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A ROBUST OVERLAPPING SCHWARZ METHOD FOR A SINGULARLY PERTURBED SEMILINEAR REACTION-DIFFUSION PROBLEM WITH MULTIPLE SOLUTIONS

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Abstract. An overlapping Schwarz domain decomposition is applied to a semilinear reaction-diffusion two-point boundary value problem with multiple solutions. Its diffusion parameter ε^2 is arbitrarily small, which induces boundary layers. The Schwarz method invokes two boundary-layer subdomains and an interior subdomain, the narrow overlapping regions being of width $O(\varepsilon|\ln\varepsilon|)$. Constructing sub- and super-solutions, we prove existence and investigate the accuracy of discrete solutions in particular subdomains. It is shown that when $\varepsilon \le CN^{-1}$ and layer-adapted meshes of Bakhvalov and Shishkin types are used, one iteration is sufficient to get second-order convergence (with, in the case of the Shishkin mesh, a logarithmic factor) in the maximum norm uniformly in ε, where N is the number of mesh intervals in each subdomain. Numerical results are presented to support our theoretical conclusions.

Key Words. semilinear reaction-diffusion, singularly perturbed, boundary layers, domain decomposition, overlapping Schwarz method.

1. Introduction

Consider the singularly perturbed semilinear reaction-diffusion boundary value problem

$$Fu := -\varepsilon^2 u''(x) + f(x,u) = 0, \qquad x \in \Omega = (0,1), \tag{1a}$$

$$u(0) = g_0, \qquad u(1) = g_1, \tag{1b}$$

where ε is a small positive parameter, f is a sufficiently smooth function, and g_0 and g_1 are given constants. This is a one-dimensional version of the multidimensional reaction-diffusion equation $-\varepsilon^2\triangle u + f(x,u) = 0$, which we will consider in a future paper [10] (when posed in a smooth two-dimensional domain).

We shall examine solutions of (1) that exhibit sharp boundary layers, which are narrow regions where solutions change rapidly. In general, solutions of (1) may also have interior transition layers [12]. To obtain reliable numerical approximations of layer solutions in an efficient way, one has to use locally refined meshes that are fine and anisotropic in layer regions and standard outside. When multidimensional meshes of different nature are introduced in different subdomains, it might be rather inconvenient to match them; see, e.g., [7] for non-matching meshes used to solve

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a two-dimensional problem of type (1). Furthermore, different discretizations of differential equations might be used in layer regions and outside, in which case they should be matched along the interface boundaries; see, e.g., [9].

Handling non-matching meshes and matching different discretizations along the interface boundaries can be entirely avoided by invoking iterative overlapping domain decomposition methods of Schwarz-Chimera type; see, e.g., [18, §1.5]. Note that non-overlapping domain decomposition methods, at best, have conventional geometric rates of convergence when applied to singularly perturbed problems of type (1). In contrast, overlapping methods, with the overlapping regions being as narrow as $O(\varepsilon|\ln\varepsilon|)$, might enjoy much faster convergence. To be more precise, we prove in this paper that one iteration is sufficient to achieve second-order accurate computed solutions when $\varepsilon \le CN^{-1}$, where N is the number of mesh intervals in each subdomain; see Theorem 5.5 for details.

When considering semilinear problems of type (1), it is frequently assumed in the numerical analysis literature that $f_u(x,u) > \gamma^2 > 0$ for all $(x,u) \in \Omega\times\mathbb{R}$ and some positive constant γ. Under this assumption, our problem (1) and the associated reduced problem

$$f(x,u_0(x)) = 0 \qquad \text{for all} \quad x\in\Omega, \tag{2}$$

defined by setting $\varepsilon = 0$ in (1), have unique solutions u and u_0. This global assumption is however rather restrictive. E.g., mathematical models of biological and chemical processes frequently involve problems related to (1) with $f(x,u)$ that is non-monotone with respect to u. Therefore, we examine problem (1) under the following weaker assumptions also used in [4, 6, 11, 17, 19, 20]:

- it has a *stable reduced solution*, i.e. there exists a sufficiently smooth solution u_0 of (2) such that

$$f_u(x,u_0(x)) > \gamma^2 > 0 \qquad \text{for all} \;\; x\in\Omega; \tag{3a}$$

- the boundary data g_l, for $l=0,1$, satisfy

$$\int_{u_0(l)}^{v} f(l,s)\,ds > 0 \qquad \text{for all} \;\; v\in(u_0(l),\, g_l\,]'. \tag{3b}$$

  Here the notation $(a,b]'$ is defined to be $(a,b]$ when $a<b$ and $[b,a)$ when $a>b$, while $(a,b]'=\emptyset$ when $a=b$.

Conditions (3) intrinsically arise from the asymptotic analysis of problem (1) and guarantee that there exists a boundary-layer solution u such that $u\approx u_0$ in the interior part of Ω, while the boundary layers are of width $O(\varepsilon|\ln\varepsilon|)$; see, e.g., [6, 17, 20]. Note that assumption (3a) is local, i.e. the reduced problem (2) is permitted to have more than one stable solution. Furthermore, if multiple stable solutions of the reduced problem satisfy (3b), then problem (1) has multiple boundary-layer solutions.

We shall now present a continuous version of the discrete Schwarz method that we investigate in this paper. Consider the overlapping subdomains

$$\Omega_{\mathrm{L}} = (0,2\sigma), \qquad \Omega_{\mathrm{C}} = (\sigma,1-\sigma), \qquad \Omega_{\mathrm{R}} = (1-2\sigma,1), \tag{4}$$

where $\sigma\in(0,\,1/4]$ is a parameter, which throughout the paper will satisfy $\sigma \ge (2/\gamma)\,\varepsilon\ln N$. Let u_{L}, u_{R}, and then u_{C} be solutions of the following boundary value problems

$$\text{(5a)}\quad Fu_{\mathrm{L}} = 0 \ \text{ for } x \in \Omega_{\mathrm{L}}, \qquad u_{\mathrm{L}}(0) = g_0, \qquad u_{\mathrm{L}}(2\sigma) = g_{2\sigma},$$

$$\text{(5b)}\quad Fu_{\mathrm{R}} = 0 \ \text{ for } x \in \Omega_{\mathrm{R}}, \qquad u_{\mathrm{R}}(1-2\sigma) = g_{1-2\sigma}, \qquad u_{\mathrm{R}}(1) = g_1,$$

$$\text{(5c)}\quad Fu_{\mathrm{C}} = 0 \ \text{ for } x \in \Omega_{\mathrm{C}}, \qquad u_{\mathrm{C}}(\sigma) = u_{\mathrm{L}}(\sigma), \qquad u_{\mathrm{C}}(1-\sigma) = u_{\mathrm{R}}(1-\sigma).$$

Here $g_0 = u(0)$ and $g_1 = u(1)$ are the boundary data of our original problem (1), while $g_{2\sigma}$ and $g_{1-2\sigma}$ should be appropriately chosen (they should satisfy (3b) with $l = 2\sigma$, $1 - 2\sigma$; otherwise the semilinear problems (5a), (5b) might have no solutions). One can take $g_{2\sigma} = g_0$ and $g_{1-2\sigma} = g_1$ when σ is sufficiently small. Now, the first-iteration approximation $u^{[1]}$ is defined by

$$u^{[1]}(x) := \begin{cases} u_{\mathrm{L}}(x), & x \in \bar{\Omega}_{\mathrm{L}} \backslash \Omega_{\mathrm{C}}, \\ u_{\mathrm{C}}(x), & x \in \bar{\Omega}_{\mathrm{C}}, \\ u_R(x), & x \in \bar{\Omega}_{\mathrm{R}} \backslash \Omega_{\mathrm{C}}. \end{cases} \tag{6}$$

Further iterations, that consist of successfully solving similar problems in the subdomains Ω_{L}, Ω_{R} and Ω_{C}, are described in Remark 5.1.

Our discrete Schwarz method is a domain decomposition version of the numerical method considered in [11]; see also [4, 19]. It invokes special layer-adapted meshes of Bakhvalov and Shishkin type in the boundary-layer subdomains Ω_{L} and Ω_{R}. Problems in the overlapping subdomains Ω_{L}, Ω_{R} and Ω_{C} are discretized by a standard three-point finite-difference scheme. To estimate the Schwarz method errors, we extend the analysis of [11] to discrete problems in particular subdomains. Compared to [11], our problems might be posed in very narrow subdomains and therefore require a more intricate analysis.

When the Shishkin mesh is used, our discrete Schwarz method is identical to the one studied in [15] for a linear version of (1). Note that the principal analysis technique in [15] is the discrete maximum principle, which cannot be extended to our more general semilinear problem (1) under conditions (3). Furthermore, we particularly address faster convergence of the algorithm when $\varepsilon \le CN^{-1}$. We also refer the reader to [2, 3, 5], where Schwarz alternating techniques were applied to semilinear problems of type (1) under the condition $f_u(x,u) > \gamma^2 > 0$ for all (x,u); these iterative algorithms used either overlapping subdomains, or, to facilitate parallel computations, two overlapping sets of subdomains with no subdomain overlap within each set.

Our paper is organized as follows. In §2 we discuss asymptotic properties of solutions to the differential equation (1a) posed in an arbitrary particular subdomain and construct its sub- and super-solutions. These results are applied in §3 to problems (5); this section culminates in an error estimate for the continuous first-iteration approximation $u^{[1]}$ from (6) to a certain solution u of our original problem (1). In §4 we introduce the finite difference scheme on an arbitrary mesh and then consider a discrete problem in a particular subdomain, establish existence and investigate accuracy of its discrete solutions. The discrete Schwarz method is described in §5 and its error estimates are derived on Bakhvalov and Shishkin meshes. Finally, numerical results of §6 illustrate our theoretical conclusions.

