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MECHANISM OF THE FORMATION OF SINGULARITIES FOR QUASILINEAR HYPERBOLIC SYSTEMS

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Abstract. In this paper the mechanism and the character of the formation of singularities caused by eigenvalues or (and) eigenvectors, respectively, will be discussed for 1-D quasilinear hyperbolic systems.

Key Words. Mechanism, singularities, quasilinear hyperbolic systems.

1. Introduction

We consider the following Cauchy problem for the first order quasilinear hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \tag{1.1}$$

$$t = 0: \quad u = \varphi(x), \tag{1.2}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown vector function of (t, x), $A(u) = (a_{ij}(u))$ is an $n \times n$ matrix with suitably smooth entries $a_{ij}(u)$ $(i, j = 1, \cdots, n)$, and $\varphi(x)$ is C^1 vector function of x with bounded C^1 norm.

By strict hyperbolicity, on the domain under consideration $A(u)$ has n distinct real eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{1.3}$$

For $i = 1, \cdots, n$, let $l_i(u) = (l_{1i}(u), \cdots, l_{ni}(u))$ (resp., $r_i(u) = (r_{i1}(u), \cdots, r_{in}(u))^T$) be a left (resp., right) eigenvector corresponding to $\lambda_i(u)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad (\text{resp., } A(u)r_i(u) = \lambda_i(u)r_i(u)). \tag{1.4}$$

We have

$$\det|l_{ij}(u)| \neq 0 \quad (\text{resp., } \det|r_{ij}(u)| \neq 0), \tag{1.5}$$

and all $\lambda_i(u)$, $l_{ij}(u)$ and $r_{ij}(u)$ $(i, j = 1, \cdots, n)$ have the same regularity as $a_{ij}(u)$ $(i, j = 1, \cdots, n)$. Without loss of generality, we assume that

$$l_i(u)r_j(u) = \delta_{ij} \quad (i, j = 1, \cdots, n), \tag{1.6}$$

where (δ_{ij}) stands for the Kronecker's symbol.

Using left eigenvectors $l_i(u)$ $(i = 1, \cdots, n)$, system (1.1) can be equivalently rewritten in the following characteristic form

$$l_i(u)\Big(\frac{\partial u}{\partial t} + \lambda_i(u)\frac{\partial u}{\partial x}\Big) = 0 \quad (i = 1, \cdots, n). \tag{1.7}$$

The i-th equation in (1.7) contains only the directional derivatives of u with respect to t along the i-th characteristic direction $dx/dt = \lambda_i(u)$.

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By local existence and uniqueness of C^1 solution to the Cauchy problem (cf. [11]), there exists $\delta > 0$ such that Cauchy problem (1.1)–(1.2) admits a unique C^1 solution $u = u(t, x)$ on $0 \leq t \leq \delta$; moreover, for a given system (1.1), δ may be chosen to depend only on the C^1 norm of φ:

$$\delta = \delta(\|\varphi\|_1), \tag{1.8}$$

where $\|\varphi\|_1 = \|\varphi\|_0 + \|\varphi'\|_0$ in which $\|\varphi\|_0 = \max_{x\in\mathbb{R}} |\varphi|$ is the C^0 norm of φ and $\varphi' = d\varphi/dx$.

Thus, in order to prove the global existence and uniqueness of C^1 solution to Cauchy problem (1.1)–(1.2), one should establish the following uniform a priori estimate: For any given $T_0 > 0$, if Cauchy problem (1.1)–(1.2) admits a unique C^1 solution $u = u(t, x)$ on $0 \leq t \leq T$ with $0 < T < T_0$, then

$$\|u(t, \cdot)\|_1 \triangleq \|u(t, \cdot)\|_0 + \|u_x(t, \cdot)\|_0 \leq C(T_0), \quad \forall\, 0 \leq t \leq T, \tag{1.9}$$

where $C(T_0)$ is a positive constant independent of T but possibly depending on T_0.

