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FINITE DIFFERENCE APPROXIMATION OF A PARABOLIC HEMIVARIATIONAL INEQUALITIES ARISING FROM TEMPERATURE CONTROL PROBLEM

GUANGHUI WANG AND XIAOZHONG YANG

Abstract. In this paper we study the finite difference approximation of a *hemivariational* inequality of parabolic type arising from temperature control problem. Stability and convergence of the proposed method are analyzed. Numerical results are also presented to show the effectiveness and usefulness of the discretization scheme.

Key words. temperature control problem, *hemivariational* inequality, existence, stability, convergence.

1. Introduction

The theory of inequalities has received remarkable development in both pure and applied mathematics as well as in mechanics, engineering sciences and economics. This theory has been a key feature in the understanding and solution of many practical problem such as market price equilibria, heat control, elastic contact and so on (cf.[6],[10],[15]). The constitutional law of these problems is usually given by a non-monotone, possibly multi-valued mapping. Such problems is described by the so-called *hemivariational* inequality, which can been viewed to be the week formulation of a certain differential inclusion. The concept of a *hemivariational* inequality is introduced by Panagiotopoulos in [12]. In the static case the *hemivariational* inequality is often equivalent to the problem of finding all sub-stationary points of a super-potential Φ which is non-convex and non-smooth, in general, provided our problem is of potential type. There is a number of results on the existence and the approximation of elliptic *hemivariational* inequalities (cf.[1],[7],[12]), however, much fewer results on the existence and the approximation of the solution of the dynamic *hemivariational* inequalities.

In this paper we shall consider a discontinuous non-linear non-monotone parabolic initial boundary value problem, i.e., a parabolic *hemivariational* inequality.

$$\begin{cases} u'(t) - Au(t) + \Xi(t) = g(t) \\ u(0) = u_0 \ \text{ and } u(t) = 0 \text{ on } \partial\Omega \ for\ a.a\ t \in (0,T) \\ \Xi(x,t) \in \partial j(u(x,t)) \quad a.e.(x,t) \in \Omega_T. \end{cases} \tag{1.1}$$

The non-linearity and the discontinuity only lie in the lower order term $\partial j(u(x,t))$, and the operator A is linear and continuous. This kind of stationary problems have been studied, for example, in (cf. [1],[7],[12]), and dynamic problems in (see [3], [5],[8],[10]-[15]). As an important application of (1.1), We shall discuss the finite difference approximation [6] and numerical modeling of temperature control problem. To the best of our knowledge, there are relatively few papers in

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which numerical methods and modeling were studied for parabolic *hemivariational* inequalities(cf.[1],[8],[11],[14]) . These papers mainly discussed the finite element numerical methods and the proof of the existence, stability and convergence of the solution of these methods, but there are nearly not papers to deal with the numerical implementation of these methods. The main difficulties in numerical modeling include: (1) the solution of the parabolic *hemivariational* inequalities is not unique, i.e., there exist more than one solution under certain conditions; (2) for numerical implementation of parabolic *hemivariational* inequalities using the finite element method, we must first transform parabolic *hemivariational* inequalities into a sub-stationary point type problem, and then solve a non-smooth and non-convex optimization problem. These need much computation. To bypass these difficulties, we adopt the finite difference method based on Galerkin variational principle to approximate the parabolic *hemivariational* inequalities. We have not found similar work in published papers. Our method is an exterior approximate method and its finite dimensional space is generated by characteristic functions. In this paper, we first analyzed the existence of solution, stability and convergence of the finite difference scheme based on the finite dimensional space, and then discussed the numerical implementation of this method. Finally, as numerical examples, figures of solutions generated by multi-value functions are presented for several cases. In contrast to finite element method, finite difference method is simple and effective for solving numerically the parabolic *hemivariational* inequalities.

The outline of this paper is as follows. In section 2, we formulate the problem, and state the main assumptions of this paper. In section 3, we construct the finite difference scheme to approximate *hemivariational* inequality arising from temperature control problem. The existence of the solution, stability and convergence of the finite difference scheme are proven in Section 4. Numerical results are reported in Section 5. Finally, we give concluding remarks.

2. The description of problem

We consider a heat conduction problem with a non-monotone relation (a temperature control problem without assuming any monotonicity for the control device). Let $\Omega \subset R^2$ be a bounded domain with the Lipschitz boundary $\partial\Omega$, representing a body, in which the temperature distribution is governed by the time dependent heat equation (cf. [2],[7])

$$u'(t) - \Delta u(t) = g(t), \qquad \text{in } \Omega, \text{ for } a.a.\ t \in (0, T)$$

with g decomposed as follows:

$$\begin{cases} g = f - \Xi, \\ f \text{ is given and } \Xi(x,t) \in \partial j(u(x,t)) \text{ for } a.e.\ (x,t) \in Q_T = \Omega \times (0,T). \end{cases}$$

Where

$$j(u) = \begin{cases} g_1(u - s_1), & \text{if } u < s_1, \\ 0, & \text{if } s_1 \le u \le s_2, \\ g_2(u - s_2), & \text{if } u \ge s_2. \end{cases}$$

then

$$\partial j(u) = \begin{cases} g_1, & \text{if } u \le s_1, \\ [g_1, 0], & \text{if } u = s_1, \\ 0, & \text{if } s_1 < u < s_2, \\ [0, g_2], & \text{if } u = s_2, \\ g_2, & \text{if } u \ge s_2, \end{cases}$$

where s_1, s_2 are two reference temperature and $s_1 < s_2$, $g_1, g_2 (g_1 < 0 < g_2)$ constants.

On the boundary $\partial\Omega$ the temperature u satisfies the homogenous Dirichlet boundary condition

$$u(t) = 0, \quad on\ \partial\Omega\ \ for\ a.a.\ t \in (0, T).$$

Moreover, at $t = 0$ the temperature is given by $u(x, 0) = u_0(x)$.

Let $V = H_0^1(\Omega), H = L^2(\Omega)$, then $V \subset H \subset L^2(\Omega) \subset V^*$, V^* is the dual space of V. We denote by $\|\cdot\|_1$, $\|\cdot\|_*$ and $\|\cdot\|_0$ the norms in V, V^* and H, respectively. The duality pairing between V and V^* is denoted by $< \cdot, \cdot >$ and the inner product in $L^2(\Omega)$ by $(\cdot, \cdot)$. Finally, let

$$W(V) \equiv \{v \in L^2(0, T; V) : v' \in L^2(0, T; V^*)\}$$

and

$$\|v\|_{W(V)} = \|v\|_{L^2(0,T;V)} + \|v'\|_{L^2(0,T;V^*)},\ T > 0.$$

Define

$$a(u, v) = (Au, v) = \int_\Omega \nabla u \cdot \nabla v dx,\ here\ \ A = -\Delta, \tag{2.1}$$

$$< f(t), v >= \int_\Omega f(t) v dx, \quad g \in L^2(0, T; L^2(\Omega)). \tag{2.2}$$

and the corresponding weak solution problem reads as follows ([9]):