Throughout our analysis we make a simplifying *assumption* that

$$\varepsilon \le CN^{-1}. \tag{7}$$

This is not a practical restriction, and from a theoretical viewpoint the analysis of a semilinear problem such as (1) would be very different if ε were not small. Note that the error estimate for the linear case [15] and our numerical results of

§6 suggest that for $\varepsilon > CN^{-1}$ the considered Schwarz method remains convergent, although might require more iterations.

Notation. Throughout the paper, C, C', $\bar{C}$ will denote generic positive constants that may take different values in different formulas, but are independent of ε and N. A subscripted C (e.g., C_1) denotes a constant that is also independent of ε and N, but takes a fixed value. For any two quantities z_1 and z_2, the notation $z_1 = O(z_2)$ is equivalent to $|z_1| \le Cz_2$. When choosing N sufficiently large independently of ε, we shall mean that $N \ge C$ for some sufficiently large constant C.

2. Continuous problem in a particular subdomain

Since our method involves the numerical solution of the differential equation (1a) in various subdomains, we shall first consider this equation and asymptotic properties of its solutions in an arbitrary particular subdomain (a, b). Here and in §4 below, we extend the asymptotic and numerical analysis of [11], now allowing very narrow subdomains, which requires more elaborate estimates.

Let $u_{[a,b]}(x)$ be a solution of the problem

$$Fu_{[a,b]} = 0 \quad \text{for } x \in (a, b), \qquad u_{[a,b]}(a) = g_a, \qquad u_{[a,b]}(b) = g_b, \tag{8}$$

where $(a, b) \subset \Omega$, and the boundary data g_l, for $l = a, b$, satisfy condition (3b). Furthermore, only to avoid considering cases, we also assume that $g_l \ge u_0(l)$ for $l = a, b$.

Then $u_{[a,b]}$ typically exhibits boundary layers and its standard first-order asymptotic expansion $u_{\mathrm{as}\,;\,[a,b]}$ is given by

$$u_{\mathrm{as}\,;\,[a,b]}(x) := u_0(x) + \left[v_{0;\,a}(\xi^+) + \varepsilon v_{1;\,a}(\xi^+)\right] + \left[v_{0;\,b}(\xi^-) + \varepsilon v_{1;\,b}(\xi^-)\right]. \tag{9}$$

Here the components $[v_{0;\,a} + \varepsilon v_{1;\,a}]$ and $[v_{0;\,b} + \varepsilon v_{1;\,b}]$ describe the boundary layers at $x = a$ and $x = b$ respectively. They use the stretched variables $\xi^+ = \xi_a^+ := \frac{x-a}{\varepsilon}$ and $\xi^- = \xi_b^- := \frac{b-x}{\varepsilon}$. More generally,

$$\xi_l^\pm := \pm(x - l)/\varepsilon.$$

When there is no ambiguity, as, e.g., in (9), the notation $\xi^\pm$ is used for ξ_a^+ and ξ_b^-. Note that $\xi_l^\pm = 0$ corresponds to $x = l$, and ξ_l^+ has the same positive direction as the x-axis, while ξ_l^- has the opposite direction.

The boundary-layer functions $v_{0;\,l}$ and $v_{1;\,l}$ in (9), with $l = a, b$, satisfy

$$-\left(\tfrac{d}{d\xi^\pm}\right)^2 v_{0;\,l} + f(l, u_0(l) + v_{0;\,l}) = 0\,, \tag{10a}$$

$$\left[-\left(\tfrac{d}{d\xi^\pm}\right)^2 + f_u(l, u_0(l) + v_{0;\,l})\right] v_{1;\,l} = \mp\xi^\pm \; \tfrac{d}{dx} f(x, u_0(x) + s)\Big|_{\substack{x=l \\ s=v_{0;\,l}(\xi^\pm)}}\,, \tag{10b}$$

with the boundary conditions

$$v_{0;\,l}(0) = g_l - u_0(l), \qquad v_{1;\,l}(0) = v_{0;\,l}(\infty) = v_{1;\,l}(\infty) = 0. \tag{10c}$$

To construct sub- and super-solutions for problem (8), we introduce a perturbation $\beta_{[a,b]}$ of the asymptotic expansion (9):

$$\beta_{[a,b]}(x; p) := u_0(x) + \left[\tilde{v}_{0;\,a}(\xi^+; p) + \varepsilon v_{1;\,a}(\xi^+)\right] + \left[\tilde{v}_{0;\,b}(\xi^-; p) + \varepsilon v_{1;\,b}(\xi^-)\right] + C_0 p. \tag{11}$$

Here p is a small real number that will be chosen later and is typically $o(N^{-1})$; for some small $p > 0$ the functions $\beta_{[a,b]}(x; -p)$ and $\beta_{[a,b]}(x; p)$ will serve as sub- and super-solutions. Relation (11) involves auxiliary functions $\tilde{v}_{0;\,l}(\xi^\pm; p)$, for $l = a, b$, that are defined by generalizing equations (10a) with the boundary conditions (10c):

$$-\left(\tfrac{d}{d\xi^\pm}\right)^2 \tilde{v}_{0;\,l} + f(l, u_0(l) + \tilde{v}_{0;\,l}) = p\tilde{v}_{0;\,l}, \;\; \tilde{v}_{0;\,l}(0; p) = g_l - u_0(l), \;\; \tilde{v}_{0;\,l}(\infty; p) = 0. \tag{12}$$

Clearly, we have $\tilde{v}_{0;\,l}(\xi^{\pm};0) = v_{0;\,l}(\xi^{\pm})$ for $l = a, b$.

The following lemma combines the results of [11, Lemma 2.1], [11, Lemma 2.3] and [11, (2.15)]. The proof invokes dynamical systems techniques to show that problems (10) and (12) have solutions and then obtain bounds on these solutions and their derivatives.

Lemma 2.1. [11] *Set* $\gamma_0^2 = \min_{x=a,b} f_u(x, u_0(x)) > \gamma^2$, *where* $\gamma > 0$ *is from (3a). Given assumption* (3b) *with* $l = a, b$, *there exists* $p_0 \in (0, \gamma_0^2)$ *such that for all* $|p| \leq p_0$, *problems* (10) *and* (12) *have solutions* $v_{0;\,a}(\xi^+)$, $v_{0;\,b}(\xi^-)$, $v_{1;\,a}(\xi^+)$, $v_{1;\,b}(\xi^-)$, $\tilde{v}_{0;\,a}(\xi^+;p)$ *and* $\tilde{v}_{0;\,b}(\xi^-;p)$. *Furthermore,*

$$v_{0;\,l}(\xi^{\pm}) \geq 0, \qquad \tfrac{\partial}{\partial p}\tilde{v}_{0;\,l}(\xi^{\pm};p) \geq 0, \qquad \textit{where} \quad l = a, b. \tag{13}$$

Moreover, for an arbitrarily small but fixed $\delta \in (0, \gamma_0 - \sqrt{p_0})$, *there is a positive constant* C_δ *such that*

$$\left|\left(\tfrac{\partial}{\partial \xi^{\pm}}\right)^k \tilde{v}_{0;\,l}\right| + \left|\left(\tfrac{d}{d\xi^{\pm}}\right)^k v_{1;\,l}\right| + \left|\tfrac{\partial}{\partial p}\,\tilde{v}_{0;\,l}\right| \leq C_\delta\,|g_l - u_0(l)|\,e^{-(\gamma_0-\sqrt{p_0}-\delta)\,\xi^{\pm}} \tag{14}$$

for $l = a, b$, $0 \leq \xi^{\pm} \leq \infty$ *and* $k = 0, 1, \ldots, 4$.

Remark 2.2. As $\gamma_0 > \gamma$, choosing p_0 and δ in Lemma 2.1 sufficiently small, we can make $\gamma_0 - \sqrt{p_0} - \delta$ in (14) satisfy $\gamma_0 - \sqrt{p_0} - \delta > \gamma$, which then yields $e^{-(\gamma_0-\sqrt{p_0}-\delta)\,\xi^{\pm}} \leq e^{-\gamma\,\xi^{\pm}}$ Consequently, we have

$$e^{-(\gamma_0-\sqrt{p_0}-\delta)\,\xi_a^+} \leq e^{-\gamma\,(x-a)/\varepsilon}, \qquad e^{-(\gamma_0-\sqrt{p_0}-\delta)\,\xi_b^-} \leq e^{-\gamma\,(b-x)/\varepsilon}. \tag{15}$$

Similarly, we can choose p_0 and δ so that $\gamma_0 - \sqrt{p_0} - \delta > \tilde{\gamma}$ for any $\tilde{\gamma} < \gamma_0$, which then yields (15) with γ replaced by $\tilde{\gamma}$.

Next we shall investigate the perturbation $\beta_{[a,b]}(x;p)$ of our asymptotic expansion. Introduce the notation

$$\hat{u}_{\mathrm{as}\,;\,[a,b]}(x) := u_0(x) + \left[v_{0;\,a}(\xi^+) \quad + \varepsilon v_{1;\,a}(\xi^+)\right], \tag{16}$$

$$\hat{\beta}_{[a,b]}(x;p) := u_0(x) + \left[\tilde{v}_{0;\,a}(\xi^+;p) + \varepsilon v_{1;\,a}(\xi^+)\right] + C_0 p. \tag{17}$$

Compared to $u_{\mathrm{as}\,;\,[a,b]}$ and $\beta_{[a,b]}$ defined in (9) and (11), the component $[v_{0;\,b} + \varepsilon v_{1;\,b}]$ and its analogue, which describe the boundary layer at $x = b$, are skipped here.

