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SUPERCONVERGENCE BY L^2-PROJECTIONS FOR STABILIZED FINITE ELEMENT METHODS FOR THE STOKES EQUATIONS

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Abstract. A general superconvergence result is established for the stabilized finite element approximations for the stationary Stokes equations. The superconvergence is obtained by applying the L^2 projection method for the finite element approximations and/or their close relatives. For the standard Galerkin method, existing results show that superconvergence is possible by projecting directly the finite element approximations onto properly defined finite element spaces associated with a mesh with different scales. But for the stabilized finite element method, the authors had to apply the L^2 projection on a trivially modified version of the finite element solution. This papers shows how the modification should be made and why the L^2 projection on the modified solution has superconvergence. Although the method is demonstrated for one class of stabilized finite element methods, it can certainly be extended to other type of stabilized schemes without any difficulty. Like other results in the family of L^2 projection methods, the superconvergence presented in this paper is based on some regularity assumption for the Stokes problem and is valid for general stabilized finite element method with regular but non-uniform partitions.

Key words. Stokes equations, Stabilized finite element method, Superconvergence, L^2 projection, least-squares method

1. Introduction

In the analysis and practice of employing finite element methods in solving the Navier-Stokes equations, the inf-sup condition has played an important role because it ensures a stability and accuracy of the underlying numerical schemes. A pair of finite element spaces that are used to approximate the velocity and the pressure unknowns are said to be stable if they satisfy the inf-sup condition. Intuitively speaking, the inf-sup condition is something that enforces a certain correlation between two finite element spaces so that they both have the required properties when employed for the Navier-Stokes or Stokes equations. It is well known that the two simplest elements P_1/P_0 (i.e., linear/constant) on triangle and Q_1/P_0 (i.e., bilinear/constant) on quadrilateral are not stable, and therefore can not be trusted when employed in practical computation. In contrast, many known stable elements do not look natural because their construction involves artificial or non-standard functions which are not commonly used/implemented in popular engineering code packages. To eliminate the inf-sup condition so that simpler and more natural finite element spaces can be used, stabilized finite element methods have been developed for the Stokes equations in the last two decades [14, 4, 15, 9, 16]. These methods are gaining more and more popularity in computational fluid dynamics.

The goal of this paper is to explore ways that improve the accuracy of the approximate solutions resulted from the stabilized finite element formulations for the Stokes equations. In particular, we are curious about postprocessing techniques that lead to new approximations with superconvergence. In the literature, there are number of techniques in the content of superconvergence [8, 10, 27, 24, 22, 18,

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31, 32, 30]. The main idea behind them is “cancelation”, which, at least in all the existing results, are possible only for certain model problems with some strong, and perhaps impractical assumptions on the geometry of the finite element partitions, see for example [20, 21]. One exception in techniques of superconvergence is the L^2-projection method proposed and analyzed by Wang [28] for the standard Galerkin method. The relaxation on the mesh uniformity is the key difference between the L^2-projection method and all other methods in superconvergence. The L^2-projection method has been extended by Wang and Ye [29] to the Stokes equations, but only for finite element methods based on stable pairs. This paper aims at a study of superconvergence by using the L^2-projection method for the stabilized finite element method.

Briefly speaking, the result to be presented in this paper shows that it is possible to obtain numerical solutions with superconvergence for the stabilized finite element methods. However, there are essential differences between the stabilized finite element method and the standard Galerkin method. For example, one can obtain superconvergence for the L^2 projection of the pressure approximation, but not for the velocity approximation as one would get in the standard Galerkin method. However, the L^2-projection of a modified or corrected form of the velocity approximation is of superconvergent to the exact velocity. Our analysis shows that the correction comes from a scaled version of the residual which is exactly the stability term added to the Galerkin formula. If similar stability terms were added to the mass conservation equation, one would need to modify the pressure approximation as well in order to obtain superconvergence by using the L^2-projection method. The main contribution of the paper is that it provides a systematic approach for obtaining superconvergence when non-standard Galerkin methods are used.

The paper is organized as follows. In Section 2, we review a stabilized finite element formulation for the Stokes equations. In Section 3, we describe the general idea of the L^2 projection method in superconvergence. In Sections 4 and 5, we establish two super-approximation properties: one for the pressure and the other for the velocity unknown by using L^2-projections. Finally, in Section 6, we derive some new superconvergent results for the Stokes equations when approximated by using stabilized finite element methods.

2. Preliminaries and the stabilized finite element method

For simplicity, we consider the homogeneous Dirichlet boundary value problem for the Stokes equations. This model problem seeks unknown functions $\mathbf{u} \in H^1(\Omega)^d$ and $p \in L^2(\Omega)$ satisfying

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{2}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \tag{3}$$

where Ω is an open bounded domain in the Euclidean space $\mathbf{R}^d (d = 2, 3)$ with a Lipschitz continuous boundary $\partial\Omega$; $\mathbf{f}$ is a given function in $H^{-1}(\Omega)^d$; Δ, ∇, and $\nabla\cdot$ denote the Laplacian, gradient, and divergence operators respectively; $\nu > 0$ is a given constant representing the viscosity of the fluid. The given function/distribution $\mathbf{f} = \mathbf{f}(x)$ is the unit external volumetric force acting on the fluid at $x \in \Omega$. Without loss of generality, we assume that $\nu = 1$, $d = 2$, and Ω is polygonal in the rest of the paper.

The above description of the Stokes problem has assumed the standard notation for the Sobolev spaces $H^s(\Omega)$ which is the collection of distributions whose weak derivatives of order up to s are square integrable functions over the domain Ω. Denote by $(\cdot,\cdot)_s$ the inner product associated with $H^s(\Omega)$, with a norm notation $\|\cdot\|_s$, and semi-norm notation $|\cdot|_s$ for non-negative integers $s \geq 0$. The Sobolev space $H^0(\Omega)$ coincides with the space of square integrable functions $L^2(\Omega)$, in which case the norm and inner product are denoted by $\|\cdot\|$ and $(\cdot,\cdot)$, respectively. In addition, denote by $L_0^2(\Omega)$ the subspace of $L^2(\Omega)$ consisting of all the functions in $L^2(\Omega)$ with vanishing mean value, and $H_0^1(\Omega)$ stands for the closed subspace of $H^1(\Omega)$ with vanishing boundary values on Ω. In general, $\|\phi\|_D$ denotes the L^2 norm of $\phi \in L^2(D)$ for any domain D.

