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SUPERCONVERGENCE OF GALERKIN SOLUTIONS FOR HAMMERSTEIN EQUATIONS

QIUMEI HUANG AND HEHU XIE

Abstract. In the present paper, we discuss the superconvergence of the interpolated Galerkin solutions for Hammerstein equations. With the interpolation post-processing for the Galerkin approximation x_h, we get a higher order approximation $I_{2h}^{2r-1}x_h$, whose convergence order is the same as that of the iterated Galerkin solution. Such an interpolation post-processing method is much simpler than the iterated method especially for the weak singular kernel case. Some numerical experiments are carried out to demonstrate the effectiveness of the interpolation post-processing method.

Key words. superconvergence, interpolation post-processing, iterated Galerkin method, Hammerstein equations, smooth and weakly singular kernels.

1. Introduction

In this paper, we investigate the superconvergence of the interpolated Galerkin solutions for Hammerstein equations with smooth and weakly singular kernels. As for Hammerstein equations, various numerical methods have been used to get the approximations. A variation of Nyström's method was proposed by Lardy [18]. Two different discrete collocation methods were proposed by Kumar [17] and Atkinson and Flores [3]. Brunner [7] discussed the connection between implicitly linear collocation methods and iterated spline collocation methods for Hammerstein equations, and then extended the results to a class of nonlinear Volterra-Fredholm integral equations. A degenerated kernel method for Hammerstein equations was introduced by Kaneko and Xu [14]. Kaneko, Noren, and Xu [13] used the product integration method and the collocation method to solve Hammerstein equations with weakly singular kernels, and got some superconvergence properties. A survey paper by Atkinson [2] gave more information about numerical solutions of Hammerstein equations. The superconvergence of the iterated Galerkin solutions for Hammerstein equations with smooth as well as weakly singular kernels was probed by Kaneko and Xu [16]. Moreover, the superconvergence of the iterated collocation method for Hammerstein equations with smooth as well as weakly singular kernels was studied by Kaneko, Noren, and Padila [11].

For Hammerstein equations, generally, the iterated post-processing method (see, for example, [4, 7, 11, 16]) is used to accelerate the approximation. If the kernel is sufficiently smooth, it is very easy to get the iterated Galerkin solutions. But if the kernel is weakly singular, there are many difficulties to

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get the iterated Galerkin solutions since the classical numerical quadrature is no longer valid.

In this paper, we use another type of acceleration method, the interpolation post-processing method, to get the same superconvergence. Applying the interpolation post-processing to the Galerkin approximation x_h, we get a higher accuracy approximation $I_{2h}^{2r-1}x_h$ (which is named the interpolated Galerkin solution throughout this paper), whose convergence order is the same as that of the iterated Galerkin solution. Furthermore, the interpolation post-processing method is simpler than the iterated post-processing method since we just need to interpolate x_h at some nearby points to get the interpolated Galerkin solution instead of computing a nonlinear integral for each subinterval which is especially difficult for the weakly singular kernel.

The interpolation post-processing technique can be used to improve the approximate rate of finite element solutions for various partial differential equations, integral equations, and integro-differential equations, and the corresponding work has been contained in some papers (such as [22, 26]) and some monographs (see [20, 21] for example). It has been found that this technique is both simple and of higher accuracy. For Hammerstein equations, Huang and Zhang [10] applied the interpolation post-processing to collocation solutions and obtained the same superconvergence as that of the iterated collocation method.

Here is the outline of the remaining sections. The Galerkin method and the iterated Galerkin method for Hammerstein equations are presented in Section 2. And some materials for the approximation theory are also reviewed in this section to make the paper self-contained. In Section 3, main results about the superconvergence of interpolated Galerkin solutions, instead of the iterated collocation solutions, are obtained. Finally, numerical experiments are listed in Section 4 to show the efficiency of the interpolation post-processing method.

2. The Iterated Galerkin Method

In this section, the Galerkin method and the iterated Galerkin method are considered for the following Hammerstein equation

$$x(t) - \int_0^1 k(t,s)\psi(s,x(s))ds = f(t), \qquad 0 \le t \le 1, \tag{2.1}$$

where k, f and ψ are known functions and x is the function to be determined. Define $k_t(s) \equiv k(t,s)$ for $t,s \in [0,1]$ to be the t section of k. We assume throughout this paper unless stated otherwise, the following conditions on k, f, and ψ hold:

1. $\lim\limits_{t\to\tau} \|k_t - k_\tau\|_\infty = 0, \qquad \tau \in [0,1];$
2. $M \equiv \sup\limits_{0\le t\le 1} \int_0^1 |k(t,s)|ds < \infty;$
3. $f \in C[0,1];$

4. $\psi(s,x)$ and its partial derivative $\psi^{(0,1)}$ with respect to the second variable are continuous in $s \in [0,1]$ and Lipschitz continuous in $x \in (-\infty, \infty)$, i.e., there exists a constant $C_1 > 0$ such that

$$\begin{aligned} |\psi(s,x_1) - \psi(s,x_2)| &\leq C_1|x_1 - x_2|, \quad \text{for all } x_1, x_2 \in (-\infty,\infty); \\ |\psi^{(0,1)}(t,x_1) - \psi^{(0,1)}(t,x_2)| &\leq C_2|x_1 - x_2|, \quad \text{for all } x_1, x_2 \in (-\infty,\infty); \end{aligned} \tag{2.2}$$

5. for $x \in C[0,1]$, $\psi(\cdot, x(\cdot)), \psi^{(0,1)}(\cdot, x(\cdot)) \in C[0,1]$.