Lemma 2.3. *Under the assumptions of Lemma 2.1, for* $\hat{\beta}_{[a,b]}(x;p)$ *of (17) we have*

$$F\hat{\beta}_{[a,b]}(x;p) = C_0 p\, f_u(x, u_0) + [1 + C_0\lambda(x)]\, p\, v_{0;\,a}(\xi^+) + O(\varepsilon^2 + p^2),$$

where $x \in (a, b)$, *while* $\lambda(x) := f_{uu}\big(x, u_0 + \vartheta v_{0;\,a}\big)$ *and* $\vartheta = \vartheta(x) \in (0, 1)$.

Proof. Similar estimates were obtained, e.g., in the proof of [11, Lemma 3.2] and in [9]; we will sketch the proof here for completeness following the argument of [9, Lemma 2.8].

Using (16), it is convenient to rewrite $\hat{\beta}_{[a,b]}$ as

$$\hat{\beta}_{[a,b]}(x;p) = \hat{u}_{\mathrm{as}\,;\,[a,b]}(x) + w(\xi^+;p) + C_0 p, \qquad w(\xi^+;p) := \tilde{v}_{0;\,a}(\xi^+;p) - v_{0;\,a}(\xi^+).$$

Note that $v_{0;\,a}(\xi^+) = \tilde{v}_{0;\,a}(\xi^+;0)$ implies $w = p\frac{\partial \tilde{v}_0}{\partial p}\Big|_{p=\theta}$; combining this with (14) yields

$$(1 + \xi^+)|w| \leq Cp. \tag{18}$$

Throughout the proof we shall use the abbreviations u_0, $v_{0;\,a}$ and $\tilde{v}_{0;\,a}$ for $u_0(x)$, $v_{0;\,a}(\xi^+)$ and $\tilde{v}_{0;\,a}(\xi^+;p)$ respectively.

Since $F\hat{u}_{\mathrm{as}\,;\,[a,b]} = O(\varepsilon^2)$ [20], then we get $\varepsilon^2 \frac{d^2}{dx^2}\hat{u}_{\mathrm{as}\,;\,[a,b]} = f(x, \hat{u}_{\mathrm{as}\,;\,[a,b]}) + O(\varepsilon^2)$. Combining this with $\varepsilon^2 \frac{d^2}{dx^2} w = \big(\frac{d}{d\xi^+}\big)^2 w$, we arrive at

$$F\hat{\beta}_{[a,b]} = -f(x, \hat{u}_{\mathrm{as}\,;\,[a,b]}) - \big(\tfrac{d}{d\xi^+}\big)^2 w + f(x, \hat{u}_{\mathrm{as}\,;\,[a,b]} + w + C_0 p) + O(\varepsilon^2). \tag{19}$$

Now, by (10a), (12) combined with $\tilde{v}_{0;\,a} = v_{0;\,a} + w$ and $w = O(p)$, we get

$$-\big(\tfrac{d}{d\xi^+}\big)^2 w = -[f(a, u_0(a) + v_{0;\,a} + w) - f(a, u_0(a) + v_{0;\,a})] + pv_{0;\,a} + O(p^2).$$

Substituting this into (19) we obtain

$$\begin{aligned} F\hat{\beta}_{[a,b]} \;&=\; -f(x, \hat{u}_{\mathrm{as}\,;\,[a,b]}) - [f(a, u_0(a) + v_{0;\,a} + w) - f(a, u_0(a) + v_{0;\,a})] + pv_{0;\,a} \\ &\quad + f(x, \hat{u}_{\mathrm{as}\,;\,[a,b]} + w + C_0 p) + O(p^2 + \varepsilon^2). \end{aligned}$$

Introducing the function $\mu(t) := f(x, \hat{u}_{\mathrm{as}\,;\,[a,b]} + t) - f(a, u_0(a) + v_{0;\,a} + t).$ we can rewrite the above relation as

$$\begin{aligned} F\hat{\beta}_{[a,b]} \;&=\; [f(x, \hat{u}_{\mathrm{as}\,;\,[a,b]} + w + C_0 p) - f(x, \hat{u}_{\mathrm{as}\,;\,[a,b]} + w)] \\ &\quad + [\mu(w) - \mu(0)] + pv_{0;\,a} + O(p^2 + \varepsilon^2). \end{aligned} \tag{20}$$

Since $\hat{u}_{\mathrm{as}\,;\,[a,b]} = u_0 + v_{0;\,a} + O(\varepsilon)$ and $w = O(p)$, we have

$$f(x, \hat{u}_{\mathrm{as}\,;\,[a,b]} + w + C_0 p) - f(x, \hat{u}_{\mathrm{as}\,;\,[a,b]} + w) = [f_u(x, u_0 + v_{0;\,a}) + O(\varepsilon + p)]C_0 p, \tag{21}$$

where

$$f_u(x, u_0 + v_{0;\,a})\, C_0 p = [f_u(x, u_0) + \lambda(x) v_{0;\,a}]C_0 p. \tag{22}$$

Furthermore,

$$\mu(w) - \mu(0) = w\mu'(\bar{\vartheta} w) = [f_u(x, \hat{u}_{\mathrm{as}\,;\,[a,b]} + \bar{\vartheta} w) - f_u(a, u_0(a) + v_{0;\,a} + \bar{\vartheta} w)]w,$$

where $\bar{\vartheta} = \bar{\vartheta}(x) \in (0, 1)$. Now by (18), we get

$$\mu(w) - \mu(0) = O(\varepsilon)\,[1 + \xi^+]\,|w| = O(\varepsilon p). \tag{23}$$

Combining (20)-(23), we complete the proof. □

Corollary 2.4. *Under the assumptions of Lemma 2.1, for $\beta_{[a,b]}(x; p)$ from (11) we have*

$$F\beta_{[a,b]}(x; p) = C_0 p\, f_u(x, u_0) + [1 + C_0 \lambda(x)]\, p\, [v_{0;\,a} + v_{0;\,b}] + O(\varepsilon 2 + p2 + e^{-\gamma(b-a)/(2\varepsilon)}),$$

where $x \in (a, b)$, while $\lambda(x) := f_{uu}\big(x, u_0 + \vartheta[v_{0;\,a} + v_{0;\,b}]\big)$ and $\vartheta = \vartheta(x) \in (0, 1)$.

Proof. Consider only the case of $x \in (a, (a+b)/2]$ as the other case is similar. Thus it suffices to prove that $|F\beta_{[a,b]}(x; p) - F\hat{\beta}_{[a,b]}(x; p)| \le Ce^{-\gamma\,(b-x)/\varepsilon} \le Ce^{-\gamma(b-a)/(2\varepsilon)}$ for all $x \in (a, (a + b)/2]$. This follows from

$$F\beta_{[a,b]}(x; p) - F\hat{\beta}_{[a,b]}(x; p) = -\big(\tfrac{d}{d\xi^+}\big)^2 [\beta_{[a,b]} - \hat{\beta}_{[a,b]}] + O(\beta_{[a,b]} - \hat{\beta}_{[a,b]})$$

combined with $\beta_{[a,b]} - \hat{\beta}_{[a,b]} = \tilde{v}_{0;\,b} + \varepsilon v_{1;\,b}$, for which we have (14) and (15). □

Remark 2.5. Setting $p = 0$ in Corollary 2.4 and recalling that $u_{as;[a,b]}(x) = \beta_{[a,b]}(x; 0)$, we observe that the first-order asymptotic expansion $u_{as;[a,b]}(x)$ defined in (9) satisfies the estimate $|Fu_{as;[a,b]}| \le C[\varepsilon^2 + e^{-\gamma(b-a)/(2\varepsilon)}]$ for all $x \in (a, b)$. Since one expects $Fu_{as;[a,b]} \approx Fu_{[a,b]} = 0$, the subdomain width $(b - a)$ should be sufficiently large, as in the following corollary. Furthermore, unless we have $b - a \gg \varepsilon$, our problem (8) is no longer singularly perturbed and its asymptotic analysis is no longer valid.

Corollary 2.6. *Let $b-a \geq (4/\gamma)\varepsilon \ln N$. Then there exists $C_0 > 0$ such that for $\beta_{[a,b]}(x;p)$ from (11) for all $|p| \leq p_0$ we have*

$$\begin{aligned} F\beta_{[a,b]}(x;p) &\geq C_0 p\,\gamma^2 + O(\varepsilon^2 + p^2 + N^{-2}), &\quad \text{if} \quad p > 0,\\ F\beta_{[a,b]}(x;p) &\leq -C_0 |p|\,\gamma^2 + O(\varepsilon^2 + p^2 + N^{-2}), &\quad \text{if} \quad p < 0. \end{aligned}$$

Proof. Recall (3a) and that $v_{0;\,a}(\xi^+) \geq 0$ and $v_{0;\,b}(\xi^-) \geq 0$, by (13). Now invoke Corollary 2.4 choosing $0 < C_0 \leq |\lambda(x)|^{-1}$ for all x so that $1 + C_0\lambda(x) \geq 0$. Finally note that $e^{-\gamma(b-a)/(2\varepsilon)} \leq N^{-2}$. □

Theorem 2.7. *Suppose that $b-a \geq (4/\gamma)\varepsilon \ln N$, where $\varepsilon + N^{-1} \leq \tilde{C}_0$ for some sufficiently small $\tilde{C}_0$, and the boundary data g_l, where $l = a, b$, of problem* (8) *satisfy (3b). Then this problem has a solution $u_{[a,b]}$ such that*

$$\big|(u_{[a,b]} - u_{\mathrm{as}\,;\,[a,b]})(x)\big| \leq C(\varepsilon^2 + N^{-2}) \quad \textit{for all} \quad x \in [a,b], \tag{24}$$

where $u_{\mathrm{as}\,;\,[a,b]}$ is defined in (9).

