|\mathbf{curl}\mathbf{v}_h^n\| \,.
\end{aligned}$$

Finally, we come to estimate F_1. By the Cauchy-Schwarz inequality we derive

$$
\begin{aligned}
|F_1| &\le \frac{1}{2}\tau\sum_{m=2}^{n}\|(\mu(\mathbf{x})-\mu_h(\mathbf{x}))\mathbf{curl}P_h\partial_\tau\mathbf{E}^m(\mu)\|^2+\frac{1}{2}\tau\sum_{m=2}^{n}\|\mathbf{curl}\mathbf{v}_h^{m-1}\|^2\\
&\le \tau\sum_{m=2}^{n}\|(\mu(\mathbf{x})-\mu_h(\mathbf{x}))(\mathbf{curl}P_h\partial_\tau\mathbf{E}^m(\mu)-\mathbf{curl}\partial_\tau\mathbf{E}^m(\mu))\|^2\\
&\quad+\tau\sum_{m=2}^{n}\|(\mu(\mathbf{x})-\mu_h(\mathbf{x}))\mathbf{curl}\partial_\tau\mathbf{E}^m(\mu)\|^2+\frac{1}{2}\tau\sum_{m=2}^{n}\|\mathbf{curl}\mathbf{v}_h^{m-1}\|^2\\
&\le C\tau\sum_{m=2}^{n}\|P_h\partial_\tau\mathbf{E}^m(\mu)-\partial_\tau\mathbf{E}^m(\mu)\|^2_{0,\mathbf{curl}}+\frac{1}{2}\tau\sum_{m=2}^{n}\|\mathbf{curl}\mathbf{v}_h^{m-1}\|^2\\
&\quad+\tau\sum_{m=2}^{n}\|\mu(\mathbf{x})-\mu_h(\mathbf{x})\|^2_{L^4(\Omega)}\|\mathbf{curl}\partial_\tau\mathbf{E}^m(\mu)\|^2_{L^4(\Omega)}\,,
\end{aligned}
$$

then we further deduce using (4.15) and Lemma 4.1 that

$$
\begin{aligned}
|F_1| &\le C\tau h^2\sum_{m=2}^{n}\|\partial_\tau\mathbf{E}^m(\mu)\|^2_{1,\mathbf{curl}}+\frac{1}{2}\tau\sum_{m=2}^{n}\|\mathbf{curl}\mathbf{v}_h^{m-1}\|^2\\
&\quad+C\tau\sum_{m=2}^{n}\|\mu(\mathbf{x})-\mu_h(\mathbf{x})\|_{L^2}\|\partial_\tau\mathbf{E}^m(\mu)\|^2_{1,\mathbf{curl}}\\
&\le C(h^2+\|\mu(\mathbf{x})-\mu_h(\mathbf{x})\|)\int_0^T\|\mathbf{E}_t(t)\|^2_{1,\mathbf{curl}}dt+\frac{1}{2}\tau\sum_{m=2}^{n}\|\mathbf{curl}\mathbf{v}_h^{m-1}\|^2\,.
\end{aligned}
$$

Now the desired estimate (4.20) follows from (4.21) and the previous estimates for F_1, F_2 and F_3. ♯

The following lemma will be essential to the convergence of the finite element approximation (3.1)-(3.3) to the continuous minimization problem (2.3)-(2.5).

Lemma 4.4. *Let $\{\mu_h\}$ be a sequence such that $\mu_h \in K_h$ and μ_h converges to some $\mu \in K$ in $L^2(\Omega)$ as h tends to 0. Let $\{\mathbf{E}_h^m(\mu_h)\}$ be the solution of (3.2)-(3.3), and $\mathbf{E}(\mu)$ be the solution of (2.4)-(2.5), satisfying the regularity (4.19). Then the following convergence holds*

$$
\lim_{h,\tau\to 0}\tau\sum_{m=0}^{M}\alpha_m\int_\omega|\mathbf{E}_h^m(\mu_h)-\mathbf{E}^\delta|^2d\mathbf{x}=\int_0^T\int_\omega|\mathbf{E}(\mu)-\mathbf{E}^\delta|^2d\mathbf{x}dt. \tag{4.22}
$$

