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# $N$-SIMPLEX CROUZEIX-RAVIART ELEMENT FOR THE SECOND-ORDER ELLIPTIC/EIGENVALUE PROBLEMS

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**Abstract.** We study the $n$-simplex nonconforming Crouzeix-Raviart element in approximating the $n$-dimensional second-order elliptic boundary value problems and the associated eigenvalue problems. By using the second Strang Lemma, optimal rate of convergence is established under the discrete energy norm. The error bound is also valid for the eigenfunction approximations. In addition, when eigenfunctions are singular, we prove that the Crouzeix-Raviart element approximates exact eigenvalues from below. Moreover, our numerical experiments demonstrate that the lower bound property is also valid for smooth eigenfunctions, although a theoretical justification is lacking.

**Key words.** $n$-simplex, nonconforming Crouzeix-Raviart element, second order elliptic equation, error estimates, eigenvalues, lower bound.

## 1. Introduction

Nonconforming finite elements have attracted much attention in scientific computing community. In some recent works, Morley element, Adini element, Bogner-Fox-Schmit element, and Zienkiewicz-type element have been extended into arbitrary dimensions by Wang, Shi, and Xu [13, 14]. In this paper, we study the $n$-simplex nonconforming Crouzeix-Raviart element.

The triangular Crouzeix-Raviart element was first introduced in 1973 [6] to solve the stationary Stokes equation. This element was also used to solve the second-order elliptic problems [12] and linear elasticity equations [3, 7]. Recently, Armentano and Durán proved that the triangular Crouzeix-Raviart element approximates the eigenvalue of the Laplace operator from below under certain conditions [1]. All above mentioned works are in the two dimensional setting. Indeed, the Crouzeix-Raviart element has its $n$-dimensional extension [5]. We shall apply it to solve higher-dimensional second-order elliptic equations here. With help of the second Strang lemma, we establish the optimal rate of convergence in the discrete energy norm. This result is then extended to eigenfunctions of the associated eigenvalue problems. We prove that when the eigenfunction is singular, the numerical eigenvalue obtained by the Crouzeix-Raviart element approximates the exact one from below. This theoretical result is illustrated by numerical examples. Moreover, our numerical experiments indicate that the lower bound property is also valid for smooth eigenfunctions, at least for the Laplace operator on the cube.

By the min-max principle, a conforming finite element results in an upper bound for eigenvalue problems associated with second-order elliptic operators. The fact that the nonconforming Crouzeix-Raviart element provides a lower bound has a significant impact from the *a posteriori* error control view point. By comparing the two, we are able to control the error by a given tolerance. That is why the

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subject of non-conforming elements approximating exact eigenvalues from below has attracted much attention in scientific community. Other than [1] mentioned above, Rannacher [11] gave numerical examples about Morley element and Adini element in approximating the exact eigenvalues from below for a plate vibration problem; Yang [15] proved that Adini element approximates the exact eigenvalues from below for the plate vibration problem; Lin and Lin [9] proved that the non-conforming $EQ_1^{rot}$ approximates the exact eigenvalues of the Laplace operator from below; Zhang, Yang and Chen [17] proved that the non-conforming Wilson element approximates the exact eigenvalues of the Laplace operator from below. Again, all above works are for the two dimensions. The results in this paper are for any $n$-dimension.

## 2. Approximation of second-order elliptic problems

Consider the second-order elliptic boundary value problem on a polygonal domain $\Omega \subset R^n$,

$$Lu \equiv -\sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j u) + au = f, \quad \text{in } \Omega; \qquad u = 0, \quad \text{on } \partial\Omega. \tag{2.1}$$

We assume that $a_{ij} = a_{ji}, a_{ij} \in W_{1,\infty}(\Omega)$, $a \in L_\infty(\Omega)$, $a \geq 0$, $f \in L_2(\Omega)$, and there exists a constant $\beta > 0$, such that $\sum\limits_{i,j=1}^{n} a_{ij}\xi_j\xi_i \geq \beta \sum\limits_{i=1}^{n} \xi_i^2$ a.e. in $\Omega$ for all $(\xi_1, \xi_2, \cdots, \xi_n) \in R^n$.

The weak form of (2.1) is to seek $u \in H_0^1(\Omega)$ such that

$$a(u, v) = b(f, v), \quad \forall v \in H_0^1(\Omega), \tag{2.2}$$

where,

$$a(u, v) = \int_\Omega \{\sum_{i,j=1}^{n} a_{ij}\partial_j u \partial_i v + auv\}dx, \quad b(f, v) = \int_\Omega fvdx, \quad \|u\|_b = \|u\|_{0,2}.$$

Then the bilinear form $a(\cdot, \cdot)$ is $H_0^1(\Omega)$-elliptic, continuous, and symmetric over the product space $H_0^1(\Omega) \times H_0^1(\Omega)$.

In the sequel, we need the following *a prior* estimate:

$$\|u\|_{2,p} \leq C(p)\|f\|_{0,p}, \quad p \in (1, \infty). \tag{2.3}$$

**Remark**. It is well known that (2.3) is valid when $\partial\Omega$ is $C^{1,1}$. When $\Omega$ is an $n$-cube, (2.3) is valid for $c(p) = \max(p, p/(p-1))$, see [4, Theorem 2.3.4], for the proof.