Proof. Set $\bar{p} := \bar{C}(\varepsilon^2 + N^{-2}) \leq p_0$ for some sufficiently large $\bar{C}$ so that applying Corollary 2.6 yields $F\beta_{[a,b]}(x;-\bar{p}) \leq 0 \leq F\beta_{[a,b]}(x;\bar{p})$ (this is possible as $\varepsilon + N^{-1}$ is sufficiently small). Furthermore, since (13) implies that $\beta_{[a,b]}(x;p)$ is increasing in p, while $\beta_{[a,b]}(x;0) = u_{\mathrm{as}\,;\,[a,b]}(x)$, we get $\beta_{[a,b]}(x;-\bar{p}) \leq u_{\mathrm{as}\,;\,[a,b]}(x) \leq \beta_{[a,b]}(x;\bar{p})$. Thus $\beta_{[a,b]}(x;-\bar{p})$ and $\beta_{[a,b]}(x;\bar{p})$ are sub- and super-solutions for problem (8). Therefore, this problem has a solution $u_{[a,b]}$ such that $\beta_{[a,b]}(x;-\bar{p}) \leq u_{[a,b]}(x) \leq \beta_{[a,b]}(x;\bar{p})$ and hence for this solution we obtain the desired bound

$$|u_{[a,b]}(x) - u_{\mathrm{as}\,;\,[a,b]}(x)| \leq \beta_{[a,b]}(x;\bar{p}) - \beta_{[a,b]}(x;-\bar{p}) \leq C\bar{p}. \tag{25}$$

The final estimate here follows from $\beta_{[a,b]}(x;\bar{p}) - \beta_{[a,b]}(x;-\bar{p}) = 2\bar{p}\,(\frac{\partial}{\partial p}\,\tilde{v}_{0;\,a} + \frac{\partial}{\partial p}\,\tilde{v}_{0;\,b} + C_0)\big|_{p=\theta}$, where we used (11) and (14). □

3. Error in the continuous Schwarz method

In this section we estimate the error in the first iteration (6) of the continuous Schwarz method and show that under condition (7), one iteration is sufficient for second-order accuracy. First, we consider problems (5a) and (5b) in the subdomains Ω_{L} and Ω_{R}.

Lemma 3.1. *Suppose that σ satisfies $(2/\gamma)\varepsilon \ln N \leq \sigma \leq 1/4$, where $\varepsilon + N^{-1} \leq \tilde{C}_0$ for some sufficiently small $\tilde{C}_0$, and the boundary data g_l, for $l = 0,\, 2\sigma,\, 1-2\sigma,\, 1$, of problems* (1), (5a), (5b) *satisfy (3b). Then there exist solutions $u = u_{[0,1]}$, $u_{\mathrm{L}} = u_{[0,2\sigma]}$ and $u_{\mathrm{R}} = u_{[1-2\sigma,1]}$ of these problems such that*

$$|(u_{\mathrm{L}} - u)(x)| \leq C(\varepsilon^2 + N^{-2}) \qquad \textit{for all} \quad x \in \bar{\Omega}_{\mathrm{L}} \setminus \Omega_{\mathrm{C}} = [0,\sigma], \tag{26a}$$

$$|(u_{\mathrm{R}} - u)(x)| \leq C(\varepsilon^2 + N^{-2}) \qquad \textit{for all} \quad x \in \bar{\Omega}_{\mathrm{R}} \setminus \Omega_{\mathrm{C}} = [1-\sigma,1]. \tag{26b}$$

Furthermore,

$$|(u_{\mathrm{L}} - u_0)(\sigma)| + |(u_{\mathrm{R}} - u_0)(1-\sigma)| \leq C(\varepsilon^2 + N^{-2}). \tag{27}$$

Proof. Applying Theorem 2.7 to problems (1), (5a), (5b) immediately yields existence of their solutions. It suffices now to prove estimates (26a) and (27) for u_{L}, as the required estimates for u_{R} are obtained similarly. Estimate (24) for problems (1) and (5a) yields

$$u_{\mathrm{L}} - u = u_{\mathrm{as}\,;\,[0,2\sigma]} - u_{\mathrm{as}\,;\,[0,1]} + O(\varepsilon^2 + N^{-2}), \quad u_{\mathrm{L}} - u_0 = u_{\mathrm{as}\,;\,[0,2\sigma]} - u_0 + O(\varepsilon^2 + N^{-2}).$$

Thus to get the desired estimates, it remains to show that

$$u_{\mathrm{as}\,;\,[0,2\sigma]} - u_{\mathrm{as}\,;\,[0,1]} = O(N^{-2}) \ \text{ for } x \in [0,\sigma], \quad (u_{\mathrm{as}\,;\,[0,2\sigma]} - u_0)\big|_{x=\sigma} = O(N^{-2}). \tag{28}$$

Note that, by (9), we have

$$u_{\mathrm{as}\,;\,[0,2\sigma]} - u_{\mathrm{as}\,;\,[0,1]} = \big[v_{0;\,2\sigma}(\xi_{2\sigma}^-) + \varepsilon v_{1;\,2\sigma}(\xi_{2\sigma}^-)\big] - \big[v_{0;\,1}(\xi_1^-) + \varepsilon v_{1;\,1}(\xi_1^-)\big],$$
$$u_{\mathrm{as}\,;\,[0,2\sigma]} - u_0 = \big[v_{0;\,0}(\xi_0^+) + \varepsilon v_{1;\,0}(\xi_0^+)\big] + \big[v_{0;\,2\sigma}(\xi_{2\sigma}^-) + \varepsilon v_{1;\,2\sigma}(\xi_{2\sigma}^-)\big].$$

Combining these with (14), (15) and $\xi_1^- = \frac{1-x}{\varepsilon}$, $\xi_{2\sigma}^- = \frac{2\sigma-x}{\varepsilon}$, $\xi_0^+ = \frac{x}{\varepsilon}$, we obtain (28) and thus complete the proof. □

Lemma 3.2. *Under the conditions of Lemma 3.1, there exist solutions u and u_{C} of problems* (1) *and* (5c) *such that*

$$|(u-u_{\mathrm{C}})(x)| \le C(\varepsilon^2 + N^{-2}) \quad \textit{for all} \quad x \in \bar{\Omega}_{\mathrm{C}} = [\sigma, 1-\sigma].$$

Proof. Applying Theorem 2.7 to problem (1) and then imitating the proof of (28) in Lemma 3.1, we get $u - u_{\mathrm{as}\,;\,[0,1]} = O(\varepsilon^2 + N^{-2})$ and $u_{\mathrm{as}\,;\,[0,1]} = u_0 + O(N^{-2})$ for $x \in [\sigma, 1-\sigma]$. Thus it remains to show that there exists a solution u_{C} of (5c) such that $u_{\mathrm{C}} - u_0 = O(\varepsilon^2 + N^{-2})$.

Combining (27) with the boundary conditions from (5c), we note that

$$u_{\mathrm{C}}(l) = g_l \qquad \text{at } \ l = \sigma,\ 1-\sigma, \qquad \text{where } \ g_l - u_0(l) = O(\varepsilon^2 + N^{-2}), \tag{29}$$

so one can easily check that the boundary conditions of problem (5c) satisfy assumption (3b). Now, Theorem 2.7, applied to this problem, implies existence of a solution u_{C} such that $u_{\mathrm{C}} - u_{\mathrm{as}\,;\,[\sigma,1-\sigma]} = O(\varepsilon^2 + N^{-2})$. Furthermore, using (14) to estimate the boundary-layer components of $u_{\mathrm{as}\,;\,[\sigma,1-\sigma]}$, we observe that they do not exceed $C_\delta |g_l - u_0(l)| = O(\varepsilon^2 + N^{-2})$. This implies that $u_{\mathrm{as}\,;\,[\sigma,1-\sigma]} = u_0 + O(\varepsilon^2 + N^{-2})$ and hence $u_{\mathrm{C}} - u_0 = O(\varepsilon^2 + N^{-2})$. □

We now obtained all the results that we need to bound the error in the continuous Schwarz method.

Theorem 3.3. *Under the conditions of Lemma 3.1, there exist a solution u of problem (1) and a first-iteration approximation $u^{[1]}$ defined in* (6) *such that*

$$|(u-u^{[1]})(x)| \le C(\varepsilon^2 + N^{-2}), \qquad \textit{for all } \ x \in \bar{\Omega}.$$

Proof. Combining (6) with Lemma 3.1 and Lemma 3.2 yields the required result. □

4. Discrete problem in a particular subdomain

4.1. Z-fields. In our analysis of nonlinear discrete problems, we shall invoke the theory of Z-fields, which we now briefly describe.

Definition 4.1. An operator $H : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a Z-field if for all $i \ne j$ the mapping $x_j \mapsto (H(x_0, x_1, \dots, x_n))_i$ is a monotonically decreasing function from $\mathbb{R}$ to $\mathbb{R}$ when $x_0, \dots, x_{j-1}, x_{j+1}, \dots x_n$ are fixed.

Remark 4.2. If H is differentiable, then H is a Z-field if and only if its Jacobian matrix has non-positive off-diagonal entries.

We shall use the following unpublished result from Lorenz [14]; see also [11].

Lemma 4.3. [14] *Let $H : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be continuous and a Z-field. Let $r \in \mathbb{R}^{n+1}$ be given. Assume that there exist α, $\beta \in \mathbb{R}^{n+1}$ such that $\alpha \le \beta$ and $H\alpha \le r \le H\beta$ (these inequalities are understood to hold true component-wise). Then the equation $Hy = r$ has a solution $y \in \mathbb{R}^{n+1}$ with $\alpha \le y \le \beta$.*

Remark 4.4. The functions α and β of Lemma 4.3 are called sub- and super-solutions of the discrete problem $Hy = r$.