Let

$$(K\Psi)(x)(t) \equiv \int_a^b k(t,s)\psi(s,x(s))ds.$$

We get the corresponding operator form of (2.1)

$$x - K\Psi x = f. \tag{2.3}$$

For any positive integer n, let

$$T_h: \ 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

be a quasi-uniform partition of [0,1]. Namely, it satisfies the condition that there exists a constant $C > 0$, independent of n, such that

$$\frac{\max\limits_{1\le i\le n}(t_i - t_{i-1})}{\min\limits_{1\le i\le n}(t_i - t_{i-1})} \le C, \quad \text{for all } n.$$

The subintervals generated by this partition of T_h are denoted by I_i, i.e., $I_1 = [t_0, t_1]$, $I_i = (t_{i-1}, t_i]$ $(i = 2, 3, \cdots, n)$. Let $h_i = t_i - t_{i-1}$ $(i = 1, 2, \cdots, n)$ and $h = \max\limits_{1\le k\le n} h_k$. We assume that the mesh size h of the partition tends to zero as $n \to \infty$. With r, a positive integer, let $S_r^0(T_h)$ be the space of all piecewise polynomials of order r (i.e., of degree at most $r-1$) on each subinterval I_i

$$S_r^0(T_h) = \{x \in L^2[0,1] : \ x|_{I_i} \in P_{r-1}, \text{ for each } i = 1, 2, \cdots, n\},$$

where the superscript 0 denotes there is no continuity condition imposed at the breakpoints, P_{r-1} denotes the space of polynomials of degree not exceeding $r-1$. It is evident that $N := \dim(S_r^0(T_h)) = nr$.

Let $P_h : C[0,1] + S_r^0(T_h) \to S_r^0(T_h)$ be an orthogonal projection operator with respect to the L_2 inner product

$$(u,v) = \int_0^1 u(s)v(s)ds$$

satisfying that, for $u \in S_r^0(T_h)$

$$(u, x) = (u, P_h x). \tag{2.4}$$

It is known that the projection P_h when restricted to $C[0,1]$ is uniformly bounded, i.e.,

$$C := \sup_h \|P_h|_{C[0,1]}\| < \infty, \tag{2.5}$$

and $P_h \to I$ pointwisely in $C[0,1]$ as $h \to 0$.

In this paper, C denotes a generic constant which may takes different values at its different occurrences, but will be independent of n.

In many cases, equation (2.1) possesses multiple solutions. Therefore, we assume for the remainder of this paper that we only treat an isolated solution x_0 of (2.1).

Let φ_{ij} $(i = 1, \cdots, n; j = 1, \cdots, r)$ be the basis function of each subinterval I_i and $S_r^0(T_h) = \text{span}\{\varphi_{ij}\}$. Then the Galerkin method is to find

$$x_h = \sum_{i=1}^{n} \sum_{j=1}^{r} b_{ij} \varphi_{ij} \tag{2.6}$$

that satisfies

$$x_h - P_h K \Psi x_h = P_h f. \tag{2.7}$$

Equivalently, we need to find the unknown coefficients $\{b_{ij}\}$ $(i = 1, \cdots, n; j = 1, \cdots, r)$ from the following system of nonlinear equations

$$\begin{aligned} &\sum_{i=1}^{n} \sum_{j=1}^{r} b_{ij}(\varphi_{ij}, \varphi_{kl}) - (\int_0^1 k(t,s)\psi(s, \sum_{i=1}^{n} \sum_{j=1}^{r} b_{ij}\varphi_{ij}(s))ds, \varphi_{kl}) \\ &= (f, \varphi_{kl}),\ 1 \leq k \leq n,\ 1 \leq l \leq r. \end{aligned} \tag{2.8}$$

Let's define

$$x_h^I = f + K \Psi x_h. \tag{2.9}$$

Applying P_h to the both sides of (2.9), we have

$$P_h x_h^I = P_h f + P_h K \Psi x_h. \tag{2.10}$$

Comparing (2.10) with (2.7), we see that

$$x_h = P_h x_h^I, \tag{2.11}$$

and the iterated Galerkin approximation x_h^I satisfies

$$x_h^I = f + K \Psi P_h x_h^I. \tag{2.12}$$

Let $I = [0, 1]$. We define $W_p^m(I)$, for $1 \leq p \leq \infty$ and m (a nonnegative integer), to be the Sobolev space of functions g such that $g^{(k)} \in L_p(I)$ for $k = 0, 1, \cdots, m$, where $g^{(k)}$ is the $k-$th order distributional derivative of g. The space $W_p^m(I)$ is equipped with the norm

$$\|g\|_{m,p,I}^p = \sum_{k=0}^{m} \|g^{(k)}\|_{0,p,I}^p.$$

For simplicity, we write $\|\cdot\|_{m,p,I}$ as $\|\cdot\|_{m,p}$ when $I = [0, 1]$. If $p = \infty$, the norm of the space $W_\infty^m(I)$ is defined by

$$\|g\|_{m,\infty} = \max_{0 \leq i \leq m} \{\|g^{(i)}\|_\infty\}.$$

We recall the following convergence and superconvergence results from [16].