Problem 2.1. *Find $(u, \Xi) \in W(V) \times L^2(Q_T)$ such that*

$$\begin{cases} \displaystyle\int_0^T < u'(t), v(t) > dt + \int_0^T a(u(t), v(t))dt + \int_0^T (\Xi(t), v(t))dt \\ \displaystyle= \int_0^T < f(t), v(t) > dt, \quad \forall v \in L^2(0, T; V^*), \\ \Xi(x, t) \in \partial j(u(x, t))\ for\ a.a.(x, t) \in Q_T \\ and\ u(0) = u_0, \end{cases}$$

We assume that the initial state u_0 is an element of H and the right hand side f belongs to $L^2(0, T; V^*)$. The function $\partial j(\xi) : R \longrightarrow R$ is defined as a multi-valued non-monotone relation that satisfies the following growth conditions: There exists a positive constant C such that

$$\eta \in \partial j(\xi) \Rightarrow \|\eta\|_0 \le C(1 + \|\xi\|_0), \qquad \forall \xi \in R. \tag{2.3}$$

In addition, the generalized directional derivative $j^0(\xi, \tau)$ at ξ in the direction τ is defined by (see [12])

$$j^0(\xi, \tau) = \limsup_{\bar{\xi} \to \xi, t \to 0^+} \frac{j(\bar{\xi} + t\pi) - j(\bar{\xi})}{t}. \tag{2.4}$$

Obviously, the bilinear form $a(u, v)$ defined in $L^2(0, T; V) \times L^2(0, T; V)$ has following properties:

$$\exists \alpha > 0 : a(w, v) \le \alpha ||w||_1 \cdot ||v||_1, \quad \forall u, v \in V. \tag{2.5}$$

$$\exists \beta > 0 : a(v, v) \ge \beta ||v||_1^2, \quad \forall v \in V. \tag{2.6}$$

According to above discussion we can get the following existence results of the solution to problem2.1 (see [9], [11]).

Theorem 2.1 Let the conditions (2.3), (2.5),(2.6) and the assumptions on u_0, f be satisfied, Then there exists at least one solution of problem2.1.

3. Approximation of parabolic hemivariational inequalities

We now consider the finite difference approximation of the solution of problem 2.1. The approximation of the solution to problem 2.1 by a finite difference method is based on a linear combination of a set of characteristic functions that do not belong to $H^1(\Omega)$. To analyze the finite difference scheme, we need to use the idea of *Exterior Approximation.* This is a difference scheme based on the variational principle.

3.1. The construction of finite dimensional space.

For brevity we only consider the uniform meshes in the following discussion. All results can be equally extended to non-uniform meshes.

Let Ω be triangulated by a uniform mesh with mesh size $h = \{h_1, h_2\} \in R^2$, we define the grid:

$$R_h := \{M | M \in R^2, M = \{m_1 h_1, m_2 h_2\}, m_1, m_2 \in \mathbb{Z}\}.$$

With each node M of R_h, we associate the *panel* with center $M = (x_1, x_2)$.

$$\omega_h^0(M) = \left[(m_1 - \frac{1}{2})h_1, (m_1 + \frac{1}{2})h_1\right] \times \left[(m_2 - \frac{1}{2})h_2, (m_2 + \frac{1}{2})h_2\right],$$

and the *cross* (with center M)

$$\omega_h^1(M) = \omega_h^0(M \pm h_1/2e_1) \cup \omega_h^0(M \pm h_2/2e_2),$$

where $e_i (i = 1, 2)$ denotes the i the unit vector in R^2.

Define

$$\Omega_h = \{M | \omega_h^0(M) \subset \Omega\}.$$

Let θ_h^M be the characteristic functions of $\omega_h^0(M)$, i.e.

$$\theta_h^M = \begin{cases} 1, & x \in \omega_h^0(M), \\ 0, & \text{otherwise}, \end{cases}$$

for $i = 0, 1, \dots, m_1 - 1$, $j = 0, 1, ..., m_2 - 1$.

Using these characteristic functions, we define the finite dimensional space V_h by $V_h = \text{Span}\{\theta_h^M\}$, i.e., V_h is the span of the basis set θ_h^M. Each $v_h \in V_h$ can be expressed as

$$v_h = \sum_{i=1}^{m_1 - 1} \sum_{j=1}^{m_2 - 1} \theta_h^0(M),$$

where $M = \{m_1 h_1, m_2 h_2\}$.

The function θ_h^M are not in $H^1(\Omega)$, so that V_h is not a subspace of $H^1(\Omega)$. Similarly, let H_h be a family of the finite dimensional subspace of H.

For the time domain, let $t_n = nk$ for $n = 0, 1, ..., N$, where N denotes a positive integer and $k = T/N$. For any admissible function $\zeta(t)$, we let $\zeta^n = \zeta(t_n)$ for $n = 0, 1, ..., N$. On this mesh for the time domain, we define

$$\chi^n(t) = \begin{cases} 1, & t \in [t_{n+1}, t_n), \\ 0, & \text{otherwise}, \end{cases}$$

for $n = 0, 1, ..., N - 1$.

We start with the approximations of the space $L^2(0,T;V)$, $L^2(0,T;L^2(\Omega))$, which will be denoted by $L^2(\Delta_k;V_h), L^2(\Delta_k;H_h)$, respectively:

$$L^2(\Delta_k;V_h) = \{v_h \in L^\infty(0,T;V_h) | v_h = \sum_{n=1}^{N} v^n\chi^n, v^n \in V_h\},$$

$$L^2(\Delta_k;Y_h) = \{v_h \in L^\infty(0,T;H_h) | v_h = \sum_{n=1}^{N} v^n\chi^n, v^n \in H_h\},$$

i.e. both consist of functions that are piecewise constant in time on the partition Δ_k and take their values in V_h, H_h, respectively.

3.2. The finite difference scheme.

Using this characteristic function of the intervals $(nk,(n+1)k]$, we define

$$v_{h,k} = \sum_{n=0}^{N-1} v_h^n \chi^n(t) \tag{3.1}$$

for any $v_h^n \in V_h, n = 0,1,...,N-1$.

Define the following 1st and 2nd order finite differences for any piecewise constant function of the form (3.1):

$$\delta_i v_{h,k} = \frac{v_{h,k}(x+h_i) - v_{h,k}(x)}{h_i}, \tag{3.2}$$

$$\delta_i^2 v_{h,k} = \frac{v_{h,k}(x+h_i) - 2v_{h,k}(x) + v_{h,k}(x-h_i)}{h_i^2}, \tag{3.3}$$

$$\bar{\delta}_t v_{h,k} = \frac{v_{h,k}(t) - v_{h,k}(t-k)}{k}. \tag{3.4}$$

Clearly, δ_i and $\bar{\delta}_t$ denote respectively the 1st order forward difference operator in x_i and backward difference operator in t, and δ_i^2 denotes the 2nd order central difference in x_i. We comment that δ_i^2 in (3.3) will not be used in the rest of discussion, but it is closed related to the operator A_h to be defined later in this section.