Let $V_h \subset \left[H_0^1(\Omega)\right]^2$ and $W_h \subset L_0^2(\Omega)$ be two finite element spaces consisting of piecewise polynomials for the velocity and pressure unknowns, respectively, associated with a prescribed finite element partition $\mathcal{T}_h$ with mesh size h. Let Γ denote the union of boundaries of the elements (e.g., triangles in the case that $\mathcal{T}_h$ is a triangulation of Ω) $T \in \mathcal{T}_h$ and $\Gamma_0 := \Gamma\backslash\partial\Omega$. Let $e \in \Gamma_0$ be an interior edge shared by two elements T_1 and T_2 in $\mathcal{T}_h$. We denote by $[q]$ the jump of q on e:

$$[q](x) = q|_{T_1}(x) - q|_{T_2}(x), \quad \forall x \in e,$$

where $q|_{T_i}(x)$ is the value of q at x as seen from the element T_i for $i = 1, 2$. It should be pointed out that interchanging the role of T_1 and T_2 in the jump definition will have no effect on the finite element scheme to be described shortly in this section.

The finite element spaces V_h and W_h are assumed to have the following approximation properties:

$$\inf_{\mathbf{v}\in V_h}\left(h^{-1}\|\mathbf{u}-\mathbf{v}\| + \|\nabla(\mathbf{u}-\mathbf{v})\| + (\sum_{T\in\mathcal{T}_h} h_T^2\|\Delta(\mathbf{u}-\mathbf{v})\|_T^2)^{\frac{1}{2}}\right) \leq Ch^i\|\mathbf{u}\|_{i+1}, \tag{4}$$

$$\inf_{q\in W_h}\left(\|p-q\| + (\sum_{T\in\mathcal{T}_h} h_T^2\|\nabla(p-q)\|_T^2)^{\frac{1}{2}} + (\sum_{e\in\Gamma_0} h_e\|[p-q]\|_e^2)^{\frac{1}{2}}\right) \leq Ch^i\|p\|_i, \tag{5}$$

for $\mathbf{u} \in [H^{i+1}(\Omega)\cap H_0^1(\Omega)]^2$ and $p \in H^i(\Omega)\cap L_0^2(\Omega)$, where $1 \leq i \leq k$, k is the order of polynomial space employed in constructing the finite element space V_h, and h_T and h_e are the diameters of the element T and edge e respectively. For compatibility of approximation accuracy, we have assumed that W_h was constructed by using polynomials of order no less than $k-1$.

Let β and γ be two parameters to be determined later and $\tau = \pm 1$. Define a bilinear form as follows:

$$\begin{aligned}\Phi(\mathbf{w}, r; \mathbf{v}, q) \;&=\; (\nabla\mathbf{w}, \nabla\mathbf{v}) - (\nabla\cdot\mathbf{v}, r) - (\nabla\cdot\mathbf{w}, q)\\ &-\; \gamma\sum_{T\in\mathcal{T}_h} h_T^2(\nabla r - \Delta\mathbf{w}, \nabla q - \tau\Delta\mathbf{v})_T - \beta\sum_{e\in\Gamma_0} h_e([r],[q])_e,\end{aligned}$$

where $(p,q)_e = \int_e pq ds$ is the L^2-inner product in $L^2(e)$. The corresponding stabilized finite element formulation for the Stokes equations seeks $(\mathbf{u}_h; p_h) \in V_h \times W_h$ such that for all $(\mathbf{v}; q) \in V_h \times W_h$

$$\Phi(\mathbf{u}_h, p_h; \mathbf{v}, q) = (\mathbf{f}, \mathbf{v}) - \gamma\sum_{T\in\mathcal{T}_h} h_T^2(\mathbf{f}, \nabla q - \tau\Delta\mathbf{v})_T. \tag{6}$$

It is not hard to see that the exact solution $(\mathbf{u}; p)$ of the Stokes equations also satisfies (6) for any values of β and γ. Thus, the following error equation is easy to be verified:

$$\Phi(\mathbf{u}-\mathbf{u}_h, p-p_h; \mathbf{v}, q) = 0, \qquad \forall\mathbf{v}\in V_h,\ \forall q\in W_h. \tag{7}$$

Observe that the bilinear form $\Phi(\cdot;\cdot)$ is symmetric for $\tau = 1$, and is nonsymmetric for $\tau = -1$. It was proved that the symmetric formulation is conditionally stable with respect to the positive parameter values of γ and β where β could assume arbitrary values. The nonsymmetric scheme is absolutely stable with respect to positive parameter values of γ and β. Details can be found in [9].

This paper aims at an establishment of superconvergence for the symmetric formulation by using the L^2 projection idea with respect to a finite element subspace on a mesh coarser than $\mathcal{T}_h$. For the nonsymmetric formulation (i.e, when $\tau = -1$), we find that the forthcoming superconvergent algorithm and analysis are difficult to apply. Therefore, this case is left to interested readers as an open problem. From now on, we shall assume that $\tau = \beta = 1$ and $\gamma > 0$ is a parameter to be determined later.