Proof. We first show that

$$
\lim_{\tau\to 0}\tau\sum_{m=0}^{M}\alpha_m\int_\omega|\mathbf{E}^m(\mu)-\mathbf{E}^\delta|^2d\mathbf{x}=\int_0^T\int_\omega|\mathbf{E}(\mu)-\mathbf{E}^\delta|^2d\mathbf{x}dt. \tag{4.23}
$$

In fact we can derive this convergence as follows:

$$
\begin{aligned}
&\Big|\tau\sum_{m=0}^{M}\alpha_m\int_\omega|\mathbf{E}^m(\mu)-\mathbf{E}^\delta|^2d\mathbf{x}-\int_0^T\int_\omega|\mathbf{E}(\mu)-\mathbf{E}^\delta|^2d\mathbf{x}dt\Big|\\
=\;&\Big|\sum_{m=1}^{M}\int_{t^{m-1}}^{t^m}\int_\omega\frac{|\mathbf{E}^{m-1}(\mu)-\mathbf{E}^\delta|^2+|\mathbf{E}^m(\mu)-\mathbf{E}^\delta|^2}{2}d\mathbf{x}dt\\
&-\sum_{m=1}^{M}\int_{t^{m-1}}^{t^m}\int_\omega|\mathbf{E}(\mu)-\mathbf{E}^\delta|^2d\mathbf{x}dt\Big|\\
=\;&\frac{1}{2}\Big|\sum_{m=1}^{M}\int_{t^{m-1}}^{t^m}\int_\omega\Big\{(\mathbf{E}^{m-1}(\mu)-\mathbf{E}(\mu))\cdot((\mathbf{E}^{m-1}(\mu)+\mathbf{E}(\mu)-2\mathbf{E}^\delta)\\
&+(\mathbf{E}^m(\mu)-\mathbf{E}(\mu))\cdot((\mathbf{E}^m(\mu)+\mathbf{E}(\mu)-2\mathbf{E}^\delta)\Big\}d\mathbf{x}dt\Big|\\
\le\;&C(\sum_{m=1}^{M}\int_{t^{m-1}}^{t^m}\int_\Omega|\mathbf{E}^{m-1}(\mu)-\mathbf{E}(\mu)|^2d\mathbf{x}dt)^{\frac{1}{2}}\\
&+C(\sum_{m=1}^{M}\int_{t^{m-1}}^{t^m}\int_\Omega|\mathbf{E}^m(\mu)-\mathbf{E}(\mu)|^2d\mathbf{x}dt)^{\frac{1}{2}}\,,
\end{aligned}
$$

which converges to zero by means of Lemma 4.2.

Using the convergence in (4.23), (4.22) follows immediately if we can derive

$$
\lim_{h,\tau\to0}\tau\sum_{m=0}^{M}\alpha_m\int_\omega|\mathbf{E}_h^m(\mu_h)-\mathbf{E}^\delta|^2d\mathbf{x}=\lim_{\tau\to0}\tau\sum_{m=0}^{M}\alpha_m\int_\omega|\mathbf{E}^m(\mu)-\mathbf{E}^\delta|^2d\mathbf{x}\,, \tag{4.24}
$$

which is to be verified below.

By means of the Cauchy-Schwarz inequality and the a priori estimates of $\mathbf{E}_h^m(\mu_h)$ and $\mathbf{E}^m(\mu)$ (similar to the ones in (3.6) and (3.7)), we can deduce