Lemma 2.1 *Let x_0 be an isolated solution of equation (2.3) and x_h the solution of equation (2.7) in a neighborhood of x_0. Assume that 1 is not an eigenvalue of $(K\Psi)'(x_0)$, where $(K\Psi)'(x_0)$ denotes the Fréchet derivative of $K\Psi$ at x_0. If $x_0 \in W^l_\infty(I)\,(0 \le l \le r)$, then*

$$\|x_0 - x_h\|_\infty = O(h^\mu),$$

where $\mu = \min\{l, r\}$. If $x_0 \in W^l_p(I)$ $(0 < l \le r,\ 1 \le p < \infty)$, then

$$\|x_0 - x_h\|_\infty = O(h^\nu),$$

where $\nu = \min\{l-1, r\}$.

Then, let's introduce the following Lemma 2.2 which establishes the superconvergence of the iterated Galerkin method in a general setting .

Lemma 2.2 *Let $x_0 \in C[0,1]$ be an isolated solution of equation (2.3), x_h be the unique solution of (2.7) in the sphere $B(x_0, \delta_1)$ for some $\delta_1 > 0$ and x^I_h be defined by the iterated scheme (2.9) (or (2.12)). Assume that 1 is not an eigenvalue of $(K\Psi)'(x_0)$. Then, for all $1 \le p \le \infty$,*

$$\|x_0 - x^I_h\|_\infty \le C\left\{\|x_0 - P_h x_0\|^2_\infty + \sup_{0\le t\le 1}\inf_{u\in S^0_r(T_h)} \|k(t,\cdot)\psi^{(0,1)}(\cdot, x_0(\cdot)) - u\|_{0,q}\|x_0 - P_h x_0\|_{0,p}\right\},$$

where $1/p + 1/q = 1$ and C is a constant independent of h.

The following two lemmas are obtained from the results of Lemma 2.2. First, when both the kernels and the solutions of equation (2.1) are smooth, the following result holds.

Lemma 2.3 *Let $x_0 \in W^l_p(I)\,(0 < l \le r)$ be an isolated solution of equation (2.3) and x_h the unique solution of (2.7) in the sphere $B(x_0, \delta_1)$ for some $\delta_1 > 0$. Let x^I_h be defined by the iterated scheme (2.9) (or (2.12)). Assume that 1 is not an eigenvalue of $(K\Psi)'(x_0)$ and for all $t \in [0,1]$, $k_t(\cdot)\psi^{(0,1)}(\cdot, x_0(\cdot)) \in W^m_q\,(0 \le m \le r)$. Then*

$$\|x_0 - x^I_h\|_\infty = O(h^{\mu+\min\{\mu,\nu\}}),$$

where $1/p + 1/q = 1$, $\mu = \min\{l, r\}$, and $\nu = \min\{m, r\}$.

When the kernel k is of weakly singular type, we consider the following special form

$$k(t,s) = m(t,s)g_\alpha(|t-s|), \tag{2.13}$$

where $m \in C^{\mu+1}(I \times I)$ and

$$g_\alpha(s) = \begin{cases} s^{\alpha-1}, & 0 < \alpha < 1, \\ \log s, & \alpha = 1. \end{cases} \tag{2.14}$$

Generally, the solution x_0 of equation (2.3) does not belong to $W^m_p(I)$. Let S be a finite set in $[0,1]$ and define the function $\omega_S(t) = \inf\{|t-s| : s \in S\}$. A function is said to be of $Type(\alpha, k, S)$, for $-1 < \alpha < 0$, if

$$|x^{(k)}(t)| \le C[\omega_S(t)]^{\alpha-k}, \quad t \notin S,$$

and for $\alpha > 0$, if the above condition holds and $x \in \mathrm{Lip}(\alpha)$. Here $\mathrm{Lip}(\alpha) = \{x : |x(t) - x(s)| \le C|t-s|^{\alpha}\}$. It was proved by Kaneko, Noren and Xu [12] that if f is of $Type(\beta, \mu, \{0,1\})$, then a solution of equation (2.1) with the kernel defined by (2.13) and (2.14) is of $Type(\gamma, \mu, \{0,1\})$, where $\gamma = \min\{\alpha, \beta\}$. The optimal convergence rate of the Galerkin solution x_h to x_0 can be recovered by selecting the knots of [0,1] that are defined by

$$\begin{aligned} t_i &= (1/2)(2i/n)^q, && 0 \le i \le n/2, \\ t_i &= 1 - t_{n-i}, && n/2 < i \le n, \end{aligned} \tag{2.15}$$

where $q = r/\gamma$ denotes the index of singularity(see [12] for details).

Applying Lemma 2.2 to equation (2.1) with kernels given by (2.13) and (2.14) and use $S_r^0(T_h)$ (where $S_r^0(T_h)$ of splines with nonuniform knots is defined as (2.15)) as approximate spaces, the authors got the following result.