4.2. Computed solution in a particular subdomain: existence and accuracy. Following the discussion of problem (8) in a particular subdomain $[a, b]$ presented in §2, we shall now focus on its numerical solution.

For a given positive integer N, introduce an arbitrary nonuniform mesh $\bar{\Omega}^N_{[a,b]} := \{x_i\}_{i=0}^N$ in $[a, b]$ with $x_0 = a$, $x_N = b$, $h_i := x_i - x_{i-1} > 0$ for $i = 1, \dots, N$, and $\hbar_i := (h_i + h_{i+1})/2$ for $i = 1, \dots, N-1$. Let $\Omega^N_{[a,b]} := \{x_i\}_{i=1}^{N-1}$ be the corresponding mesh of interior nodes.

The computed solution $u^N_{[a,b],i}$, $i = 0, \dots, N$, is required to satisfy the standard three-point difference scheme:

$$F^N u^N_{[a,b],i} := -\varepsilon^2 \delta^2\, u^N_{[a,b],i} + f(x_i, u^N_{[a,b],i}) = 0 \qquad \text{for } x_i \in \Omega^N_{[a,b]}, \tag{30a}$$

$$u^N_{[a,b],0} = g^N_a, \qquad u^N_{[a,b],N} = g^N_b, \tag{30b}$$

where

$$\delta^2 v_i := \frac{1}{\hbar_i}\left(\frac{v_{i+1} - v_i}{h_{i+1}} - \frac{v_i - v_{i-1}}{h_i}\right). \tag{30c}$$

Remark 4.5. The mapping $(x_0, \dots, x_N) \mapsto (g^N_a, F^N(u^N)_1, \dots, F^N(u^N)_{N-1}, g^N_b)$ is a Z-field.

We want to investigate the existence and accuracy of solutions to problem (30). As with the analysis of the discretisation of (1) in [11], this is done by constructing discrete sub- and super-solutions and then invoking the theory of Z-fields.

Lemma 4.6. *Let $b - a \ge (4/\gamma)\varepsilon \ln N$ and $\beta_{[a,b]}(x; p)$, for $|p| \le p_0$, be defined by (11) with p_0 from Lemma 2.1. Suppose that the truncation error*

$$r_i\left[\beta_{[a,b]}(x; p)\right] := F^N \beta_{[a,b]}(x_i; p) - F\beta_{[a,b]}(x; p)\big|_{x=x_i} = O(N^{-2} \ln^q N) \tag{31a}$$

for all $|p| \le p_0$ and $x_i \in \Omega^N_{[a,b]}$ and for some $q > 0$. Furthermore, suppose that

$$|g^N_a - g_a| + |g^N_b - g_b| \le C N^{-2} \ln^q N. \tag{31b}$$

Then, for sufficiently large N, there exists a solution $u^N_{[a,b],i}$ of (30) such that

$$|u^N_{[a,b],i} - u_{[a,b]}(x_i)| \le C N^{-2} \ln^q N \qquad \text{for} \quad x_i \in \bar{\Omega}^N_{[a,b]}, \tag{32}$$

where $u = u_{[a,b]}$ is a solution of (8).

Proof. Set $\bar{p} := \bar{C} N^{-2} \ln^q N \le p_0$ for some sufficiently large $\bar{C}$ so that, by Corollary 2.6, we have

$$F\beta_{[a,b]}(x; -\bar{p}) \le -(\bar{C}/2) C_0 \gamma^2 N^{-2} \ln^q N, \qquad F\beta_{[a,b]}(x; \bar{p}) \ge (\bar{C}/2) C_0 \gamma^2 N^{-2} \ln^q N$$

(this is possible since $\varepsilon \le C N^{-1}$, by (7), and N is sufficiently large). Combining these with (31a) and choosing $\bar{C}$ sufficiently large, yields

$$F^N \beta_{[a,b]}(x_i; -\bar{p}) \le 0 \le F^N \beta_{[a,b]}(x_i; \bar{p}) \qquad \text{for} \quad x_i \in \Omega^N_{[a,b]}.$$

Furthermore, by (31b), a sufficiently large $\bar{C}$ provides $\beta_{[a,b]}(l; -\bar{p}) \le g^N_l \le \beta_{[a,b]}(l; \bar{p})$ for $l = a, b$. Finally note that $\beta_{[a,b]}(x; -\bar{p}) \le \beta_{[a,b]}(x; \bar{p})$; see the proof of Theorem 2.7. Thus $\beta_{[a,b]}(x_i; -\bar{p})$ and $\beta_{[a,b]}(x_i; \bar{p})$ are sub- and super-solutions for the discrete problem (30) and, by Lemma 4.3, there exists a discrete solution $u^N_{[a,b],i}$ between $\beta_{[a,b]}(x_i; -\bar{p})$ and $\beta_{[a,b]}(x_i; \bar{p})$. Now, imitating the argument used in the proof of Theorem 2.7 to establish (25), we see that $|u^N_{[a,b],i} - u_{\mathrm{as}\,;\,[a,b]}(x_i)| \le C\bar{p}$. Combining this with (25), we get the desired error estimate (32). $\square$

5. Discrete Schwarz method. Error analysis

Now we introduce a discrete Schwarz method for problem (1) by discretizing the continuous problems (5) in the overlapping subdomains $\Omega_{\rm L}$, $\Omega_{\rm C}$ and $\Omega_{\rm R}$ described in (4).

Choose $\sigma \geq (2/\gamma)\,\varepsilon \ln N$ and define the meshes $\bar\Omega_{\rm L}^N$, $\bar\Omega_{\rm C}^N$ and $\bar\Omega_{\rm R}^N$ in these subdomains as follows. Let the mesh $\bar\Omega_{\rm C}^N = \{x_{{\rm C},i}\}_{i=0}^N$ in the interior region $\bar\Omega_{\rm C}$ be uniform with $x_{{\rm C},i} = \sigma + i(1-2\sigma)/N$. In the boundary-layer regions $\bar\Omega_{\rm L}$ and $\bar\Omega_{\rm R}$ we use certain layer-adapted meshes $\bar\Omega_{\rm L}^N = \{x_{{\rm L},i}\}_{i=0}^N$ and $\bar\Omega_{\rm R}^N = \{x_{{\rm R},i}\}_{i=0}^N$, which are specified below. Note that the numbers of mesh nodes in these three meshes should not be necessarily equal, but only of the same order, and are chosen equal here only to simplify the presentation.

Each of the three problems in (5) is discretized by the finite difference scheme (30). Thus we require the computed solutions $u_{\rm L}^N$, $u_{\rm R}^N$ and $u_{\rm C}^N$, associated with the meshes $\bar\Omega_{\rm L}^N$, $\bar\Omega_{\rm R}^N$ and $\bar\Omega_{\rm C}^N$, respectively, to satisfy

$$F^N u_{{\rm L},i}^N = 0 \quad \text{for } x_{{\rm L},i} \in \Omega_{\rm L}^N,\ \ u_{\rm L}^N(0) = g_0, \qquad u_{\rm L}^N(2\sigma) = g_{2\sigma}, \tag{33a}$$

$$F^N u_{{\rm R},i}^N = 0 \quad \text{for } x_{{\rm R},i} \in \Omega_{\rm R}^N,\ \ u_{\rm R}^N(1-2\sigma) = g_{1-2\sigma}, \quad u_{\rm R}(1) = g_1, \tag{33b}$$

$$F^N u_{{\rm C},i}^N = 0 \quad \text{for } x_{{\rm C},i} \in \Omega_{\rm C}^N,\ \ u_{\rm C}^N(\sigma) = u_{\rm L}^N(\sigma), \quad u_{\rm C}^N(1-\sigma) = u_{\rm R}^N(1-\sigma), \tag{33c}$$

where $g_0 = u(0)$ and $g_1 = u(1)$. Next, the discrete first-iteration approximation $u^{N,[1]}$ is defined, similarly to (6), by

$$u^{N,[1]}(x_i) := \begin{cases} u_{\rm L}^N(x_i), & x_i \in \bar\Omega_{\rm L}^N \backslash \Omega_{\rm C}, \\ u_{\rm C}^N(x_i), & x_i \in \bar\Omega_{\rm C}^N, \\ u_R^N(x_i), & x_i \in \bar\Omega_{\rm R}^N \backslash \Omega_{\rm C}. \end{cases} \tag{34}$$

Remark 5.1. Relations (33), (34) describe the first iteration of the Schwarz iterative procedure, which, as we shall show below, is sufficient for second-order accuracy in the case of $\varepsilon \leq CN^{-1}$. Consequently, we do not theoretically investigate the accuracy of further iterations, which, for the alternating version of the Schwarz method, consist of successfully solving the following discrete problems: $F^N u_{\rm L,R,C}^{N,[k]} = 0$ in $\Omega_{\rm L,R,C}^N$ subject to $u_{\rm L}^{N,[k]}(0) = g_0$ and $u_{\rm L}^{N,[k]}(2\sigma) = u_{\rm C}^{N,[k-1]}(2\sigma)$ for $u_{\rm L}^{N,[k]}$, subject to $u_{\rm R}^{N,[k]}(1-2\sigma) = u_{\rm C}^{N,[k-1]}(1-2\sigma)$ and $u_{\rm R}^{N,[k]}(1) = g_1$ for $u_{\rm R}^{N,[k]}$, and subject to $u_{\rm C}^{N,[k]}(\sigma) = u_{\rm L}^{N,[k]}(\sigma)$ and $u_{\rm C}^{N,[k]}(1-\sigma) = u_{\rm R}^{N,[k]}(1-\sigma)$ for $u_{\rm C}^{N,[k]}$, where $k = 2, 3, \ldots$. Here if, e.g., 2σ is not on the mesh $\Omega_{\rm C}^N$, then the standard linear interpolation is applied to the computed solution $u_{\rm C}^{N,[k-1]}$ to evaluate the boundary condition at 2σ for $u_{\rm L}^{N,[k]}$.

