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NUMERICAL SOLUTIONS OF STOCHASTIC DIFFERENTIAL DELAY EQUATIONS WITH JUMPS

GUIHUA ZHAO, MINGHUI SONG AND MINGZHU LIU

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Abstract. In this paper, the semi-implicit Euler (SIE) method for the stochastic differential delay equations with Poisson jump and Markov switching (SDDEwPJMSs) is developed. We show that under global Lipschitz assumptions the numerical method is convergent and SDDEwPJMSs is exponentially stable in mean-square if and only if for some sufficiently small step-size Δ the SIE method is exponentially stable in mean-square. We then replace the global Lipschitz conditions with local Lipschitz conditions and the assumptions that the exact and numerical solution have a bounded pth moment for some $p > 2$ and give the convergence result.

Key Words. Poisson jump, Lipschitz condition, semi-implicit Euler method, exponential stability, convergence.

1. Introduction

Stochastic modeling has come to play an important role in many branches of science and industry and there are significant literatures that have been done concerning approximate schemes for stochastic differential equations (SDEs) with Markov switching [8, 12] or SDEs with Poisson jump [5, 6, 7].

In general, the future state of a system depends on the present and past states. Hence, it is more significant to consider stochastic differential delay equations with Poisson jump and Markov switching (SDDEwPJMSs). As many other equations, SDDEwPJMSs cannot be solved analytically. Thus, it is necessary to develop numerical methods and to study the properties of these methods. Finite time convergence analysis of an Euler scheme is given in [13]. In this work, we consider the finite time convergence of SIE method, the exponential mean-square stability of analytic and SIE numerical solutions.

Throughout this paper, we let $W(t)$ be a d-dimensional Brownian motion, $N(t)$ be a scalar Poisson process with intensity λ and independent of the Brownian motion. Also we let $r(t)$, $t \geq 0$ be a right-continuous Markov chain taking values in a finite state space $\mathbb{S} = \{1, 2, \ldots, N\}$. The corresponding generator is denoted $\Gamma = (\gamma_{ij})_{N\times N}$, so that

$$\mathbb{P}\{r(t+\delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\delta + o(\delta): & \text{if } i \neq j, \\ 1 + \gamma_{ij}\delta + o(\delta): & \text{if } i = j, \end{cases}$$

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where $\delta > 0$. Here γ_{ij} is the transition rate from i to j satisfying $\gamma_{ij} \geq 0$ if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. Assume the Markov chain $r(\cdot)$ is independent of the Brownian motion $W(\cdot)$ and Poisson jump $N(\cdot)$. We note that almost every sample path of $r(\cdot)$ is right continuous step function with a finite number of sample jumps in any finite subinterval of $\mathbb{R}_+ := [0, \infty)$.

In this paper, we need to work on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions. To construct such a filtration, we denote by $\mathcal{N}$ the collection of $\mathbb{P}$-null sets, that is $\mathcal{N} = \{A \in \mathscr{F} : \mathbb{P}(A) = 0\}$, For each $t \geq 0$, define $\mathscr{F}_t = \sigma(\mathcal{N} \cup \sigma(B(s), r(s), N(s) : 0 \leq s \leq t))$.

We will use $|\cdot|$ to denote the Euclidean norm of a vector and the trace norm of a matrix and $< \cdot, \cdot >$ to denote the scalar product. We will denote the indicator function of a set G by I_G and denote by $L^2_{\mathscr{F}_t}([-\tau, 0]; \mathbb{R}^n)$ the family of $\mathscr{F}_t$-measurable, $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables $\varphi = \{\varphi(t) : -\tau \leq t \leq 0\}$ such that

$$||\varphi||^2_{\mathbb{E}} := \sup_{-\tau\leq u\leq 0} \mathbb{E}|\varphi(u)|^2 < \infty.$$

For $\mu \in \mathbb{R}$, $\mathrm{In}[\mu]$ denote the integer part of μ. In this paper we consider the following n-dimensional SDDEwPJMSs

$$\begin{cases} dx(t) = f(t, x(t), x(\tau(t)), r(t))dt + g(t, x(t), x(\tau(t)), r(t))dW(t) \\ \qquad\qquad + h(t, x(t), x(\tau(t)), r(t))dN(t), & t \geq 0, \\ x(t) = \varphi(t), \ r(0) = r_0, & t \in [-\tau, 0], \end{cases} \tag{1.1}$$

where $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times d}$, $h : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\tau(t)$ satisfy:
there exists a positive constant ρ such that

$$-\tau \leq \tau(t) < t, \text{ and } |\tau(t) - \tau(s)| \leq \rho|t - s|, \ \forall t, s \geq 0, \tag{1.2}$$

and $\varphi(t) \in L^2_{\mathscr{F}_0}([-\tau, 0]; \mathbb{R}^n)$ which is uniformly Hölder continuous with exponent $\gamma \in (0, 1]$, that is, there exists a constant $M > 0$ such that for all $-\tau \leq s < t \leq 0$

$$\mathbb{E}|\varphi(t) - \varphi(s)|^2 \leq M(t - s)^\gamma. \tag{1.3}$$

We also assume that

$$a(t, 0, 0, i) = 0 \ \forall i \in \mathbb{S}, \ a = f, g, h, \tag{1.4}$$

so Eq. (1.1) admits the zero solution $x(t) = 0$.

To define the SIE approximate solution, we will need the following lemma (see [1]).

Lemma 1.1. *Given $\Delta > 0$, let $r^\Delta_k = r(k\Delta)$ for $k \geq 0$. Then $\{r^\Delta_k, k = 0, 1, 2, \ldots\}$ is a discrete Markov chain with the one-step transition probability matrix*

$$\mathbb{P}(\Delta) = (\mathbb{P}_{ij}(\Delta))_{N\times N} = e^{\Delta\Gamma}. \tag{1.5}$$

Given a fixed step size $\Delta > 0$ and the one-step transition probability matrix $\mathbb{P}(\Delta)$ in (1.5), the discrete Markov chain $\{r^\Delta_k, k = 0, 1, 2, \ldots\}$ can be simulated as follows [8]. Let $r^\Delta_0 = r_0$ and compute a pseudo-random number ξ_1 from the uniform

$(0,1)$ distribution. Define

$$r_1^\Delta = \begin{cases} i : \text{if } i \in \mathbb{S} - \{N\} \text{ such that} \\ \qquad \displaystyle\sum_{j=1}^{i-1} \mathbb{P}_{r_0^\Delta, j}(\Delta) \le \xi_1 < \sum_{j=1}^{i} \mathbb{P}_{r_0^\Delta, j}(\Delta), \\ N : \text{if } \displaystyle\sum_{j=1}^{N-1} \mathbb{P}_{r_0^\Delta, j}(\Delta) \le \xi_1, \end{cases}$$

where we set $\sum_{j=1}^{0} \mathbb{P}_{r_0^\Delta, j}(\Delta) = 0$ as usual. In other words, we ensure that the probability of state s being chosen is given by $\mathbb{P}(r_1^\Delta = s) = \mathbb{P}_{r_0^\Delta, s}(\Delta)$. Generally, having computed $r_0^\Delta, r_1^\Delta, \dots, r_k^\Delta$, we compute r_{k+1}^Δ by drawing a uniform $(0,1)$ pseudo-random number ξ_{k+1} and setting

$$r_{k+1}^\Delta = \begin{cases} i : \text{if } i \in \mathbb{S} - \{N\} \text{ such that} \\ \qquad \displaystyle\sum_{j=1}^{i-1} \mathbb{P}_{r_k^\Delta, j}(\Delta) \le \xi_{k+1} < \sum_{j=1}^{i} \mathbb{P}_{r_k^\Delta, j}(\Delta), \\ N : \text{if } \displaystyle\sum_{j=1}^{N-1} \mathbb{P}_{r_k^\Delta, j}(\Delta) \le \xi_{k+1}. \end{cases}$$

This procedure can be carried out independently to obtain more trajectories.

Having explained how to simulate the discrete Markov chain, we now define the SIE approximation for Eq. (1.1).

Let the step-size $\Delta \in (0,1)$ be $\frac{\tau}{m}$ for some positive integer m, $t_k = k\Delta$. The SIE method applied to (1.1) computer approximations $Y_k \simeq x(t_k)$, by setting $Y_k = \varphi(t_k)$ for $-m \le k \le 0$, $r_0^\Delta = r_0$ and forming

(1.6)
$$\begin{aligned} Y_{k+1} = &Y_k + [(1-\theta) f(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta) + \theta f(t_{k+1}, Y_{k+1}, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]}, r_k^\Delta)]\Delta \\ &+ g(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta)\Delta W_k + h(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta)\Delta N_k, \qquad k \ge 0, \end{aligned}$$

where $\Delta W_k = W(t_{k+1}) - W(t_k)$, $\Delta N_k = N(t_{k+1}) - N(t_k)$. Let

$$\left.\begin{array}{lll} z_1(t) = Y_k, & \hat{z}_1(t) = Y_{k+1}, & \\ z_2(t) = Y_{\mathrm{In}[\tau(t_k)/\Delta]}, & \hat{z}_2(t) = Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]}, & \\ \underline{t} = t_k, & \overline{t} = t_{k+1}, & \bar{r}(t) = r_k^\Delta, \end{array}\right\} \quad \text{for } t \in [t_k, t_{k+1})$$

and define the continuous SIE approximate solution by

$$\begin{aligned} Y(t) := Y_0 &+ \int_0^t (1-\theta) f(\underline{s}, z_1(s), z_2(s), \bar{r}(s)) + \theta f(\overline{s}, \hat{z}_1(s), \hat{z}_2(s), \bar{r}(s)) ds \\ &+ \int_0^t g(\underline{s}, z_1(s), z_2(s), \bar{r}(s)) dW(s) + \int_0^t h(\underline{s}, z_1(s), z_2(s), \bar{r}(s)) dN(s), \end{aligned} \tag{1.7}$$

with $Y(t) = \varphi(t)$ on $t \in [-\tau, 0]$.

A key component in our analysis is the compensated Poisson process

$$\tilde{N}(t) := N(t) - \lambda t \tag{1.8}$$

which is a martingale.

Throughout this paper, we usually use the following equalities.

$$\mathbb{E}\Big|\int_0^t F(s) d\tilde{N}(s)\Big|^2 = \lambda \int_0^t \mathbb{E}|F(s)|^2 ds, \tag{1.9}$$

$$\mathbb{E}|\int_0^t F(s)dW(s)|^2 = \int_0^t \mathbb{E}|F(s)|^2 ds \tag{1.10}$$

which hold for integrated functions F (in particular, $F(s)$ is $\mathscr{F}_s$-measurable, if it is random).