$$
\begin{aligned}
&|\tau\sum_{m=0}^{M}\alpha_m\int_\omega|\mathbf{E}_h^m(\mu_h)-\mathbf{E}^\delta|^2d\mathbf{x}-\tau\sum_{m=0}^{M}\alpha_m\int_\omega|\mathbf{E}^m(\mu)-\mathbf{E}^\delta|^2d\mathbf{x}|\\
=\;&\tau|\sum_{m=0}^{M}\alpha_m\int_\omega(\mathbf{E}_h^m(\mu_h)-\mathbf{E}^m(\mu))\cdot(\mathbf{E}_h^m(\mu_h)+\mathbf{E}^m(\mu)-2\mathbf{E}^\delta)d\mathbf{x}|\\
\le\;&(\tau\sum_{m=0}^{M}\int_\omega|\mathbf{E}_h^m(\mu_h)-\mathbf{E}^m(\mu)|^2d\mathbf{x})^{\frac{1}{2}}(\tau\sum_{m=0}^{M}\int_\omega|\mathbf{E}_h^m(\mu_h)+\mathbf{E}^m(\mu)-2\mathbf{E}^\delta|^2d\mathbf{x})^{\frac{1}{2}}\\
\le\;&C(\tau\sum_{m=0}^{M}\int_\Omega|\mathbf{E}_h^m(\mu_h)-\mathbf{E}^m(\mu)|^2d\mathbf{x})^{\frac{1}{2}}\,.
\end{aligned}
$$

Now the convergence in (4.24) would follow if we have

$$
\lim_{h,\tau\to0}\{\max_{1\le m\le M}\|\mathbf{E}_h^m(\mu_h)-\mathbf{E}^m(\mu)\|_0\}=0. \tag{4.25}
$$

For the purpose, we first analysis the error function $\eta_h^m=\mathbf{E}_h^m(\mu_h)-P_h\mathbf{E}^m(\mu)$ for $m=1,2,\cdots,M$. Taking $\mathbf{v}=\tau^{-1}\mathbf{v}_h\in\mathbf{V}_h^0$ in (2.5), then integrating with respect to t over the interval I^m, we obtain

(4.26)

$$
\int_\Omega\varepsilon(\mathbf{x})\partial_\tau\mathbf{E}_t^m(\mu)\cdot\mathbf{v}_hd\mathbf{x}+\frac{1}{\tau}\int_\Omega\mu(\mathbf{x})\int_{t^{m-1}}^{t^m}\mathbf{curl}\mathbf{E}dt\cdot\mathbf{curl}\mathbf{v}_hd\mathbf{x}=\int_\Omega\partial_\tau\mathbf{J}^m\cdot\mathbf{v}_hd\mathbf{x}\,.
$$

Now subtracting (4.26) from (3.3) and making some rearrangements, we have

$$
\begin{aligned}
&\int_\Omega \varepsilon(\mathbf{x})\partial_\tau^2\eta_h^m\cdot\mathbf{v}_h d\mathbf{x}+\int_\Omega \mu_h(\mathbf{x})\mathbf{curl}\eta_h^m\cdot\mathbf{curl}\mathbf{v}_h d\mathbf{x}\\
= &\int_\Omega \varepsilon(\mathbf{x})\partial_\tau(\mathbf{E}_t^m(\mu)-\partial_\tau P_h\mathbf{E}^m(\mu))\cdot\mathbf{v}_h d\mathbf{x}\\
&+\frac{1}{\tau}\int_\Omega\mu(\mathbf{x})\int_{t^{m-1}}^{t^m}\mathbf{curl}(\mathbf{E}-P_h\mathbf{E}^m(\mu))dt\cdot\mathbf{curl}\mathbf{v}_h d\mathbf{x}\\
&+\int_\Omega(\mu(\mathbf{x})-\mu_h(\mathbf{x}))\mathbf{curl}P_h\mathbf{E}^m(\mu)\cdot\mathbf{curl}\mathbf{v}_h d\mathbf{x}\,,
\end{aligned}
$$

letting $\mathbf{v}_h=\tau\partial_\tau\eta_h^m=\eta_h^m-\eta_h^{m-1}$, we further deduce