Let $\pi_h$ be an $n$-simplex partition for $\Omega$, and let the barycenters of the $n+1$-faces of an $n$-simplex be $z_1, z_2, z_3, \cdots, z_{n+1}$. Then the non-conforming Crouzeix-Raviart finite element space is,

$S^h = \{v \in L_2(\Omega) : v\mid_K \in P_1(K),\ \forall K \in \pi_h,\ v$ is continuous at $z_j$, and $v = 0$ at barycenters on $\partial\Omega\}$. Clearly, $S^h \not\subset H_0^1(\Omega)$.

In this paper, we suppose that the family of triangulations $\pi_h$ is regular (see [5, P131]).

The non-conforming Crouzeix-Raviart finite element approximation of (2.1) is to seek $u_h \in S^h$ such that

$$a_h(u_h, v) = b(f, v),\ \forall v \in S^h, \tag{2.4}$$

where,

$$a_h(u_h, v) = \sum_{K\in\pi_h} \int_K \{\sum_{i,j=1}^{n} a_{ij}\partial_j u_h \partial_i v + a u_h v\} dx.$$

Define $\|\cdot\|_h = (\sum_{K\in\pi_h} |\cdot|_{1,K}^2)^{\frac{1}{2}}$ for $a(x) = 0$, and $\|\cdot\|_h = (\sum_{K\in\pi_h} \|\cdot\|_{1,K}^2)^{\frac{1}{2}}$ for $a(x) \geq \delta > 0$. Then $\|\cdot\|_h$ is a norm over the finite element space $S^h$, and it is not difficult to verify that $a_h(\cdot,\cdot)$ is continuous and uniformly $S^h$-elliptic, namely there exists constants $M, \alpha > 0$ independent of $S^h$ such that

$$| a_h(u,v) | \leq M\|u\|_h\|v\|_h, \ \forall u, v \in S^h.$$

$$a_h(v,v) \geq \alpha\|v\|_h^2, \ \forall v \in S^h.$$

Now we turn to error estimates.

**Lemma 1**. Let $W_{1,l}(\hat{K}) \hookrightarrow L_g(\partial\hat{K})$, $\hat{w} \in W_{1,l}(\hat{K})$, then the following inequality is valid.

$$\int_{\partial K} |w|^g ds \leq C\{h_K^{n-\frac{gn}{l}-1}\|w\|_{0,l,K}^g + h_K^{g+n-\frac{gn}{l}-1}|w|_{1,l,K}^g\}, \ \forall K \in \pi_h,$$

where $\hat{K}$ is a reference element, $K$ and $\hat{K}$ are affine-equivalent, $C$ is a positive constant independent of $w$, the diameter of $K$.

*Proof.* It is an application of the trace theorem (see, e.g., [12]). □

Define

$$P_0^F f = \frac{1}{meas(F)} \int_F f ds, \quad R_0^F f = f - P_0^F f, \tag{2.5}$$

$$P_0^K f = \frac{1}{meas(K)} \int_K f dx, \quad R_0^K f = f - P_0^K f, \tag{2.6}$$

where $K \in \pi_h$ and $F$ is an arbitrary element side of $\pi_h$.

**Lemma 2**. Let $f \in W_{1,p}(K)$, then it holds

$$\|R_0^K f\|_{0,p,K} \leq C h_K |f|_{1,p,K}. \tag{2.7}$$

*Proof.* By [5, Theorem 15.3], or by [8, (7.45)], we can obtain (2.7).□

**Lemma 3**. For $q \in [1, \infty]$, there hold

$$\|P_0^F f\|_{0,q,F} \leq \|f\|_{0,q,F}, \tag{2.8}$$

$$\|R_0^F f\|_{0,q,F} \leq 2\|f - v\|_{0,q,F}, \quad \forall v \in P_0(K). \tag{2.9}$$

*Proof.* a) $1 \leq q < \infty$. By the definition, we have,

$$\begin{aligned} | P_0^F f |^q &= \frac{1}{meas(F)^q} | \int_F f ds |^q \leq \frac{1}{meas(F)^q} (\int_F |f|^q ds)^{\frac{1}{q}\times q} (\int_F 1^p ds)^{\frac{1}{p}\times q} \\ &\leq meas(F)^{-q} meas(F)^{\frac{q}{p}} |f|_{0,q,F}^q, \end{aligned}$$

perform integration on both sides yields,

$$\|P_0^F f\|_{0,q,F}^q = \int_F | P_0^F f |^q ds \leq \int_F meas(F)^{-q} meas(F)^{\frac{q}{p}} |f|_{0,q,F}^q ds = |f|_{0,q,F}^q.$$

b) $q = \infty$. Again, from the definition,

$$\|P_0^F f\|_{0,\infty,F} = |P_0^F f| = |\frac{1}{meas(F)} \int\limits_F f ds| \leq \|f\|_{0,\infty,F}.$$

Therefore, we establish (2.8).