2. Convergence with the global Lipschitz condition

In this section we shall show the strong convergence of the SIE approximate solution to the exact solution under the following global Lipschitz conditions:
(GL) There is a constant $C > 0$ such that

$$|a(t,x,y,i) - a(t,\bar{x},\bar{y},i)|^2 \le C(|x-\bar{x}|^2 + |y-\bar{y}|^2),\ a = f,g,h, \tag{2.1}$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$, $t \in \mathbb{R}_+$ and $i \in \mathbb{S}$;
there is a constant $K > 0$ such that

$$|a(t,x,y,i) - a(s,x,y,i)|^2 \le K(1 + |x|^2 + |y|^2)|t-s|,\ a = f,g,h, \tag{2.2}$$

for $\forall x,\ y \in \mathbb{R}^n$, $\forall t,s \in [-\tau,\infty)$, $i \in \mathbb{S}$.

Recall (1.4) we observe from (2.1) that the linear growth condition

$$|a(t,x,y,i)|^2 \le C(|x|^2 + |y|^2),\ a = f,g,h \tag{2.3}$$

hold for all $(t,x,y,i) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}$. Let us now present a number of lemmas that will lead to our convergence result. First, we consider the existence of a solution to (1.6).

Lemma 2.1. *Under* (2.1), *if Δ is sufficiently small such that $\Delta\theta\sqrt{C} < 1$, then equation* (1.6) *can be solved uniquely for Y_{k+1} given $Y_{In[\tau(t_k)/\Delta]}$, $Y_{In[\tau(t_{k+1})/\Delta]}$, Y_k, with probability 1.*

Proof. Define, for $u \in \mathbb{R}^n$

$$\begin{aligned} F(u) =& Y_k + [(1-\theta)f(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^{\Delta}) + \theta f(t_{k+1}, u, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]}, r_k^{\Delta})]\Delta \\ &+ g(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^{\Delta})\Delta W_k + h(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^{\Delta})\Delta N_k. \end{aligned}$$

Using (GL), we have

$$\begin{aligned} |F(u) - F(v)| &= \theta\Delta|f(t_{k+1}, u, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]}, r_k^{\Delta}) - f(t_{k+1}, v, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]}, r_k^{\Delta})| \\ &\le \theta\Delta\sqrt{C}|u-v|. \end{aligned}$$

By the classical Banach contraction mapping theorem, $F(u)$ has a unique fixed point, which is Y_{k+1}. $\square$

Lemma 2.2. *If* (2.3) *holds, then for all sufficiently small $\Delta (< 1/(3+3C))$ the SIM approximate solution* (1.7) *satisfies, for any $T > 0$,*

$$\sup_{-\tau \le t \le T} \mathbb{E}|Y(t)|^2 \le \alpha_T, \tag{2.4}$$

where $\alpha_T = [6 + 10TC(2 + 3T + 8CT + 6C\lambda T + 2\lambda^2 T)(1 + T + 2\lambda + 2\lambda^2 T) \times e^{(8+15C+12C\lambda+3\lambda^2+3\lambda)(T+1)}]\|\varphi\|_{\mathbb{E}}^2$. *Moreover, the true solution of* (1.1) *also obeys*

$$\sup_{-\tau \le t \le T} \mathbb{E}|x(t)|^2 \le \bar{B}_T, \tag{2.5}$$

where $\bar{B}_T = [5 + 4C(1 + 2\lambda + T + 2\lambda^2 T)]e^{8C(1+2\lambda+T+2\lambda^2 T)}\|\varphi\|_{\mathbb{E}}^2$. *For convenience, we denote* $B = \max\{\alpha_T, \bar{B}_T\}$.

Proof. It follows from (1.6) that

$$\begin{aligned}\mathbb{E}|Y_{k+1}|^2 =&\mathbb{E}|Y_k|^2 + \mathbb{E}|[(1-\theta)f(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta) + \theta f(t_{k+1}, Y_{k+1}, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]}, r_k^\Delta)]\Delta \\ &+ g(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta)\Delta W_k + h(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta)\Delta N_k|^2 \\ &+ 2\Delta\mathbb{E}(< Y_k, (1-\theta)f(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta) + \theta f(t_{k+1}, Y_{k+1}, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]}, r_k^\Delta) > \\ &+ 2\mathbb{E} < Y_k, g(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta)\Delta W_k > + 2\mathbb{E} < Y_k, h(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta)\Delta N_k > .\end{aligned}$$

Noting that ΔW_k and ΔN_k are independent of $\mathscr{F}_{t_k}$, $\mathbb{E}\Delta W_k = 0$, $\mathbb{E}\Delta N_k = \lambda\Delta$. Hence

$$\begin{aligned}\mathbb{E}|Y_{k+1}|^2 =&\mathbb{E}|Y_k|^2 + \mathbb{E}|[(1-\theta)f(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta) + \theta f(t_{k+1}, Y_{k+1}, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]}, r_k^\Delta)]\Delta \\ &+ g(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta)\Delta W_k + h(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta)\Delta N_k|^2 \\ &+ 2\Delta\mathbb{E}(< Y_k, (1-\theta)f(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta) + \theta f(t_{k+1}, Y_{k+1}, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]}, r_k^\Delta) > \\ &+ 2\lambda\Delta\mathbb{E} < Y_k, h(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta) > .\end{aligned}$$

Using elementary inequalities

$$2 < u, v > \leq |u|^2 + |v|^2 \text{ and } |(1-\theta)u + \theta v|^2 \leq |u|^2 + |v|^2, \ \forall\ u, v \in \mathbb{R}^n,$$

$$\mathbb{E}[|g(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta)|^2|\Delta W_k|^2] = \Delta\mathbb{E}|g(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta)|^2,$$

$$\mathbb{E}[|h(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta)|^2|\Delta N_k|^2] = (\lambda^2\Delta^2 + \lambda\Delta)\mathbb{E}|h(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta)|^2,$$

$\tau(t) < t$ and (2.3), we then compute

$$\begin{aligned}\mathbb{E}|Y_{k+1}|^2 \leq&\mathbb{E}|Y_k|^2 + 3\mathbb{E}[|f(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta)|^2\Delta^2 + |f(t_{k+1}, Y_{k+1}, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]}, r_k^\Delta)|^2\Delta^2 \\ &+ |g(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta)|^2|\Delta W_k|^2 + |h(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta)|^2|\Delta N_k|^2] \\ &+ \Delta\mathbb{E}(|Y_k|^2 + |f(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta)|^2 + |f(t_{k+1}, Y_{k+1}, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]}, r_k^\Delta)|^2) \\ &+ \lambda\Delta\mathbb{E}(|Y_k|^2 + |h(t_k, Y_k, Y_{\mathrm{In}[\tau(t_k)/\Delta]}, r_k^\Delta)|^2) \\ \leq&\mathbb{E}|Y_k|^2 + (3C\Delta^2 + 3C\Delta + 3C(\lambda^2\Delta^2 + \lambda\Delta) + \Delta + \lambda\Delta + C\Delta + \lambda C\Delta)\mathbb{E}|Y_k|^2 \\ &+ (3C\Delta^2 + 3C\Delta + 3C(\lambda^2\Delta^2 + \lambda\Delta) + C\Delta + \lambda C\Delta)\mathbb{E}|Y_{\mathrm{In}[\tau(t_k)/\Delta]}|^2 \\ &+ (3C\Delta^2 + C\Delta)\mathbb{E}|Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]}|^2 + (3C\Delta^2 + C\Delta)\mathbb{E}|Y_{k+1}|^2 \\ \leq&\mathbb{E}|Y_k|^2 + \Delta(2 + 4C + 4C\lambda + \lambda^2 + \lambda)\mathbb{E}|Y_k|^2 \\ &+ \Delta(1 + 4C + 4C\lambda + \lambda^2)\mathbb{E}|Y_{\mathrm{In}[\tau(t_k)/\Delta]}|^2 \\ &+ \Delta(1 + C)\mathbb{E}|Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]}|^2 + \Delta(1 + C)\mathbb{E}|Y_{k+1}|^2,\end{aligned}$$

where we have noted that $3C\Delta < 1$. Let $\tilde{M}$ be any positive integer such that $\tilde{M} \le \mathrm{In}[T/\Delta]+1$. Summing the inequality above for k from 0 to $\tilde{M}-1$, we obtain

$$\begin{aligned}
\mathbb{E}|Y_{\tilde{M}}|^2 \le & \mathbb{E}|Y_0|^2 + \Delta(2+4C+4C\lambda+\lambda^2+\lambda)\sum_{k=0}^{\tilde{M}-1}\mathbb{E}|Y_k|^2 \\
& + \Delta(1+4C+4C\lambda+\lambda^2)\sum_{k=0}^{\tilde{M}-1}\mathbb{E}|Y_{\mathrm{In}[\tau(t_k)/\Delta]}|^2 \\
& + \Delta(1+C)\sum_{k=0}^{\tilde{M}-1}\mathbb{E}|Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]}|^2 + \sum_{k=0}^{\tilde{M}-1}\Delta(1+C)\mathbb{E}|Y_{k+1}|^2 \\
\le & (1+2T+5CT+4C\lambda T+\lambda^2 T)\|\varphi\|_{\mathbb{E}}^2 + \Delta(1+C)\mathbb{E}|Y_{\tilde{M}}|^2 \\
& + \Delta(5+10C+8C\lambda+2\lambda^2+2\lambda)\sum_{k=0}^{\tilde{M}-1}\mathbb{E}|Y_k|^2.
\end{aligned}$$

Noting that $(1+C)\Delta < 1/3$, we have

$$\mathbb{E}|Y_{\tilde{M}}|^2 \le (2+3T+8CT+6C\lambda T+2\lambda^2T)\|\varphi\|_{\mathbb{E}}^2+(8+15C+12C\lambda+3\lambda^2+3\lambda)\Delta\sum_{k=0}^{\tilde{M}-1}\mathbb{E}|Y_k|^2$$

Using the discrete Gronwall inequality [3] and recalling that $\tilde{M}\Delta \le T+1$, we obtain

$$\mathbb{E}|Y_{\tilde{M}}|^2 \le \bar{\alpha}_T\|\varphi\|_{\mathbb{E}}^2,$$

where $\bar{\alpha}_T = (2+3T+8CT+6C\lambda T+2\lambda^2T)e^{(8+15C+12C\lambda+3\lambda^2+3\lambda)(1+T)}$. Recalling the definition of $z_1(t)$, $z_2(t)$, $\hat{z}_1(t)$ and $\hat{z}_2(t)$ we see that

$$\sup_{0\le t\le T}\mathbb{E}|z_j(t)| \le \bar{\alpha_T}\|\varphi\|_{\mathbb{E}}^2,\ \sup_{0\le t\le T}\mathbb{E}|\hat{z}_j(t)| \le \bar{\alpha_T}\|\varphi\|_{\mathbb{E}}^2,\ j=1,2.$$