$$
\begin{aligned}
&\frac{1}{2}\int_\Omega\varepsilon(\mathbf{x})(|\partial_\tau\eta_h^m|^2-|\partial_\tau\eta_h^{m-1}|^2)d\mathbf{x}\\
&+\frac{1}{2}\int_\Omega\mu_h(\mathbf{x})(|\mathbf{curl}\eta_h^m|^2-|\mathbf{curl}\eta_h^{m-1}|^2)d\mathbf{x} \qquad (4.27)\\
\le\ &\tau\int_\Omega\varepsilon(\mathbf{x})(\partial_\tau\mathbf{E}_t^m(\mu)-\partial_\tau^2\mathbf{E}^m(\mu))\cdot\partial_\tau\eta_h^m d\mathbf{x}\\
&+\tau\int_\Omega\varepsilon(\mathbf{x})(\partial_\tau^2\mathbf{E}^m(\mu)-\partial_\tau^2P_h\mathbf{E}^m(\mu))\cdot\partial_\tau\eta_h^m d\mathbf{x}\\
&+\int_\Omega\mu(\mathbf{x})\int_{t^{m-1}}^{t^m}\mathbf{curl}(\mathbf{E}-\mathbf{E}^m(\mu))dt\cdot\mathbf{curl}\partial_\tau\eta_h^m d\mathbf{x}\\
&+\tau\int_\Omega\mu(\mathbf{x})\mathbf{curl}(\mathbf{E}^m(\mu)-P_h\mathbf{E}^m(\mu))\cdot\mathbf{curl}\partial_\tau\eta_h^m d\mathbf{x}\\
&+\tau\int_\Omega(\mu(\mathbf{x})-\mu_h(\mathbf{x}))\mathbf{curl}P_h\mathbf{E}^m(\mu)\cdot\mathbf{curl}\partial_\tau\eta_h^m d\mathbf{x}\\
\equiv\ &\sum_{i=1}^5 A_i\,. \qquad (4.28)
\end{aligned}
$$

Next, we will estimate the terms A_i's above one by one. Using the Cauchy-Schwarz inequality and (4.18), A_1 can be estimated by

$$
\begin{aligned}
A_1 &\le \frac{1}{2}\varepsilon_1\tau\|\partial_\tau\eta_h^m\|_0^2+\frac{1}{2}\varepsilon_1\tau\|\partial_\tau\mathbf{E}_t^m(\mu)-\partial_\tau^2\mathbf{E}^m(\mu)\|_0^2\\
&\le \frac{1}{2}\varepsilon_1\tau\|\partial_\tau\eta_h^m\|_0^2+C\varepsilon_1\tau^2\int_{t^{m-2}}^{t^m}\|\mathbf{E}_{ttt}\|_0^2dt\,,
\end{aligned}
$$

while A_2 can be bounded by using the Cauchy-Schwarz inequality and Lemma 4.1 and the estimates (4.15)-(4.17) as follows:

$$
\begin{aligned}
A_2 &\le \frac{1}{2}\varepsilon_1\tau\|\partial_\tau\eta_h^m\|_0^2+\frac{1}{2}\varepsilon_1\tau\|\partial_\tau^2\mathbf{E}^m(\mu)-P_h\partial_\tau^2\mathbf{E}^m(\mu)\|_{0,\mathbf{curl}}^2\\
&\le \frac{1}{2}\varepsilon_1\tau\|\partial_\tau\eta_h^m\|_0^2+C\varepsilon_1\tau h^2\|\partial_\tau^2\mathbf{E}^m(\mu)\|_{1,\mathbf{curl}}^2\\
&\le \frac{1}{2}\varepsilon_1\tau\|\partial_\tau\eta_h^m\|_0^2+C\varepsilon_1 h^2\int_{t^{m-2}}^{t^m}\|\mathbf{E}_{tt}(t)\|_{1,\mathbf{curl}}^2dt.
\end{aligned}
$$