However, it is well-known (cf. [5, 6]) that, generically speaking, the C^1 solution $u = u(t, x)$ to Cauchy problem (1.1)–(1.2) exists only locally in time and the singularity may occur in a finite time, i.e., there exists $t* > 0$ such that as $t \uparrow t*$,

$$\|u(t, \cdot)\|_1 = \|u(t, \cdot)\|_0 + \|u_x(t, \cdot)\|_0 \text{ becomes unbounded.} \tag{1.10}$$

If the C^1 solution $u = u(t, x)$ to Cauchy problem (1.1)–(1.2) blows up in a finite time, we say that there is a formation of singularities. The problem we would like to study is what is the mechanism and the character of the formation of singularities for quasilinear hyperbolic systems. That is to say, in what follows we don't pay our attention on studying if there is a global C^1 solution or if the C^1 solution blows up in a finite time (This is another business on which there are already many results), we study only the mechanism and the character of the formation of singularities under the hypothesis that the formation of singularities occurs.

Obviously, if all eigenvalues λ_i and all left (resp., right) eigenvectors l_i (resp., r_i) $(i = 1, \cdots, n)$ are independent of u, system (1.1) or (1.7) reduces to a linear hyperbolic system with constant coefficients and then there is no singularity at all. Hence, in order that the singularity occurs, it is necessary to have the dependence of eigenvalues or (and) eigenvectors on u.

2. Singularity caused by eigenvalues

In the special case that all left (resp., right) eigenvectors are independent of u, the singularity (if any!) should be caused only by eigenvalues. By the invertible linear transformation

$$\bar{u} = Lu, \tag{2.1}$$

where $L = (l_{ij})$ denotes the matrix of left eigenvectors, system (1.1) can be reduced to a diagonal form, Thus, we may suppose that

$$A(u) = \mathrm{diag}\{\lambda_1(u), \cdots, \lambda_n(u)\}. \tag{2.2}$$

In this diagonal case, it is not difficult to get the following conclusions (cf. [6]):

(A) The solution $u = u(t, x)$ itself always remains bounded, while the first order derivative u_x of the C^1 solution becomes unbounded along a characteristic at the starting point of singularities on the (t, x)-plane.

(B) The singularity occurs at the starting point of the envelope of characteristics of the same family, i.e., at the point with minimum t-value on the envelope. Here, the characteristics of the i-th family are the integral curves of

$$\frac{dx}{dt} = \lambda_i(u(t,x)) \tag{2.3}$$

on the (t,x)-plane

This kind of singularity is called to be “the shock formation” or “the geometric singularity”.

(C) There is no singularity for the linearly degenerate (LD) system,namely, the Cauchy problem for any given LD system always admits a unique global C^1 solution for all $t \in \mathbb{R}$, provided that the initial data have a bounded C^1 norm.

(D) If $\lambda_i(u)$ is LD, then the family of the i-th characteristics never forms any envelope at least up to the blow-up time.

Here, $\lambda_i(u)$ is linearly degenerate (LD) means that on the domain under consideration

$$\bigtriangledown\lambda_i(u) r_i(u) \equiv 0. \tag{2.4}$$

For the system of diagonal form, it simply means that

$$\frac{\partial\lambda_i(u)}{\partial u_i} \equiv 0. \tag{2.5}$$

The system is LD means that all the eigenvalues are LD.

Moreover, even though the eigenvectors may depend on u, if the initial data have the following small and decaying property: there exists a constant $\mu > 0$ such that

$$\sup_{x\in\mathbb{R}}\{(1+|x|)^{1+\mu}(|\varphi(x)| + |\varphi'(x)|)\} \ll 1, \tag{2.6}$$

the previous four conclusions (A)–(D) are still valid for Cauchy problem (1.1)–(1.2) (cf. [8, 12]). Therefore, in this case the singularity is essentially caused by the dependence of eigenvalues on u again.