Lemma 2.4 *Let x_0 be an isolated solution of equation (2.3) with kernels given by (2.13) and (2.14), x_h be the unique solution of (2.7) in the sphere $B(x_0, \delta_1)$ with some $\delta_1 > 0$ and knots defined by (2.15) and x_h^I be defined by the iterated scheme (2.9) or (2.12). Assume that 1 is not an eigenvalue of $(K\Psi)'(x_0)$ and that $\psi^{(0,1)}(\cdot, x_0(\cdot))$ is of $Type(\alpha, r, \{0,1\})$ for $\alpha > 0$ whenever x_0 is of the same type. Then*

$$\|x_0 - x_h^I\|_\infty = O(h^{r+\alpha}). \tag{2.16}$$

The following lemma is concerned with spline approximation in L_p spaces (see [9] for details).

Lemma 2.5 *Let $1 \le p \le \infty$, $g \in W_p^m(I)$, $m \ge 0$. Then for each $n \ge 1$, there exists $\phi_h \in S_r^0(T_h)$ such that*

$$\|g - \phi_h\|_{0,p} \le ch^{m^*}\|g\|_{m^*,p},$$

where $m^* = \min\{m, r\}$.

3. Superconvergence of Interpolated Galerkin Method

In this section, we apply the interpolation post-processing technique to the Galerkin solution to get a superconvergent approximation.

Theorem 3.1 *Let $x_0 \in C[0,1]$ be an isolated solution of equation (2.3) and x_h be the unique solution of (2.7) in the sphere $B(x_0, \delta_1)$ for some $\delta_1 > 0$. P_h is the orthogonal projection operator defined by (2.4). Assume that 1 is not an eigenvalue of $(K\Psi)'(x_0)$. Then, for all $1 \le p \le \infty$,*

$$\begin{aligned} \|x_h - P_h x_0\|_\infty \le C\Big\{&\|x_0 - P_h x_0\|_\infty^2 \\ &+ \sup_{0\le t\le 1} \inf_{u \in S_r^0(T_h)} \|k(t,\cdot)\psi^{(0,1)}(\cdot, x_0(\cdot)) - u\|_{0,q}\|x_0 - P_h x_0\|_{0,p}\Big\}, \end{aligned} \tag{3.1}$$

where $1/p + 1/q = 1$.

Proof. For simplicity, we use the result of Lemma 2.2. In fact, from (2.11), (2.5), and Lemma 2.2, we obtain

$$\begin{aligned} & \|x_h - P_h x_0\|_\infty = \|P_h x_h^I - P_h x_0\|_\infty \\ & \le \sup_h \|P_h\| \cdot \|x_h^I - x_0\|_\infty \le C\|x_h^I - x_0\|_\infty \\ \le \ & C\left\{\|x_0 - P_h x_0\|_\infty^2 \right. \\ & \left. + \sup_{0\le t\le 1} \inf_{u\in S_r^0(T_h)} \|k(t,\cdot)\psi^{(0,1)}(\cdot, x_0(\cdot)) - u\|_{0,q} \|x_0 - P_h x_0\|_{0,p}\right\}. \end{aligned}$$

This completes the proof. □

Remark 3.1 *In fact, the interpolated post-processing has nothing to do with the iterated post-processing. That is, these two kinds of post-processing are independent with each other. We can also prove Theorem 3.1 without the results of Lemma 2.2. See Appendix for details.*

(3.1) shows that x_h (the Galerkin approximation) is closer to $P_h x_0$ (the orthogonal projection of x_0) than to the solution x_0 itself, which is called the superclose. From (3.1), we can obtain global superconvergence of the interpolated Galerkin solutions by applying the interpolation post-processing to the Galerkin solutions.

We assume that T_h is gained from T_{2h} with mesh size $2h$ by subdividing each element into two equal elements, so that the number of elements N for T_h is a even number. Then we define a higher order interpolation operator I_{2h}^{2r-1} of degree $(2r-1)$ on each bigger element $I_i \cup I_{i+1}$ $(i = 1, 3, 5, \cdots, N-1)$ associated with T_h according to the following conditions:

$$I_{2h}^{2r-1} x|_{I_i\cup I_{i+1}} \in P_{2r-1}, \quad i = 1, 3, 5, \cdots, N-1, \tag{3.2}$$

and

$$\int_{I_l} (x - I_{2h}^{2r-1} x) v ds = 0, \ \forall v \in P_{r-1}(I_l), \quad l = i, i+1. \tag{3.3}$$

It is easy to check that

$$I_{2h}^{2r-1} P_h = I_{2h}^{2r-1} \quad \text{and} \quad \|I_{2h}^{2r-1} v\|_\infty \le C\|v\|_\infty, \quad \forall v \in S_r^0(T_h). \tag{3.4}$$

Then, we get the global superconvergence by interpolation post-processing method.

When both the kernels and the solutions of equation (2.1) are smooth, the following result holds.

Theorem 3.2 *Let $x_0 \in W_p^l(I) \cap W_\infty^{2r}(I)$ $(0 < l \le r)$ be an isolated solution of equation (2.3) and x_h the unique solution of (2.7) in the sphere $B(x_0, \delta_1)$ for some $\delta_1 > 0$. The interpolation operator I_{2h}^{2r-1} of degree (2r-1) is defined by (3.2) and(3.3). Assume that 1 is not an eigenvalue of $(K\Psi)'(x_0)$ and for all $t \in [0,1]$, $k_t(\cdot)\psi^{(0,1)}(\cdot, x_0(\cdot)) \in W_q^m$ $(0 \le m \le r)$. Then*

$$\|I_{2h}^{2r-1} x_h - x_0\|_\infty = O(h^{\mu+\min\{\mu,\nu\}}),$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $\mu = \min\{l, r\}$ and $\nu = \min\{m, r\}$.