If $\{u^n\}_{n=0}^N$ is the set of value of a sufficiently smooth function u at the time levels $t_n = nk$:

$$u^n \equiv u(nk), \quad n = 0,1,...,N,$$

then the symbol $u^{n+\theta}, \theta \in [0,1]$, stands for the convex combination of the value at two successive time steps n and $(n+1)$:

$$u^n \equiv (1-\theta)u^n + \theta u^{n+1}, \quad n = 0,1,...,N-1.$$

Similarly to the static case, we shall study the full approximation of problem2.1, including an approximation of the bilinear form $a(\cdot,\cdot)$ and of the linear functional Ξ and f.

Define the following inner and operator

(1) Define the inner product and the norm on V_h by

$$(u_h, v_h)_h = (u_h, v_h) + \sum_{i=1}^{2} (\delta_i u_h, \delta_i v_h),$$

$$\|v_h\|_h = \{\|v_h\|_0^2 + \sum_{i=1}^{2} \|\delta_i v_h\|_0^2\}^{1/2},$$

where δ_i is the operators defined in (3.2).

(2) $p_h : v_h \to \{v_h, \delta_1 v_h, \delta_2 v_h\}$, a linear operator from $V_h \to F$, satisfying the stability condition:

$$\|p_h v_h\|_F \le C\|v_h\|_h,$$

for some constant $C > 0$, independent of h and v_h. $F = L^2(\Omega) \times L^2(\Omega)$.

Because V_h is not the subspace of $H^1(\Omega)$, hence, we cannot define $a(u,v)$ on V_h, and we must modify $a(u,v)$. For this we replace the derivatives by differential quotients. Using (3.2), for $u_h, v_h \in V_h$ we define

$$a_h(u_h, v_h) = \int_\Omega \sum_{i=1}^{2} \delta_i u_h \delta_i v_h dx. \tag{3.5}$$

It is easy to prove that $a_h(u_h, v_h)$ is continuous and coercive in V_h, as given in the following lemma.

Lemma 3.1. *There exist positive constants C_3 and C_4, independent of h and v_h, such that for any $u_h,\ v_h \in V_h$*

$$a_h(u_h, v_h) \le C_3\|u_h\|_h\|v_h\|_h, \tag{3.6}$$

$$a_h(u_h, u_h) \ge C_4\|u_h\|_h^2. \tag{3.7}$$

PROOF. This proof is standard, □

Similar to lemma 3.2 and lemma 3.4 in [4], we state the following two lemmas.

Lemma 3.2. *The space V_h and V defined, respectively, in sections 3 and 2 satisfy that, $\forall v \in L^2(0,T;V)$, $\exists v_{h,k} \in L^2(0,T;V_h)$ such that*

$$v_{h,k} \longrightarrow v \text{ strongly in } L^2(0,T;H),$$

$$p_h v_{h,k} \longrightarrow \left\{v, \tfrac{\partial v}{\partial x_1}, \tfrac{\partial v}{\partial x_2}\right\} \text{ strongly in } L^2(0,T;F).$$

Lemma 3.3. *Let $a_h(v_{h,k}, w_{h,k})$ and $a(v,w)$ be defined by (3.5) and (2.1), respectively. Given $v \in L^2(0,T;V)$ and $w \in L^2(0,T;V)$, a.e. $t \in [0,T]$ such that*

$$p_h v_{h,k} \longrightarrow \left\{v, \tfrac{\partial v}{\partial x_1}, \tfrac{\partial v}{\partial x_2}\right\} \text{ weakly in } L^2(0,T;F),$$

$$p_h w_{h,k} \longrightarrow \left\{w, \tfrac{\partial w}{\partial x_1}, \tfrac{\partial w}{\partial x_2}\right\} \text{ strongly in } L^2(0,T;F),$$

we have

$$\int_0^T a_h(v_{h,k}, w_{h,k})dt \to \int_0^T a(v,w)dt,$$

$$\int_0^T a_h(w_{h,k}, v_{h,k})dt \to \int_0^T a(w,v)dt$$

when $h, k \to 0^+$.

The proofs to the two lemmas are very similar to those for lemma 3.2 and lemma 3.4 in [4], and we refer the reader to [4]. In addition, using the definitions of $\|v_h\|_0$ and $\|v_h\|_h$ we can show the following lemma without difficulty.

Lemma 3.4. *For any $v_h \in V_h$, we have*

$$\|v_h\|_0 \le C_5\|v_h\|_h, \tag{3.8}$$

$$\|v_h\|_h \le C_6 s(h)\|v_h\|_0, \tag{3.9}$$

for some positive constants C_5, C_6 , independent of h and v_h, $s(h) \to +\infty$, as $h \to 0^+$.

Similar to $a(u,v)=(Au,v)$, as in problem2.1, we define $a_h(u_h,v_h)=(A_hu_h,v_h)$, then A_h is second order differential quotients defined in (3.3). Evidently, A_h is an operator $V_h\to V_h$.

Using a pair of function $(u^\theta_{h,k},\Xi^\theta_{h,k})\in L^2(\Delta_k;V_h)\times L^2(\Delta_k;H_h)$ of the form

$$u^\theta_{h,k}=\sum_{n=0}^{N-1}u^{n+\theta}_{h,k}\chi^{n+1},\quad \Xi^\theta_{h,k}=\sum_{n=0}^{N-1}\Xi^{n+\theta}_{h,k}\chi^{n+1}, \tag{3.10}$$

we define the following difference scheme approximating problem 2.1 for all $n=0,1,...,N-1$.

Problem 3.1. *Find $(u^{n+\theta}_{h,k},\Xi^{n+\theta}_{h,k})\in V_h\times H_h$ such that*

$$\begin{cases}\left(\dfrac{u^{n+1}_{h,k}-u^n_{h,k}}{k},v_h\right)+a_h(u^{n+\theta}_{h,k},v_h)+(\Xi^{n+\theta}_{h,k},v_h)\\ =(f^{n+\theta}_{h,k},v_h),\quad\forall v_h\in V_h,\\ \Xi^{n+\theta}_{h,k}\in\partial j(u^{n+1}_{h,k}(x))\ for\ a.a.\ x\in\Omega,\ and\ n=0,...,m-1.\\ and\ u^0_{h,k}=u_{0h}.\end{cases} \tag{3.11}$$

The approximation schemes corresponding to $\theta=1,\frac12,0$ are termed: *implicit, Crank-Nicholson* and *explicit*, respectively.

Theorem 3.1. *Let all the assumptions concerning of $a_h,\{f^n_{h,k}\},u_{0h},p_h$ and ∂j be satisfied. Then problem3.1 has at least one solution $(\Xi^\theta_{h,k},u^\theta_{h,k})\in L^2(\Delta_k;V_h)\times L^2(\Delta_k;H_h)$ for any $\theta\in[0,1],h>0$ and sufficiently small $k>0$.*

PROOF. The idea is to transform problem3.1 to the discrete elliptic problem for each time step and to use the results of ([9]). We rewrite problem3.1 as follows:

Find $u^{i+\theta}_{h,k}\in V_h$ and $\Xi^{i+\theta}\in H_h$ for all $i=0,...,m-1$ such that, $\forall v_h\in V_h$

$$\begin{cases}kA_hu^{i+1\theta}_{h,k}+\dfrac1\theta u^{i+\theta}_{h,k}+k\Xi^{i+\theta}_{h,k}=kf^{i+\theta}_{h,k}+\dfrac1\theta u^i_{h,k},\quad in\ V^*_h\\ \Xi^{i+\theta}_{h,k}(x)\in\partial j(u^{i+\theta}_{h,k}(x)),\quad a.e.\ x\in\Omega,\end{cases} \tag{3.12}$$

where V^* is the dual space of V_h.