More precisely, a complete category has been given for each eigenvalue $\lambda_i(u)$ and for the whole system (1.1) in this situation. For this purpose, we introduce the following

Definition 1: $\lambda_i(u)$ is weakly linearly degenerate (WLD) if along the i-th characteristic trajectory $u = u^{(i)}(s)$ passing through $u = 0$ in the u-space, defined by

$$\begin{cases} \dfrac{du}{ds} = r_i(u), \\ s = 0: \ u = 0, \end{cases} \tag{2.7}$$

we have

$$\bigtriangledown\lambda_i(u) r_i(u) \equiv 0, \quad \forall |u| \text{ small}, \tag{2.8}$$

i.e.,

$$\lambda_i(u^{(i)}(s)) \equiv \lambda_i(0), \quad \forall |s| \text{ small}. \tag{2.9}$$

If all eigenvalues $\lambda_i(u)$ $(i = 1, \cdots, n)$ are weakly linearly degenerate (WLD), system (1.1) is called to be WLD.

Obviously, if $\lambda_i(u)$ is LD, then $\lambda_i(u)$ is WLD. Thus, a LD system is a WLD system.

By definition, if $\lambda_i(u)$ is not WLD, then either there exists an integer $\alpha_i \geq 0$ such that

$$\frac{d^l\lambda_i(u^{(i)}(s))}{ds^l}\Big|_{s=0} = 0 \ (l = 1, \cdots, \alpha_i), \text{ but } \frac{d^{\alpha_i+1}\lambda_i(u^{(i)}(s))}{ds^{\alpha_i+1}}\Big|_{s=0} \neq 0 \tag{2.10}$$

or

$$\frac{d^l \lambda_i(u^{(i)}(s))}{ds^l}\Big|_{s=0} = 0 \ (l = 1, 2, \cdots), \ \text{but } \lambda_i(u^{(i)}(s)) \not\equiv \lambda_i(0), \tag{2.11}$$

$$\forall |s| \text{ small, denoted by } \alpha_i = +\infty.$$

α_i is the index corresponding to non-WLD $\lambda_i(u)$.

Hence we have

<table>
<tr><th colspan="3">$\boldsymbol{\lambda_i(u)}$</th></tr>
<tr><td colspan="2">not WLD</td><td rowspan="2">WLD</td></tr>
<tr><td>$\alpha_i = \overbrace{0, 1, 2, \cdots}^{\text{finte}}$
GN</td><td>$+\infty$
critical</td></tr>
</table>

When $\alpha_i = 0$, $\lambda_i(u)$ is genuinely nonlinear (GN) is a neighbourhood of $u = 0$, i.e.,

$$\nabla \lambda_i(u) r_i(u) \neq 0, \ \forall |u| \text{ small.} \tag{2.12}$$

Moreover, when α_i is getting larger and larger, $\lambda_i(u)$ is closer and closer to the WLD situation.

If system (1.1) is not WLD, then there exists a nonempty set $J \subseteq \{1, 2, \cdots, n\}$ such that $\lambda_i(u)$ is not WLD if and only if $i \in J$.

For each $i \in J$, there is an index α_i which is an integer ≥ 0 or $+\infty$. Let

$$\alpha = \min\{\alpha_i \big| i \in J\}. \tag{2.13}$$

We have

<table>
<tr><th colspan="3">System (1.1)</th></tr>
<tr><td colspan="2">not WLD</td><td rowspan="2">WLD</td></tr>
<tr><td>$\alpha = \overbrace{0, 1, 2, \cdots}^{\text{finte}}$</td><td>$+\infty$
critical</td></tr>
</table>

For small and decaying initial data, we can prove that if system (1.1) is WLD, in particular, if system (1.1) is LD, then the Cauchy problem (1.1)–(1.2) always admits a global C^1 solution $u = u(t, x)$ for all $t \in \mathbb{R}$, namely, there is no singularity at all; while, if system (1.1) is not WLD (α is finite or $+\infty$), the singularity must occur at least for a part of initial data. Moreover, we can prove that if the singularity occurs, namely, if the lifespan $\widetilde{T}$ of the corresponding C^1 solution $u = u(t, x)$ is finite, then

(1). On the existence domain $o \leq t < \widetilde{T}$, the solution itself remains bounded and small, while the first order derivative u_x becomes unbounded along a characteristic at the starting point of singularities on $t = \widetilde{T}$.

(2). The singularity occurs at the starting point of the envelope of characteristics of the same family, i.e., at the point with minimum t-value on the envelope.