Using the Hölder inequality, (1.8)-(1.10) and (2.3), we derive from (1.7) that

$$\begin{aligned}
\mathbb{E}|Y(t)|^2 \le & 5\mathbb{E}|Y_0|^2 + 5(1-\theta)^2T\mathbb{E}\int_0^t |f(\underline{s}, z_1(s), z_2(s), \bar{r}(s))|^2ds \\
& + 5\mathbb{E}\int_0^t \theta^2T|f(\overline{s}, \hat{z}_1(s), \hat{z}_2(s), \bar{r}(s))|^2 + |g(\underline{s}, z_1(s), z_2(s), \bar{r}(s))|^2ds \\
& + (10\lambda+10\lambda^2T)\mathbb{E}\int_0^t |h(\underline{s}, z_1(s), z_2(s), \bar{r}(s))|^2ds \\
\le & 5\mathbb{E}|Y_0|^2 + 5\theta^2TC\mathbb{E}\int_0^t |\hat{z}_1(s)|^2 + |\hat{z}_2(s)|^2ds \\
& + [5(1-\theta)^2T+5+10\lambda+10\lambda^2T]C\mathbb{E}\int_0^t |z_1(s)|^2 + |z_2(s)|^2ds \\
\le & [5+10\bar{\alpha}_TTC(1+T+2\lambda+2\lambda^2T)]\|\varphi\|_{\mathbb{E}}^2.
\end{aligned}$$

Hence

$$\sup_{-\tau\le t\le T}\mathbb{E}|Y(t)|^2 \le [6+10\bar{\alpha}_TTC(1+T+2\lambda+2\lambda^2T)]\|\varphi\|_{\mathbb{E}}^2,$$

which is the required assertion (2.4). Next, we prove (2.5). Using the Hölder inequality, (1.8)-(1.10) and (2.3), we derive from (1.1) that

$$\begin{aligned}\mathbb{E}|x(t)|^2 \leq & 4\mathbb{E}|x_0|^2 + 4\mathbb{E}\int_0^t T|f(s,x(s),x(\tau(s)),r(s))|^2 ds \\ & + |g(s,x(s),x(\tau(s)),r(s))|^2 ds \\ & + (8\lambda + 8\lambda^2 T)\mathbb{E}\int_0^t |h(s,x(s),x(\tau(s)),r(s))|^2 ds \\ \leq & 4\mathbb{E}|x_0|^2 + 4C(1+2\lambda+T+2\lambda^2 T)\mathbb{E}\int_0^t |x(s)|^2 + |x(\tau((s))|^2 ds \\ \leq & 4\mathbb{E}|x_0|^2 + 4C(1+2\lambda+T+2\lambda^2 T)||\varphi||_{\mathbb{E}}^2 \\ & + 8C(1+2\lambda+T+2\lambda^2 T)\int_0^t \sup_{0\leq\xi\leq s}\mathbb{E}|x(\xi)|^2 ds.\end{aligned}$$

Since the right-hand side term is non-decreasing in t, we have

$$\begin{aligned}\sup_{0\leq t\leq t_1}\mathbb{E}|x(t)|^2 \leq & [4 + 4C(1+2\lambda+T+2\lambda^2 T)]\|\varphi\|_{\mathbb{E}}^2 \\ & + 8C(1+2\lambda+T+2\lambda^2 T)\int_0^{t_1} \sup_{0\leq\xi\leq s}\mathbb{E}|x(\xi)|^2 ds.\end{aligned}$$

The continuous Gronwall inequality [9] yields

$$\sup_{0\leq t\leq T}\mathbb{E}|x(t)|^2 \leq [4 + 4C(1+2\lambda+T+2\lambda^2 T)]e^{8C(1+2\lambda+T+2\lambda^2 T)}\|\varphi\|_{\mathbb{E}}^2.$$

Hence

$$\sup_{-\tau\leq t\leq T}\mathbb{E}|x(t)|^2 \leq [5 + 4C(1+2\lambda+T+2\lambda^2 T)]e^{8C(1+2\lambda+T+2\lambda^2 T)}\|\varphi\|_{\mathbb{E}}^2.$$

□

Next, we shall employ the technique in [8] to bound the effect of replacing the right-continuous Markov chain by the interpolation of the discrete time Markov chain.

Lemma 2.3. *If* (2.3) *holds, then for all sufficiently small* Δ,

$$\mathbb{E}\int_0^T |a(\underline{s}, z_1(s), z_2(s), r(s)) - a(\underline{s}, z_1(s), z_2(s), \bar{r}(s))|^2 ds \leq C_1\Delta \tag{2.6}$$

and

$$\mathbb{E}\int_0^T |f(\overline{s}, \hat{z}_1(s), \hat{z}_2(s), r(s)) - f(\overline{s}, \hat{z}_1(s), \hat{z}_2(s), \bar{r}(s))|^2 ds \leq C_1\Delta \tag{2.7}$$

for any $T > 0$, *where* a *is* f, g, *or* h *and* $C_1 = 8C\hat{\gamma}TB$, $\hat{\gamma} = N[\max_{1\leq i\leq N}(-\gamma_{ii}) + 1]$.

Proof. Let $l = [T/\Delta]$, then

$$
\begin{aligned}
&\mathbb{E}\int_0^T |f(\underline{s}, z_1(s), z_2(s), r(s)) - f(\underline{s}, z_1(s), z_2(s), \bar{r}(s))|^2 ds \\
=&\sum_{k=0}^{l} \mathbb{E}\int_{t_k}^{t_{k+1}} |f(\underline{s}, z_1(s), z_2(s), r(s)) - f(\underline{s}, z_1(s), z_2(s), \bar{r}(s))|^2 ds \\
\le& 2\sum_{k=0}^{l} \mathbb{E}\int_{t_k}^{t_{k+1}} [|f(\underline{s}, z_1(s), z_2(s), r(s))|^2 \\
&+ |f(\underline{s}, z_1(s), z_2(s), \bar{r}(s))|^2] I_{\{r(s)\neq r(t_k)\}} ds \\
\le& 4C\sum_{k=0}^{l} \mathbb{E}\int_{t_k}^{t_{k+1}} (|z_1(s)|^2 + |z_2(s)|^2) I_{\{r(s)\neq r(t_k)\}} ds
\end{aligned}
\tag{2.8}
$$

with, for convenience, t_{l+1} being redefined as T. By the property of conditional expectation in [9], we have

$$
\begin{aligned}
&\mathbb{E}\int_{t_k}^{t_{k+1}} (|z_1(s)|^2 + |z_2(s)|^2) I_{\{r(s)\neq r(t_k)\}} ds \\
=& \int_{t_k}^{t_{k+1}} \mathbb{E}[\mathbb{E}[(|Y_k|^2 + |Y_{\mathrm{In}[\tau(t_k)/\Delta]}|^2) I_{\{r(s)\neq r(t_k)\}} | r(t_k)]] ds \\
=& \int_{t_k}^{t_{k+1}} \mathbb{E}[\mathbb{E}[(|Y_k|^2 + |Y_{\mathrm{In}[\tau(t_k)/\Delta]}|^2) | r(t_k)] \mathbb{E}[I_{\{r(s)\neq r(t_k)\}} | r(t_k)]] ds,
\end{aligned}
\tag{2.9}
$$

where in the last step we used the fact that Y_k and $Y_{\mathrm{In}[\tau(t_k)/\Delta]}$ are independent of $I_{\{r(s)\neq r(t_k)\}}$ with respect to the $\sigma-$algebra generated by $r(t_k)$. By the Markov property

$$
\begin{aligned}
\mathbb{E}[I_{\{r(s)\neq r(t_k)\}} | r(t_k)] &= \sum_{i\in\mathbb{S}} I_{\{r(t_k)=i\}} \mathbb{P}(r(s)\neq i | r(t_k) = i) \\
&= \sum_{i\in\mathbb{S}} I_{\{r(t_k)=i\}} \sum_{j\neq i} (\gamma_{ij}(s-t_k) + o(s-t_k)) \\
&\le (\max_{1\le i\le N}(-\gamma_{ii})\Delta + o(\Delta)) \sum_{i\in\mathbb{S}} I_{\{r(t_k)=i\}} \\
&\le \hat{\gamma}\Delta,
\end{aligned}
$$

where $\hat{\gamma} = N[1 + \max_{1\le i\le N}(-\gamma_{ii})]$. Substituting this into (2.9) gives

$$
\begin{aligned}
&\mathbb{E}\int_{t_k}^{t_{k+1}} (|z_1(s)|^2 + |z_2(s)|^2) I_{\{r(s)\neq r(t_k)\}} ds \\
\le& \hat{\gamma}\Delta \int_{t_k}^{t_{k+1}} \mathbb{E}[|Y_k|^2 + |Y_{\mathrm{In}[\tau(t_k)/\Delta]}|^2] ds.
\end{aligned}
$$

Putting it into (2.8)

$$
\begin{aligned}
&\mathbb{E}\int_0^T |f(\underline{s}, z_1(s), z_2(s), r(s)) - f(\underline{s}, z_1(s), z_2(s), \bar{r}(s))|^2 ds \\
\le& 4C\hat{\gamma}\Delta \sum_{k=0}^{l} \int_{t_k}^{t_{k+1}} \mathbb{E}[|Y_k|^2 + |Y_{\mathrm{In}[\tau(t_k)/\Delta]}|^2] ds \\
\le& 8C\hat{\gamma}TB\Delta.
\end{aligned}
$$

We can show $a = g$ and $a = h$ similarly. We have used that $z_1(t)$ is $\mathscr{F}_t$-measurable. However, $\hat{z}_1(t)$ is not $\mathscr{F}_t$-measurable, so assertion (2.7) requires a more careful treatment. By (2.3), it is easy to show that

$$\begin{aligned}
&\mathbb{E}\int_0^T |f(\overline{s}, \hat{z}_1(s), \hat{z}_2(s), r(s)) - f(\overline{s}, \hat{z}_1(s), \hat{z}_2(s), \bar{r}(s))|^2 ds \\
&= \sum_{k=0}^{l} \mathbb{E}\int_{t_k}^{t_{k+1}} |f(\overline{s}, \hat{z}_1(s), \hat{z}_2(s), r(s)) - f(\overline{s}, \hat{z}_1(s), \hat{z}_2(s), \bar{r}(s))|^2 ds \\
&\le 4C \sum_{k=0}^{l} \mathbb{E}\int_{t_k}^{t_{k+1}} (|Y_{k+1}|^2 + |Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]}|^2) I_{\{r(s)\neq r(t_k)\}} ds.
\end{aligned} \tag{2.10}$$