Let us assume that problem3.1 has been already solved for $i=0,...,n-2$. Hence, the functions $u^1_{h,k},...,u^{n-1}_{h,k}\in V_h$ and $\Xi^\theta_{h,k},...,\Xi^{n-2+\theta}_{h,k}\in H_h$ are know. Then, we define

$$\begin{cases}\bar Av\equiv kA_hv+\dfrac{1}{2\theta}v,\\ \bar j\equiv kj(\xi)+\dfrac{1}{4\theta}\xi^2,\\ \bar jv\equiv kf^{n-1+\theta}_{h,k}+\dfrac1\theta u^{n-1}_{h,k}.\end{cases}$$

Using these notation, problem (3.12) can be written in the following form:

$$\begin{cases}(\bar Au^{i+\theta}_{h,k},v)+(\Xi^{i+\theta}_{h,k},v)=(\bar f,v),\quad\forall v\in V_h,\\ \Xi^{n+\theta}_{h,k}(x)\in\partial\bar j(u^{n+\theta}_{h,k}(x)),\quad for\ a.a.x\in\Omega.\end{cases}$$

Due to $(\bar Av,v)\ge C\|v\|^2_h$, $(C>0)$, and $\bar j^0(\xi,-\xi)=kj^0(\xi,-\xi)-\frac{1}{2\theta}\xi^2$. According to the Appendix of [9], the solvability of (3.12) can been guaranteed.

If $\theta=0$ the problem3.1 is equivalent to the following problem:

Find $u_{h,k}^{i+1} \in V_h$ for all $i = 1, ..., n-1$ such that

$$\begin{cases} u_{h,k}^{i+1} = -kA_h u_{h,k}^i + u_{h,k}^i - k\Xi_{h,k}^i + kf_{h,k}^i \\ \Xi_{h,k}^i(x) \in \partial\bar{j}(u_{h,k}^i(x)), \quad for\ a.a.x \in \Omega. \end{cases}$$

which is trivially solvable. □

4. Stability and convergence of the method

We shall show that the finite difference method defined in problem 3.1 is stable and the solution to problem 2.1 converges to the solution of problem 3.1.

4.1. Stability.

The stability of the finite difference method is established in the following theorem (cf. [4],[9])

Theorem 4.1. *(i) For any $\theta \in [\frac{1}{2}, 1]$, let $(u_{h,k}^\theta, \Xi_{h,k}^\theta)$ be the solution to problem 3.1, then we have the following conclusions*

$$(4.1)\qquad \begin{cases} \max\limits_{1\le i\le m} \|u_{h,k}^i\|_0, \|u_{h,k}^{i-1+\theta}\|_0 \le C, \\ \sum\limits_{i=0}^{m-1} k\|u_{h,k}^{i+\theta}\|_h^2 \le C, \\ (2\theta-1)\sum\limits_{i=0}^{m-1} k\|u_{h,k}^{i+1} - u_{h,k}^i\|_0^2 \le C \\ \|\Xi_{h,k}\|_{L^2(Q_T)}^2 \le C. \end{cases}$$

(ii) For any $\theta \in [0, \frac{1}{2})$, let all the assumption of the case (i) be satisfied. If, moreover, h, k are such that the following stability condition

$$(4.2)\qquad 1 - 2(1-\theta)ks(h)^2 C_0 \ge c > 0,$$

where c is a positive constant, is satisfied, then the conclusions of the case (i) hold true.

PROOF. (i) For any $\theta \in [\frac{1}{2}, 1]$, substituting $v_h = u_{h,k}^{i+\theta}$ into (3.11) we have that

$$(4.3)\qquad \left(\frac{u_{h,k}^{i+1} - u_{h,k}^i}{k}, u_{h,k}^{i+\theta}\right) + a_h(u_{h,k}^{i+\theta}, u_{h,k}^{i+\theta}) + (\Xi_{h,k}^{i+\theta}, u_{h,k}^{i+\theta}) = (f_{h,k}^{i+\theta}, u_{h,k}^{i+\theta}).$$

Using the following classical relation, we can rewrite the first term of (4.3) as follows

$$\frac{1}{k}(u_{h,k}^{i+1} - u_{h,k}^i, u_{h,k}^{i+\theta}) = \frac{1}{2k}(\|u_{h,k}^{i+1}\|_0^2 - |u_{h,k}^i\|_0^2) + \frac{(2\theta-1)}{2k}\|u_{h,k}^{i+1} - u_{h,k}^i\|_0^2.$$

Due to the following equality

$$(4.4)\qquad (a-b)a = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a-b)^2, \qquad \forall a, b \in \mathbb{R},$$

the second the term of (4.3) can be estimated from below

$$a_h(u_{h,k}^{i+\theta}, u_{h,k}^{i+\theta}) \ge C_1\|u_{h,k}^{i+\theta}\|_h^2.$$

In addition, because of the Growth condition (2.3) and the bounded-ness of p_h and the following young inequality

$$ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2, \qquad \forall a, b \in \mathbb{R},\ \forall \varepsilon > 0 \tag{4.5}$$

we obtain an upper estimate for the third term of (4.3)

$$\begin{aligned} |(\Xi_{h,k}^{i+\theta}, u_{h,k}^{i+\theta})| &\leq \int_\Omega C(1 + \|u_{h,k}^{i+\theta}(x)\|_0)\|u_{h,k}^{i+\theta}(x)\|_0 dx \\ &\leq C + \bar{c}(h)\|u_{h,k}^{i+\theta}\|_h^2 + C\|u_{h,k}^{i+\theta}\|_0^2, \end{aligned} \tag{4.6}$$

where $\bar{c}(h)$ is a positive constant satisfying $\bar{c}(h) \to 0$ as $h \to 0_+$. By young inequality we get the upper bound for the right hand side of (4.3)

$$|(f_{h,k}^{i+\theta}, u_{h,k}^{i+\theta})| \leq \varepsilon \|u_{h,k}^{i+\theta}\|_0^2 + C(\varepsilon)\|f_{h,k}^{i+\theta}\|_*^2,$$

for all $\varepsilon > 0$. Summing up the both sides of (4.3) from $i = 0$ to $i = n-1, n \leq m$, multiplying it by $2k$ and taking into account above estimate and f_h we conclude that

$$\begin{aligned} &\sum_{i=0}^{n-1}\{\|u_{h,k}^{i+1}\|_0^2 - \|u_{h,k}^i\|_0^2\} + \sum_{i=0}^{n-1}(2\theta - 1)\|u_{h,k}^{i+1} - u_{h,k}^i\|_0^2 \\ &+2\sum_{i=0}^{n-1} k(C - (\varepsilon + \bar{c}(h)))\|u_{h,k}^{i+\theta}\|_h^2 \leq C\sum_{i=0}^{n-1} k\|u_{h,k}^i\|_0^2. \end{aligned}$$