Proof. From (3.4), the interpolation error estimates, and Theorem 3.1, we have

$$\begin{aligned}(3.5)\ \|I_{2h}^{2r-1}x_h - x_0\|_\infty \\ \leq\ & \|I_{2h}^{2r-1}x_h - I_{2h}^{2r-1}P_h x_0\|_\infty + \|I_{2h}^{2r-1}P_h x_0 - x_0\|_\infty \\ \leq\ & C\|x_h - P_h x_0\|_\infty + \|I_{2h}^{2r-1}x_0 - x_0\|_\infty \\ \leq\ & C\Big\{\|x_0 - P_h x_0\|_\infty^2 \\ & + \sup_{0\leq t\leq 1}\inf_{u\in S_r^0(T_h)}\|k(t,\cdot)\psi^{(0,1)}(\cdot,x_0(\cdot)) - u\|_{0,q}\|x_0 - P_h x_0\|_{0,p}\Big\} + O(h^{2r}).\end{aligned}$$

It follows from Lemma 2.5 that, for all $u\in S_r^0(T_h)$,

$$\begin{aligned}\|x_0 - P_h x_0\|_{0,p} \ \leq\ & \|x_0 - P_h x_0\|_\infty \leq \|x_0 - u\|_\infty + \|P_h(u - x_0)\|_\infty \\ \leq\ & (1+C)\inf_{u\in S_r^0(T_h)}\|x_0 - u\|_\infty \leq O(h^\mu),\end{aligned}$$

where $\mu = \min\{l, r\}$. Similarly, Lemma 2.5 leads to that

$$\sup_{0\leq t\leq 1}\inf_{u\in S_r^0(T_h)}\|k(t,\cdot)\psi^{(0,1)}(\cdot,x_0(\cdot)) - u\|_{0,q} \leq O(h^\nu),$$

where $\nu = \min\{m, r\}$. Then, from Theorem 3.1 and Lemma 2.1, we complete the proof of Theorem 3.2. □

The superconvergence of the interpolated Galerkin solutions of Hammerstein equations with weakly singular kernels is also considered.

The following theorem gives the superconvergence for the equation (2.3) with the kernels given by (2.13) and (2.14).

Theorem 3.3 *Let $x_0 \in C[0,1]\cap W_\infty^{r+\alpha}(I)$ be an isolated solution of the exact equation (2.3) with kernels defined by (2.13) and (2.14), x_h be the unique solution of the Galerkin equation (2.6) in the sphere $B(x_0,\delta_1)$. The interpolation operator I_{2h}^{2r-1} of degree (2r-1) is defined by (3.2) and (3.3). Assume that 1 is not an eigenvalue of $(K\Psi)'(x_0)$ and $\psi^{(0,1)}(\cdot,x_0(\cdot))$ is of $Type(\alpha, r, \{0,1\})$ for $\alpha > 0$ whenever x_0 is of the same type. Then we have*

$$\|I_{2h}^{2r-1}x_h - x_0\|_\infty = O(h^{r+\alpha}). \tag{3.6}$$

Proof. From (3.4), (2.10), and the interpolation error estimates, we have

$$\begin{aligned}\|I_{2h}^{2r-1}x_h - x_0\|_\infty \ \leq\ & \|I_{2h}^{2r-1}x_h - I_{2h}^{2r-1}P_h x_0\|_\infty + \|I_{2h}^{2r-1}P_h x_0 - x_0\|_\infty \\ \leq\ & C\|x_h - P_h x_0\|_\infty + \|I_{2h}^{2r-1}x_0 - x_0\|_\infty \\ \leq\ & C\sup_h\|P_h\|\|x_h^I - x_0\|_\infty + \|I_{2h}^{2r-1}x_0 - x_0\|_\infty \\ \leq\ & C\|x_h^I - x_0\|_\infty + O(h^{r+\alpha}),\end{aligned}$$

which, together with Lemma 2.4, leads to (3.6). □

Since the theoretical results of the superconvergence of the interpolated Galerkin solutions are the same as that of the iterated Galerkin solutions, why we use the interpolation post-processing? What are the advantages of this post-processing? We then give a comparison between these two post-processing methods.

- Iterated Galerkin method

  Since we must compute the integral

$$x_h^I(t) = f(t) + \int_0^1 k(t,s)\psi(s,x(s))ds := f + U,$$

  the computational complexity of x_h^I is determined by the the computational complexity of U.

  (1) For the smooth kernel case.
  The nonlinear integral U can be approximated directly by the classical numerical quadrature.

  (2) For the weakly singular kernel case.
  The classical numerical quadrature is no longer valid since it won't converge the true solution any more. We need to use the graded mesh generated by the singularity of the kernel to approximate the nonlinear integral which may increase greatly the complexity of the numerical quadrature (see [15] for details).

- Interpolated Galerkin method

  There is no such difficulty since we just need to interpolate x_h at some nearby points to get the interpolated Galerkin solution.

Therefore, we conclude that the interpolation post-processing method is not only effective but also simple.

Remark 3.2 *By the linear transformation Φ from [0,1] to [a,b], $\Phi(t) = (b-a)t + a$, $t \in [0,1]$, all the lemmas and theorems can be extended from [0,1] to [a,b].*

4. Numerical Experiments

In this section, two examples are given to illustrate the theory established in the previous sections.