Lemma 1. There exists a constant C independent of h such that for any $(\mathbf{v}; q) \in V_h \times W_h$, one has

$$\Phi(\mathbf{v}, q; \mathbf{v}, -q) \geq C |\!|\!| (\mathbf{v}; q) |\!|\!| \tag{8}$$

where $|\!|\!|(\mathbf{v}; q)|\!|\!|$ is a norm in $V_h \times W_h$ defined as follows:

$$|\!|\!|(\mathbf{v}; q)|\!|\!|^2 = \|\nabla \mathbf{v}\|^2 + \sum_{T\in\mathcal{T}_h} h_T^2 \|\nabla q\|_T^2 + \sum_{e\in\Gamma_0} h_e([q],[q])_e.$$

Proof. It follows from the definition of $\Phi(\cdot;\cdot)$ that

$$\begin{aligned}\Phi(\mathbf{v}, q; \mathbf{v}, -q) &= (\nabla\mathbf{v}, \nabla\mathbf{v}) - \gamma \sum_{T\in\mathcal{T}_h} h_T^2 (\nabla q - \Delta\mathbf{v}, -\nabla q - \Delta\mathbf{v})_T + \beta \sum_{e\in\Gamma_0} h_e([q],[q])_e \\ &= \|\nabla\mathbf{v}\|^2 - \gamma \sum_{T\in\mathcal{T}_h} h_T^2 \|\Delta\mathbf{v}\|_T^2 + \gamma \sum_{T\in\mathcal{T}_h} h_T^2 \|\nabla q\|_T^2 + \beta \sum_{e\in\Gamma_0} h_e([q],[q])_e.\end{aligned}$$

Now we use the standard inverse inequality to estimate the second term on the right-hand side of the above identify. With $\gamma \in (0, \alpha_0)$ for a sufficiently small, but fixed α_0, we easily obtain the desired estimate (8). □

An application of Lemma 1 and the standard error equation (7) for the stabilized finite element scheme (6) yields the following result of error estimate.

Theorem 1. Let $(\mathbf{u}_h;\ p_h) \in V_h \times W_h$ and $(\mathbf{u};\ p) \in (H^{k+1}(\Omega) \cap H_0^1(\Omega))^2 \times H^k(\Omega) \cap L_0^2(\Omega)$ be the solutions of (6) and (1)-(3) respectively. Then there exists a constant C independent of h such that

$$|\!|\!|(\mathbf{u} - \mathbf{u}_h; p - p_h)|\!|\!| \leq Ch^k(\|\mathbf{u}\|_{k+1} + \|p\|_k). \tag{9}$$

A similar result can be found in [4, 15].

The superconvergence analysis to be presented in next section requires a certain regularity for the Stokes problem. To this end, we consider a more general Stokes problem which seeks $(\mathbf{u}; p) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ satisfying

$$(\nabla\mathbf{u}, \nabla\mathbf{v}) - (\nabla\cdot\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}), \quad \forall\, \mathbf{v} \in H_0^1(\Omega)^2, \tag{10}$$

$$(\nabla\cdot\mathbf{u}, q) = (g, q), \quad \forall\, q \in L_0^2(\Omega), \tag{11}$$

where $g \in L_0^2(\Omega)$ is a given function. Assume that the domain Ω is so regular that ensures a $H^s, s \geq 1$ regularity for the solution of (10) and (11). In other words, for any $\mathbf{f} \in H^{s-2}(\Omega)^2$ and $g \in H^{s-1}(\Omega) \cap L_0^2(\Omega)$, the problem (10) and (11) has a unique solution $\mathbf{u} \in H_0^1(\Omega)^2 \cap H^s(\Omega)^2$ and $p \in H^{s-1}(\Omega) \cap L_0^2(\Omega)$ satisfying the following a priori estimate:

$$\|\mathbf{u}\|_s + \|p\|_{s-1} \leq C(\|f\|_{s-2} + \|g\|_{s-1}), \tag{12}$$

where C is a constant independent of the data $\mathbf{f}$ and g.

3. L^2-Projection: a general idea for superconvergence

L^2-Projection is a postprocessing technique introduced by Wang [28] for standard Galerkin methods. The basic idea is to project the finite element solution to another finite element space with a different, but coarser mesh. The scale difference in the two meshes is the key for achieving a superconvergence after the postprocessing.

For the stabilized finite element method, in addition to the finite element partition $\mathcal{T}_h$ that was used to produce the finite element approximation $(\mathbf{u}_h; p_h)$ from (6), we introduce another finite element partition $\mathcal{T}_\tau$ with mesh size τ, where $h << \tau$. Assume that the scales τ and h have the following relationship:

$$\tau \approx h^\alpha \tag{13}$$

with $\alpha \in (0,1)$. It will be seen that the parameter α plays an important role in the post processing. For now, let V_τ and W_τ be any two finite element spaces consisting of piecewise polynomials of degree r and t respectively associated with the partition $\mathcal{T}_\tau$. Define Q_τ and R_τ to be the L^2 projectors from $L^2(\Omega)$ onto the finite element spaces V_τ and W_τ respectively. Roughly speaking, the postprocessing of the finite element approximation $(\mathbf{u}_h; p_h)$ is simply given by their L^2 projections:

$$\text{postprosessed } (\mathbf{u}_h; p_h) \approx (Q_\tau \mathbf{u}_h; R_\tau p_h).$$

We will show that $R_\tau p_h$ is indeed a new approximation of the pressure variable with superconvergence. But the same assertion can not be made for the velocity approximation under the above framework. However, the same L^2 projection for a slightly modified finite element approximation $\mathbf{u}_h$ still gives an approximation with superconvergence. Details are presented in forthcoming sections.