By the Markov property

$$\begin{aligned}
&\mathbb{E}[|Y_{k+1}|^2 I_{\{r(s)\neq r(t_k)\}}] \\
&= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \sum_{i\in\mathbb{S}} \mathbb{E}[|Y_{k+1}|^2 I_{\{r(s)\neq i\}} | Y_k = x_1, Y_{\mathrm{In}[\tau(t_k)/\Delta]} = x_2, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]} = x_3, r(t_k) = i] \\
&\times \mathbb{P}\{Y_k = dx_1, Y_{\mathrm{In}[\tau(t_k)/\Delta]} = dx_2, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]} = dx_3, r(t_k) = i\}.
\end{aligned}$$

Given that $Y_k = x_1$, $Y_{\mathrm{In}[\tau(t_k)/\Delta]} = x_2$, $Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]} = x_3$, $r(t_k) = i$, we see from (1.7) that

$$\begin{aligned}
Y_{k+1} =& x_1 + [(1-\theta) f(t_k, x_1, x_2, i) + \theta f(t_{k+1}, Y_{k+1}, x_3, i)]\Delta \\
&+ g(t_k, x_1, x_2, i)\Delta W_k + h(t_k, x_1, x_2, i)\Delta N_k.
\end{aligned}$$

For $\tau(t) < t$, $\mathrm{In}[\tau(t_{k+1})/\Delta] \le k$. Hence, Y_{k+1} depends on ΔW_k, ΔN_k which are independent of the Markov chain. In other words, Y_{k+1} and $I_{\{r(s)\neq i\}}$ are independent given $Y_k = x_1$, $Y_{\mathrm{In}[\tau(t_k)/\Delta]} = x_2$, $Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]} = x_3$, $r(t_k) = i$. Hence

$$\begin{aligned}
&\mathbb{E}[|Y_{k+1}|^2 I_{\{r(s)\neq r(t_k)\}}] \\
&= \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \sum_{i\in\mathbb{S}} \mathbb{E}[|Y_{k+1}|^2 | Y_k = x_1, Y_{\mathrm{In}[\tau(t_k)/\Delta]} = x_2, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]} = x_3, r(t_k) = i] \\
&\times \mathbb{P}\{r(s) \neq i | Y_k = x_1, Y_{\mathrm{In}[\tau(t_k)/\Delta]} = x_2, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]} = x_3, r(t_k) = i\} \\
&\times \mathbb{P}\{Y_k = dx_1, Y_{\mathrm{In}[\tau(t_k)/\Delta]} = dx_2, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]} = dx_3, r(t_k) = i\}.
\end{aligned} \tag{2.11}$$

We compute that

$$\begin{aligned}
&\mathbb{P}\{r(s) \neq i | Y_k = x_1, Y_{\mathrm{In}[\tau(t_k)/\Delta]} = x_2, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]} = x_3, r(t_k) = i\} \\
&= \frac{\mathbb{P}\{r(s) \neq i, Y_k = x_1, Y_{\mathrm{In}[\tau(t_k)/\Delta]} = x_2, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]} = x_3, r(t_k) = i\}}{\mathbb{P}\{Y_k = x_1, Y_{\mathrm{In}[\tau(t_k)/\Delta]} = x_2, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]} = x_3, r(t_k) = i\}} \\
&= \frac{\mathbb{P}\{r(s) \neq i, Y_k = x_1, Y_{\mathrm{In}[\tau(t_k)/\Delta]} = x_2, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]} = x_3 | r(t_k) = i\}}{\mathbb{P}\{Y_k = x_1, Y_{\mathrm{In}[\tau(t_k)/\Delta]} = x_2, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]} = x_3 | r(t_k) = i\}}.
\end{aligned} \tag{2.12}$$

Noting that given $r(t_k) = i$, then event $r(s) \neq i$ is independent of $Y_k = x_1$, $Y_{\mathrm{In}[\tau(t_k)/\Delta]} = x_2$, $Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]} = x_3$, we have

$$\begin{aligned}
&\mathbb{P}\{r(s) \neq i, Y_k = x_1, Y_{\mathrm{In}[\tau(t_k)/\Delta]} = x_2, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]} = x_3 | r(t_k) = i\} \\
&= \mathbb{P}\{r(s) \neq i | r(t_k) = i\} \mathbb{P}\{Y_k = x_1, Y_{\mathrm{In}[\tau(t_k)/\Delta]} = x_2, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]} = x_3 | r(t_k) = i\}.
\end{aligned}$$

Putting this into (2.12) we obtain

$$\begin{aligned}&\mathbb{P}\{r(s)\neq i|Y_k=x_1, Y_{\mathrm{In}[\tau(t_k)/\Delta]}=x_2, Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]}=x_3, r(t_k)=i\}\\ &=\mathbb{P}\{r(s)\neq i|r(t_k)=i\}\leq \hat{\gamma}\Delta.\end{aligned}\tag{2.13}$$

Using this in (2.11)

$$\mathbb{E}[|Y_{k+1}|^2 I_{\{r(s)=r(t_k)\}}]\leq \hat{\gamma}\Delta\mathbb{E}|Y_{k+1}|^2.\tag{2.14}$$

Noting that $|Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]}|^2$ and $I_{\{r(s)\neq r(t_k)\}}$ are conditionally independent with respect to the σ-algebra generated by $r(t_k)$, we have

$$\begin{aligned}&\mathbb{E}[|Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]}|^2 I_{\{r(s)\neq r(t_k)\}}]\\ &=\mathbb{E}[\mathbb{E}[|Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]}|^2 I_{\{r(s)\neq r(t_k)\}}|r(t_k)]]\\ &=\mathbb{E}[\mathbb{E}[|Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]}|^2|r(t_k)]\mathbb{E}[I_{\{r(s)\neq r(t_k)\}}|r(t_k)]]\\ &\leq \hat{\gamma}\mathbb{E}|Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]}|^2\Delta.\end{aligned}\tag{2.15}$$

Substituting (2.15) and (2.14) into (2.10) we obtain

$$\begin{aligned}&\mathbb{E}\int_0^T |f(\overline{s},\hat{z}_1(s),\hat{z}_2(s),r(s))-f(\overline{s},\hat{z}_1(s),\hat{z}_2(s),\bar{r}(s))|^2 ds\\ &\leq 8C\hat{\gamma}TB\Delta.\end{aligned}$$

□

Lemma 2.4. *If* (2.3) *holds, for all sufficiently small* Δ *we have*

$$\mathbb{E}|Y(t)-z_1(t)|^2\leq C_2\Delta,\ \forall t\in[0,T],$$

for any $T>0$*, where* $C_2=16(1+\lambda+\lambda^2)CB$ *is a constant independent of* Δ*.*

Proof. For any $t\in[0,T]$, there exists a k such that $t\in[t_k,t_{k+1})$. Then

$$\begin{aligned}&Y(t)-z_1(t)\\ &=[(1-\theta)f(t_k,Y_k,Y_{\mathrm{In}[\tau(t_k)/\Delta]},r(t_k))+\theta f(t_{k+1},Y_{k+1},Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]},r(t_k))](t-t_k)\\ &\quad+g(t_k,Y_k,Y_{\mathrm{In}[\tau(t_k)/\Delta]},r(t_k))(W(t)-W(t_k))\\ &\quad+h(t_k,Y_k,Y_{\mathrm{In}[\tau(t_k)/\Delta]},r(t_k))(N(t)-N(t_k)).\end{aligned}$$

For $g(t_k,Y_k,Y_{\mathrm{In}[\tau(t_k)/\Delta]},r(t_k))$ and $g(t_k,Y_k,Y_{\mathrm{In}[\tau(t_k)/\Delta]},r(t_k))$ are $\mathscr{F}_{t_k}$-measurable, $W(t)-W(t_k)$ and $N(t)-N(t_k)$ are independent of $\mathscr{F}_{t_k}$, we have

$$\mathbb{E}|g(t_k,Y_k,Y_{\mathrm{In}[\tau(t_k)/\Delta]},r(t_k))(W(t)-W(t_k))|^2=(t-t_k)\mathbb{E}|g(t_k,Y_k,Y_{\mathrm{In}[\tau(t_k)/\Delta]},r(t_k))|^2,$$

$$\begin{aligned}\mathbb{E}|g(t_k,Y_k,Y_{\mathrm{In}[\tau(t_k)/\Delta]},r(t_k))(N(t)-N(t_k))|^2&=\mathbb{E}|g(t_k,Y_k,Y_{\mathrm{In}[\tau(t_k)/\Delta]},r(t_k))|^2|N(t)-N(t_k)|^2\\ &\leq(2\lambda^2\Delta^2+2\lambda\Delta)\mathbb{E}|g(t_k,Y_k,Y_{\mathrm{In}[\tau(t_k)/\Delta]},r(t_k))|^2.\end{aligned}$$

By (2.3), we have

$$\begin{aligned}\mathbb{E}|Y(t)-z_1(t)|^2\leq&4(1-\theta)^2C\mathbb{E}[|Y_k|^2+|Y_{\mathrm{In}[\tau(t_k)/\Delta t]}|^2]\Delta^2\\ &+4\theta^2C\mathbb{E}[|Y_{k+1}|^2+|Y_{\mathrm{In}[\tau(t_{k+1})/\Delta]}|^2]\Delta^2\\ &+4C\Delta\mathbb{E}[|Y_k|^2+|Y_{\mathrm{In}[\tau(t_k)/\Delta t]}|^2]\\ &+8\lambda\Delta(1+\lambda\Delta)\mathbb{E}[|Y_k|^2+|Y_{\mathrm{In}[\tau(t_k)/\Delta t]}|^2]\\ \leq&16(1+\lambda+\lambda^2)CB\Delta.\end{aligned}$$

□

We will use the technique in [10] to prove the following lemma.