Note that $R_0^F v = 0$ for $v \in P_0(K)$. The estimate (2.9) follows from (2.8) by replacing $f$ with $f - v$. □

**Lemma 4**. Let $E$ be a simplex in $R^s$, $z$ is the barycenter of $E$, $P_1(x)$ is a polynomial of degree 1 on $E$, then we have

$$\int\limits_E P_1(x)dx = P_1(z)meas(E).$$

*Proof.* See [5,P187]. □

Define $A = [\frac{2n}{n+2}, 2]$ for $n \geq 3$ and $A = (1, 2]$ for $n = 2$.

Next we will use Lemma 1 and the error estimate of interpolation. In order to satisfy the conditions of Lemma 1 and [5,Theorem 15.3], therefore we suppose $p \in A$ in the rest of the paper.

**Theorem 1**. Let $u \in W_{2,p}(\Omega)$ be the solution of (2.2), $p \in A$. Then the consistent term $E_h(u, w_h) = a_h(u, w_h) - (f, w_h)$ (for the Crouzeix-Raviart finite element) can be estimated by

$$| E_h(u, w_h) | \leq Ch^{1+\frac{n}{2}-\frac{n}{p}} | u |_{2,p} \|w_h\|_h, \quad \forall w_h \in H_0^1(\Omega) \oplus S^h. \tag{2.10}$$

*Proof.* We extend the proof for the triangular Crouzeix-Raviart element (cf., e.g., [12, §7.2.1]) to the $n$-dimensional setting. By Green's formula, we obtain

$$\begin{aligned} & E_h(u, w_h) = a_h(u, w_h) - (f, w_h) \\ = & \sum_{K\in\pi_h} \int\limits_K \{ \sum_{i,j=1}^n a_{ij}\partial_j u \partial_i w_h + auw_h\}dx - \int\limits_\Omega f w_h dx \\ = & \int\limits_\Omega (Lu - f)w_h dx + \sum_{K\in\pi_h} \int\limits_{\partial K} \frac{\partial u}{\partial\nu} w_h ds, \qquad \frac{\partial u}{\partial \nu} = \sum_{i,j=1}^n a_{ij}\partial_j u \nu_i. \end{aligned}$$

Since $u$ is a solution of (2.1), we have $\int\limits_\Omega (Lu - f)w_h dx = 0$, therefore,

$$\begin{aligned} E_h(u, w_h) &= \sum_{K\in\pi_h} \int\limits_{\partial K} \frac{\partial u}{\partial \nu} w_h ds = \sum_{K\in\pi_h} \int\limits_{\partial K} \sum_{i,j=1}^n a_{ij}\partial_j u w_h \nu_i ds \\ &= \sum_{i,j=1}^n \sum_{F\not\subset\partial\Omega} \int\limits_F (a_{ij}\partial_j u w_h |_{K^+} - a_{ij}\partial_j u w_h |_{K^-})\nu_i^+ ds \\ &+ \sum_{i,j=1}^n \sum_{F\subset\partial\Omega} \int\limits_F a_{ij}\partial_j u w_h \nu_i ds. \end{aligned} \tag{2.11}$$

We denote that $w_h|_{K^+} = w_h^+, w_h|_{K^-} = w_h^-$ and the jump of $w_h$ on $F$ as $[w_h]_F = (w_h^+ - w_h^-) |_F$.

We estimate $\int_F a_{ij}\partial_j u[w_h]\nu_i^+ ds$ for $F \not\subset \partial\Omega$ first. By (2.6),

$$\begin{aligned} \int_F a_{ij}\partial_j u[w_h]\nu_i^+ ds &= \nu_i^+ \int_F a_{ij}\partial_j u[w_h] ds \\ &= \nu_i^+ \int_F R_0^K(a_{ij}\partial_j u)[w_h] ds + \nu_i^+ \int_F P_0^K(a_{ij}\partial_j u)[w_h] ds. \end{aligned} \tag{2.12}$$

Applying Lemma 4 and the trace theorem, we obtain

$$\int_F P_0^K(a_{ij}\partial_j u)[w_h] ds = P_0^K(a_{ij}\partial_j u) \int_F [w_h] ds = 0.$$

Substituting this into (2.12), we get

$$\begin{aligned} \int_F a_{ij}\partial_j u[w_h]\nu_i^+ ds &= \nu_i^+ \int_F R_0^K(a_{ij}\partial_j u)[w_h] ds \\ &= \nu_i^+ \int_F R_0^K(a_{ij}\partial_j u) R_0^F([w_h]) ds + \nu_i^+ \int_F R_0^K(a_{ij}\partial_j u) P_0^F([w_h]) ds. \end{aligned} \tag{2.13}$$

Since

$$P_0^F([w_h]) = \frac{1}{meas(F)} \int_F [w_h] ds = 0,$$

by (2.13), we have

$$\begin{aligned} \mid \int_F a_{ij}\partial_j u[w_h]\nu_i^+ ds \mid &= \mid \nu_i^+ \int_F R_0^K(a_{ij}\partial_j u) R_0^F([w_h]) ds \mid \\ &\le \{\int_F \mid R_0^K(a_{ij}\partial_j u) \mid^{q'} ds\}^{\frac{1}{q'}} \{\int_F \mid R_0^F([w_h]) \mid^{q} ds\}^{\frac{1}{q}}, \end{aligned} \tag{2.14}$$

where $\frac{1}{q'} + \frac{1}{q} = 1$, $q' = \max\{p, 2 - \frac{2}{n}\}$.