These two points show that the singularity always corresponds to a shock formation, namely, a geometric singularity.

(3) If $\lambda_i(u)$ is WLD, in particular, if $\lambda_i(u)$ is LD, then the family of the i-th characteristics never forms any envelope on the domain $0 \leq t \leq \widetilde{T}$.

Thus, for small and decaying initial data, the singularity is essentially caused by the dependence of eigenvalues on u, the dependence of eigenvectors on u gives no influence on the formation of singularities.

3. Singularity caused by eigenvectors

For arbitrary C^1 initial data with bounded C^1 norm, the situation changes tremendously.

For instance, we consider the system given by A. Jeffrey [4]:

$$\begin{cases} \frac{\partial u_1}{\partial t} - \cosh(2u_2)\frac{\partial u_1}{\partial x} - \sinh(2u_2)\frac{\partial u_3}{\partial x} = 0, \\ \frac{\partial u_2}{\partial t} + \cosh u_2 \frac{\partial u_1}{\partial x} + \sinh u_2 \frac{\partial u_3}{\partial x} = 0, \\ \frac{\partial u_3}{\partial t} + \sinh(2u_2)\frac{\partial u_1}{\partial x} + \cosh(2u_2)\frac{\partial u_3}{\partial x} = 0. \end{cases} \tag{3.1}$$

It is easy to see that it is a strictly hyperbolic system with constant eigenvalues

$$\lambda_1 = -1 < \lambda_2 = 0 < \lambda_3 = 1. \tag{3.2}$$

The characteristic form of (3.1) can be written as

$$\begin{cases} \cosh u_2\Big(\frac{\partial u_1}{\partial t} - \frac{\partial u_1}{\partial x}\Big) + \sinh u_2\Big(\frac{\partial u_3}{\partial t} - \frac{\partial u_3}{\partial x}\Big) = 0, \\ \cosh u_2 \frac{\partial u_1}{\partial t} + \frac{\partial u_2}{\partial t} + \sinh u_2 \frac{\partial u_3}{\partial t} = 0, \\ \sinh u_2\Big(\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial x}\Big) + \cosh u_2\Big(\frac{\partial u_3}{\partial t} + \frac{\partial u_3}{\partial x}\Big) = 0. \end{cases} \tag{3.3}$$

System (3.1) or (3.3) is obviously LD, then WLD; moreover, all the characteristics of the same kind are parallel, then never form any envelope. However, for the following initial data

$$t = 0: \quad u_1 = \frac{x}{\alpha}, \quad u_2 = 0, \quad u_3 = -\frac{x}{\alpha}, \quad x \in [-1, 1], \tag{3.4}$$

on the maximum determinate domain $\{(t,x) | 0 \le t \le 1,\ t-1 \le x \le 1-t\}$ the solution to Cauchy problem (3.1) and (3.4) can be expressed explicitly as

$$\begin{cases} u_1 = \frac{\alpha}{\alpha - t} + \frac{x}{\alpha} - 1, \\ u_2 = \ln\Big(1 - \frac{t}{\alpha}\Big), \\ u_3 = \frac{\alpha}{\alpha - t} - \frac{x}{\alpha} - 1. \end{cases} \tag{3.5}$$

Therefore, if $0 < \alpha < 1$ (namely, the initial data are not small), as $t \uparrow \alpha$, the solution u itself and its first order derivative $\frac{\partial u}{\partial t}$ go to the infinity on the line $t = \alpha$ (however, $\frac{\partial u}{\partial x}$ remains bounded in this case).

This example shows that for arbitrary (quite large) C^1 initial data,

(1) The formation of singularities may not be due to the envelope of characteristics of the same kind, but due to the dependence of left (resp., right) eigenvectors on u;

(2) The singularity of C^1 solution may occur in a finite time even for the LD system (with constant eigenvalues);

(3) At the starting point of singularities, the solution itself may not keep the boundedness so that the solution and its first order derivatives become unbounded simultaneously.

From this example, we see that the previous conclusions (A)–(C) fail in this situation, then the mechanism and the character of the formation of singularities produced by eigenvectors are quite different from those produced by the envelope of characteristics of the same kind. It gives us a new kind of singularity (cf. [7, 9]).