We shall now elaborate on the layer-adapted meshes $\bar\Omega_{\rm L}^N$ and $\bar\Omega_{\rm R}^N$, whose choice is crucial for the accuracy of the method. Note that our results hold true on general layer-adapted meshes such as those considered in [13], but for clarity we shall discuss only two examples: Bakhvalov and Shishkin meshes. Let the mesh $\bar\Omega_{\rm R}^N$ reflect $\bar\Omega_{\rm L}^N$ in $x = 1/2$, i.e. $x_{{\rm R},i} = 1 - x_{{\rm L},N-i}$ for $i = 0, \ldots, N$, and define $\bar\Omega_{\rm L}^N$ as follows.

5(a) Bakhvalov mesh. [1] Set $\sigma := (2/\gamma)\varepsilon|\ln\varepsilon|$ and let the mesh $\bar\Omega_{\rm L}^N$ on $[0, 2\sigma]$ have the nodes $x_{{\rm L},i} := x(t_i)$, where $t_i = i[2(1-\varepsilon)/N]$ for $i = 0\ldots, N$. Here we used the mesh-generating function

$$x(t) \;\; := \;\; -(2/\gamma)\varepsilon \ln(1-t) \qquad \text{for} \quad t \in [0, 1-\varepsilon],$$

while for $t \in [1-\varepsilon, 2(1-\varepsilon)]$ we set $x(t) = 2\sigma - x(2(1-\varepsilon) - t)$ so that the sub-mesh $\{x_{{\rm L},i}\}_{i=N/2}^N$ on $[\sigma, 2\sigma]$ reflects the sub-mesh $\{x_{{\rm L},i}\}_{i=0}^{N/2}$ on $[0, \sigma]$ in $x = \sigma$.

5(b) Shishkin mesh. [16] Set $\sigma := (2/\gamma)\,\varepsilon \ln N$ and introduce a uniform mesh $\bar\Omega^N_{\rm L}$ on $[0, 2\sigma]$ with the nodes $x_{{\rm L},i} := i(2\sigma/N)$ for $i = 0, \ldots, N$.

Remark 5.2. In view of Remark 2.2, an inspection of our further analysis shows that in the mesh definitions 5(a),(b), the parameter γ from (3a) can be replaced by an arbitrary, possibly larger, value $\tilde\gamma$ such that $0 < \tilde\gamma^2 < \gamma_0^2 = \min_{x=0,1} f_u(x, u_0(x))$.

Lemma 5.3. *Suppose that the boundary data g_l, for $l = 0,\ 2\sigma,\ 1-2\sigma,\ 1$, of problems (5a) and (5b) satisfy (3b), and, under condition (7), the mesh $\bar\Omega^N_{\rm L}$ is either the Bakhvalov mesh of §5(a) or the Shishkin mesh of §5(b). Then there exist solutions $u^N_{\rm L}$ and $u^N_{\rm R}$ of discrete problems (33a) and (33b) such that, for sufficiently large N,*

$$|u^N_{\rm L,R} - u_{\rm L,R}| \le CN^{-2}\ln^q N \qquad \textit{for}\ \ x_i \in \bar\Omega^N_{\rm L,R}, \tag{35}$$

where $q = 0$ for the Bakhvalov mesh and $q = 2$ for the Shishkin mesh.

Proof. It suffices to estimate the error in the computed solution $u^N_{\rm L}$, as the analogous estimate for $u^N_{\rm R}$ is similar. The desired estimate for $u^N_{\rm L}$ is obtained by applying Lemma 4.6 to the continuous problem (5a) and its discretization (33a). Hypothesis (31b) of this lemma is straightforward. Thus, it remains to check the other hypothesis (31a) for $\beta_{[0,2\sigma]}$. Furthermore, we shall establish this hypothesis only for the case of $x_i \in [0, \sigma]$ as the other case of $x_i \in [\sigma, 2\sigma]$ is analogous.

Let $z := \beta_{[0,2\sigma]} - \hat\beta_{[0,2\sigma]}$, where $\beta_{[0,2\sigma]}$ and $\hat\beta_{[0,2\sigma]}$ are defined in (11), (17), so $z(x) = \tilde v_{0;\,2\sigma}(\xi^-_{2\sigma}; p) + \varepsilon v_{1;\,2\sigma}(\xi^-_{2\sigma})$; note that $\beta_{[0,2\sigma]} \approx \hat\beta_{[0,2\sigma]}$ on $[0, \sigma]$. The definition of the truncation error r_i from (31a) implies that $r_i\,[\beta_{[0,2\sigma]}] = r_i\,[\hat\beta_{[0,2\sigma]}] + r_i\,[z]$. Furthermore,

$$\bigl|r_i\,[z]\bigr| = \varepsilon^2 \bigl|\delta^2 z(x_i) - \tfrac{d^2}{dx^2} z(x_i)\bigr| \le 2\varepsilon^2 \max_{x\in[0,\sigma+h]} \bigl|\tfrac{d^2}{dx^2} z\bigr| = 2 \max_{x\in[0,\sigma+h]} \bigl|\bigl(\tfrac{d}{d\xi^-_{2\sigma}}\bigr)^2 z\,\bigr|. \tag{36}$$

Here $x_i \in [0, \sigma]$ and h is the maximum mesh size of $\bar\Omega^N_{\rm L}$, for which a calculation shows that $2\sigma - (\sigma + h) = \sigma - h = x_{N/2-1} \ge (2/\gamma)\varepsilon \ln(CN)$. Thus, invoking (14) combined with (15), we get $r_i\,[z] = O(N^{-2}\ln^q N)$. Finally, imitating the proof of [11, Lemma 3.3 and §3.4.2] yields $r_i\,[\hat\beta_{[0,2\sigma]}] = O(N^{-2}\ln^q N)$. Combining this with our estimate for $r_i\,[z]$, we obtain the required estimate (31a) for $r_i\,[\beta_{[0,2\sigma]}]$ and thus complete the proof. $\square$

Lemma 5.4. *Under the conditions of Lemma 5.3, there exists a solution $u^N_{\rm C}$ of discrete problem (33c) such that*

$$|u^N_{\rm C} - u_{\rm C}| \le CN^{-2}\ln^q N \qquad \textit{for}\ \ x_i \in \bar\Omega^N_{\rm C},$$

where $u_{\rm C}$ is a solution of problem (5c), while $q = 0$ for the Bakhvalov mesh and $q = 2$ for the Shishkin mesh.

Proof. The desired estimate for $u^N_{\rm C}$ follows from Lemma 4.6 applied to the continuous problem (5c) and its discretization (33b). Hypothesis (31b) of this lemma follows from (35) combined with the boundary conditions in problems (5c) and (33c). Therefore it remains to check the other hypothesis (31a) for $\beta_{[\sigma,1-\sigma]}$.

By (11), we have $\beta_{[\sigma,1-\sigma]} = u_0(x) + z_1 + z_2 + C_0 p$, where $z_1(x) := \tilde v_{0;\,\sigma}(\xi^+_\sigma; p) + \varepsilon v_{1;\,\sigma}(\xi^+_\sigma)$ and $z_2(x) := \tilde v_{0;\,1-\sigma}(\xi^-_{1-\sigma}; p) + \varepsilon v_{1;\,1-\sigma}(\xi^-_{1-\sigma})$. Now, imitating (36) yields

$$\bigl|r_i\,[\beta_{[\sigma,1-\sigma]}]\bigr| \le 2\varepsilon^2 \max_{x\in[\sigma,1-\sigma]} \bigl|\tfrac{d^2}{dx^2}\beta_{[\sigma,1-\sigma]}\bigr| = 2 \max_{x\in[\sigma,1-\sigma]} \bigl|\varepsilon^2 u_0'' + \bigl(\tfrac{d}{d\xi^+_\sigma}\bigr)^2 z_1 + \bigl(\tfrac{d}{d\xi^-_{1-\sigma}}\bigr)^2 z_2\,\bigr|.$$

Noting that $\varepsilon^2 u_0'' = O(\varepsilon^2)$ and estimating the derivatives of z_1 and z_2 by combining (14) with (15) and (29), we arrive at $r_i\,[\beta_{[\sigma,1-\sigma]}] = O(\varepsilon^2 + N^{-2}) = O(N^{-2})$. Here we also used (7). Thus we obtained the required estimate (31a) for $r_i\,[\beta_{[\sigma,1-\sigma]}]$. $\square$

Theorem 5.5. *Suppose that the boundary data g_l, for $l = 0, 2\sigma, 1-2\sigma, 1$, of problems (5a) and (5b) satisfy (3b), and, under condition (7), the mesh $\bar\Omega^N_{\rm L}$ is either the Bakhvalov mesh of §5(a) or the Shishkin mesh of §5(b). Then there exist a solution u of problem (1) and a discrete first-order approximation $u^{N,[1]}$ defined in (33), (34), such that, for sufficiently large N, we have*

$$|u_i^{N,[1]} - u(x_i)| \le CN^{-2}\ln^q N \qquad \text{for } x_i \in \bar\Omega^N := (\bar\Omega^N_{\rm L} \setminus \bar\Omega_{\rm C}) \cup \bar\Omega^N_{\rm C} \cup (\bar\Omega^N_{\rm R} \setminus \bar\Omega_{\rm C}),$$

where $q = 0$ for the Bakhvalov mesh and $q = 2$ for the Shishkin mesh.