Using Lemma 1 with $g = q', l = p$ and (2.7), we derive

$$\begin{aligned} &\int_F \mid R_0^K(a_{ij}\partial_j u) \mid^{q'} ds \\ \le\ & C\{h_K^{n-\frac{q'n}{p}-1} \|R_0^K(a_{ij}\partial_j u)\|_{0,p,K}^{q'} + h_K^{q'+n-\frac{q'n}{p}-1} |R_0^K(a_{ij}\partial_j u)|_{1,p,K}^{q'}\} \\ \le\ & C h_K^{q'+n-\frac{q'n}{p}-1} |u|_{2,p,K}^{q'}. \end{aligned} \tag{2.15}$$

On the other hand, from (2.9), Lemma 1 with $g = q, l = 2$ and (2.7), we have

$$\begin{aligned} & \int_F | R_0^F([w_h]) |^q \, ds = \int_F | [w_h] - P_0^F([w_h]) |^q \, ds \\ = & \int_F | (w_h^+ - P_0^F w_h^+) - (w_h^- - P_0^F w_h^-) |^q \, ds \\ \le & \; 2^q \{ \int_F | (w_h^+ - P_0^F w_h^+) |^q \, ds + \int_F | (w_h^- - P_0^F w_h^-) |^q \, ds \} \\ \le & \; 2^{q+q} \{ \int_F | (w_h^+ - P_0^{K^+} w_h^+) |^q \, ds + \int_F | (w_h^- - P_0^{K^-} w_h^-) |^q \, ds \} \quad (2.16) \\ \le & \; C\{ (h_{K^+}^{n-\frac{qn}{2}-1} \|R_0^{K^+} w_h^+\|_{0,2,K^+}^q + h_{K^+}^{q+n-\frac{qn}{2}-1} |R_0^{K^+} w_h^+|_{1,2,K^+}^q) \\ + & \; (h_{K^-}^{n-\frac{qn}{2}-1} \|R_0^{K^-} w_h^-\|_{0,2,K^-}^q + h_{K^-}^{q+n-\frac{qn}{2}-1} |R_0^{K^-} w_h^-|_{1,2,K^-}^q) \} \\ \le & \; C\{ h_{K^+}^{q+n-\frac{qn}{2}-1} |w_h^+|_{1,2,K^+}^q + h_{K^-}^{q+n-\frac{qn}{2}-1} |w_h^-|_{1,2,K^-}^q \}. \end{aligned}$$

Combining (2.14)-(2.16) yields

$$| \int_F a_{ij} \partial_j u [w_h] \nu_i^+ ds | \le C h_K^{1+\frac{n}{2}-\frac{n}{p}} |u|_{2,p,K} |w_h|_{1,2,K^+ \cup K^-}. \quad (2.17)$$

The second term on the right hand side of (2.11) can be estimated analogously. Observe that $\int_F w_h ds = 0$ for $F \subset \partial\Omega$, and we have

$$| \int_F a_{ij} \partial_j u w_h \nu_i ds | \le C h_K^{1+\frac{n}{2}-\frac{n}{p}} |u|_{2,p,K} |w_h|_{1,2,K}. \quad (2.18)$$

Substituting (2.17) and (2.18) into the right-hand side of (2.11), and then applying the Hölder inequality and the Jensen inequality on the right hand side of (2.11), we prove (2.10). $\square$

We define an interpolation operator of face average $I_K : H_0^1(K) \to P_1(K)$:

$$\int_F I_K u = \int_F u \;\forall F, \;\forall u \in H_0^1(K),$$

where $F$ is a face of an arbitrary element $K$ in $\pi_h$, and define an interpolation operator $I_h : H_0^1(\Omega) \to S^h$:

$$(I_h u) |_K = I_K(u |_K), \;\forall K \in \pi_h.$$

**Lemma 5**. Let $u \in W_{2,p}(\Omega)$, $p \in A$, then the following inequalities are valid:

$$\|u - I_h u\|_{0,2} \le C h^{2+\frac{n}{2}-\frac{n}{p}} \|u\|_{2,p}, \quad (2.19)$$

$$\|u - I_h u\|_{0,p} \le C h^2 \|u\|_{2,p}, \quad (2.20)$$

$$\|u - I_h u\|_h \le C h^{1+\frac{n}{2}-\frac{n}{p}} \|u\|_{2,p}, \quad (2.21)$$

$$\|I_h u\|_{0,q} \le C \|u\|_{2,p}, \quad (2.22)$$

where $\frac{1}{q} + \frac{1}{p} = 1$.