For A_3, we can use the integration by parts formula to derive

$$\begin{aligned}A_3 &= \int_{t^{m-1}}^{t^m}\int_\Omega \mathbf{curl}(\mu(\mathbf{x})\mathbf{curl}(\mathbf{E}-\mathbf{E}^m(\mu)))\cdot\partial_\tau\eta_h^m d\mathbf{x}dt\\ &= -\int_{t^{m-1}}^{t^m}\int_t^{t^m}\int_\Omega \mathbf{curl}(\mu(\mathbf{x})\mathbf{curl}\mathbf{E}_t)\cdot\partial_\tau\eta_h^m d\mathbf{x}dsdt\\ &\le \int_{t^{m-1}}^{t^m}\int_t^{t^m}(\frac{1}{2\tau}\|\partial_\tau\eta_h^m\|_0^2+\frac{\tau}{2}\|\mathbf{curl}(\mu(\mathbf{x})\mathbf{curl}\mathbf{E}_t)\|_0^2)dsdt\\ &\le \frac{1}{2}\tau\|\partial_\tau\eta_h^m\|_0^2+\frac{1}{2}\tau^2\int_{t^{m-1}}^{t^m}\|\mathbf{curl}(\mu(x)\mathbf{curl}\mathbf{E}_t)\|_0^2dt.\end{aligned}$$

For the estimate of A_4, we use the definition of P_h in (4.14) and Lemma 4.1 to derive that

$$\begin{aligned}A_4 &= \tau\int_\Omega \mu(\mathbf{x})\mathbf{curl}(\mathbf{E}^m(\mu)-P_h\mathbf{E}^m(\mu))\cdot\mathbf{curl}\partial_\tau\eta_h^m d\mathbf{x}\\ &= -\tau\int_\Omega \mu(\mathbf{x})(\mathbf{E}^m(\mu)-P_h\mathbf{E}^m(\mu))\cdot\partial_\tau\eta_h^m d\mathbf{x}\\ &\le \frac{1}{2}\tau\mu_1^2\|\partial_\tau\eta_h^m\|_0^2+\frac{\tau}{2}\|\mathbf{E}^m(\mu)-P_h\mathbf{E}^m(\mu)\|_{0,\mathbf{curl}}^2\\ &\le \frac{1}{2}\tau\mu_1^2\|\partial_\tau\eta_h^m\|_0^2+C\tau h^2\|\mathbf{E}^m(\mu)\|_{1,\mathbf{curl}}^2\,.\end{aligned}$$

Summing up both sides of (4.28) from 1 to n ($n\le M$) with respect to m, and using the previous estimates of A_i's, we obtain

$$\begin{aligned}&\frac{1}{2}\varepsilon_0\|\partial_\tau\eta_h^n\|_0^2+\frac{1}{2}\mu_0\|\mathbf{curl}\eta_h^n\|_0^2\\ \le\ &\frac{1}{2}\varepsilon_1\|\partial_\tau\eta_h^0\|_0^2+\frac{1}{2}\mu_1\|\mathbf{curl}\eta_h^0\|_0^2+C\tau\sum_{m=1}^n\|\partial_\tau\eta_h^m\|_0^2+C(\tau^2+h^2)+\sum_{m=1}^n A_5\,.\end{aligned}\tag{4.29}$$

By the definition of $\mathbf{E}_h^0$ and P_h and their approximation properties, we have

$$\|\eta_h^0\|_{0,\mathbf{curl}}=\|P_h(\Pi_h\mathbf{E}_0-\mathbf{E}_0)\|_{0,\mathbf{curl}}\le C\|\Pi_h\mathbf{E}_0-\mathbf{E}_0\|_{0,\mathbf{curl}}\le Ch\|\mathbf{E}_0\|_{1,\mathbf{curl}}.\tag{4.30}$$

And by the definition of η_h^0 , $\mathbf{E}_h^0$ and $\mathbf{E}_h^{-1}$, we derive

$$\begin{aligned}\partial_\tau\eta_h^0 &= \partial_\tau(\mathbf{E}_h^0-P_h\mathbf{E}^0)=\frac{1}{\tau}(\mathbf{E}_h^0-\mathbf{E}_h^{-1}-P_h\mathbf{E}^0+P_h\mathbf{E}^{-1})\\ &= P_h(\Pi_h\mathbf{E}_1-\mathbf{E}_1)+\frac{1}{\tau}P_h(\mathbf{E}(-\tau)-\mathbf{E}(0)+\tau\mathbf{E}_t(0))\,,\end{aligned}$$