Example 4.1 The equation

$$x(t) = t^2 + (\sin t) \cdot \int_{-1}^1 \exp(-2s)x^2(s)ds, \quad t \in [-1,1]$$

can be proved to have two solutions, one of which is

$$x(t) = t^2 + c\sin t,$$

where $c = 1.9577839864709\ldots$.

We choose uniform partition with mesh $h = 2/n\,(n = 8, 16, 32, 64, 128)$. The basis functions φ_{ij} $(i = 1, \cdots, n; j = 1, \cdots, r)$ are generated by selecting the r Gaussian points in each subinterval as interpolated points. Results are obtained by using piecewise linear functions ($r = 2$) and piecewise quadratic functions ($r = 3$). The spline coefficients in (2.8) are obtained by using Newton-Raphson algorithm. The tabulated errors are estimated by taking the largest of the computed errors at $z_i = -1 + i/50$ $(i = 0, 1, \cdots, 100)$.

$e_h = \|x - x_h\|_\infty$, $e'_h = \|x - I_{2h}^{2r-1} x_h\|_\infty$, and $e''_h = \|x - x_h^I\|_\infty$ denote the approximate errors which are defined by

$$\max\{|x(z_i) - x_h(z_i)| : i = 0, 1, \cdots, 100\},$$
$$\max\{|x(z_i) - I_{2h}^{2r-1} x_h(z_i)| : i = 0, 1, \cdots, 100\},$$

and

$$\max\{|x(z_i) - x_h^I(z_i)| : i = 0, 1, \cdots, 100\}.$$

For simplicity, we introduce some notations as follows:

$$R_h = \log_2(e_h/e_{h/2}), \quad R'_h = \log_2(e'_h/e'_{h/2}), \quad R''_h = \log_2(e''_h/e''_{h/2}).$$

Numerical results are listed in the following tables.

Table 1 The errors of the approximate solutions for $r = 2$

n	e_h	R_h	e'_h	R'_h	e''_h	R''_h
16	4.6878e-3		7.3792e-6		7.9916e-6	
32	1.1799e-3	1.9903	4.6929e-7	3.9749	5.0759e-7	3.9767
64	2.9592e-4	1.9954	2.9646e-8	3.9846	3.1854e-8	3.9941
128	7.4094e-5	1.9977	1.8575e-9	3.9964	2.0255e-9	3.9751

Table 2 The errors of the approximate solutions for $r = 3$

n	e_h	R_h	e'_h	R'_h	e''_h	R''_h
8	2.5361e-4		5.6658e-7		1.3429e-6	
16	3.1824e-5	2.9944	9.2967e-9	5.9294	2.1982e-8	5.9329
32	3.9818e-6	2.9986	1.4819e-10	5.9711	3.4824e-10	5.9800

The results displayed in the above table show that the convergence order of the two post-processing are all $O(h^{2r})$ which support the theory of Theorem 3.2. Here, because the kernel function $k(t, s)$ is of the degenerate from $k(t, s) = g(s) \cdot h(t)$, the computational cost of the two post-processing methods are all very small, we didn't list the elapsed time.

Example 4.2. Let's consider the equation

$$x(t) - \int_0^1 \frac{x^2(s)}{\sqrt{|t - s|}} ds = f(t), \quad t \in [0, 1], \tag{4.1}$$

where f is selected so that $x(t) = \sqrt{t}$ is the solution. The splines of order 1 ($q = 3$, see [19] for details) with knots defined by equation (2.15) in terms of q is used in the computation.

The interpolated points are M_i $(i = 1, \cdots, n)$ (one Gaussian point as interpolated point in each subinterval). Let $l_i(t)$ be the corresponding Lagrange basis function and $x_h(t) = \sum_{j=1}^{n} a_j l_j(t)$. Then the iterated Galerkin solution satisfies:

$$\begin{aligned} x_h^I(z_i) &= \int_0^1 \frac{(\sum\limits_{j=1}^{n} a_j l_j(s))^2}{\sqrt{|z_i - s|}} ds + f(z_i) = \sum_{j=1}^{n} a_j^2 \int_{t_{j-1}}^{t_j} \frac{1}{\sqrt{|z_i - s|}} ds + f(z_i) \\ &:= \sum_{j=1}^{n} a_j^2 \cdot A(i,j) + f(z_i), \end{aligned}$$

in which $z_i = i/100$ $(i = 0, \cdots, 100)$.

Here, because of the singularity of the kernel, $A(i,j)$ can not be obtained directly the classical numerical quadrature. We approximate each $A(i,j)$ $(i = 0, \cdots, 100; j = 1, \cdots, n)$ on the graded mesh corresponding to the original point z_i which increases greatly the complexity of the numerical integration.

However, there is no such difficulty for the interpolated Galerkin method, since we just need to use two adjacent points to get a linear interpolation.

We denote the elapsed time of the interpolated Galerkin solutions by time1 and the elapsed time of the iterated Galerkin solutions by time2. The results are listed in the following table.