*Proof.* Let $K$ and $\hat{K}$ be two affine-equivalent bounded open subsets of $R^n$, and let $\hat{F}_i$, $i = 1, 2, \cdots, n+1$, be faces of $\hat{K}$. For any $f \in W_{2,p}(K)$, by the definition of $I_K$

and the trace theorem, we deduce that

$$\int\limits_{\hat{K}} \mid I_{\hat{K}}\hat{f} \mid^q d\hat{x} = \int\limits_{\hat{K}} \mid \sum_{i=1}^{n+1} \frac{1}{meas(\hat{F_i})} \int\limits_{\hat{F}_i} \hat{f}|_{\hat{F}_i} d\hat{s}\hat{\phi}_i \mid^q d\hat{x} \leq C\|\hat{f}\|^q_{2,p,\hat{K}},$$

where the functions $\hat{\phi}_i$, $1 \leq i \leq n+1$, are the basis functions of the Crouzeix-Raviart finite element. In other words, $I_{\hat{K}} : W_{2,p}(\hat{K}) \to L_q(\hat{K})$ is a linear bounded operator. Since embedding theorem, if $\frac{1}{p} - \frac{2}{n} > 0$, we obtain

$$W_{2,p}(\hat{K}) \hookrightarrow L_{p^*}(\hat{K}) \ (\frac{1}{p^*} = \frac{1}{p} - \frac{2}{n}),$$

and according to assumption $p \in A$, we see that

$$\frac{2}{p} - \frac{2}{n} - 1 \leq 0, i.e., -\frac{1}{q} + \frac{1}{p} - \frac{2}{n} \leq 0, \text{ namely } q \leq p^*,$$

therefore

$$W_{2,p}(\hat{K}) \hookrightarrow L_q(\hat{K}). \tag{2.23}$$

In addition, (2.23) is obviously valid as to $\frac{1}{p} - \frac{2}{n} \leq 0$.
Observe that

$$I_{\hat{K}}\hat{P} = \sum_{i=1}^{n+1} \frac{1}{meas(\hat{F_i})} \int\limits_{\hat{F}_i} \hat{P}|_{\hat{F}_i} d\hat{s}\hat{\phi}_i = \hat{P} \quad \forall \hat{P} \in P_1(\hat{K}).$$

Therefore, note $p \leq 2 \leq q$, from [5, Theorem 15.3], we obtain (2.19), (2.20) and

$$\|u - I_h u\|_{0,q} \leq Ch^{n+2-\frac{2n}{p}}\|u\|_{2,p}. \tag{2.24}$$

Recalling the following inequality

$$\|I_h u\|_{0,q} \leq \|I_h u - u\|_{0,q} + \|u\|_{0,q},$$

we see that the proposition (2.22) is proved.

The (2.21) is able to be concluded analogously. □

**Lemma 6**. Let $u \in W_{2,p}(\Omega)$, $p \in A$, then the following inequality is valid:

$$\mid a_h(u - I_h u, v) \mid \leq Ch^{2+\frac{n}{2}-\frac{n}{p}}|u|_{2,p}\|v\|'_{1,2}, \ \forall v \in S^h. \tag{2.25}$$

*Proof.* By the definition of $I_K$, for any constant $C_F$, we see that

$$\int_K \partial_j(u - I_h u)C_F dx = \int_{\partial K} (u - I_h u)C_F \nu_j ds = 0.$$

Given $v \in S^h$, $\partial v$ is a constant over an arbitrary element $K$. Therefore,

$$\begin{aligned}
& \mid a_h(u - I_h u, v) \mid \\
= & \sum_{K\in\pi_h} \int_K \{ \sum_{i,j=1}^{n} a_{ij}\partial_j(u - I_h u)\partial_i v + a(u - I_h u)v \} dx \\
= & \sum_{K\in\pi_h} \int_K \{ \sum_{i,j=1}^{n} (a_{ij} - I_0 a_{ij})\partial_j(u - I_h u)\partial_i v + a(u - I_h u)v \} dx \\
+ & \sum_{K\in\pi_h} \int_K \{ \sum_{i,j=1}^{n} I_0 a_{ij}\partial_j(u - I_h(u))\partial_i v dx \\
= & \sum_{K\in\pi_h} \int_K \{ \sum_{i,j=1}^{n} (a_{ij} - I_0 a_{ij})\partial_j(u - I_h u)\partial_i v + a(u - I_h u)v \} dx \\
\leq & \; C(\|a_{ij} - I_0 a_{ij}\|_{0,\infty}\|\partial_j(u - I_h u)\|_{0,2} + \|u - I_h u\|_{0,2})\|v\|'_{1,2} \\
\leq & \; Ch^{2+\frac{n}{2}-\frac{n}{p}}|u|_{2,p}\|v\|'_{1,2},
\end{aligned}$$

where $I_0$ is the piecewise constant projection operator. □

**Theorem 2**. Let $u \in W_{2,p}(\Omega)$ be the solution of (2.2), $p \in A$, and $u_h \in S^h$ be the solution of (2.4), then

$$\|u_h - u\|_h \leq Ch^{1+\frac{n}{2}-\frac{n}{p}} \mid u \mid_{2,p}, \tag{2.26}$$

further, let (2.3) be valid, then

$$\|u_h - u\|_{0,2} \leq Ch^{2+n-\frac{2n}{p}} \mid u \mid_{2,p} . \tag{2.27}$$

*Proof.* Recall the second Strang Lemma (see [5, 10]),

$$\|u - u_h\|_h \leq C(\inf_{v\in S^h} \|u - v\|_h + \sup_{w_h\in S^h, w_h\neq 0} \frac{\mid E_h(u, w_h) \mid}{\|w_h\|_h}). \tag{2.28}$$

According to the interpolation error estimate (2.21), we have

$$\inf_{v\in S^h} \|u - v\|_h \leq Ch^{1+\frac{n}{2}-\frac{n}{p}}|u|_{2,p}. \tag{2.29}$$

Recalling (2.10), we derive

$$\mid E_h(u, w_h) \mid \leq Ch^{1+\frac{n}{2}-\frac{n}{p}} \mid u \mid_{2,p} \|w_h\|_h.$$

Therefore,

$$\sup_{w_h\in S^h, w_h\neq 0} \frac{\mid E_h(u, w_h) \mid}{\|w_h\|_h} \leq Ch^{1+\frac{n}{2}-\frac{n}{p}} \mid u \mid_{2,p} . \tag{2.30}$$

Applying (2.29) and (2.30) to (2.28) yields (2.26).