Lemma 2.5. *If* (2.3) *hold, then for all sufficiently small* Δ

$$\mathbb{E}|Y(\tau(t)) - z_2(t)|^2 \le C_3\Delta^\gamma, \ \forall t \in [0,T],$$

for any $T > 0$*, where* $C_3 = 2M(\rho+1)^\gamma + 16(\rho+1)[(\rho+1)(1+2\lambda^2) + 1 + 2\lambda]CB$ *is a constant independent of* Δ*.*

Proof. For any $t \in [0,T]$, there exists a k such that $t \in [t_k, t_{k+1})$. Then

$$Y(\tau(t)) - z_2(t) = Y(\tau(t)) - Y(\mathrm{In}[\tau(t_k)/\Delta]\Delta). \tag{2.16}$$

It is also useful to note that

$$\tau(t_k) - \Delta \le \mathrm{In}[\tau(t_k)/\Delta]\Delta \le \tau(t_k). \tag{2.17}$$

To show the desired result, let us consider the following five possible cases:

- if $0 \le \mathrm{In}[\tau(t_k)/\Delta]\Delta \le \tau(t)$, then by (2.17) and (1.2)

$$\tau(t) - \mathrm{In}[\tau(t_k)/\Delta]\Delta \le \tau(t) - \tau(t_k) + \Delta \le (\rho+1)\Delta.$$

  Using Hölder inequality, (1.8)-(1.10), (2.16) and (2.3), we have that

$$\begin{aligned}
&\mathbb{E}|Y(\tau(t)) - z_2(t)|^2 \\
&= \mathbb{E}|\int_{\mathrm{In}[\tau(t_k)/\Delta]\Delta}^{\tau(t)} (1-\theta)f(\underline{s}, z_1(s), z_2(s), \bar{r}(s)) \\
&\quad + \theta f(\overline{s}, \hat{z}_1(s), \hat{z}_2(s), \bar{r}(s))ds \\
&\quad + \int_{\mathrm{In}[\tau(t_k)/\Delta]\Delta}^{\tau(t)} g(\underline{s}, z_1(s), z_2(s), \bar{r}(s))dW(s) \\
&\quad + \int_{\mathrm{In}[\tau(t_k)/\Delta]\Delta}^{\tau(t)} h(\underline{s}, z_1(s), z_2(s), \bar{r}(s))dN(s)|^2 \\
&\le 4C(1-\theta)^2(\rho+1)\Delta\mathbb{E}\int_{\mathrm{In}[\tau(t_k)/\Delta]\Delta}^{\tau(t)} |z_1(s)|^2 + |z_2(s)|^2 ds \\
&\quad + 4C\theta^2(\rho+1)\Delta\mathbb{E}\int_{\mathrm{In}[\tau(t_k)/\Delta]\Delta}^{\tau(t)} |\hat{z}_1(s)|^2 + |\hat{z}_2(s)|^2 ds \\
&\quad + 4C\mathbb{E}\int_{\mathrm{In}[\tau(t_k)/\Delta]\Delta}^{\tau(t)} |z_1(s)|^2 + |z_2(s)|^2 ds \\
&\quad + 8C\lambda[(\rho+1)\lambda\Delta + 1]\mathbb{E}\int_{\mathrm{In}[\tau(t_k)/\Delta]\Delta}^{\tau(t)} |z_1(s)|^2 + |z_2(s)|^2 ds \\
&\le 8(\rho+1)[(\rho+1)(1+2\lambda^2)\Delta + 1 + 2\lambda]CB\Delta.
\end{aligned}$$

- if $0 \le \tau(t) \le \mathrm{In}[\tau(t_k)/\Delta]\Delta$, then by (2.17) and (1.2)

$$\mathrm{In}[\tau(t_k)/\Delta]\Delta - \tau(t) \le \tau(t_k) - \tau(t) \le \rho\Delta.$$

  Hence, it follows from (2.16) and (2.3) that

$$\mathbb{E}|\bar{Y}(\tau(t)) - z_2(t)|^2 \le 8\rho[\rho(1+2\lambda^2)\Delta + 1 + 2\lambda]CB\Delta.$$

- if $\tau(t) \le \mathrm{In}[\tau(t_k)/\Delta]\Delta \le 0$ or $\mathrm{In}[\tau(t_k)/\Delta]\Delta \le \tau(t) \le 0$, then by (2.17) and (2.16)

$$|\tau(t) - \mathrm{In}[\tau(t_k)/\Delta]\Delta| \le (\rho+1)\Delta.$$

  So by (1.3)

$$\begin{aligned}
\mathbb{E}|Y(\tau(t)) - z_2(t)|^2 &= \mathbb{E}|\varphi(\tau(t)) - \varphi(\mathrm{In}[\tau(t_k)/\Delta]\Delta)|^2 \\
&\le M|\tau(t) - \mathrm{In}[\tau(t_k)/\Delta]\Delta|^\gamma \le M(1+\rho)^\gamma\Delta^\gamma.
\end{aligned}$$

- if $\text{In}[\tau(t_k)/\Delta]\Delta \leq 0 \leq \tau(t)$, then

$$-\text{In}[\tau(t_k)/\Delta]\Delta \leq (\rho+1)\Delta, \text{ since } \tau(t) \leq \tau(t) - \text{In}[\tau(t_k)/\Delta]\Delta \leq (\rho+1)\Delta.$$

Hence

$$\begin{aligned}&\mathbb{E}|Y(\tau(t)) - z_2(t)|^2 \leq 2\mathbb{E}|Y(\tau(t)) - Y(0)|^2 + 2\mathbb{E}|\varphi(0) - \varphi(\text{In}[\tau(t_k)/\Delta]\Delta)|^2 \\ &\leq 16(\rho+1)[(\rho+1)(1+2\lambda^2)\Delta + 1 + 2\lambda]CB\Delta + 2M(1+\rho)^\gamma\Delta^\gamma.\end{aligned}$$

- if $\tau(t) \leq 0 \leq \text{In}[\tau(t_k)/\Delta]\Delta$, then

$$-\tau(t) \leq \rho\Delta, \text{ since } \text{In}[\tau(t_k)/\Delta]\Delta \leq \text{In}[\tau(t_k)/\Delta]\Delta - \tau(t) \leq \rho\Delta.$$

Hence

$$\begin{aligned}&\mathbb{E}|Y(\tau(t)) - z_2(t)|^2 \leq 2\mathbb{E}|\varphi(\tau(t)) - \varphi(0)|^2 + 2\mathbb{E}|Y(0) - Y(\text{In}[\tau(t_k)/\Delta]\Delta)|^2 \\ &\leq 16\rho[\rho(1+2\lambda^2)\Delta + 1 + 2\lambda]CB\Delta + 2M\rho^\gamma\Delta^\gamma.\end{aligned}$$

□

Corollary 2.6. *If* (2.3) *holds, for all sufficiently small* Δ

$$\mathbb{E}|Y(t) - \hat{z}_1(t)|^2 \leq 4C_2\Delta, \ \forall t \in [0,T]$$

for any $T > 0$*, where* C_2 *is defined in Lemma 2.4.*

Proof. For any $t \in [0,T]$, there exists a k such that $t \in [t_k, t_{k+1})$, then

$$\begin{aligned}|Y(t) - \hat{z}_1(t)|^2 &\leq 2|Y(t) - z_1(t)|^2 + 2|\hat{z}_1(t) - z_1(t)|^2 \\ &\leq 2|Y(t) - z_1(t)|^2 + 2|Y_{k+1} - Y_k|^2.\end{aligned}$$

It is easy to get the result from Lemma 2.4. □

Corollary 2.7. *If* (2.3) *hold, then for all sufficiently small* Δ

$$\mathbb{E}|Y(\tau(t)) - \hat{z}_2(t)|^2 \leq 4C_3\Delta^\gamma, \ \forall t \in [0,T]$$

for any $T > 0$*, where* C_3 *is defined in Lemma 2.5.*

Proof. For any $t \in [0,T]$, there exists a k such that $t \in [t_k, t_{k+1})$, then

$$\begin{aligned}&|Y(\tau(t)) - \hat{z}_2(t)|^2 \\ &\leq 2|Y(\tau(t)) - z_2(t)|^2 + 2|\hat{z}_2(t) - z_2(t)|^2 \\ &\leq 2|Y(\tau(t)) - z_2(t)|^2 + 2|Y(\text{In}[\tau(t_k+1)/\Delta]\Delta) - Y(\text{In}[\tau(t_k)/\Delta]\Delta)|^2.\end{aligned} \tag{2.18}$$

Noting that

$$0 \leq \text{In}[\tau(t_k+1)/\Delta]\Delta - \text{In}[\tau(t_k)/\Delta]\Delta \leq \tau(t_{k+1}) - \tau(t_k) + \Delta \leq (\rho+1)\Delta.$$

Clearly, we get the result from Lemma 2.5. □

Theorem 2.8. *If* (2.1), (2.2) *hold, then for all sufficiently small* Δ

$$\mathbb{E}[\sup_{0\leq t\leq T}|x(t) - Y(t)|^2] \leq \hat{C}\Delta^\gamma,$$

for any $T > 0$*, where* $\hat{C} = 12[T + 1 + 2\lambda(1+\lambda T)][K(1+2B)T + C_1 + 8C(C_2 + C_3)T]e^{48C[T+1+2\lambda(1+\lambda T)]}$ *which is independent of* Δ*, where* B*,* C_1*,* C_2*,* C_3 *are defined in Lemma 2.2, Lemma 2.3, Lemma 2.4, Lemma 2.5 respectively.*

Proof.

$$
\begin{aligned}
x(t)-Y(t)=&(1-\theta)\int_0^t f(s,x(s),x(\tau(s)),r(s))-f(\underline{s},z_1(s),z_2(s),\bar{r}(s))ds\\
&+\theta\int_0^t f(s,x(s),x(\tau(s)),r(s))-f(\overline{s},\hat{z}_1(s),\hat{z}_2(s),\bar{r}(s))ds\\
&+\int_0^t g(s,x(s),x(\tau(s)),r(s))-g(\underline{s},z_1(s),z_2(s),\bar{r}(s))dW(s)\\
&+\int_0^t h(s,x(s),x(\tau(s)),r(s))-h(\underline{s},z_1(s),z_2(s),\bar{r}(s))dN(s).
\end{aligned}
$$

By Hölder inequality, (1.8)-(1.10), we have

$$
\begin{aligned}
&\mathbb{E}[\sup_{0\le s\le t}|x(s)-Y(s)|^2]\\
&\le 4T(1-\theta)^2\int_0^t \mathbb{E}|f(s,x(s),x(\tau(s)),r(s))-f(\underline{s},z_1(s),z_2(s),\bar{r}(s))|^2ds\\
&\quad+4T\theta^2\int_0^t \mathbb{E}|f(s,x(s),x(\tau(s)),r(s))-f(\overline{s},\hat{z}_1(s),\hat{z}_2(s),\bar{r}(s))|^2ds\\
&\quad+4\int_0^t \mathbb{E}|g(s,x(s),x(\tau(s)),r(s))-g(\underline{s},z_1(s),z_2(s),\bar{r}(s))|^2ds\\
&\quad+8\lambda\int_0^t \mathbb{E}|h(s,x(s),x(\tau(s)),r(s))-h(\underline{s},z_1(s),z_2(s),\bar{r}(s))|^2ds\\
&\quad+8\lambda^2T\int_0^t \mathbb{E}|h(s,x(s),x(\tau(s)),r(s))-h(\underline{s},z_1(s),z_2(s),\bar{r}(s))|^2ds.
\end{aligned}
\tag{2.19}
$$