4. A super-approximation for the pressure

For the pressure unknown, the postprocessed approximate solution is given by the L^2 projection of the finite element solution p_h:

$$\text{postprossed pressure} = R_\tau p_h.$$

The rest of this section is devoted to a mathematical analysis on a super-approximation property for the postprocessed pressure approximation $R_\tau p_h$. To this end, we introduce the following notation

$$a(\mathbf{w}, \mathbf{v}) := (\nabla \mathbf{w}, \nabla \mathbf{v}), \quad b(\mathbf{v}, q) := (\nabla \cdot \mathbf{v}, q)$$

and

$$d(\mathbf{w}, r; \mathbf{v}, q) := \gamma \sum_{T \in \mathcal{T}^h} h_T^2 (\nabla r - \Delta \mathbf{w}, \nabla q - \Delta \mathbf{v})_T.$$

The error equation (7) is equivalent to the following two equations:

$$a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) - b(\mathbf{v}, p - p_h) - d(\mathbf{u} - \mathbf{u}_h, p - p_h; \mathbf{v}, 0) = 0 \tag{14}$$

$$b(\mathbf{u} - \mathbf{u}_h, q) + d(\mathbf{u} - \mathbf{u}_h, p - p_h; 0, q) + \beta \sum_{e \in \Gamma_0} h_e([p - p_h], [q])_e = 0. \tag{15}$$

The following lemma provides an estimate for $R_\tau p - R_\tau p_h$.

Lemma 2. Assume that the regularity result (12) holds true with $1 \le s \le k+1$ and $W_\tau \subset H^{s-1}(\Omega)$. Then there is a constant C independent of h and τ such that

$$\|R_\tau p - R_\tau p_h\| \le C h^{k+s-1+\alpha \min(0,s-1)} (\|\mathbf{u}\|_{k+1} + \|p\|_k) \tag{16}$$

where $\alpha \in (0,1)$ is a parameter as defined in (13).

Proof. It follows from the definition of $\|\cdot\|$ and R_τ that

$$\|R_\tau p - R_\tau p_h\| = \sup_{\phi\in L^2(\Omega),\|\phi\|=1} |(R_\tau p - R_\tau p_h, \phi)|$$

and

$$(R_\tau p - R_\tau p_h, \phi) = (p - p_h, R_\tau \phi).$$

Thus,

$$\|R_\tau p - R_\tau p_h\| = \sup_{\phi\in L^2(\Omega),\|\phi\|=1} |(p - p_h, R_\tau \phi)|.$$

Consider the following problem: find $(\omega, \xi) \in (H^{k+1}(\Omega)\cap H_0^1(\Omega))^2\times(H^k(\Omega)\cap L_0^2(\Omega))$ with $k \geq 1$ such that

$$a(\omega, \mathbf{v}) - b(\mathbf{v}, \xi) = 0, \quad \forall\, \mathbf{v} \in H_0^1(\Omega)^2, \tag{17}$$

$$b(\omega, q) = (R_\tau \phi, q), \quad \forall\, q \in L_0^2(\Omega). \tag{18}$$

The solution of the above problem implies the following

$$-\Delta\omega + \nabla\xi = 0.$$

Furthermore, we have

$$d(\omega, \xi; \mathbf{v}, q) = 0. \tag{19}$$

It is not hard to see that the solution of the problem (17)-(18) satisfies

$$\Phi(\omega, \xi; \mathbf{v}, q) = (R_\tau \phi, q), \quad \forall\, \mathbf{v} \in H_0^1(\Omega)^2,\ \forall\, q \in L_0^2(\Omega). \tag{20}$$

By setting $\mathbf{v} = \mathbf{u} - \mathbf{u}_h$ and $q = p - p_h$ in (20) we obtain

$$\begin{aligned}
(p - p_h, R_\tau\phi) &= \Phi(\omega, \xi; \mathbf{u} - \mathbf{u}_h, p - p_h) \\
&= \Phi(\mathbf{u} - \mathbf{u}_h, p - p_h; \omega, \xi) \\
&= \Phi(\mathbf{u} - \mathbf{u}_h, p - p_h; \omega - \omega_I, \xi - \xi_I). \\
&= a(\mathbf{u} - \mathbf{u}_h, \omega - \omega_I) - b(\omega - \omega_I, p - p_h) \\
&- b(\mathbf{u} - \mathbf{u}_h, \xi - \xi_I) - \beta \sum_{e\in\Gamma_0} h_e([p - p_h], [\xi - \xi_I])_e \\
&- d(\mathbf{u} - \mathbf{u}_h, p - p_h; \omega - \omega_I, \xi - \xi_I) \\
&:= A + B + C + D + E
\end{aligned} \tag{21}$$

where ω_I and ξ_I are interpolates of ω and ξ in V_h and W_h, respectively. The terms A, B, C, D, and E are defined according to their appearance in order, and their estimates are given as follows.

For the first term A, it follows from the Schwarz inequality, (4), and (12) that

$$\begin{aligned}
|A| &= |a(\mathbf{u} - \mathbf{u}_h, \omega - \omega_I)| \leq \|\nabla(\mathbf{u} - \mathbf{u}_h)\|\|\nabla(\omega - \omega_I)\| \\
&\leq Ch^{s-1}\|R_\tau\phi\|_{s-1}|\!|\!|(\mathbf{u} - \mathbf{u}_h; p - p_h)|\!|\!| \\
&\leq Ch^{s-1}\tau^{\min(0,1-s)}|\!|\!|(\mathbf{u} - \mathbf{u}_h; p - p_h)|\!|\!|\|\phi\|.
\end{aligned}$$

As to the second term B, we use the integration by parts, trace theorem, (4), and (12), to come up with the following estimates

$$
\begin{aligned}
|B| &= |b(\omega-\omega_I, p-p_h)| \\
&= \left| -\sum_{T\in\mathcal{T}_h}(\omega-\omega_I, \nabla(p-p_h)) + \sum_{e\in\Gamma_0}\int_e (\omega-\omega_I)\cdot\mathbf{n}[p-p_h]ds \right| \\
&\le |\!|\!|(\mathbf{u}-\mathbf{u}_h; p-p_h)|\!|\!|(h^{-1}\|\omega-\omega_I\| + \|\nabla(\omega-\omega_I)\|) \\
&\le Ch^{s-1}\|\omega\|_s |\!|\!|(\mathbf{u}-\mathbf{u}_h; p-p_h)|\!|\!| \\
&\le Ch^{s-1}\|R_\tau\phi\|_{s-1} |\!|\!|(\mathbf{u}-\mathbf{u}_h; p-p_h)|\!|\!| \\
&\le Ch^{s-1}\tau^{\min(0,1-s)} |\!|\!|(\mathbf{u}-\mathbf{u}_h; p-p_h)|\!|\!| \|\phi\|.
\end{aligned}
$$