Next, we establish the $L_2$ error bound using the duality argument used by Nitsche (1974), Lascaux and Lesaint (1975), see, e.g., [5].

$$\begin{aligned}
\|u - u_h\|_{0,2} \leq C\|u - u_h\|_h & \sup_{\zeta\in L_2(\Omega)} \{ \frac{1}{\|\zeta\|_{0,2}} \inf_{\varphi_h\in S^h} \|\varphi - \varphi_h\|_h \} \\
& + \sup_{\zeta\in L_2(\Omega)} \{ \frac{1}{\|\zeta\|_{0,2}} \inf_{\varphi_h\in S^h} (E_h(u, \varphi - \varphi_h) + E_h(\varphi, u - u_h)) \},
\end{aligned} \tag{2.31}$$

where $\varphi \in W_{2,p}(\Omega)$ is the unique solution of the following variational problem:

$$a(v, \varphi) = (\zeta, v), \; \forall v \in H_0^1(\Omega).$$

Combining the error estimate of interpolation with (2.3), we derive

$$\|\varphi - I_h\varphi\|_h \le Ch^{1+\frac{n}{2}-\frac{n}{p}}|\varphi|_{2,p} \le Ch^{1+\frac{n}{2}-\frac{n}{p}}\|\zeta\|_{0,p}. \tag{2.32}$$

Substituting above the inequality into (2.10), we obtain $\forall\zeta \in L_2(\Omega)$

$$\begin{aligned} | E_h(u, \varphi - I_h\varphi) | &\le Ch^{1+\frac{n}{2}-\frac{n}{p}} \,| u |_{2,p} \,\|\varphi - I_h\varphi\|_h \\ &\le Ch^{2+n-\frac{2n}{p}} \,| u |_{2,p} \,\|\zeta\|_{0,p}. \end{aligned} \tag{2.33}$$

By (2.10) and (2.26) we derive

$$| E_h(\varphi, u - u_h) | \le Ch^{2+n-\frac{2n}{p}} \,| u |_{2,p} \,\|\zeta\|_{0,p}. \tag{2.34}$$

Therefore, applying (2.26), (2.32), (2.33) and (2.34) to (2.31) gives (2.27). □

## 3. Approximation of the eigenvalue problem

Consider the corresponding eigenvalue problem of (2.1)

$$Lu = \lambda\rho u, \ \text{in}\Omega; \qquad u = 0, \ \text{on}\partial\Omega. \tag{3.1}$$

Here $\rho \in L_\infty(\Omega)$, and there exists a constant $d > 0$, such that $\rho \ge d > 0$ a.e. in $\Omega$. The weak form of (3.1) is: Find $\lambda \in R$ and $u \in H_0^1(\Omega)$ with $\|u\|_b = 1$, such that

$$a(u, v) = \lambda b(u, v), \ \forall v \in H_0^1(\Omega), \tag{3.2}$$

where $a(u, v)$ is defined by (2.2), $b(u, v) = \int_\Omega \rho uv dx$, $\|\cdot\|_b = b(\cdot,\cdot)^{\frac{1}{2}}$.

The Crouzeix-Raviart finite element approximation for (3.1) is: Find $\lambda_h \in R$ and $u_h \in S^h$ with $\|u_h\|_b = 1$, such that

$$a_h(u_h, v) = \lambda_h b(u_h, v), \ \forall v \in S^h, \tag{3.3}$$

where $a_h(u, v)$ is defined by (2.4).

It is clear that $\|\cdot\|_b$ and $\|\cdot\|_{0,2}$ are two equivalent norms on $L_2(\Omega)$. Introduce the operators T and $T_h$ by $T : L_2(\Omega) \to L_2(\Omega)$ and $T_h : L_2(\Omega) \to S^h$, respectively:

$$\begin{aligned} a(Tf, v) &= b(f, v), \ \forall f \in L_2(\Omega), \ \forall v \in H_0^1(\Omega), \\ a_h(T_hf, v) &= b(f, v), \ \forall f \in L_2(\Omega), \ \forall v \in S^h. \end{aligned}$$

In the rest of this paper, we denote $\lambda_j$ as the jth eigenvalue (counting multiplicities) of (3.1).