Proof. The desired estimate immediately follows from Theorem 3.3, Lemma 5.3 and Lemma 5.4 combined with (6), (34) and then (7). ☐

Corollary 5.6. *Under the conditions of Theorem 5.5, for the linear interpolant $u^{N,[1]}(x)$ of the discrete solution $\{u_i^{N,[1]}\}$ on the mesh $\bar\Omega^N = (\bar\Omega^N_{\rm L} \setminus \bar\Omega_{\rm C}) \cup \bar\Omega^N_{\rm C} \cup (\bar\Omega^N_{\rm R} \setminus \bar\Omega_{\rm C})$, we have*

$$|u^{N,[1]} - u(x)| \le CN^{-2}\ln^q N \qquad \text{for } x \in [0,1].$$

Proof. The desired estimate follows from the analogous estimate for $u - u^I$, where $u^I(x)$ is a linear interpolant of the exact solution on the mesh $\bar\Omega^N$. ☐

6. Numerical results

Consider the following version of a problem of Herceg [8]:

$$-\varepsilon^2 u'' + (u^2 + u - 3.75)(u - 0.5)(u + 2 - \cos x) = 0 \quad \text{for } x \in (0,1), \tag{37}$$

with $u(0) = u(1) = 0$. Here $f(x,u) = (u^2+u-3.75)(u-0.5)(u+2-\cos x)$, and the reduced problem $f(x,u_0) = 0$ has four solutions $u_1 = -2.5$, $u_2 = \cos x - 2$, $u_3 = 0.5$ and $u_4 = 1.5$ with $f_u(x,u_{1,3}) < 0$ and $f_u(x,u_{2,4}) > 0$ for $x \in [0,1]$. By (3a), the reduced solutions u_1 and u_3 are not stable, while u_2 and u_4 are stable and satisfy conditions (3). We shall present numerical results for the solutions of (37) close to $u_2 = \cos x - 2$. By (3a), we choose the Bakhvalov/Shishkin mesh parameter $\gamma = 2$ so that $\gamma < \min_{x\in[0,1]} [f_u(x, \cos x - 2)]^{1/2} \approx 2.37$.

The test problem (37) was solved numerically using the alternating Schwarz iterative procedure (see Remark 5.1) with $g_{2\sigma} := g_0 = 0$ and $g_{1-2\sigma} := g_1 = 0$ in (33), and the stopping criterion

$$\max_{x_i\in\bar\Omega^N} |u^{N,[k+1]}(x_i) - u^{N,[k]}(x_i)| \le C^* N^{-2}\ln^q N, \tag{38}$$

where $q = 0$ for the Bakhvalov mesh and $q = 2$ for the Shishkin mesh, and we shall usually take $C^* = 0.2$. Here the tolerance of $C^*N^{-2}\ln^q N$ is motivated by the error estimate of Theorem 5.5.

Tables 1 and 4 list the numbers of Schwarz iterations needed until the stopping criterion (38) is satisfied. We observe that whenever $\varepsilon \le N^{-1}$, one iteration is required, i.e. $u^{N,[k]} = u^{N,[1]}$.

To computationally investigate $|u^{N,[k]} - u|$ (equal to $|u^{N,[1]} - u|$ for $\varepsilon \le N^{-1}$), note that the considered Schwarz domain decomposition method may be interpreted as an iterative solver for the discretization [11] in the nondecomposed domain $[0,1]$:

$$F^N u^N = 0 \qquad \text{for } x_i \in \Omega^N := (\Omega^N_{\rm L} \setminus \bar\Omega_{\rm C}) \cup \bar\Omega^N_{\rm C} \cup (\Omega^N_{\rm R} \setminus \bar\Omega_{\rm C}), \tag{39}$$

with $u^N(0) = u^N(1) = 0$. As it was proved in [11] that $|u_i^N - u(x_i)| \le CN^{-2}\ln^q N$, we list the values of $\max_{x_i\in\bar\Omega^N} |u_i^{N,[k]} - u_i^N|$ in Tables 2 and 5. Furthermore, in Tables 3 and 6 we present approximate values of $\max_{x_i\in\bar\Omega^N} |u_i^N - u(x_i)|$, computed as described in [11], in order to compare these values with $\max_{x_i\in\bar\Omega^N} |u_i^{N,[k]} - u_i^N|$.

N	$\varepsilon=1$	$\varepsilon=10^{-1}$	$\varepsilon=10^{-2}$	$\varepsilon=10^{-3}$	$\varepsilon=10^{-4}$	$\varepsilon=10^{-5}$	$\varepsilon=10^{-6}$	$\varepsilon=10^{-7}$	$\varepsilon=10^{-8}$
16	9	2	1	1	1	1	1	1	1
32	11	2	1	1	1	1	1	1	1
64	13	2	1	1	1	1	1	1	1
128	15	2	2	1	1	1	1	1	1
256	17	2	2	1	1	1	1	1	1
512	19	2	2	1	1	1	1	1	1

TABLE 1. Bakhvalov mesh: number k of Schwarz iterations needed to satisfy the stopping criterion (38) with $C^* = 0.2$ and $q = 0$; $\gamma = 2$.

N	$\varepsilon=1$	$\varepsilon=10^{-1}$	$\varepsilon=10^{-2}$	$\varepsilon=10^{-3}$	$\varepsilon=10^{-4}$	$\varepsilon=10^{-6}$	$\varepsilon=10^{-8}$
16	9.77e-4	7.22e-4	2.11e-4	4.95e-5	1.76e-5	4.23e-6	1.61e-6
32	2.61e-4	3.01e-4	8.05e-5	1.39e-5	4.45e-6	9.62e-7	3.48e-7
64	6.98e-5	9.57e-5	3.83e-5	4.14e-6	1.17e-6	2.25e-7	7.71e-8
128	1.86e-5	2.59e-5	1.71e-6	1.33e-6	3.16e-7	5.31e-8	1.71e-8
256	4.98e-6	6.58e-6	1.16e-6	4.78e-7	8.85e-8	1.27e-8	3.83e-9
512	1.33e-6	1.60e-6	4.57e-7	2.09e-7	2.59e-8	3.09e-9	8.62e-10

TABLE 2. Bakhvalov mesh: maximum nodal values of $|u^{N,[k]} - u^N|$, where $u^{N,[k]}$ satisfies the stopping criterion (38) with $C^* = 0.2$ and $q = 0$, and u^N is a solution of (39).

N	$\varepsilon=1$	$\varepsilon=10^{-1}$	$\varepsilon=10^{-2}$	$\varepsilon=10^{-3}$	$\varepsilon=10^{-4}$	$\varepsilon=10^{-6}$	$\varepsilon=10^{-8}$	$\max\limits_{\substack{\varepsilon=10^{-m}\\ m=0,\dots,8}}$
16	2.00	1.98	2.00	2.00	2.00	2.00	2.00	2.00
32	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
64	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
128	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
256	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
16	4.35e-5	5.88e-3	7.24e-3	7.38e-3	7.39e-3	7.39e-3	7.39e-3	7.39e-3
32	1.09e-5	1.50e-3	1.81e-3	1.85e-3	1.85e-3	1.85e-3	1.85e-3	1.85e-3
64	2.73e-6	3.74e-4	4.54e-4	4.62e-4	4.63e-4	4.63e-4	4.63e-4	4.63e-4
128	6.82e-7	9.36e-5	1.14e-4	1.16e-4	1.16e-4	1.16e-4	1.16e-4	1.16e-4
256	1.70e-7	2.34e-5	2.84e-5	2.89e-5	2.90e-5	2.90e-5	2.90e-5	2.90e-5
512	4.26e-8	5.85e-6	7.10e-6	7.23e-6	7.24e-6	7.24e-6	7.24e-6	7.24e-6

TABLE 3. Bakhvalov mesh: computational convergence rates r for the conventional nondecomposed-domain method (39) assuming that the error $u^N - u$ is CN^{-r} (upper part); approximate maximum nodal errors $|u^N - u|$ (lower part).

Discrete non-linear problems involved in our method, were solved using Newton iterations with initial guess equal to u_2 at all mesh nodes. In all our computations 5 Newton iterations were sufficient to get discrete solutions within the tolerance of 10^{-8}.

6.1. Bakhvalov mesh. The numerical results for the Bakhvalov mesh are given in Tables 1–3. The mesh definition of §5(a) is valid only when $\varepsilon \le 1/e$ and $\sigma \le 1/4$; otherwise, to accommodate larger values of $\varepsilon \in (0, 1]$, we set $\sigma := 1/4$ and use uniform meshes $\bar{\Omega}^N_{\mathrm{L}}$ and $\bar{\Omega}^N_{\mathrm{R}}$.

N	$\varepsilon = 1$	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-7}$	$\varepsilon = 10^{-8}$
16	6	1	1	1	1	1	1	1	1
32	7	2	1	1	1	1	1	1	1
64	9	2	1	1	1	1	1	1	1
128	10	2	1	1	1	1	1	1	1
256	12	2	1	1	1	1	1	1	1
512	14	2	1	1	1	1	1	1	1

TABLE 4. Shishkin mesh: number k of Schwarz iterations needed to satisfy the stopping criterion (38) with $C^* = 0.2$ and $q = 2$; $\gamma = 2$.