For the term C, we use the Schwarz inequality, (5), and (12) to obtain

$$
\begin{aligned}
|C| &= |b(\mathbf{u}-\mathbf{u}_h, \xi-\xi_I)| \le \|\nabla(\mathbf{u}-\mathbf{u}_h)\|\|\xi-\xi_I\| \\
&\le Ch^{s-1}\tau^{\min(0,1-s)} |\!|\!|(\mathbf{u}-\mathbf{u}_h; p-p_h)|\!|\!| \|\phi\|.
\end{aligned}
$$

Next, it follows from the trace inequality, (5), and (12) that

$$
\begin{aligned}
|D| &= |\beta \sum_{e\in\Gamma_0} h_e([p-p_h], [\xi-\xi_I])_e| \\
&\le |\beta| \left(\sum_{e\in\Gamma_0} h_e\|[p-p_h]\|_e^2\right)^{\frac{1}{2}} \left(\sum_{e\in\Gamma_0} h_e\|[\xi-\xi_I]\|_e^2\right)^{\frac{1}{2}} \\
&\le Ch^{s-1}\tau^{\min(0,1-s)} |\!|\!|(\mathbf{u}-\mathbf{u}_h; p-p_h)|\!|\!| \|\phi\|.
\end{aligned}
$$

To deal with the last term E, we first establish an estimate for local integrals by using the standard triangle and the inverse inequalities:

$$
\begin{aligned}
&\sum_{T\in\mathcal{T}_h} h_T^2\|\Delta(\mathbf{u}-\mathbf{u}_h)\|_T^2 \le 2\sum_{T\in\mathcal{T}_h} h_T^2\|\Delta(\mathbf{u}-\mathbf{u}_I)\|_T^2 + 2\sum_{T\in\mathcal{T}_h} h_T^2\|\Delta(\mathbf{u}_I-\mathbf{u}_h)\|^2 \\
&\le C(\sum_{T\in\mathcal{T}_h} h_T^2\|\Delta(\mathbf{u}-\mathbf{u}_I)\|_T^2 + \|\nabla(\mathbf{u}_I-\mathbf{u}_h)\|^2) \\
&\le C(|\!|\!|(\mathbf{u}-\mathbf{u}_h; p-p_h)|\!|\!|^2 + \sum_{T\in\mathcal{T}_h} h_T^2\|\Delta(\mathbf{u}-\mathbf{u}_I)\|_T^2 + \|\nabla(\mathbf{u}-\mathbf{u}_I)\|^2).
\end{aligned}
$$

Using the above inequality, (4)-(5) and (12) one arrives at the following estimate

$$
\begin{aligned}
|E| &= |d(\mathbf{u}-\mathbf{u}_h, p-p_h; \omega-\omega_I, \xi-\xi_I)| \\
&\le |\gamma| \left(\sum_{T\in\mathcal{T}_h} h_T^2\|\nabla(p-p_h)-\Delta(\mathbf{u}-\mathbf{u}_h)\|_T^2\right)^{\frac{1}{2}} \left(\sum_{T\in\mathcal{T}_h} h_T^2\|\nabla(\xi-\xi_I)-\Delta(\omega-\omega_I)\|_T^2\right)^{\frac{1}{2}} \\
&\le Ch^{s-1}\tau^{\tilde{s}}\|\phi\| \left((\sum_{T\in\mathcal{T}_h} h_T^2\|\nabla(p-p_h)\|_T^2)^{\frac{1}{2}} + (\sum_{T\in\mathcal{T}_h} h_T^2\|\Delta(\mathbf{u}-\mathbf{u}_h)\|_T^2)^{\frac{1}{2}} \right) \\
&\le Ch^{s-1}\tau^{\tilde{s}}\|\phi\| \left(|\!|\!|(\mathbf{u}-\mathbf{u}_h; p-p_h)|\!|\!| + (\sum_{T\in\mathcal{T}_h} h_T^2\|\Delta(\mathbf{u}-\mathbf{u}_I)\|_T^2)^{\frac{1}{2}} + \|\nabla(\mathbf{u}-\mathbf{u}_I)\| \right),
\end{aligned}
$$

where $\tilde{s} = \min(0, 1-s)$.

Collectively, the above estimates for the terms A, B, C, D, and E, together with an application of (9) and (21), yield the following inequality

$$
|(p-p_h, R_\tau\phi)| \le Ch^{k+s-1+\alpha\min(0,1-s)}(\|\mathbf{u}\|_{k+1} + \|p\|_k)\|\phi\|,
$$

which implies the following

$$\|R_\tau p - R_\tau p_h\| \le Ch^{k+s-1+\alpha\min(0,1-s)}(\|\mathbf{u}\|_{k+1} + \|p\|_k).$$

This completes the proof of the lemma. □

5. A super-approximation for the velocity

Let $(\mathbf{u}_h; p_h)$ be the finite element approximation of the Stokes problem arising from the stabilized finite element formulation (6). A straightforward application of the projection method to the velocity approximation would involve a direct L^2 projection of $\mathbf{u}_h$ into a finite element space V_τ defined on a coarse level $\mathcal{T}_\tau$. But we were not able to establish any super-approximation theory for such postprocessed approximations. Yet, our numerical experiments did not provide any strong evidence for possible superconvergence when $\mathbf{u}_h$ was projected to V_τ without any modification. Note that no numerical experiments are reported in this paper.