Then using the fact that $u_{h,k}^{i+\theta} \equiv (1-\theta)u_{h,k}^i + \theta u_{h,k}^{i+1}$we obtain

$$\begin{aligned} &(1 - Ck)\|u_{h,k}^n\|_0^2 + \sum_{i=0}^{n-1}(2\theta - 1)\|u_{h,k}^{i+1} - u_{h,k}^i\|_0^2 \\ &+2\sum_{i=0}^{n-1} k(C - (\varepsilon + \bar{c}(h)))\|u_{h,k}^{i+\theta}\|_h^2 \leq C\sum_{i=0}^{n-1} k\|u_{h,k}^i\|_0^2. \end{aligned}$$

for all $n = 1, ..., m$. Next, we take h, k and ε small enough that the both coefficients $(1 - Ck)$ and $(C_1 - (\varepsilon + \bar{c}))$ are positive. This fixes the constant $C(\varepsilon)$. Then by the discrete $Gronwall's$ lemma we get

$$\begin{cases} \max\limits_{1\leq i\leq m} \|u_{h,k}^i\|_0, \|u_{h,k}^{i-1+\theta}\|_0 \leq C, \\ \sum\limits_{i=0}^{m-1} k\|u_{h,k}^{i+\theta}\|_h^2 \leq C, \\ (2\theta - 1)\sum\limits_{i=0}^{m-1} k\|u_{h,k}^{i+1} - u_{h,k}^i\|_0^2 \leq C. \end{cases}$$

It remains to prove that $\{\Xi_{h,k}\}$ is bounded in $L^2(Q_T)$. From the growth condition of $\partial j(\xi)$ it follow

$$\|\Xi_{h,k}\|^2_{L^2(Q_T)} = \sum_{i=0}^{m-1} k \int_\Omega \|\Xi^{i+\theta}_{h,k}(x)\|^2_0 dx \tag{4.7}$$

$$\leq \sum_{i=0}^{m-1} k \int_\Omega (C(1+\|u^{i+\theta}_{h,k}(x)\|_0))^2 dx$$

$$\leq C\left(1+\sum_{i=0}^{m-1} k\|u^{i+\theta}_{h,k}\|^2_0\right) \leq C\left(1+\sum_{i=0}^{m-1} k\|u^{i+\theta}_{h,k}\|^2_h\right), \tag{4.8}$$

which implies together with (4.1) the desired result.

(ii) For any $\theta \in [0, \frac{1}{2})$, we use a similar approach as in (i). First, we substitute $v_h = u^{i+1}_{h,k}$ in (3.11) giving

$$\left(\frac{u^{i+1}_{h,k} - u^i_{h,k}}{k}, u^{i+1}_{h,k}\right) + a_h(u^{i+\theta}_{h,k}, u^{i+1}_{h,k}) + (\Xi^{i+\theta}_{h,k}, u^{i+1}_{h,k}) = (f^{i+\theta}_{h,k}, u^{i+1}_{h,k}). \tag{4.9}$$

Using (4.4), the first term in (4.9) can be written

$$\frac{1}{k}(u^{i+1}_{h,k} - u^i_{h,k}, u^{i+1}_{h,k}) = \frac{1}{2k}(\|u^{i+1}_{h,k}\|^2_0 - \|u^i_{h,k}\|^2_0 + \|u^{i+1}_{h,k} - u^i_{h,k}\|^2_0)$$

By means of (3.6), (3.9) and (4.5) we obtan

$$\begin{aligned} a_h(u^{i+\theta}_{h,k}, u^{i+1}_{h,k}) &= \theta a_h(u^{i+1}_{h,k}, u^{i+1}_{h,k}) + (1-\theta)a_h(u^i_{h,k}, u^{i+1}_{h,k}) \\ &= \theta a_h(u^{i+1}_{h,k}, u^{i+1}_{h,k}) + (1-\theta)a_h(u^i_{h,k}, u^i_{h,k}) + (1-\theta)a_h(u^i_{h,k}, u^{i+1}_{h,k} - u^i_{h,k}) \\ &\geq \theta\alpha\|u^{i+1}_{h,k}\|^2_h + (1-\theta)\alpha\|u^i_{h,k}\|^2_h - \frac{1}{4}\alpha(1-\theta)\|u^i_{h,k}\|^2_h \\ &\quad -(1-\theta)\frac{\gamma}{\alpha}s(h)^2\|u^{i+1}_{h,k} - u^i_{h,k}\|^2_0. \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} (f^{i+\theta}_{h,k}, u^{i+1}_{h,k}) &= \theta(f^{i+\theta}_{h,k}, u^{i+1}_{h,k}) \\ &\quad +(1-\theta)(f^{i+\theta}_{h,k}, u^i_{h,k}) + (1-\theta)(f^{i+\theta}_{h,k}, u^{i+1}_{h,k} - u^i_{h,k}) \\ &\leq c(\varepsilon)\|f^{i+\theta}_{h,k}\|^2_* + \frac{1}{4}\theta\alpha\|u^{i+1}_{h,k}\|^2_h + \frac{1}{4}(1-\theta)\alpha\|u^i_{h,k}\|^2_h \\ &\quad +(1-\theta)\varepsilon s(h)^2\|u^{i+1}_{h,k} - u^i_{h,k}\|^2_0. \end{aligned} \tag{4.11}$$

By the growth condition (2.3), the third term in (4.9) can be estimated as follows:

$$|(\Xi^{i+\theta}_{h,k}, u^{i+1}_{h,k})| \leq C + C\|u^{i+1}_{h,k})\|^2_0 + C\|u^i_{h,k}\|^2_0. \tag{4.12}$$

Summing up (4.9) for $i = 0$ to $i = n-1, n \leq m$, multiplying it by $2k$ and making use of (4.10)-(4.12) and the property of $\|f^\theta_{h,k}\|_{L^2(0,T;V^*_h)}$ we obtain

$$\begin{aligned} &(1-Ck)\|u^n_{h,k}\|^2_0 + \theta C\sum_{i=0}^{n-1} k\|u^{i+1}_{h,k}\|^2_h + (1-\theta)C\sum_{i=0}^{n-1} k\|u^i_{h,k}\|^2_h \\ &\quad + \big(1 - 2(1-\theta)ks(h)^2(C_0+\varepsilon)\big)\sum_{i=0}^{n-1}\|u^{i+1}_{h,k} - u^i_{h,k}\|^2_0 \\ &\leq C(\varepsilon) + C\sum_{h,k}^{n-1} k\|u^i_{h,k}\|^2_0. \end{aligned}$$

By the stability assumption in (ii). Choosing $h,\ k$ such that the coefficient $(1-Ck)>0$ and the stability assumption hold then the discrete $Gronwall's$ lemma implies

$$\left\{\begin{array}{l}\max\limits_{1\le i\le m}\|u^i_{h,k}\|_0, \|u^{i-1+\theta}_{h,k}\|_0\le C,\\ \sum\limits_{i=0}^{m-1}k\|u^{i+\theta}_{h,k}\|_h^2\le C,\\ \sum\limits_{i=0}^{m-1}k\|u^{i+1}_{h,k}-u^i_{h,k}\|_0^2\le C.\end{array}\right. \tag{4.13}$$

In a similar way as in (4.6) we can deduce that

$$\|\Xi_{h,k}\|_{L^2(Q_T)}\le C \tag{4.14}$$

□

Remark 4.1 From (4.1) and (4.2) we see that the finite difference scheme (3.11) is unconditionally stable if $\theta\in[\frac{1}{2},1]$. The method is conditionally stable if $\theta\in[0,\frac{1}{2})$.