By (2.1) and (2.2), we have

$$
\begin{aligned}
&|f(s,x(s),x(\tau(s)),r(s))-f(\overline{s},\hat{z}_1(s),\hat{z}_2(s),\bar{r}(s))|^2\\
&\le 3|f(s,x(s),x(\tau(s)),r(s))-f(\overline{s},x(s),x(\tau(s)),r(s))|^2\\
&\quad+3|f(\overline{s},x(s),x(\tau(s)),r(s))-f(\overline{s},\hat{z}_1(s),\hat{z}_2(\tau(s)),r(s))|^2\\
&\quad+3|f(\overline{s},\hat{z}_1(s),\hat{z}_2(\tau(s)),r(s))-f(\overline{s},\hat{z}_1(s),\hat{z}_2(s),\bar{r}(s))|^2\\
&\le 3K(1+|x(s)|^2+|x(\tau(s))|^2)|s-\overline{s}|+3C(|x(s)-\hat{z}_1(s)|^2\\
&\quad+|x(\tau(s))-\hat{z}_2(s)|^2)+3|f(\overline{s},\hat{z}_1(s),\hat{z}_2(\tau(s)),r(s))\\
&\quad-f(\overline{s},z_1(s),z_2(s),\bar{r}(s))|^2\\
&\le 3K(1+|x(s)|^2+|x(\tau(s))|^2)|s-\overline{s}|+6C(|x(s)-Y(s)|^2\\
&\quad+|x(\tau(s))-Y(\tau(s))|^2+|Y(s)-\hat{z}_1(s)|^2+|Y(\tau(s))-\hat{z}_2(s)|^2)\\
&\quad+3|f(\overline{s},\hat{z}_1(s),\hat{z}_2(\tau(s)),r(s))-f(\overline{s},\hat{z}_1(s),\hat{z}_2(s),\bar{r}(s))|^2.
\end{aligned}
$$

From Lemma 2.2, Lemma 2.3, Corollary 2.6 and Corollary 2.7,

$$
\begin{aligned}
&\int_0^t \mathbb{E}|f(s,x(s),x(\tau(s)),r(s)) - f(\overline{s},\hat{z}_1(s),\hat{z}_2(s),\bar{r}(s))|^2 ds \\
&\leq \int_0^t 3K(1+2B)\Delta + 6C\mathbb{E}(|x(s)-Y(s)|^2 + |x(\tau(s))-Y(\tau(s))|^2 \\
&\quad + 24CC_2\Delta + 24CC_3\Delta^\gamma ds + 3C_1\Delta \\
&\leq [3K(1+2B)T + 3C_1 + 24CC_2T + 24CC_3T]\Delta^\gamma \\
&\quad + 12C\mathbb{E}\int_0^t \sup_{0\leq\xi\leq s}|x(\xi)-Y(\xi)|^2 ds.
\end{aligned}
\tag{2.20}
$$

By (2.1) and (2.2), we have

$$
\begin{aligned}
&|f(s,x(s),x(\tau(s)),r(s)) - f(\underline{s},z_1(s),z_2(s),\bar{r}(s))|^2 \\
&\leq 3K(1+|x(s)|^2+|x(\tau(s))|^2)|s-\underline{s}| + 6C(|x(s)-Y(s)|^2 \\
&\quad + |x(\tau(s))-Y(\tau(s))|^2 + |Y(s)-z_1(s)|^2 + |Y(\tau(s))-z_2(s)|^2) \\
&\quad + 3|f(\underline{s},z_1(s),z_2(\tau(s)),r(s)) - f(\underline{s},z_1(s),z_2(s),\bar{r}(s))|^2.
\end{aligned}
$$

From Lemma 2.2, Lemma 2.3, Lemma 2.4, Lemma 2.5,

$$
\begin{aligned}
&\int_0^t \mathbb{E}|f(s,x(s),x(\tau(s)),r(s)) - f(\underline{s},z_1(s),z_2(s),\bar{r}(s))|^2 ds \\
&\leq \int_0^t 3K(1+2B)\Delta + 6C\mathbb{E}(|x(s)-Y(s)|^2 + |x(\tau(s))-Y(\tau(s))|^2 \\
&\quad + 6CC_2\Delta + 6CC_3\Delta^\gamma ds + 3C_1\Delta \\
&\leq [3K(1+2B)T + 3C_1 + 6CC_2T + 6CC_3T]\Delta^\gamma \\
&\quad + 12C\mathbb{E}\int_0^t \sup_{0\leq\xi\leq s}|x(\xi)-Y(\xi)|^2 ds.
\end{aligned}
\tag{2.21}
$$

Similarly,

$$
\begin{aligned}
&\int_0^t \mathbb{E}|a(s,x(s),x(\tau(s)),r(s)) - a(\underline{s},z_1(s),z_2(s),\bar{r}(s))|^2 ds \\
&\leq [3K(1+2B)T + 3C_1 + 6CC_2T + 6CC_3T]\Delta^\gamma \qquad a = g,\ h. \\
&\quad + 12C\mathbb{E}\int_0^t \sup_{0\leq\xi\leq s}|x(\xi)-Y(\xi)|^2 ds
\end{aligned}
\tag{2.22}
$$

Putting (2.20)-(2.22) into (2.19), we have

$$
\begin{aligned}
&\mathbb{E}[\sup_{0\leq s\leq t}|x(s)-Y(s)|^2] \\
&\leq 12[T+1+2\lambda(1+\lambda T)][K(1+2B)T + C_1 + 8C(C_2+C_3)T]\Delta^\gamma \\
&\quad + 48C[T+1+2\lambda(1+\lambda T)]\mathbb{E}\int_0^t \sup_{0\leq\xi\leq s}|x(\xi)-Y(\xi)|^2 ds.
\end{aligned}
\tag{2.23}
$$

By the continuous Gronwall inequality

$$\mathbb{E}[\sup_{0\leq s\leq t}|x(s)-Y(s)|^2] \leq \hat{C}\Delta^\gamma,$$

where $\hat{C} = 12[T+1+2\lambda(1+\lambda T)][K(1+2B)T+C_1+8C(C_2+C_3)T]e^{48C[T+1+2\lambda(1+\lambda T)]}$.

□

3. Stability with global Lipschitz

In this section, we shall extend the results in [4, 11] to SDDEwPJMSs. We will consider the autonomous case of (1.1). We show that under a global Lipschitz assumption the SDDEwPJMSs is exponentially stable in mean-square if and only if for some sufficeiently small step-size Δ the SIE approximation is exponentially stable in mean-square.

$$\begin{cases} dx(t) = f(x(t), x(t-\tau), r(t))dt + g(t, x(t), x(t-\tau), r(t))dW(t) \\ \qquad\qquad + h(x(t), x(t-\tau), r(t))dN(t), & t \geq 0, \\ x(t) = \varphi(t), & t \in [-\tau, 0], \end{cases} \tag{3.1}$$

The solution of (3.1) is denoted by $x(t) := x(t; 0, \varphi)$. The discrete SIE approximation of (3.1) is

$$\begin{aligned} Y_{k+1} =& Y_k + [(1-\theta) f(Y_k, Y_{k-m}, r_k^\Delta) + \theta f(Y_{k+1}, Y_{k+1-m}, r_k^\Delta)]\Delta \\ &+ g(Y_k, Y_{k-m}, r_k^\Delta)\Delta W_k + h(Y_k, Y_{k-m}, r_k^\Delta)\Delta N_k, \ k \geq 0. \end{aligned} \tag{3.2}$$

The continuous SIE approximation of (3.2) is

$$\begin{aligned} Y(t) =& Y_0 + \int_0^t [(1-\theta) f(z_1(s), z_2(s), r_k^\Delta) + \theta f(\hat{z}_1(s), \hat{z}_2(s), r_k^\Delta)] ds \\ &+ \int_0^t g(z_1(s), z_2(s), r_k^\Delta) dW(s) + \int_0^t h(z_1(s), z_2(s), r_k^\Delta) dN(s). \end{aligned} \tag{3.3}$$

Denote by $Y(t) := Y(t; 0, \varphi)$ the SIE approximation. In this section we shall write $L^2_{\mathscr{F}_t}([-\tau; 0]; \mathbb{R}^n) = L^2_{\mathscr{F}_t}$ for simplicity. Next we will show that SIE (3.3) shares stability with (3.1) under the (GL). First of all, we give some definitions.

Definition 3.1. *The SDDEwPJMSs* (3.1) *is said to be exponentially stable in mean square, if there is a pair of positive constants $\bar{\lambda}$ and $\bar{M}$ such that for any initial data $\varphi \in L^2_{\mathscr{F}_0}$*

$$\mathbb{E}|x(t)|^2 \leq \bar{M} ||\varphi||^2_{\mathbb{E}} e^{-\bar{\lambda} t}, \ \forall t \geq 0.$$

We refer to $\bar{\lambda}$ as a rate constant and $\bar{M}$ as a growth constant.

Definition 3.2. *Given a step-size $\Delta = \tau/m$ for some positive integer m, the discrete SIE* (3.2) *is said to be exponentially stable in mean square on the SDDEw-PJMSs* (3.1)*, if there is a pair of positive constants μ and $\bar{H}$ such that for any initial data $\varphi \in L^2_{\mathscr{F}_0}$*

$$\mathbb{E}|Y_k|^2 \leq \bar{H} |\varphi||^2_{\mathbb{E}} e^{-k\mu\Delta}, \ \forall k \geq 0. \tag{3.4}$$

We refer to μ as a rate constant and $\bar{H}$ as a growth constant.

Definition 3.3. *Given a step-size $\Delta = \tau/m$ for some positive integer m, the continuous SIE* (3.3) *is said to be exponentially stable in mean square on the SD-DEwPJMSs* (3.1)*, if there is a pair of positive constants μ and H such that for any initial data $\varphi \in L^2_{\mathscr{F}_0}$*

$$\mathbb{E}|Y(t)|^2 \leq H ||\varphi||^2_{\mathbb{E}} e^{-\mu t}, \ \forall t \geq 0. \tag{3.5}$$

We refer to mu as a rate constant and H as a growth constant.

Proposition 3.4. *Under (GL), the discrete SIE method on the SDDEwPJMSs* (3.2) *is exponentially stable in mean square with rate constant* μ *and growth constant* $\bar{H}$ *if and only if the continuous SIE method* (3.3) *is exponentially stable in mean square with the same rate constant* μ *but may be a different growth constant* H.