The goal of this section is to present a procedure which results in a super-approximation property for the velocity. This new procedure continues our exploration of the L^2 projection method, in which the projection shall be applied to a modified or corrected finite element approximation $\mathbf{u}_h$. To this end, we modify or correct the velocity approximation to the Stokes problem as follows:

$$\mathbf{u}_h^* = \mathbf{u}_h + \psi_h, \tag{22}$$

where ψ_h is defined by assuming the value $\gamma h_T^2(f - \nabla p_h + \Delta \mathbf{u}_h)$ on each element T. We claim that the L^2 projection of this modified velocity approximation $\mathbf{u}_h^*$ has superconvergence.

Lemma 3. Assume that (12) holds true with $1 \le s \le k+1$ and $V_\tau \subset H^{s-2}(\Omega)^2$. Then, there is a constant C independent of h and τ such that

$$\|Q_\tau \mathbf{u} - Q_\tau \mathbf{u}_h^*\| \le Ch^{k+s-1+\alpha\min(0,s-2)}(\|\mathbf{u}\|_{k+1} + \|p\|_k) \tag{23}$$

where $\alpha \in (0,1)$ is a parameter specified as in (13).

Proof. Using the definition of $\|\cdot\|$ and Q_τ we have

$$\begin{aligned} \|Q_\tau \mathbf{u} - Q_\tau \mathbf{u}_h^*\| &= \sup_{\phi\in L^2(\Omega)^2, \|\phi\|=1} |(Q_\tau \mathbf{u} - Q_\tau \mathbf{u}_h^*, \phi)| \\ &= \sup_{\phi\in L^2(\Omega)^2, \|\phi\|=1} |(\mathbf{u} - \mathbf{u}_h^*, Q_\tau \phi)|. \end{aligned}$$

Consider the following problem: find $(\mathbf{w}; \lambda) \in (H^{k+1}(\Omega) \cap H_0^1(\Omega))^2 \times (H^k(\Omega) \cap L_0^2(\Omega))$ with $k \ge 1$ such that

$$a(\mathbf{w}, \mathbf{v}) - b(\mathbf{v}, \lambda) = (Q_\tau \phi, \mathbf{v}) \quad \forall\, \mathbf{v} \in H_0^1(\Omega)^2, \tag{24}$$

$$b(\mathbf{w}, q) = 0 \quad \forall\, q \in L_0^2(\Omega). \tag{25}$$

It can be seen that for any $\mathbf{v} \in H_0^1(\Omega)^2$ and $q \in L_0^2(\Omega)$ which are locally smooth (meaning that $\mathbf{v}$ must be locally on H^2 and q must be locally on H^1 over each element), we have

$$\Phi(\mathbf{w}, \lambda; \mathbf{v}, q) = (Q_\tau \phi, \mathbf{v}) - \gamma \sum_{T\in\mathcal{T}_h} h_T^2 (Q_\tau \phi, \nabla q - \Delta \mathbf{v})_T. \tag{26}$$

By setting $\mathbf{v} = \mathbf{u} - \mathbf{u}_h$ and $q = p - p_h$, the right-hand side of (26) becomes

$$\begin{aligned}(27)\qquad & (Q_\tau\phi, \mathbf{v}) - \gamma \sum_{T\in\mathcal{T}_h} h_T^2 (Q_\tau\phi, \nabla q - \Delta\mathbf{v})_T \\ &= (Q_\tau\phi, \mathbf{u} - \mathbf{u}_h) - \gamma \sum_{T\in\mathcal{T}_h} h_T^2 (Q_\tau\phi, \nabla(p - p_h) - \Delta(\mathbf{u} - \mathbf{u}_h))_T \\ &= (Q_\tau\phi, \mathbf{u} - \mathbf{u}_h) - \gamma \sum_{T\in\mathcal{T}_h} h_T^2 (Q_\tau\phi, f - \nabla p_h + \Delta\mathbf{u}_h)_T \\ &= (Q_\tau\phi, \mathbf{u} - \mathbf{u}_h^*),\end{aligned}$$

where in the last step we have used the definition of $\mathbf{u}_h^*$ given in (22). Thus, with $\mathbf{v} = \mathbf{u} - \mathbf{u}_h$ and $q = p - p_h$, by substituting (27) into (26) we obtain

$$\begin{aligned}(28)\qquad (Q_\tau\phi, \mathbf{u} - \mathbf{u}_h^*) &= \Phi(\mathbf{w}, \lambda; \mathbf{u} - \mathbf{u}_h, p - p_h) \\ &= \Phi(\mathbf{u} - \mathbf{u}_h, p - p_h; \mathbf{w}, \lambda) \\ &= \Phi(\mathbf{u} - \mathbf{u}_h, p - p_h; \mathbf{w} - \mathbf{w}_I, \lambda - \lambda_I).\end{aligned}$$

Now it follows from (21) that

$$\begin{aligned}|\Phi(\mathbf{u} - \mathbf{u}_h, p - p_h; \mathbf{w} - \mathbf{w}_I, \lambda - \lambda_I)| &\leq Ch^{k+s-1}\|Q_\tau\phi\|_{s-2}(\|\mathbf{u}\|_{k+1} + \|p\|_k) \\ &\leq Ch^{k+s-1+\alpha\min(0,2-s)}(\|\mathbf{u}\|_{k+1} + \|p\|_k).\end{aligned}$$

The last estimate, together with (28), completes the proof of the lemma. $\square$

We point out that the technique in establishing Lemma 3 can be applied to derive error estimates for $\mathbf{u} - \mathbf{u}_h^*$ in negative norms. To the author's best knowledge, there are no results in existing literature for estimating the error $\mathbf{u} - \mathbf{u}_h$ in negative norms. In fact, we wonder if it is possible to have any optimal order error estimates for $\mathbf{u} - \mathbf{u}_h$ in applicable negative norms.

6. Superconvergence

The results of super-approximation developed in the previous two sections can be used to derive superconvergence for the finite element approximate solution to the Stokes equations. The following is a superconvergent result for the velocity approximation.