**Lemma 7**. Let the *a prior* estimate (2.3) be valid and $(\lambda_{j,h}, u_{j,h})$ be the jth eigenpair of (3.3) with $\|u_{j,h}\|_b = 1$. Then there is a eigen-pair $(\lambda_j, u_j)$ of (3.1) with $\|u_j\|_b = 1$ such that $\lambda_{j,h} \to \lambda_j$ as $h \to 0$, and

$$| \lambda_{j,h} - \lambda_j | \le C\|Tu_j - T_hu_j\|_b, \tag{3.4}$$

$$\|u_{j,h} - u_j\|_b \le C\|Tu_j - T_hu_j\|_b, \tag{3.5}$$

$$\|u_{j,h} - u_j\|_h \le \lambda_j\|Tu_j - T_hu_j\|_h + C\|Tu_j - T_hu_j\|_b. \tag{3.6}$$

*Proof.* See [16]. □

Lemma 7 estimates the errors in the Crouzeix-Raviart element for eigenvalue problems in terms of error estimates for the associated source problem.

**Theorem 3**. Let $u_j \in W_{2,p}(\Omega)$, $p \in A$, then we have

$$| \lambda_{j,h} - \lambda_j | \le Ch^{2+n-\frac{2n}{p}} \,| u_j |_{2,p}, \tag{3.7}$$

$$\|u_j - u_{j,h}\|_{0,2} \le Ch^{2+n-\frac{2n}{p}} \,| u_j |_{2,p}, \tag{3.8}$$

$$\|u_j - u_{j,h}\|_h \le Ch^{1+\frac{n}{2}-\frac{n}{p}} \,| u_j |_{2,p} \,. \tag{3.9}$$

*Proof.* The above error bounds are direct consequence of Lemma 7 and Theorem 2. □

## 4. Eigenvalue Approximation from below

In this section, we extend the result in [1] for the two dimensional case to the $n$-dimensional simplex Crouzeix-Raviart element in approximating the model eigenvalue problem (3.2). Towards this end, we need the following fundamental identities for non-conforming finite elements, see, [17], also see [1] for a special case of this identity.

**Lemma 8**. Let $(\lambda, u) \in R \times H_0^1(\Omega)$ be an eigenpair of (3.2) and $(\lambda_h, u_h) \in R \times S^h$ be an eigenpair of (3.3), respectively. Then the following identity is valid:

$$\begin{aligned} \lambda - \lambda_h &= \|u - u_h\|_h^2 - \lambda_h \|w_h - u_h\|_b^2 \\ &+ \lambda_h(\|w_h\|_b^2 - \|u\|_b^2) + 2a_h(u - w_h, u_h), \ \forall w_h \in S^h. \end{aligned} \tag{4.1}$$

*Proof.* The proof can be given using the similar method as in the proof of Lemma 1 in [1]. □

**Theorem 4**. Let $\lambda_j$ be the jth eigenvalue of (3.2) and $\lambda_{j,h}$ be the jth Crouzeix-Raviart finite element eigenvalue of (3.3), respectively. Assume that $u_j \in W_{2,p}(\Omega)$, $p \in A$, $p < p_0 < 2$ and the $p$ arbitrarily approaches $p_0$, $u_j \notin W_{2,p_0}(\Omega)$, and $\|u_j - u_{j,h}\|_h \geq Ch^{1+\frac{n}{2}-\frac{n}{p_0}}$. When $h$ is sufficiently small, we have

$$\lambda_{j,h} \leq \lambda_j. \tag{4.2}$$

*Proof.* We set $w_h = I_h u$ in (4.1). The first and fourth terms on the right-hand-side are estimated by (3.9) and (2.25), respectively. Using (3.8) and (2.19), we conclude that

$$\|I_h u_j - u_{j,h}\|_{0,2} \leq \|I_h u_j - u_j\|_{0,2} + \|u_j - u_{j,h}\|_{0,2} \leq Ch^{2+n-\frac{2n}{p}} \|u_j\|_{2,p}. \tag{4.3}$$

A direct consequence of (2.20) and (2.22) is the following

$$\begin{aligned} | \ \|I_h u_j\|_{0,2}^2 - \|u_j\|_{0,2}^2 \ | &= | \int_\Omega (u_j - I_h u_j)(u_j + I_h u_j) dx \ | \\ &\leq C\|u_j\|_{2,p} \|u_j - I_h u_j\|_{0,p} \leq Ch^2 \|u_j\|_{2,p}^2. \end{aligned} \tag{4.4}$$

In light of (4.3), (4.4) and (2.25), we observe that the second, third, and fourth terms on the right hand sides of (4.1) are of higher orders comparing with the first term with the assumption $\|u_j - u_{j,h}\|_h \geq Ch^{1+\frac{n}{2}-\frac{n}{p_0}}$. Therefore, the sign is determined by the first term on the right hand side of (4.1), and hence $\lambda_j - \lambda_{j,h} > 0$. □

## 5. Numerical experiments

Consider the eigenvalue problem of the Laplace operator

$$-\Delta u = \lambda u, \ \text{in } \Omega; \ u = 0, \ \text{on } \partial\Omega. \tag{5.1}$$

Here, $\Omega \subset R^3$ is the unit cube $[0,1] \times [0,1] \times [0,1]$, or an L-shaped domain $[0,1] \times [0,1] \times [0,\frac{1}{2}] \cup [0,\frac{1}{2}] \times [0,1] \times [\frac{1}{2},1]$, or a cracked domain $[0,1] \times [0,1] \times [0,1] - [\frac{1}{2},1] \times [\frac{1}{2},\frac{1}{2}] \times [0,1]$.