Table 4 The errors of the approximate solutions for $r = 1$

n	e_h	R_h	e'_h	R'_h	time1	e''_h	R''_h	time2
32	3.8004e-2		1.5698e-2		0.0000	1.6764e-2		5.5780
64	1.7258e-2	1.1388	3.3302e-3	2.2369	0.0000	5.5771e-3	1.5878	9.9849
128	8.5037e-3	1.0211	8.6349e-4	1.9473	1.50e-2	2.5618e-3	1.1223	18.703
256	4.2029e-3	1.0167	2.2754e-4	1.9240	1.60e-2	1.3308e-3	0.9449	36.562

From the table above, we conclude that the interpolated Galerkin method is not only effective but also simple.

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Appendix Here, we give a Proof of Theorem 3.1 which is independent of the iterated Galerkin solution x_h^I.

We apply the mean-value theorem to $\psi(s, y)$ at $y = y_0$ to get

$$\psi(s, y) = \psi(s, y_0) + \psi^{(0,1)}(s, y_0 + \theta(y - y_0))(y - y_0). \tag{4.2}$$

where $\theta := \theta(s, y_0, y)$ with $0 < \theta < 1$. Also let

$$g(t, s, y_0, y, \theta) = k(t, s)\psi^{(0,1)}(s, y_0 + \theta(y - y_0)), \tag{4.3}$$

$$(G_h x)(t) = \int_0^1 g(t, s, P_h x_0(s), P_h x_n^I(s), \theta) x(s) ds, \tag{4.4}$$

and

$$(Gx)(t) = \int_0^1 g_t(s) x(s) ds, \quad \text{where} \quad g_t(s) = k(t, s)\psi^{(0,1)}(s, x_0(s)).$$

Proof. From (2.3) and (2.7), we see that

$$\begin{aligned} x_h - P_h x_0 &= P_h K\Psi x_h - P_h K\Psi x_0 \\ &= P_h(K\Psi x_h - K\Psi P_h x_0) + P_h(K\Psi P_h x_0 - K\Psi x_0). \end{aligned} \tag{4.5}$$

From (2.11) and (4.4), it is evident that

$$(G_h x)(t) = \int_0^1 g(t,s,P_h x_0(s), x_h(s), \theta) x(s) ds.$$

Then, we see from (4.2) and (4.3) that

$$K\Psi x_h - K\Psi P_h x_0 = \int_0^1 k(t,s)[\psi(s,x_h(s) - \psi(s,P_h x_0(s))]ds \tag{4.6}$$

$$\begin{aligned} &= \int_0^1 k(t,s)\psi^{(0,1)}\left(s, P_h x_0(s) + \theta(x_h(s) - P_h x_0(s)\right)\left[x_h(s) - P_h x_0(s)\right]ds \\ &= G_h(x_h - P_h x_0). \end{aligned}$$

Substituting (4.6) into (4.5), we obtain

$$x_h - P_h x_0 = P_h G_h (x_h - P_h x_0) - P_h(K\Psi P_h x_0 - K\Psi x_0). \tag{4.7}$$

It follows from the Lipschitz condition (2.2) imposed on $\psi^{(0,1)}$ and condition 2 that for $x \in [0,1]$

$$\begin{aligned} &\|G_h x - Gx\|_\infty \\ \le\ & C_2 \sup_{0\le t\le 1} \int_0^1 |k(t,s)|ds \cdot \|x\|_\infty \cdot (\|P_h x_0 - x_0\|_\infty + \|x_h - P_h x_0\|_\infty) \\ \le\ & C_2 M(\|P_h x_0 - x_0\|_\infty + \|x_h - x_0\|_\infty + \|x_0 - P_h x_0\|_\infty) \cdot \|x\|_\infty \\ \le\ & C(\|P_h x_0 - x_0\|_\infty + \|x_h - x_0\|_\infty) \cdot \|x\|_\infty. \end{aligned}$$

Then, we have

$$\|G_h - G\|_\infty \to 0 \quad \text{as } h \to 0.$$

Also, for each $x \in C[0,1]$, it is easy to verify that $Gx \in C[0,1]$.

Since $P_h \to I$ pointwisely in C[0,1] as $h \to 0$, we have

$$P_h G x \to Gx \quad \text{as } h \to 0.$$

That is, $P_h G \to G$ pointwisely in $C[0,1]$ as $h \to 0$.

As P_h is uniformly bounded, we have that for each $x \in C[0,1]$

$$\begin{aligned} \|P_h G_h x - Gx\|_\infty &\le \|P_h G_h x - P_h G x\|_\infty + \|P_h G x - Gx\|_\infty \\ &\le \sup_h \|P_h\| \|G_h x - Gx\|_\infty + \|P_h G x - Gx\|_\infty. \end{aligned}$$

Thus, $P_h G_h \to G$ pointwisely in $C[0,1]$ as $h \to 0$. By conditions 4 and 5, we know that there exists a constant $C > 0$, such that for any h,

$$\begin{aligned} &|\psi^{(0,1)}(s, P_h x_0(s) + \theta(x_h(s) - P_h x_0(s)))| \\ \le\ & |\psi^{(0,1)}(s, P_h x_0(s) + \theta(x_h(s) - P_h x_0(s))) - \psi^{(0,1)}(s, x_0)| + |\psi^{(0,1)}(s, x_0)| \\ \le\ & C_2 \|P_h x_0 - x_0\|_\infty + C_2 \theta \|x_h - P_h x_0\|_\infty + M_1 \\ \le\ & C_2 \|P_h x_0 - x_0\|_\infty + C_2 \theta(\|x_h - x_0\|_\infty + \|x_0 - P_h x_0\|_\infty) + M_1 \le C. \end{aligned}$$