Theorem 2. Assume that (12) holds true with $1 \leq s \leq k+1$ and $V_\tau \subset H^{s-2}(\Omega)^2$. If $(\mathbf{u}_h; p_h)$ is the finite element approximation of the solution $(\mathbf{u}; p)$ of (6), then we have

$$\begin{aligned}&\|\mathbf{u} - Q_\tau\mathbf{u}_h^*\| + h^\alpha\|\nabla_\tau(\mathbf{u} - Q_\tau\mathbf{u}_h^*)\| \\ (29)\qquad &\leq Ch^{\alpha(r+1)}\|\mathbf{u}\|_{r+1} + Ch^\sigma(\|\mathbf{u}\|_{k+1} + \|p\|_k),\end{aligned}$$

where $\sigma = k + s - 1 + \alpha\min(0, 2-s)$.

Proof. By the definition of Q_τ and the relation (13) between τ and h, we have

$$(30)\qquad \|\mathbf{u} - Q_\tau\mathbf{u}\| \leq C\tau^{r+1}\|\mathbf{u}\|_{r+1} \leq Ch^{\alpha(r+1)}\|\mathbf{u}\|_{r+1}.$$

Combining (30) and (23) gives

$$\begin{aligned}\|\mathbf{u} - Q_\tau\mathbf{u}_h^*\| &\leq \|\mathbf{u} - Q_\tau\mathbf{u}\| + \|Q_\tau\mathbf{u} - Q_\tau\mathbf{u}_h^*\| \leq Ch^{\alpha(r+1)}\|\mathbf{u}\|_{r+1} \\ &+ Ch^{k+s-1+\alpha\min(0,2-s)}(\|\mathbf{u}\|_{k+1} + \|p\|_k),\end{aligned}$$

which completes the estimate for $\|\mathbf{u} - Q_\tau\mathbf{u}_h^*\|$ in (29). The gradient term $h^\alpha\|\nabla_\tau(\mathbf{u} - Q_\tau\mathbf{u}_h^*)\|$ can be estimated in a similar manner, and is thus omitted. $\square$

For the pressure approximation, we have the following superconvergence.

Theorem 3. Assume that (12) holds true with $1 \le s \le k+1$ and $W_\tau \subset H^{s-1}(\Omega)$. Let $(\mathbf{u}_h; p_h)$ be the finite element approximation of the solution $(\mathbf{u}; p)$ of (6). Then, we have

$$\|p - R_\tau p_h\| \le Ch^{\alpha(t+1)}\|p\|_{t+1} + Ch^{\varrho}\left(\|\mathbf{u}\|_{k+1} + \|p\|_k\right), \tag{31}$$

where $\varrho = k + s - 1 + \alpha\min(0, 1-s)$.

Proof. By the definition of R_τ and the scale relation (13), we have

$$\|p - R_\tau p\| \le C\tau^{t+1}\|p\|_{t+1} = Ch^{\alpha(t+1)}\|p\|_{t+1}. \tag{32}$$

Thus, it follows from (32) and (16) that

$$\begin{aligned} \|p - R_\tau p_h\| &\le \|p - R_\tau p\| + \|R_\tau p - R_\tau p_h\| \le Ch^{\alpha(t+1)}\|p\|_{t+1} \\ &+ Ch^{k+s-1+\alpha\min(0,1-s)}(\|\mathbf{u}\|_{k+1} + \|p - p_h\|_k), \end{aligned} \tag{33}$$

which completes the proof. □

The velocity estimate can be optimized by choosing $\alpha = \alpha_u$ such that

$$\alpha_u(r+1) = k + s - 1 + \alpha_u\min(0, 2-s). \tag{34}$$

The corresponding error estimate is given by

$$\begin{aligned} \|\mathbf{u} - Q_\tau \mathbf{u}_h^*\| &+ h^{\alpha_u}\|\nabla_\tau(\mathbf{u} - Q_\tau \mathbf{u}_h^*)\| \\ &\le Ch^{\alpha_u(r+1)}\left(\|\mathbf{u}\|_{r+1} + \|\mathbf{u}\|_{k+1} + \|p\|_k\right). \end{aligned}$$

Similarly, the pressure estimate can be optimized by choosing $\alpha = \alpha_p$ such that

$$\alpha_p(t+1) = k + s - 1 + \alpha_p\min(0, 1-s).$$

The corresponding error estimate for the post-processed pressure approximation is given by

$$\|p - R_\tau p_h\| \le Ch^{\alpha_p(t+1)}\left(\|p\|_{t+1} + \|\mathbf{u}\|_{k+1} + \|p\|_k\right).$$

The results are summarized as follows.

Theorem 4. Assume that (12) holds true with $1 \le s \le k+1$. Let the surface fitting spaces V_τ and W_τ be sufficiently smooth such that $V_\tau \subset H^{s-2}(\Omega)^2$ and $W_\tau \subset H^{s-1}(\Omega)$. Let $(\mathbf{u}_h; p_h)$ be the finite element approximation of the solution $(\mathbf{u}; p)$ of (6). Denote by $\mathbf{u}_h^*$ the modified/corrected approximation such that over each element T,

$$\mathbf{u}_h^*(x) = \mathbf{u}_h(x) + \gamma h_T^2(f(x) - \nabla p_h(x) + \Delta \mathbf{u}_h(x)), \quad x \in T.$$

Then, the post-processed velocity approximation $Q_\tau \mathbf{u}_h^*$ satisfies the following superconvergent estimate:

$$\begin{aligned} \|\mathbf{u} - Q_\tau \mathbf{u}_h^*\| &+ h^{\alpha_u}\|\nabla_\tau(\mathbf{u} - Q_\tau \mathbf{u}_h^*)\| \\ &\le Ch^{\alpha_u(r+1)}\left(\|\mathbf{u}\|_{r+1} + \|\mathbf{u}\|_{k+1} + \|p\|_k\right), \end{aligned} \tag{35}$$

where

$$\alpha_u = \frac{k + s - 1}{r + 1 - \min(0, 2-s)}. \tag{36}$$

As to the pressure unknown, the post-processed pressure approximation $Q_\tau p_h$ satisfies the following estimate

$$\|p - R_\tau p_h\| \le Ch^{\alpha_p(t+1)}\left(\|p\|_{t+1} + \|\mathbf{u}\|_{k+1} + \|p\|_k\right),$$

where

$$\alpha_p = \frac{k+s-1}{t+1-\min(0,1-s)}. \tag{37}$$

It should be pointed out that the coarse scale for postprocessing the velocity might be different from that for the pressure in the theoretical optimization.