4.2. Convergence.

We shall now show that the solution $u_{h,k}$, $\Xi_{h,k}$ of problem3.1 converge to that of problem2.1. According to theorem4.1, $\{p_h u_{h,k}\}$ and $\{\Xi_{h,k}\}$ are uniformly bounded in $L^2(0,T;F)$ and $L^2(Q_T)$, respectively, we can extract subsequences (denoted by the same symbols as original sequences) such that

$$p_h u_{h,k}\rightharpoonup\{u,\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2}\}\ \ in\ L^2(0,T;F), \tag{4.15}$$

$$\Xi_{h,k}\rightharpoonup\ \ in\ L^2(Q_T), \tag{4.16}$$

as $h,\ k\to 0^+$. We start the discussion by the following lemma.

Lemma 4.1. *Let $u_{h,k},\tilde{u}_{h,k}$ and $v_{h,k}$ be functions of the form (3.1). If*

$$\int_k^T(\bar{\delta}_t u_{h,k}+A_h u^\theta_{h,k}+\Xi^\theta_{h,k}-g^\theta_{h,k},v_{h,k}-u_{h,k})dt=0, \tag{4.17}$$

and $v_{h,k}(0)=u_{h,k}(0)$, where $\bar{\delta}_t$ denotes the finite differencing in (3.4), then we have

$$\int_k^T(\bar{\delta}_t v_{h,k}+A_h u^\theta_{h,k}+\Xi^\theta_{h,k}-g^\theta_{h,k},v_{h,k}-u_{h,k})dt\ge 0 \tag{4.18}$$

PROOF. Let $u^n_h=u_{h,k}(x,t_n)$ and $v^n_h=v_{h,k}(x,t_n)$ for any $n=0,1,...,N$. We have

$$\begin{aligned}(v^{n+1}_h-v^n_h,v^{n+1}_h-u^{n+1}_h) &= (u^{n+1}_h-u^n_h,v^{n+1}_h-u^{n+1}_h)\\ &\quad+((v^{n+1}_h-u^{n+1}_h)-(v^n_h-u^n_h),v^{n+1}_h-u^{n+1}_h).\end{aligned}$$

Dividing both sides of this equality by k and using (4.4), then

$$\begin{aligned}&\left(\frac{v^{n+1}_h-v^n_h}{k},v^{n+1}_h-u^{n+1}_h\right)=\left(\frac{u^{n+1}_h-u^n_h}{k},v^{n+1}_h-u^{n+1}_h\right)\\ &\quad+\frac{1}{2k}\|v^{n+1}_h-u^{n+1}_h\|_0^2-\frac{1}{2k}\|v^n_h-u^n_h\|_0^2+\frac{1}{2k}\|v^{n+1}_h-v^n_h-(u^{n+1}_h-u^n_h)\|_0^2.\end{aligned}$$

Now summing the above equality from $n = 0$ to $N-1$, we obtain

$$\begin{aligned}\int_k^T (\bar{\delta}_t v_{h,k}, v_{h,k} - u_{h,k})dt &= \int_k^T (\bar{\delta}_t u_{h,k}, v_{h,k} - u_{h,k})dt + \frac{1}{2}\sum_{n=0}^{N-1} \|v_h^{n+1} - u_h^{n+1}\|_0^2 \\ &\quad -\frac{1}{2}\sum_{n=0}^{N-1} \|v_h^n - u_h^n\|_0^2 + \frac{1}{2}\sum_{n=0}^{N-1} ||v_h^{n+1} - v_h^n - u_h^{n+1} + u_h^n||_0^2.\end{aligned}$$

From this we have

$$\begin{aligned}\int_k^T (\bar{\delta}_t u_{h,k}, v_{h,k} - u_{h,k})dt &\le \int_k^T (\bar{\delta}_t v_{h,k}, v_{h,k} - u_{h,k})dt \\ &\quad -\frac{1}{2}\sum_{n=1}^{N} \|v_h^n - u_h^n\|_0^2 + \frac{1}{2}\sum_{n=0}^{N-1} \|v_h^n - u_h^n\|_0^2 \\ &= \int_k^T (\bar{\delta}_t v_{h,k}, v_{h,k} - u_{h,k})dt \\ &\quad -\frac{1}{2}||v_{h,k}(T) - u_{h,k}(T)||_0^2 + \frac{1}{2}||v_{h,k}(0) - u_{h,k}(0)||_0^2.\end{aligned}$$

From the above equality and (4.17) we have

$$\begin{aligned}&\int_k^T (\bar{\delta}_t v_{h,k} + A_h u_{h,k}^\theta + \Xi_{h,k}^\theta - g_{h,k}^\theta, v_{h,k})dt \\ &\ge \frac{1}{2}\|v_{h,k}(T) - u_{h,k}(T)\|_0^2 - \frac{1}{2}\|v_{h,k}(0) - u_{h,k}(0)\|_0^2.\end{aligned} \tag{4.19}$$

Finally, (4.18) follows from (4.19) and the assumption that $v_{h,k}(0) = u_{h,k}(0)$.

□

Using this lemma, we have the following theorem.

Theorem 4.2. *Under conditions (4.15) and (4.16), if* $(1-\theta)k/h^2 \to 0^+$*, then we have*

$$p_h u_{h,k} \to \{u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\} \ \textit{strongly in } L^2(0,T;F), \tag{4.20}$$

where u *denotes the solution to problem2.1 and* p_h *is the mapping defined in section 3.*

PROOF. Let $v_{h,k} \in L^2(0,T;V_h)$ be as in lemma3.2. Multiplying (3.11) by k and summing over n gives

$$\sum_{n=0}^{N-1}\left(\frac{u_{h,k}^{n+1} - u_{h,k}^n}{k} + A_h u_{h,k}^{n+\theta} + \Xi_{h,k}^{n+\theta} - g_{h,k}^{n+\theta}, v_h - u_{h,k}^{n+1}\right) = 0, \quad \forall v_h \in V_h.$$

This can be rewritten as

$$\int_0^k \left(\bar{\delta}_t u_{h,k} + A_h u_{h,k}^\theta + \Xi_{h,k}^\theta - f_{h,k}^\theta, v_{h,k} - u_{h,k}\right) dt = 0, \quad \forall v_h \in V_h, \tag{4.21}$$

where $v_{h,k}$ is as defined in (3.1),

$$f_{h,k}^\theta = \sum_{i=1}^{N}((1-\theta)f_{h,k}^{i-1} + \theta f_{h,k}^i)\lambda^i, \tag{4.22}$$

and

$$u^{\theta}_{h,k} = u_{h,k}(t) + (\theta - 1)(u_{h,k}(t) - u_{h,k}(t-k)). \tag{4.23}$$

By lemma3.3, for $u \in L^2(0,T;V)$, there exists a sequence $\bar{u}_{h,k}$ such that

$$p_h \bar{u}_{h,k} \to \{u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\}, \ \textit{strongly in } L^2(0,T;F). \tag{4.24}$$

as $h, k \to 0^+$. We comment that one choice of the sequence is

$$\bar{u}_{h,k} = \sum_{i=0}^{n} u_l(x,t_i)\chi_i,$$

where $u_l(x,t)$ denotes the V_h-interpolant of $u(x,t)$.