which leads to

$$\begin{aligned}\|\partial_\tau\eta_h^0\|^2 &\le C\Big(\|\Pi_h\mathbf{E}_1-\mathbf{E}_1\|_{0,\mathbf{curl}}^2+\frac{1}{\tau^2}\|\mathbf{E}(-\tau)-\mathbf{E}(0)+\tau\mathbf{E}_t(0)\|_{0,\mathbf{curl}}^2\Big)\\ &\le Ch^2\|\mathbf{E}_1\|_{1,\mathbf{curl}}^2+C\tau\int_{-\tau}^0\|\mathbf{E}_{tt}\|_{0,\mathbf{curl}}^2dt\,.\qquad(4.31)\end{aligned}$$

With the above estimates, it follows from (4.29) that

$$\frac{1}{2}\varepsilon_0\|\partial_\tau\eta_h^n\|_0^2+\frac{1}{2}\mu_0\|\mathbf{curl}\eta_h^n\|_0^2\le C(\tau+h^2)+C\tau\sum_{m=1}^n\|\partial_\tau\eta_h^m\|_0^2+\sum_{m=1}^n A_5\,.$$

Then by making use of the result of Lamma 4.3 with $\mathbf{v}_h^m = \eta_h^m$ in (4.20), we obtain that

$$\begin{aligned}
&\frac{1}{2}\varepsilon_0\|\partial_\tau \eta_h^n\|_0^2 + \frac{1}{2}\mu_0\|\mathbf{curl}\eta_h^n\|_0^2 \\
\leq\ & \mu_1\|\mathbf{E}^1(\mu)\|_{0,\mathbf{curl}}\|\mathbf{curl}\eta_h^0\| + C\|\mathbf{E}^n(\mu)\|_{1,\mathbf{curl}}(\|\mu(\mathbf{x}) - \mu_h(\mathbf{x})\|_0^{\frac{1}{2}} + h)\|\mathbf{curl}\eta_h^n\| \\
&+C(h^2 + \|\mu(\mathbf{x}) - \mu_h(\mathbf{x})\|_0)\int_0^T \|\mathbf{E}_t(t)\|_{1,\mathbf{curl}}^2 dt + C(\tau + h^2) \\
&+C\tau\sum_{m=1}^n (\|\mathbf{curl}\eta_h^m\|_0^2 + \|\partial_\tau \eta_h^m\|_0^2)\,.
\end{aligned}$$

Now, using (4.30) and applying the discrete Gronwall's inequality to the above estimate we conclude that

$$\max_{1\leq n\leq M}\|\partial_\tau \eta_h^n\|_0^2 \to 0 \quad as \quad h,\tau\to 0. \tag{4.32}$$

Using this and the following estimate that

$$\begin{aligned}
\|\eta_h^n\| \ &\leq\ \|\eta_h^0\| + \|\eta_h^n - \eta_h^0\| = \|\eta_h^0\| + \tau\|\sum_{m=1}^n \partial_\tau \eta_h^m\| \leq \|\eta_h^0\| + \tau\sum_{m=1}^n \|\partial_\tau \eta_h^m\| \\
&\leq\ \|\eta_h^0\| + T\max_{1\leq m\leq n}\|\partial_\tau \eta_h^m\|\,,
\end{aligned}$$

we derive from (4.32) and (4.30) that

$$\max_{1\leq n\leq M}\|\eta_h^n\|_0^2 \to 0 \quad as \quad h,\tau\to 0. \tag{4.33}$$

Finally noting that

$$\|P_h\mathbf{E}^m(\mu) - \mathbf{E}^m(\mu)\|_0 \leq \|\Pi_h\mathbf{E}^m(\mu) - \mathbf{E}^m(\mu)\|_{0,\mathbf{curl}} \leq Ch\|\mathbf{E}^m(\mu)\|_{1,\mathbf{curl}}\,,$$

we then have by the triangle inequality that

$$\max_{1\leq m\leq M}\|\mathbf{E}_h^m(\mu_h) - \mathbf{E}^m(\mu)\| \leq \max_{1\leq m\leq M}\|\eta_h^m\| + \max_{1\leq m\leq M}\|P_h\mathbf{E}^m(\mu) - \mathbf{E}^m(\mu)\|\,,$$

which leads to the desired convergence in (4.25), hence completes the proof Lemma 4.4. $\sharp$

Finally we are ready to demonstrate the convergence of the finite element approximation (3.1)-(3.3) to the the continuous minimization problem (2.3)-(2.5).