Proof. Obviously (3.5) implies (3.4) and in this case we even have $\bar{H} = H$. So we need only to show (3.4) implies (3.5). For any $t \geq 0$ choose $k \geq 0$ such that $t \in [t_k, t_{k+1})$. Note that

$$\begin{aligned} Y(t) =& Y_k + [(1-\theta)f(Y_k, Y_{k-m}, r_k^{\Delta}) + \theta f(Y_{k+1}, Y_{k+1-m}, r_k^{\Delta})](t - t_k) \\ & + g(Y_k, Y_{k-m}, r_k^{\Delta})(W(t) - W_k) + h(Y_k, Y_{k-m}, r_k^{\Delta})(N(t) - N_k). \end{aligned}$$

By (1.8)-(1.10), it is straightforward to show that

$$\begin{aligned} \mathbb{E}|Y(t)|^2 \leq & 5\mathbb{E}|Y_k|^2 + 5\Delta^2\mathbb{E}[(1-\theta)^2|f(Y_k, Y_{k-m}, r_k^{\Delta})|^2 + \theta^2|f(Y_{k+1}, Y_{k+1-m}, r_k^{\Delta})|^2] \\ & + 5\Delta\mathbb{E}|g(Y_k, Y_{k-m}, r_k^{\Delta})|^2 + 10\lambda\Delta(1+\lambda\Delta)\mathbb{E}|h(Y_k, Y_{k-m}, r_k^{\Delta})|^2. \end{aligned}$$

Using (3.4) and (GL) we have

$$\mathbb{E}|Y(t)|^2 \leq \bar{H}||\varphi||_{\mathbb{E}}^2 e^{-\mu k\Delta}[5 + 5C\Delta(1 + e^{\mu\tau})(\Delta + 1 + 2\lambda(1 + \lambda\Delta))].$$

Consequently, (3.5) follows by setting

$$H = \bar{H}e^{\mu\Delta}[5 + 5C\Delta(1 + e^{\mu\tau})(\Delta + 1 + 2\lambda(1 + \lambda\Delta))].$$

□

Definition 3.5. [11] *Let* $\Delta > 0$. *A stochastic process* $\{y(t; s, \varphi) : s \in \mathbb{R}_+, s - \tau \leq t < \infty, \varphi \in L^2_{\mathscr{F}_t}\}$, *which will be written as* $\{y(t; s, \varphi)\}$ *thereafter for simplicity, is said to be an* $L^2_{\mathscr{F}_t}$*-related* Δ*-period stochastic flow if it satisfies the following three conditions:*

1. $\{y(s + u; s, \varphi) : -\tau \leq u \leq 0\} = \varphi,$
2. $y_t := \{y(t + u; s, \varphi) : -\tau \leq u \leq 0\} \in L^2_{\mathscr{F}_t},$ *for* $\forall t \geq s,$
3. $y(t; s, \varphi) = y(t; s + k\Delta, y_{s+k\Delta})$ *for* $\forall t \geq s + k\Delta$ *and* $k = 0, 1, 2, \cdots.$

The process is said to be an $L^2_{\mathscr{F}_t}$-related stochastic flow if it is an $L^2_{\mathscr{F}_t}$-related Δ-period stochastic flow for any $\Delta > 0$ (Δ may be not τ/m here).

For the Eq. (3.1) and Eq. (3.3) are both automatic, we have the following Property (P1) by Lemma 2.2 and Theorem 2.8:

1. there is a positive constant C_1^* independent of s, φ and Δ such that
$$\sup_{0\leq u\leq\tau}[\mathbb{E}|x(s + u; s, \varphi)|^2 \vee \mathbb{E}|Y(s + u; s, \varphi)|^2] \leq C_1^*||\varphi||_{\mathbb{E}}^2;$$
2. there is a positive constant $C_2^* = C_2^*(T)$ independent of s, φ, Δ such that
$$\sup_{\tau\leq u\leq\tau+T} \mathbb{E}|Y(s + u; s, \varphi) - x(s + u; s + \tau, Y_{s+\tau})|^2 \leq C_2^*||\varphi||_{\mathbb{E}}^2\Delta^{\gamma}$$
and
$$\sup_{\tau\leq u\leq\tau+T} \mathbb{E}|Y(s + u; s + \tau, x_{s+\tau}) - x(s + u; s, \varphi)|^2 \leq C_2^*||\varphi||_{\mathbb{E}}^2\Delta^{\gamma}.$$

3. $Y(t;s,\varphi)$ is $L^2_{\mathscr{F}_t}$-related Δ-period stochastic flow and $x(t;s,\varphi)$ is $L^2_{\mathscr{F}_t}$-related stochastic flow.

Noting that for system (3.1), the C_3 in Lemma 2.5 can be

$$\tilde{C}_3 = 16(\rho+1)[(\rho+1)(1+2\lambda^2)+1+2\lambda]CB,\ t \geq \tau,$$

and Eq. (2.2) holds with $K = 0$.

By Theorem 5.1 in [11], we obtain the following result.

Theorem 3.6. *Under (GL), the* (3.1) *is exponentially stable in mean-square if and only if for some* $\Delta > 0$*, the SIE method is exponentially stable in mean-square with rate constant* μ *and growth constant* H *satisfying*

$$\beta_3\Delta^\gamma + 2\sqrt{\beta_3 H\Delta^\gamma}e^{-1/2\mu(\nu-2\tau)} + He^{-\mu(\nu-2\tau)} \leq e^{-1/2\mu\nu},$$

where $\nu = \tau(9+In[4log(H)/\mu\tau])$*,* $\beta_3 = C_2^*(2\nu-2\tau)$ *and* $C_2^*(\cdot)$ *was given by Property (P1).*

4. Convergence with the Local Lipschitz condition

In this section we shall discuss the strong convergence of the SIE method on the SDDEwPJMSs (1.1) under the local Lipschitz condition. In many situations, the coefficients f, g and h are only locally Lipschitz continuous. It is therefore useful to establish the strong convergence of the SIE method under the local Lipschitz condition. By the local Lipschitz condition we mean:

(LL) There is a constant $C^R > 0$ such that

$$|a(t,x,y,i) - a(t,\bar{x},\bar{y},i)|^2 \leq C^R(|x-\bar{x}|^2 + |y-\bar{y}|^2),\ a = f,g,h, \tag{4.1}$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$, $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq R$, $t \in \mathbb{R}_+$ and $i \in \mathbb{S}$;

there is a constant $K^R > 0$ such that

$$|a(t,x,y,i) - a(s,x,y,i)|^2 \leq K^R(1+|x|^2+|y|^2)|t-s|,\ a = f,g,h,$$

for $\forall x, y \in \mathbb{R}^n$, $|x| \vee |y| \leq R$, $\forall t, s \in [-\tau,\infty)$. We also have the following assumption

Assumption 4.1. *There is a constant* $\hat{C} > 0$ *such that*

$$|f(t,x,y,i) - f(t,\bar{x},y,i)|^2 \leq \hat{C}|x-\bar{x}|^2 \tag{4.2}$$

for all $x, \bar{x}, y \in \mathbb{R}^n$*,* $t \in \mathbb{R}_+$ *and* $i \in \mathbb{S}$*, and*

for some $p > 2$*, there is a constant* $A > 0$ *such that*

$$\sup_{-\tau \leq t \leq T} \mathbb{E}|x(t)|^p \vee \sup_{-\tau \leq t \leq T} \mathbb{E}|Y(t)|^p < A. \tag{4.3}$$

Lemma 4.2. *Under* (4.2)*, if* $\theta\sqrt{\hat{C}}\Delta < 1$*, then equation* (1.6) *can be solved uniquely for* Y_{k+1} *given* $Y_{In[\tau(t_k)/\Delta]}$*,* $Y_{In[\tau(t_{k+1})/\Delta]}$*,* Y_k*, with probability 1.*

Proof. From the Lemma 2.1, we can easily get the result. □

Define

$$\rho_R := \inf\{t \geq 0 : |x(t)| \geq R\},\ \tau_R := \inf\{t \geq 0 : |Y(t)| \geq R\},\ \theta_R := \rho_R \wedge \tau_R.$$

Lemma 4.3. *If* (4.1) *and Assumption 4.1 hold, then for all sufficiently small* Δ *we have*

$$\mathbb{E}\int_0^{T\wedge\theta_R} |f(\overline{s},\hat{z}_1(s),\hat{z}_2(s),r(s)) - f(\overline{s},\hat{z}_1(s),\hat{z}_2(s),\bar{r}(s))|^2 ds \leq C_1^R\Delta \tag{4.4}$$

and

$$\mathbb{E}\int_0^{T\wedge\theta_R} |a(\underline{s},z_1(s),z_2(s),r(s)) - a(\underline{s},z_1(s),z_2(s),\bar{r}(s))|^2 ds \leq C_1^R\Delta \tag{4.5}$$

for any $T > 0$, where a is f, g, or h and $C_1^R = 8C^R\hat{\gamma}TA^{p/2}$, $\hat{\gamma} = N[1+\max_{1\le i\le N}(-\gamma_{ii})]$.

Proof. Let $l = [T/\Delta]$, then

$$\begin{aligned}
&\mathbb{E}\int_0^{T\wedge\theta_R} |f(\overline{s}, \hat{z}_1(s), \hat{z}_2(s), r(s)) - f(\overline{s}, \hat{z}_1(s), \hat{z}_2(s), \bar{r}(s))|^2 ds \\
&= \sum_{k=0}^{l} \mathbb{E}\int_{t_k}^{t_{k+1}} |f(\overline{s}, \hat{z}_1(s), \hat{z}_2(s), r(s)) - f(\overline{s}, \hat{z}_1(s), \hat{z}_2(s), \bar{r}(s))|^2 I_{[[0,\theta_R]]} ds \\
&\le 2\sum_{k=0}^{l} \mathbb{E}\int_{t_k}^{t_{k+1}} [|f(\overline{s}, \hat{z}_1(s), \hat{z}_2(s), r(s))|^2 + |f(\underline{s}, z_1(s), z_2(s), \bar{r}(s))|^2] I_{[[0,\theta_R]]} I_{\{r(s)\ne r(t_k)\}} ds \\
&\le 4C^R \sum_{k=0}^{l} \mathbb{E}\int_{t_k}^{t_{k+1}} (|\hat{z}_1(s)|^2 + |\hat{z}_2(s)|^2) I_{\{r(s)\ne r(t_k)\}} ds,
\end{aligned}$$

where $[[0, \theta_R]]$ is defined in [9]

$$[[0, \theta_R]] := \{(t, w) \in \mathbb{R}_+ \times \Omega : 0 \le t \le \theta_R(w)\}.$$

Then we can get (4.4) directly from Lemma 2.3. Similarly, we can get (4.5). □

Using the above technique, the following lemmas and corollaries can be derived directly by the lemmas and corollaries in section 2.