Let us consider

$$Y_{h,k} = \int_0^k (A_h u_{h,k} - A_h \bar{u}_{h,k}, u_{h,k} - \bar{u}_{h,k})dt. \tag{4.25}$$

Because of (4.21), putting $v_{h,k} = \tilde{u}_{h,k}$ in (4.18), we obtain

$$\int_k^T (A_h u^{\theta}_{h,k}, u_{h,k})dt = \int_k^T -(\bar{\delta}_t \tilde{u}_{h,k} - f_{h,k}, \tilde{u}_{h,k} - u_{h,k})dt + \int_k^T (A_h u^{\theta}_{h,k}, \tilde{u}_{h,k})dt.$$

However, from (4.25) we see that

$$Y_{h,k} = \int_k^T (A_h u_{h,k}, u_{h,k} - \tilde{u}_{h,k})dt + \int_k^T (A_h \tilde{u}_{h,k}, \tilde{u}_{h,k} - u_{h,k})dt. \tag{4.26}$$

Using (4.23) we have

$$\begin{aligned} \int_k^T (A_h u_{h,k}, u_{h,k} - \tilde{u}_{h,k})dt &= \int_k^T (A_h u^{\theta}_{h,k}, u_{h,k} - \tilde{u}_{h,k})dt \\ &\quad -(\theta - 1)\int_k^T (A_h[u_{h,k}(t+k) - u_{h,k}(t)], u_{h,k} - \tilde{u}_{h,k})dt. \end{aligned}$$

Combining this with (4.26) and using (4.18) we have

$$\begin{aligned} Y_{h,k} &\leq \int_k^T (\delta_t \tilde{u}_{h,k} + \Xi_{h,k} - f_{h,k}, \tilde{u}_{h,k} - u_{h,k})dt + \int_k^T (A_h \tilde{u}_{h,k}, \tilde{u}_{h,k} - u_{h,k})dt \\ &\quad +(\theta - 1)\int_k^T (A_h u_{h,k}(t) - A_h u_{h,k}(t-k), \tilde{u}_{h,k} - u_{h,k})dt. \\ &\leq \int_k^T (\delta_t \tilde{u}_{h,k} + \Xi_{h,k} - f_{h,k}, \tilde{u}_{h,k} - u_{h,k})dt + \int_k^T (A_h \tilde{u}_{h,k}, \tilde{u}_{h,k} - u_{h,k})dt \\ &\quad +(1-\theta)\left|\sum_{n=0}^{N-1} k(A_h(u_h^{n+1} - u_h^n), \tilde{u}_h^{n+1} - u_h^{n+1})\right| \\ &=: Z^1_{h,k} + Z^2_{h,k} + Z^3_{h,k}. \end{aligned} \tag{4.27}$$

For $Z^3_{h,k}$, we have

$$Z^3_{h,k} \leq (1-\theta)\frac{C_6}{h}\sum_{n=0}^{N-1}\sqrt{k}\|u_h^{n+1}-u_h^n\|_0\sqrt{k}\|\tilde{u}_h^{n+1}-u_h^{n+1}\|_h \tag{4.28}$$

$$\leq (1-\theta)\frac{C_6\sqrt{k}}{h}\left(\sum_{n=0}^{N-1}\|u_h^{n+1}-u_h^n\|_0^2\right)^{1/2}\left(\sum_{n=0}^{N-1}k\|\tilde{u}_h^{n+1}-u_h^{n+1}\|_h\right)^{1/2}$$

By(4.1), together with the fact that $p_h\tilde{u}_{h,k}$ and $p_h u_{h,k}$ remain bounded in $L^2(0,T;F)$, we have $Z^3_{h,k}\to 0$ as $h,k,(1-\theta)k/h^2\to 0^+$.

Moreover, using (4.15), (4.16) and lemma3.2 and 3.3, it is easy to prove

$$Z^1_{h,k}+Z^2_{h,k}\to 0$$

as $h,k\to 0^+$. Therefore, combining (4.27), (4.28) and the above we have $Y_{h,k}\to 0$ as $h,k,(1-\theta)k/h^2\to 0^+$. Now, by the definition of A_h and (3.7) we have from (4.25)

$$\int_k^T ||u_{h,k}-\tilde{u}_{h,k}||_h dt \leq C_3 Y_{h,k}\to 0,$$

and so

$$p_h(u_{h,k}-\tilde{u}_{h,k})\to 0 \text{ strongly in } L^2(0,T;F) \tag{4.29}$$

as $h,k,(1-\theta)k/h^2\to 0^+$. Finally, combining (4.29) and (4.24) we have (4.20). □

5. Numerical Experiments

In this section we demonstrate the efficiency and usefulness of the above finite difference method by solving the following model test problem. For simplicity, we consider one dimensional problem.

In problem3.1, we take the following parameters: $\Omega=[-1,1]$, $[s_1,s_2]=[-3,3]$, $[g_1,g_2]=[-2,2]$, $J=(0,T)=(0,1)$ and $u_0=\frac{1}{2}(-x^2+1)$.

To solve this problem we divide Ω and J uniformly into M_x and N_t subintervals, respectively, so that $h=1/M_x$ and $k=1/N_t$. The mesh point are

$$x_i=ih,\ i=0,1,...,M_x \quad and \quad t_n=nk, \quad n=0,1,...,N_t.$$

For $i=0,1,...,M_x$ and $n=0,1,\ldots,N_t$. Clearly, at each time step scheme (3.11) become a linear system with the unknown coefficients $\{u^{n+1}\}_{i=1}^{M_x-1}$ and $\Xi_{i=1}^{N_t+1}$. To solve this linear system, we choose $\theta=1$ and the following “decoupled” scheme of (3.11) (cf. [4]):

$$\begin{cases}\dfrac{u_i^{n+1/2}-u_i^n}{k}-\dfrac{1}{h^2}\left(u_{i+1}^{n+1/2}-2u_i^{n+1/2}+u_{i-1}^{n+1/2}\right)+\Xi_i^{n+1/2}=f_i^{n+1},\\ \Xi_i^{n+1/2}\in\partial j(u_i^{n+1/2}), \quad for\ i=1,...,M_x-1, \quad n=0,1,...,N_t-1.\end{cases} \tag{5.1}$$