Theorem 4.1. *Let $\left\{\mu_h^*\right\}_{h>0}$ be the minimizers to the discrete minimization problem (3.1)-(3.3), then there exists a subsequence of $\left\{\mu_h^*\right\}_{h>0}$ which converges strongly in $L^2(\Omega)$ to a minimizer of the continuous problem (2.3)-(2.5) as h and τ tend to 0. If the minimizers of the system (2.3)-(2.5) are unique, then the whole sequence of $\left\{\mu_h^*\right\}_{h>0}$ converges to the unique minimizer.*

Proof. As we did earlier, it is easy to see the boundedness of $\left\{\mu_h^*\right\}$ in $H^1(\Omega)$. Thus there exists a subsequence, still denoted as $\left\{\mu_h^*\right\}$, and some μ^* in K such that $\mu_h^* \rightharpoonup \mu^*$ in $H^1(\Omega)$, and $\mu_h^* \to \mu^*$ in $\mathrm{L}^2(\Omega)$. Next we prove that μ^* is a minimizer of the continuous problem (2.3)-(2.5). For any $\mu\in K$, letting $\mu_h = \pi_h\mu \in K_h$, then we have

$$G_{h,\tau}(\mu_h^*) \leq G_{h,\tau}(\pi_h\mu) \quad \text{and} \quad \lim_{h\to 0}\|\pi_h\mu - \mu\|_{H^1(\Omega)} = 0. \tag{4.34}$$

Now using (4.34), Lemma 4.4 and the lower semi-continuity of a norm, we deduce

$$\begin{aligned}
G(\mu^*) &= \int_0^T \int_\omega |\mathbf{E}(\mu^*) - \mathbf{E}^\delta|^2 d\mathbf{x} + \frac{\beta}{2}\|\nabla\mu^*\|^2_{L^2(\Omega)} \\
&= \lim_{h,\tau\to 0} \tau \sum_{m=0}^{M} \alpha_m \int_\omega |\mathbf{E}_h^m(\mu_h^*) - \mathbf{E}^\delta|^2 d\mathbf{x} + \frac{\beta}{2}\|\nabla\mu^*\|^2_{L^2(\Omega)} \\
&\le \lim_{h,\tau\to 0} \tau \sum_{m=0}^{M} \alpha_m \int_\omega |\mathbf{E}_h^m(\mu_h^*) - \mathbf{E}^\delta|^2 d\mathbf{x} + \frac{\beta}{2}\liminf_{h,\tau\to 0}\|\nabla\mu_h^*\|^2_{L^2(\Omega)} \\
&\le \liminf_{h,\tau\to 0} G_{h,\tau}(\mu_h^*) \le \liminf_{h,\tau\to 0} G_{h,\tau}(\pi_h\mu) \\
&\le \lim_{h,\tau\to 0} (\tau \sum_{m=0}^{M} \alpha_m \int_\omega |\mathbf{E}_h^m(\pi_h\mu) - \mathbf{E}^\delta|^2 d\mathbf{x} + \frac{\beta}{2}\|\nabla\pi_h\mu\|^2_{L^2(\Omega)}) \\
&= \int_0^T \int_\omega |\mathbf{E}(\mu) - \mathbf{E}^\delta|^2 d\mathbf{x} + \frac{\beta}{2}\|\nabla\mu\|^2_{L^2(\Omega)} \\
&= G(\mu)\,,
\end{aligned}$$

so μ^* is indeed a minimizer of the continuous problem (2.3)–(2.5). The second part of the theorem follows directly from the unique assumption on the minimizers to the system (2.3)-(2.5) and the previously proved result in the first part of the theorem. This completes the proof of Theorem 4.1. $\sharp$

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