Lemma 4.4. *Under* (4.1) *and Assumption 4.1, for all sufficiently small* Δ

$$\mathbb{E}|Y(t) - z_1(t)|^2 \le C_2^R\Delta,\ \forall t \in [0, T\wedge\theta_R],$$

for any $T > 0$, where $C_2^R = 16(1 + \lambda + \lambda^2)C^R A^{p/2}$ is a constant independent of Δ.

Lemma 4.5. *Under* (4.1) *and Assumption 4.1, for all sufficiently small* Δ

$$\mathbb{E}|Y(\tau(t)) - z_2(t)|^2 \le C_3^R\Delta^\gamma,\ \forall t \in [0, T\wedge\theta_R],$$

for any $T > 0$ where $C_3^R = 2M(\rho+1)^\gamma + 16(\rho+1)[(\rho+1)(1+2\lambda^2)+1+2\lambda]C^R A^{p/2}$ is a constant independent of Δ.

Corollary 4.6. *Under* (4.1) *and Assumption 4.1, for all sufficiently small* Δ

$$\mathbb{E}|Y(t) - \hat{z}_1(t)|^2 \le 4C_2^R\Delta,\ \forall t \in [0, T\wedge\theta_R]$$

for any $T > 0$, where C_2^R is defined in Lemma 4.4.

Corollary 4.7. *Under* (4.1) *and Assumption 4.1, then for all sufficiently small* Δ

$$\mathbb{E}|Y(\tau(t)) - \hat{z}_2(t)|^2 \le 4C_3^R\Delta^\gamma,\ \forall t \in [0, T\wedge\theta_R]$$

for any $T > 0$, where C_3^R is defined in Lemma 4.5.

Theorem 4.8. *If (LL), Assumption 4.1 hold and then for any $T > 0$*

$$\lim_{\Delta\to 0} \mathbb{E}[\sup_{0\le t\le T} |x(t) - Y(t)|^2] = 0.$$

Proof. We will employ the technique due to Higham [3] to prove the theorem. Let

$$e(t) := x(t) - Y(t).$$

Recall the Young inequality [3]: for $r^{-1} + q^{-1} = 1$

$$ab \le \frac{\delta}{r}a^r + \frac{1}{q\delta^{q/r}}b^q,\ \forall a, b, \delta > 0.$$

We thus have for any $\delta > 0$

$$
\begin{aligned}
E[\sup_{0\le t\le T}|e(t)|^2] \le & E[\sup_{0\le t\le T}|e(t)|^2 I_{\{\rho_R>T,\tau_R>T\}}] \\
& + E[\sup_{0\le t\le T}|e(t)|^2 I_{\{\rho_R\le T \text{ or } \tau_R\le T\}}] \\
\le & E[\sup_{0\le t\le T}|e(t\wedge\theta_R)|^2 I_{\{\theta_R>T\}}] + \frac{2\delta}{p}E[\sup_{0\le t\le T}|e(t)|^p] \\
& + \frac{1-2/p}{\delta^{2/(p-2)}}P(\rho_R\le T \text{ or } \tau_R\le T).
\end{aligned} \tag{4.6}
$$

Now

$$
P(\rho_R\le T) = E[\mathbf{1}_{\rho_R\le T}\frac{|x(\rho_R)|^p}{R^P}] \le \frac{1}{R^p}E[\sup_{-\tau\le t\le T}|x(t)|^p] \le \frac{A}{R^p}.
$$

Using Assumption 4.1. A similar result can be derived for τ_R so that

$$
P(\tau_R\le T \text{ or } \rho_R\le T) \le P(\tau_R\le T) + P(\rho_R\le T) \le \frac{2A}{R^p}.
$$

Using these bounds along with

$$
E[\sup_{0\le t\le T}|e(t)|^p] \le 2^{p-1}E[\sup_{0\le t\le T}(|x(t)|^p + |Y(t)|^p)] \le 2^pA
$$

in (4.6) gives

$$
\begin{aligned}
E[\sup_{0\le t\le T}|e(t)|^2] \le & E[\sup_{0\le t\le T}|x(t\wedge\theta_R) - Y(t\wedge\theta_R)|^2] \\
& + \frac{2^{p+1}\delta A}{p} + \frac{(p-2)2A}{p\delta^{2/(p-2)}R^p}.
\end{aligned} \tag{4.7}
$$

In the similar way as Theorem 2.8 was proved, we can show that

$$
\mathbb{E}[\sup_{0\le t\le T}|x(t\wedge\theta_R) - Y(t\wedge\theta_R)|^2] \le \hat{C}_R\Delta^\gamma, \tag{4.8}
$$

where $\hat{C}_R$ is a constant independent of Δ. Substituting this into (4.7) gives

$$
E[\sup_{0\le t\le T}|e(t)|^2] \le \hat{C}_R\Delta^\gamma + \frac{2^{p+1}\delta A}{p} + \frac{(p-2)2A}{p\delta^{2/(p-2)}R^p}.
$$

Now, given any $\epsilon > 0$, we may first choose $\delta > 0$ such that $2^{p+1}\delta A/p < \epsilon/3$. then we may choose R so that $(p-2)2A/(p\delta^{2/(p-2)}R^p) < \epsilon/3$, and finally choose Δ to ensure that $\hat{C}_R\Delta t < \epsilon/3$. Hence, in (4.8), $E[\sup_{0\le t\le T}|e(t)|^2] < \epsilon$, as required. □

Corollary 4.9. *If (LL) and (4.3) hold, the Euler method ($\theta = 0$) is strongly convergent.*

5. Numerical examples

In this section, we will illustrate the theoretical convergence of the semi-implicit Euler method. The data used in FIGURE 1 and FIGURE 2 are obtained by the mean square of data by 1000 trajectories, that is, $w_i : 1 \le i \le 1000$, $E|x(T) - Y_L|^2 = 1/1000\Sigma_{i=1}^{1000}|x(T, w_i) - Y_L(w_i)|^2$. First, we consider the following test equation.

$$
\begin{cases}
dx(t) = [ax(t) + r(t)x(t-1)]dt + [cx(t) + dx(t-1)]dW(t) & t \ge 0, \\
x(t) = t+1),\ r(0) = 1, & t \in [-1, 0],
\end{cases} \tag{5.1}
$$

where $a,\ c,\ d \in \mathbb{R}$, $w(t)$ is a scalar Brownian motion, the state space for $r(t)$ is $\mathbb{S} = \{1, 2\}$, the corresponding generator Γ is a zero matrix. By [9], we can obtain the solution of (5.1). The solution of (5.1) for $t \in [0, 1]$ is

$$x(t) = \Phi_{t,0}(x_0 + \int_0^t \Phi_{s,0}^{-1}(r(s) - cd)sds + \int_0^t \Phi_{s,0}^{-1}dsdW(s)),$$

where

$$\Phi_{t,0} = \exp\{\int_0^t a - \frac{1}{2}c^2 ds + \int_0^t cdW(s)\}.$$

For time $t \in [1, 2]$, we obtain the explicit solution by using the explicit solution given above as a new initial function. Clearly, the explicit solution of (5.1) involve a stochastic integral. Referring to [2], we take the SIE solution with $\theta = 0$ and $\Delta = 2^{-13}$ to be a good approximation of the exact solution and compare this with the SIE approximation. We illustrate the convergence of the semi-implicit Euler method for (5.1) in case I: $a = -3,\ c = 2,\ d = 2$ and case II: $a = -2,\ c = 1,\ d = 1$. We consider another test equation.

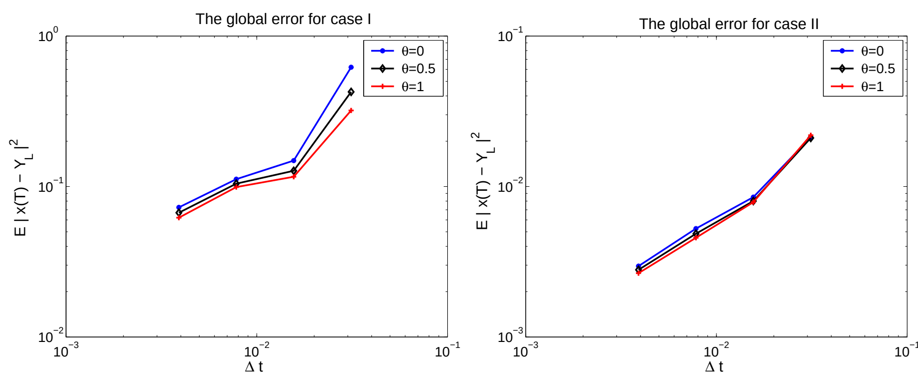

FIGURE 1. The global error of SIE for (5.1) at $T = 2$.

$$\begin{cases} dx(t) = \mu x(t)dt + \sigma x(t)dW(t) + \gamma x(t)dN(t), & t \geq 0, \\ x(0) = 1, \end{cases} \tag{5.2}$$

where $\mu,\ \sigma,\ \gamma \in \mathbb{R}$, $w(t)$ is a scalar Brownian motion, $N(t)$ is a scalar Poisson process with intensity λ. By [6], the solution of (5.2) is

$$x(t) = x(0)(1 + \gamma)^{N(t)} \exp\{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\}.$$

We illustrate the convergence of the semi-implicit Euler method for (5.2) in case I: $\mu = 1,\ \sigma = 0.5,\ \gamma = 0.1,\ \lambda = 2$ and case II: $\mu = 1,\ \sigma = 1,\ \gamma = 0.2,\ \lambda = 1$.

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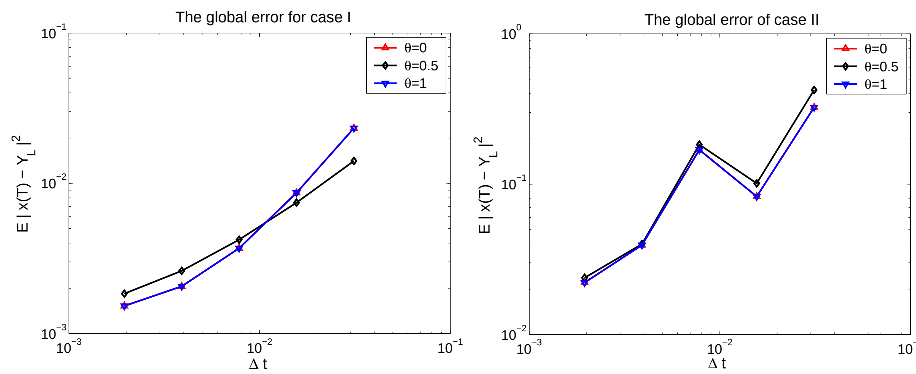

FIGURE 2. The global error of SIE for (5.2) at $T = 1$

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Department of mathematics, Harbin Institute of Technology, Harbin, 150001, P.R.China
E-mail: zgh-hit1108@163.com, songmh@lsec.cc.ac.cn and mzliu@hit.edu.cn