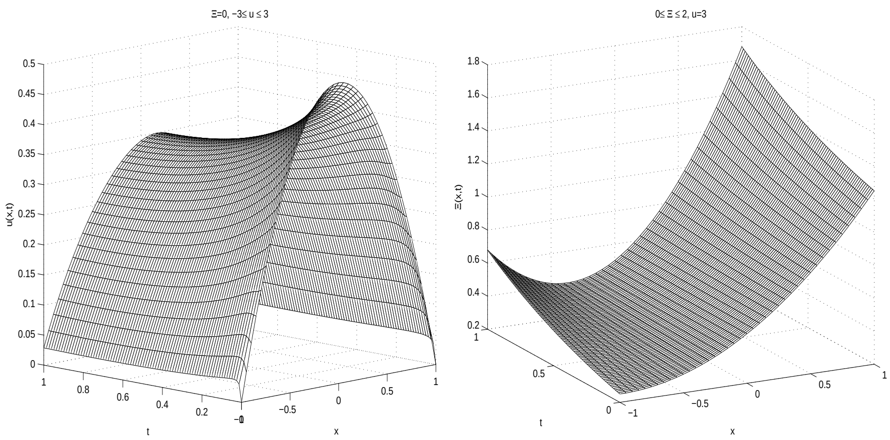
Note that this is an implicit scheme, and thus we choose $M_x=40$ and $N_t=80$.

The computed $u(x,t)$ and $\Xi(x,t)$ are depicted in Figure 5.1-5.3, when $M_x=40, N_t=80$. We take the following $f(x,t)$, respectively

$$f(x,t)=\tfrac{1}{4}x^2e^t+\tfrac{1}{2}(x+1), \tag{5.2}$$

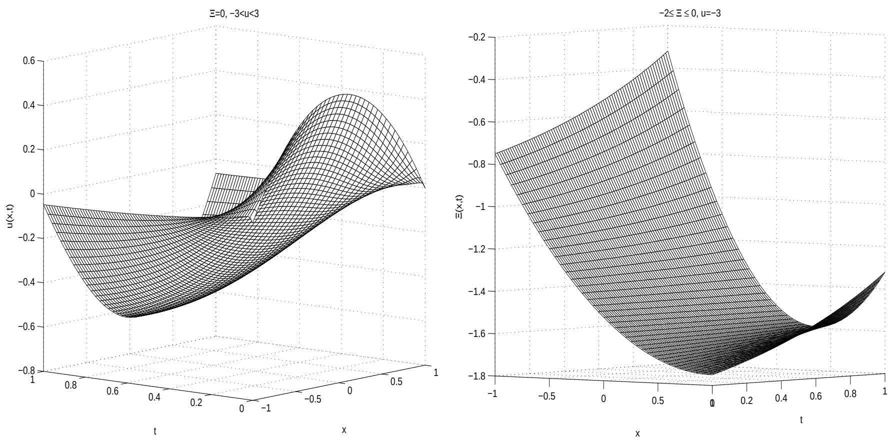
$$f(x,t)=\tfrac{1}{4}x^2e^t-\tfrac{1}{2}(x+3), \tag{5.3}$$

$$f(x,t)=\tfrac{1}{4}x^2e^t-\tfrac{1}{2}(x+12). \tag{5.4}$$



$u(x,t)$ $\Xi(x,t)$

FIGURE 5.1. Computed value function $u(x,t)$ and $\Xi(x,t)$ of temperature control problem when $f(x,t)=\frac{1}{4}x^2e^t+\frac{1}{2}(x+1)$.



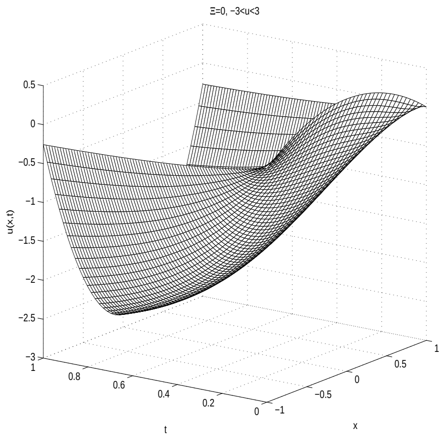
$u(x,t)$ $\Xi(x,t)$

FIGURE 5.2. Computed value function $u(x,t)$ and $\Xi(x,t)$ of temperature control problem when $f(x,t)=\frac{1}{4}x^2e^t-\frac{1}{2}(x+3)$.

Then there exist two solutions to problem 3.1 depicted by Figure 5.1 and Figure 5.2, for (5.2) and (5.3). In addition, there exists unique solution of problem 3.1 depicted by Figure 5.3 for (5.4). Thus, we see that the number of solutions depends on the the magnitude of function $f(x,t)$.

In order to test the convergence of the finite difference scheme numerically, we examine the following two discrete norms of the computed error on different partitions

$$\|u-u_{h,k}\|_0=\left(\sum_{n=0}^{N_t}\sum_{i=0}^{M_x}\|u_i^n-u(x_i,t_n)\|_0^2hk\right)^{1/2}$$



$u(x,t)$

FIGURE 5.3. Computed value function $u(x,t)$ and $\Xi(x,t)$ of temperature control problem when $f(x,t) = \frac{1}{4}x^2e^t - \frac{1}{2}(x+12)$.

M_x	N_t	$\|\cdot\|_0$	order in $\|\cdot\|_0$	$\|\cdot\|_h$	order in $\|\cdot\|_h$
10	20	1.3102	—	3.1437	—
20	40	0.6610	0.9871	1.6602	0.9211
40	80	0.3166	1.0622	0.8584	0.9517
80	160	0.1441	1.1355	0.4320	0.9832
160	320	0.0554	1.3481	0.1997	1.1204

TABLE 5.1. Computed errors in the two different norms using various meshes.

and

$$\|u - u_{h,k}\|_h = \max_{1\le n\le N_t} \|u(x,t_n) - u_h^n\|_0 + \left(\sum_{n=1}^{N_t} \|u(\cdot,t_n) - u_h^n\|_h^2 k\right)^{1/2}$$

We use the numerical solution on the uniform mesh with $M_x = 1280, N_t = 2560$ as the "exact solution", and the computed convergence histories in the two norms are listed in Table5.1. From this table we see that the computed rates of convergence in $\|\cdot\|_h$ are close to 1. (Note that it is known that the rates of convergence are normally over-estimated when the mesh approaches to the one used for the "exact solution"). This not only confirms our theoretical results in theorems 4.1 and 4.2 but also shows numerically that the rate of convergence in the discrete energy norm is at greater than 0.5 with the optimal rate being equal to 1.

6. Concluding Remarks

In this paper we have presented a finite difference approximation of the *hemivariational* inequality of parabolic type arising from temperature control problem. Stability and convergence of the discretization method have been proven, and numerical results have been presented to confirm the theoretical finding. Numerical computation confirm that finite difference method is simpler and more effective than finite element method for solving numerically parabolic *hemivariational* inequality.

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State Key Laboratory of Severe Weather (LaSW), Chinese Academy of Meteorological Sciences, Beijing 100081, China
E-mail: wanggh@cams.cma.gov.cn

School of Mathematics and Physics, North China Electric Power University, Beijing 102206, China
E-mail: xzy1018@hotmail.com