Therefore,

$$\begin{aligned}
& \|P_h G_h x\|_\infty \\
\leq\ & \sup_h \|P_h\| \cdot \|G_h x\|_\infty \leq C\|G_h x\|_\infty \\
\leq\ & C \sup_{0\leq t\leq 1} \left| \int_0^1 k(t,s)\psi^{(0,1)}(s, P_h x_0(s) + \theta(x_h(s) - P_h x_0(s)))x(s)ds \right| \\
\leq\ & C\|x\|_\infty,
\end{aligned}$$

and

$$\begin{aligned}
& |P_h G_h x(t) - P_h G_h x'(t)| \\
\leq\ & \sup_h \|P_h\| \cdot \left| \int_0^1 k(t,s)\psi^{(0,1)}(s, P_h x_0(s) + \theta(x_h(s) - P_h x_0(s)))(x(s) - x'(s))ds \right| \\
\leq\ & C\|x - x'\|_\infty.
\end{aligned}$$

This implies that $\{P_h G_h\}$ is collectively compact (see [1] for details). Since $G = (K\Psi)'(x_0)$ is compact and $(I - G)^{-1}$ exists, it follows from the theory of collectively compact operators that $(I - P_h G_h)^{-1}$ exists and is uniformly bounded for sufficiently small h. Therefore, by (4.7), we have

$$\begin{aligned}
\sup_{0\leq t\leq 1} |(x_h - P_h x_0)(t)| &\leq C \sup_{0\leq t\leq 1} |P_h(K\Psi P_h x_0 - K\Psi x_0)(t)| \\
&\leq C \sup_{0\leq t\leq 1} |(K\Psi P_h x_0 - K\Psi x_0)(t)|.
\end{aligned}$$

Let $d(t) = |(K\Psi P_h x_0 - K\Psi x_0)(t)|$. We will estimate the term $d(t)$.

Using the mean-value formula (4.2) with $y = x_0$ and $y_0 = P_h x_0$ and (4.3), we have

$$\begin{aligned}
& d(t) \\
=\ & \left| \int_0^1 K(t,s)[\psi(s, x_0(s)) - \psi(s, P_h x_0(s))]ds \right| \\
=\ & \left| \int_0^1 K(t,s)\psi^{(0,1)}(s, P_h x_0(s) + \theta(x_0(s) - P_h x_0(s)))[x_0(s) - P_h x_0(s)]ds \right| \\
=\ & \left| \int_0^1 g(t, s, P_h x_0(s), x_0(s), \theta)[x_0(s) - P_h x_0(s)]ds \right|.
\end{aligned}$$

Note that P_h is the orthogonal projection operator from $C[0,1] + S_r^0(T_h)$ onto $S_r^0(T_h)$, that is,

$$\int_0^1 u(s)[x_0(s) - P_h x_0(s)]ds = 0 \quad \text{for all}\, u \in S_r^0(T_h).$$

Thus, for all $u \in S_r^0(T_h)$

$$\begin{aligned} d(t) &= \left| \int_0^1 [g(t,s,P_h x_0(s), x_0(s), \theta) - u(s)][x_0(s) - P_h x_0(s)] ds \right| \\ &\leq \int_0^1 |g(t,s,P_h x_0(s), x_0(s), \theta) - g_t(s)| ds \|x_0 - P_h x_0\|_\infty \\ &\quad + \left| \int_0^1 [g_t(s) - u(s)][x_0(s) - P_h x_0(s)] ds \right|. \end{aligned}$$

By (2.2), we have

$$\begin{aligned} (4.8) \qquad & \int_0^1 |g(t,s,P_h x_0(s), x_0(s), \theta) - g_t(s)| ds \\ &= \int_0^1 \Big| K(t,s) \Big[\psi^{(0,1)}(s, P_h x_0(s) + \theta(x_0(s) - P_h x_0(s))) \\ &\qquad - \psi^{(0,1)}(s, x_0(s)) \Big] \Big| ds \\ &\leq C_2 \int_0^1 |K(t,s)| ds \|x_0 - P_h x_0\|_\infty \\ &\leq C_2 M \|x_0 - P_h x_0\|_\infty. \end{aligned}$$

As $\frac{1}{p} + \frac{1}{q} = 1$, we have from Cauchy-Schwarz inequality that

$$\begin{aligned} (4.9) \qquad & \left| \int_0^1 [g_t(s) - u(s)][x_0(s) - P_h x_0(s)] ds \right| \\ & \leq \|g_t - u\|_{0,q} \, \|x_0 - P_h x_0\|_{0,p}. \end{aligned}$$

Combining (4.8) and (4.9), we obtain

$$d(t) \leq C\|x_0 - P_h x_0\|_\infty^2 + \|g_t - u\|_{0,q} \cdot \|x_0 - P_h x_0\|_{0,p} \quad \text{for all } u \in S_r^0(T_h).$$

Therefore, we complete the proof of Theorem 3.1. □

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College of Applied Sciences, Beijing University of Technology, Beijing 100124, China
E-mail: qmhuang@bjut.edu.cn

LSEC, ICMSEC, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China
E-mail: hhxie@lsec.cc.ac.cn