The three-dimensional domain $\Omega$ is first decomposed into uniform cubes with edge length $l$, then each cube is divided into six congruent tetrahedrons as demonstrated in Fig.1.

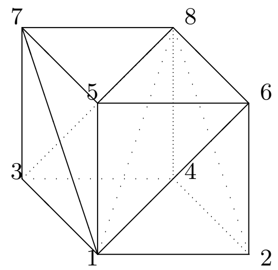


**Fig.1**

We compute some eigenvalues of the Laplace operator by the non-conforming Crouzeix-Raviart element. Numerical results are demonstrated in Tables 1, 2, and 3.

**Table 1: Approximated eigenvalues on the cube domain**

| length of edge $l$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{12}$ | $\frac{1}{16}$ | $\frac{1}{18}$ | trend |
|---|---|---|---|---|---|---|
| unitary number | 384 | 3072 | 10368 | 24576 | 48000 | |
| $\lambda_{1,h}$ | 28.3875 | 29.2948 | 29.4685 | 29.5297 | 29.5463 | $\nearrow$ |
| $\lambda_{2,h}$ | 52.0958 | 57.3250 | 53.3662 | 58.7366 | 58.8371 | $\nearrow$ |
| $\lambda_{3,h}$ | 53.2861 | 57.6705 | 58.5246 | 58.8267 | 58.9085 | $\nearrow$ |
| $\lambda_{4,h}$ | 53.2861 | 57.6705 | 58.5246 | 58.8267 | 58.9085 | $\nearrow$ |
| $\lambda_{5,h}$ | 75.2561 | 85.2691 | 87.2252 | 87.9215 | 88.1105 | $\nearrow$ |
| $\lambda_{6,h}$ | 77.1959 | 85.7913 | 87.4633 | 88.0568 | 88.2177 | $\nearrow$ |
| $\lambda_{7,h}$ | 77.1959 | 85.7913 | 87.4633 | 88.0568 | 88.2177 | $\nearrow$ |
| $\lambda_{8,h}$ | 83.0006 | 101.522 | 105.379 | 106.763 | 107.139 | $\nearrow$ |
| $\lambda_{9,h}$ | 83.0006 | 101.522 | 105.379 | 106.763 | 107.139 | $\nearrow$ |
| $\lambda_{10,h}$ | 83.4117 | 101.550 | 105.386 | 106.764 | 107.139 | $\nearrow$ |

The exact eigenvalues: $\lambda_1 = 3\pi^2 \approx 29.6088$, $\lambda_2 = \lambda_3 = \lambda_4 = 6\pi^2 \approx 59.2176$, $\lambda_5 = \lambda_6 = \lambda_7 = 9\pi^2 \approx 88.8264$, and $\lambda_8 = \lambda_9 = \lambda_{10} = 11\pi^2 \approx 108.566$.

**Table 2: Approximated eigenvalues of the L-shaped domain**

| length of edge $l$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{12}$ | $\frac{1}{16}$ | $\frac{1}{20}$ | trend |
|---|---|---|---|---|---|---|
| unitary number | 288 | 2304 | 7776 | 18432 | 36000 | |
| $\lambda_{1,h}$ | 41.8851 | 46.3055 | 47.3317 | 47.7376 | 47.9434 | $\nearrow$ |
| $\lambda_{2,h}$ | 62.7537 | 68.4872 | 69.6698 | 70.0961 | 70.2962 | $\nearrow$ |
| $\lambda_{3,h}$ | 65.8143 | 74.4059 | 76.2614 | 76.9627 | 77.3062 | $\nearrow$ |
| $\lambda_{4,h}$ | 76.7974 | 85.6299 | 87.3865 | 88.0125 | 88.3043 | $\nearrow$ |

**Table 3: Approximated eigenvalues of the crack domain**

| length of edge $l$ | $\frac{1}{4}$ | $\frac{1}{8}$ | $\frac{1}{12}$ | $\frac{1}{16}$ | $\frac{1}{20}$ | trend |
|---|---|---|---|---|---|---|
| unitary number | 384 | 3072 | 10368 | 24576 | 48000 | |
| $\lambda_{1,h}$ | 36.8840 | 40.5919 | 41.6454 | 41.9242 | 42.1262 | $\nearrow$ |
| $\lambda_{2,h}$ | 52.8762 | 57.5546 | 58.4716 | 58.6684 | 58.7966 | $\nearrow$ |
| $\lambda_{3,h}$ | 61.1370 | 68.7727 | 70.6103 | 71.0583 | 71.3707 | $\nearrow$ |
| $\lambda_{4,h}$ | 66.8307 | 73.8079 | 75.2495 | 75.5632 | 75.7685 | $\nearrow$ |

It can be seen from Tables 2 and 3 that the non-conforming Crouzeix-Raviart element for the L-shaped domain and crack domain approximates eigenvalues from below as predicted by Theorem 4, when eigenfunctions are singular. However, when eigenfunctions are smooth for the cube domain, Table 1 shows that the non-conforming Crouzeix-Raviart element approximates eigenvalues also from below, although the theoretical proof is still an open problem.

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