N	$\varepsilon = 1$	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-8}$
16	7.05e-3	3.76e-3	3.53e-3	4.05e-3	4.11e-3	4.12e-3	4.12e-3
32	3.65e-3	4.13e-8	5.54e-4	6.94e-4	7.11e-4	7.13e-4	7.13e-4
64	9.77e-4	3.85e-8	8.77e-5	1.26e-4	1.32e-4	1.33e-4	1.33e-4
128	5.05e-4	3.78e-8	1.42e-5	2.27e-5	2.49e-5	2.52e-5	2.52e-5
256	1.35e-4	3.76e-8	2.50e-6	3.98e-6	4.73e-6	4.83e-6	4.83e-6
512	3.61e-5	3.76e-8	4.67e-7	6.64e-7	8.92e-7	9.30e-7	9.30e-7

TABLE 5. Shishkin mesh: maximum nodal values of $|u^{N,[k]} - u^N|$, where $u^{N,[k]}$ satisfies the stopping criterion (38) with $C^* = 0.2$ and $q = 2$, and u^N is a solution of (39).

Tables 1 and 2 list the numbers k of Schwarz iterations needed until the stopping criterion (38) is satisfied, and the corresponding values $\max_{x_i \in \bar{\Omega}^N} |u_i^{N,[k]} - u_i^N|$. We observe that whenever $\varepsilon \le N^{-1}$, one iteration is required, i.e. $u^{N,[k]} = u^{N,[1]}$.

Table 3 gives approximate values of the convergence rates r and the errors $|u^N - u|$ for the conventional nondecomposed-domain method (39). As the exact solution is unknown, Table 3 was computed as described in [11, §4.1] (assuming that $u_i^N - u_i \approx CN^{-r}$ for some $C > 0$ and $r > 0$, and therefore using $u^N - u^{2N}$). It is clear from our numerical results that $r = 2$, i.e. u^N is an ε-uniform second-order approximation to the exact solution u.

Comparing Tables 2 and 3, we observe that $\max_i |u^{N,[k]} - u^N|$ does not exceed $\max_i |u^N - u|$ (with one exception of $\varepsilon = 1$, which we discuss below). Thus, $u^{N,[k]}$ is also an ε-uniform second-order approximation to the exact solution u. In particular, for $\varepsilon \le N^{-1}$, when we have $u^{N,[1]} = u^{N,[k]}$, this agrees with the theoretical conclusion of Theorem 5.5.

In the non-singularly-perturbed case of $\varepsilon = 1$, the numerical method (39) enjoys much faster convergence; a calculation using the first column of Table 3 shows that $\max_i |u^N - u| \approx 0.0112\, N^{-2}$. Therefore, to make $\max_i |u^{N,[k]} - u^N|$ negligible compared with $\max_i |u^N - u|$, one needs to use the stopping criterion (38) with $q = 0$ and $C^* < 0.0112$ (this is confirmed by numerical results that we do not present here).

6.2. Shishkin mesh. The numerical results for the Shishkin mesh are given in Tables 4–6. To accommodate larger values of $\varepsilon \in (0, 1]$, the definition of σ of §5(b) is slightly modified to $\sigma := \min\{(2/\gamma)\, \varepsilon \ln N,\, 1/4\}$.

Tables 4 and 5 list the numbers k of Schwarz iterations needed until the stopping criterion (38) is satisfied, and the corresponding values $\max_{x_i \in \bar{\Omega}^N} |u_i^{N,[k]} - u_i^N|$.

N	$\varepsilon = 1$	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-8}$	$\max_{\substack{\varepsilon=10^{-m}\\ m=0,\dots,8}}$
16	2.95	2.24	0.69	0.68	0.68	0.68	0.68	0.73
32	2.71	2.60	2.00	2.00	2.00	2.00	2.00	2.00
64	2.57	2.57	1.93	1.93	1.93	1.93	1.93	1.93
128	2.48	2.47	1.99	1.99	1.99	1.99	1.99	1.99
256	2.41	2.41	2.00	2.00	2.00	2.00	2.00	2.00
16	4.35e-5	7.04e-3	6.86e-3	6.87e-3	6.87e-3	6.88e-3	6.88e-3	7.04e-3
32	1.09e-5	2.45e-3	4.97e-3	4.99e-3	4.99e-3	4.99e-3	4.99e-3	4.99e-3
64	2.73e-6	6.49e-4	1.79e-3	1.80e-3	1.80e-3	1.80e-3	1.80e-3	1.80e-3
128	6.82e-7	1.63e-4	6.35e-4	6.37e-4	6.37e-4	6.37e-4	6.37e-4	6.37e-4
256	1.70e-7	4.09e-5	2.08e-4	2.09e-4	2.09e-4	2.09e-4	2.09e-4	2.09e-4
512	4.26e-8	1.02e-5	6.60e-5	6.62e-5	6.62e-5	6.62e-5	6.62e-5	6.62e-5

TABLE 6. Shishkin mesh: computational convergence rates r for the conventional nondecomposed-domain method (39) assuming that the error $u^N - u$ is $C(N^{-1}\ln N)^r$ (upper part); approximate maximum nodal errors $|u^N - u|$ (lower part).

We observe that whenever $\varepsilon \le N^{-1}$ (and occasionally even when $\varepsilon > N^{-1}$), one iteration is required, i.e. $u^{N,[k]} = u^{N,[1]}$.

Table 6 gives approximate values of the convergence rates r and the errors $|u^N - u|$ for the conventional nondecomposed-domain method (39). As the exact solution is unknown, Table 6 was computed as described in [11, §4.2] (assuming that $u_i^N - u_i \approx C(N^{-1}\ln N)^r$ for some $C > 0$ and $r > 0$, and therefore using the discrete solution $\bar{u}^{2N}$ on the auxiliary bisected mesh). It is clear from our numerical results that $r = 2$, i.e. $|u^N - u| \le CN^{-2}\ln^2 N$.

Comparing Tables 5 and 6, we observe that $\max_i |u^{N,[k]} - u^N|$ does not exceed $\max_i |u^N - u|$ (with one exception of the non-singularly-perturbed case of $\varepsilon = 1$, which we already discussed in §6.1). Thus, $u^{N,[k]}$ gives an ε-uniform almost-second-order approximation (with a logarithmic factor $\ln^2 N$) to the exact solution u. In particular, for $\varepsilon \le N^{-1}$, when we have $u^{N,[1]} = u^{N,[k]}$, this confirms the theoretical estimate of Theorem 5.5.

In summary, our numerical results agree with our theoretical conclusions.

References

1. N. S. Bakhvalov, *On the optimization of methods for solving boundary value problems with boundary layers*, (Russian) Zh. Vychisl. Mat. Mat. Fis. **9** (1969), 841–859.
2. I. Boglaev, *Domain decomposition in boundary layers for singularly perturbed problems*, Appl. Numer. Math. **34** (2000), 145–166.
3. I. P. Boglaev, *Iterative algorithms of domain decomposition for the solution of a quasilinear elliptic problem*, J. Comp. Appl. Math. **80** (1997), 299–316.
4. C. M. D'annunzio, *Numerical analysis of a singular perturbation problem with multiple solutions*, Ph.D. thesis, University of Maryland at College Park, 1986.
5. P. A. Farrell, I. P. Boglaev, and V. V. Sirotkin, *Parallel domain decomposition methods for semi-linear singularly perturbed differential equations*, Comput. Fluid Dynamics Journal **2** (1994), 423–433.
6. P. C. Fife, *Semilinear elliptic boundary value problems with small parameters*, Arch. Rational Mech. Anal. **52** (1973), 205–232.
7. B. Heinrich and K. Pönitz, *Nitsche type mortaring for singularly perturbed reaction-diffusion problems*, Computing **75** (2005), 257–279.
8. D. Herceg, *Uniform fourth order difference scheme for a singular perturbation problem*, Numer. Math. **56** (1990), 675–693.
9. N. Kopteva, *Maximum norm error analysis of a 2d singularly perturbed semilinear reaction-diffusion problem*, Math. Comp. **76** (2007), 631–646.

10. N. Kopteva and M. Pickett, *A second-order overlapping Schwarz method for a 2d singularly perturbed semilinear reaction-diffusion problem with multiple solutions*, In Preparation.
11. N. Kopteva and M. Stynes, *Numerical analysis of a singularly perturbed nonlinear reaction-diffusion problem with multiple solutions*, Appl. Numer. Math. **51** (2004), 273–288.
12. ______, *Approximation of interior-layer solutions in a singularly perturbed semilinear reaction-diffusion problem*, (submitted for publication), www.staff.ul.ie/natalia/pubs.html (2008).
13. T. Linß, *Sufficient conditions for uniform convergence on layer-adapted grids*, Appl. Numer. Math. **37** (2001), 241–255.
14. J. Lorenz, *Nonlinear singular perturbation problems and the enquist-osher scheme*, Report 8115, Mathematical Institute, Catholic University of Nijmegen (1981).
15. H. MacMullen, J. J. H. Miller, E. O'Riordan, and G. I. Shishkin, *A second-order parameter-uniform overlapping Schwarz method for reaction-diffustion problems with boundary layers*, J. Comput. Appl. Math. **130** (2001), 231–244.
16. J. J. H. Miller, E. O'Riordan, and G. I. Shishkin, *Fitted numerical methods for singular perturbation problems*, World Scientific Publishing Co. Pte. Ltd., Singapore, 1996.
17. R. E. O'Malley, *Singular perturbation methods for ordinary differential equations*, Springer-Verlag, New York, 1991.
18. A. Quarteroni and A. Valli, *Domain decomposition methods for partial differential equations*, Clarendon Press, Oxford, 1999.
19. G. Sun and M. Stynes, *A uniformly convergent method for a singularly perturbed semilinear reaction-diffusion problem with multiple solutions*, Math. Comp. **65** (1996), 1085–1109.
20. A. B. Vasil'eva, V. F. Butuzov, and L. V. Kalachev, *The boundary function method for singular perturbation problems*, SIAM Stud. Appl. Math. vol. 14, SIAM, Philadelphia, PA, 1995.

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