As an illustrative example, we consider an application of the superconvergence theory on the $P_{k+1} - P_k$ element. In this case, the finite element spaces V_h and W_h are defined as follows:

$$V_h = \{\mathbf{v} \in C^0(\Omega)^2 : \ \mathbf{v}|_T \in (P_{k+1}(T))^2, \forall T \in \mathcal{T}_h, \mathbf{v}|_{\partial\Omega} = 0\}$$

and

$$W_h = \{q \in C^0(\Omega) : \ q|_T \in P_k(T), \ \forall T \in \mathcal{T}_h\} \cap L_0^2(\Omega),$$

or

$$W_h = \{q \in L_0^2(\Omega) : \ q|_T \in P_k(T), \ \forall T \in \mathcal{T}_h\},$$

where $P_k(T)$ consists of all the polynomials with degree less or equal to k defined on the element T.

Let $(\mathbf{u}_h; p_h)$ be the solution of the stabilized finite element formulation (6) with V_h and W_h defined as above. It follows from Theorem 1 that the following error estimate holds true:

$$\|\nabla(\mathbf{u} - \mathbf{u}_h)\| + \|p - p_h\| \le Ch^k(\|\mathbf{u}\|_{k+1} + \|p\|_k).$$

But for the L^2 projected approximation, we have from Theorem 4 that

$$\|\mathbf{u} - Q_\tau \mathbf{u}_h^*\| + h^{\alpha_u}\|\nabla_\tau(\mathbf{u} - Q_\tau \mathbf{u}_h^*)\| \le Ch^{\alpha_u(r+1)}\left(\|\mathbf{u}\|_{r+1} + \|\mathbf{u}\|_{k+1} + \|p\|_k\right)$$

and

$$\|p - R_\tau p_h\| \le Ch^{\alpha_p(t+1)}\left(\|p\|_{t+1} + \|\mathbf{u}\|_{k+1} + \|p\|_k\right)$$

with α_u and α_p defined in (36) and (37) respectively.

For the $P_1 - P_0$ elements, it is well known that the velocity is over-constrained and a locking phenomenon will occur when this element is used in the standard finite element formulation. However, this simple velocity/pressure combination can be used in the stabilized finite element formulation. As to superconvergence, since the order of polynomial here is $k = 1$, it is sufficient to assume the H^2-regularity (i.e., $s = 2$ in Theorem 4). The fitting finite element space V_τ must be selected to satisfy $V_\tau \subset H^{s-2}(\Omega)$. Since $s = 2$ in the application to the $P_1 - P_0$, we see that V_τ could be chosen as a finite element space consisting of discontinuous piecewise polynomials of degree $r \ge 0$. Using Theorem 4 we obtain the following estimate for the velocity approximation:

$$\|\mathbf{u} - Q_\tau \mathbf{u}_h^*\| \le Ch^2(\|\mathbf{u}\|_{r+1} + \|\mathbf{u}\|_2 + \|p\|_1), \tag{38}$$

$$\|\nabla_\tau(\mathbf{u} - Q_\tau \mathbf{u}_h^*)\| \le Ch^{\frac{2r}{r+1}}(\|\mathbf{u}\|_2 + \|\mathbf{u}\|_{r+1} + \|p\|_1). \tag{39}$$

Therefore, we see no improvement for the velocity error in L^2-norm. But (39) is a superconvergence result for the gradient of the velocity. For example, with $r = 2$ (i.e., projection to the space of piecewise quadratic functions), the post-processed velocity approximation has the following superconvergence:

$$\|\nabla_\tau(\mathbf{u} - Q_\tau \mathbf{u}_h^*)\| \le Ch^{\frac{4}{3}}(\|\mathbf{u}\|_3 + \|p\|_1). \tag{40}$$

The above estimate is useful for an accurate determination of the fluid velocity.

As to the pressure approximation, observe that the theory presented in Theorem 4 requires that $W_\tau \subset H^{s-1}(\Omega) = H^1(\Omega)$ in the post-processing method. Let W_τ

be a surface fitting space consisting of continuous piecewise polynomials of degree $t \geq 1$. By using Theorem 4 we obtain the following estimate

$$\|p - R_\tau p_h\| \leq Ch^{\frac{2(t+1)}{t+2}}(\|\mathbf{u}\|_2 + \|\mathbf{u}\|_{t+1} + \|p\|_1). \tag{41}$$

With $s = 2$ and $t = 1$, we have the following error estimate for the pressure approximation

$$\|p - R_\tau p_h\| \leq Ch^{\frac{4}{3}}(\|\mathbf{u}\|_2 + \|p\|_2). \tag{42}$$

With $s = 2$ and $t = 2$, the pressure approximation can be improved by

$$\|p - R_\tau p_h\| \leq Ch^{\frac{3}{2}}(\|\mathbf{u}\|_2 + \|p\|_3). \tag{43}$$

Assume that the exact solution is sufficiently smooth. Then it is not hard to see that

$$\|\nabla_\tau(\mathbf{u} - Q_\tau \mathbf{u}_h^*)\| \approx O(h^2), \quad \text{as } r \to \infty$$

and

$$\|p - R_\tau p_h\| \approx O(h^2), \quad \text{as } t \to \infty. \tag{44}$$

In practical computation, there is no need to use very high order of polynomials in the L^2 projection method. The results developed in this paper are robust and applicable to finite element partitions with the usual assumption on regularity. In theory, the L^2 projection is computationally easy to implement. But it remains to numerically verify the efficiency of the superconvergent algorithms presented and analyzed in this paper.

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