On the other hand, we consider the following system

$$\begin{cases} \frac{\partial u_1}{\partial t} + (u_1 + u_2)\frac{\partial u_1}{\partial x} + (u_1 + u_2 - 1)\frac{\partial u_3}{\partial x} = 0, \\ \frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_3}{\partial x} = 0, \\ \frac{\partial u_3}{\partial t} - (1 + u_1 + u_2)\frac{\partial u_1}{\partial x} - (u_1 + u_2)\frac{\partial u_3}{\partial x} = 0. \end{cases} \tag{3.6}$$

Similar to system (3.1), this is still a strictly hyperbolic system with constant eigenvalues (3.2). However, since the corresponding characteristic form can be written as

$$
\begin{cases}
\frac{\partial(u_1+u_3)}{\partial t} - \frac{\partial(u_1+u_3)}{\partial x} = 0, \\
\frac{\partial u_2}{\partial t} + u_2 \frac{\partial(u_1+u_3)}{\partial x} = 0, \\
(1+u_1+u_2)\Big(\frac{\partial u_1}{\partial t} + \frac{\partial u_1}{\partial x}\Big) + (u_1+u_2-1)\Big(\frac{\partial u_3}{\partial t} + \frac{\partial u_3}{\partial x}\Big) = 0,
\end{cases}
\tag{3.7}
$$

using the initial data, we can first explicitly solve $u_1 + u_3$ from the first equation of (3.7), then solve u_2 from the second equation of (3.7), and finally get u_1 (or u_3) from the third equation of (3.7). Hence, for any given C^1 initial data with bounded C^1 norm, the Cauchy problem for system (3.6) always admits a unique global C^1 solution for all $t \in \mathbb{R}$. In this case, the dependence of eigenvectors on u does not cause any singularities.

Thus, it is natural to ask the following problems:

1. For what kind of eigenvectors does the singularity of C^1 solution to Cauchy problem (1.1)–(1.2) occur? For what kind of eigenvectors does Cauchy problem (1.1)–(1.2) admit a unique global C^1 solution $u = u(t, x)$ on $t \geq 0$ or for all $t \in \mathbb{R}$?

2. What is the character of singularities produced by eigenvectors?

4. Completely reducible hyperbolic systems

In order to consider the influence of the dependence of eigenvectors on u to the formation of singularities, we should try to give a complete category to the eigenvectors so that we can distinguish this influence in different levels.

In what follows we will arrange all the eigenvalues

$$
\lambda_i(u),\ \lambda_2(u), \cdots, \lambda_n(u) \tag{4.1}
$$

according to the property of eigenvectors, then it is not necessary to ask them to be in order as shown in (1.3).

If, by means of a suitable diffeomorphism of unknown variables, all the left (resp., right) eigenvectors can be taken as constant vectors $l_1, \cdots, l_n$ (resp., $r_1, \cdots, r_n$), system(1.1) can be equivalently rewritten in a diagonal form, then we have all the conclusions (A)–(D) presented above. It is the simplest case: no influence of eigenvectors on the formation of singularities. This trivial situation is called to be 1-step completely reducible (cf. [7, 9]).

More generally, we give the following (cf. [7, 9])

Definition 2: System (1.1) is called to be m-step completely reducible, if, there is a global C^2 diffeomorphism form $\mathbb{R}^n$ to $\mathbb{R}^n$:

$$
u = u(\tilde{u}) \tag{4.2}
$$

such that system (1.1) can be equivalently rewritten as

$$
\frac{\partial \tilde{u}}{\partial t} + \widetilde{A}(\tilde{u}) \frac{\partial \tilde{u}}{\partial x} = 0, \tag{4.3}
$$

where

$$
\widetilde{A}(\tilde{u}) = \begin{pmatrix}
\widetilde{\Lambda}^{(1)}(\tilde{u}) & & & \\
\widetilde{A}_{21}(\tilde{u}) & \widetilde{\Lambda}^{(2)}(\tilde{u}) & & \\
\cdots & \cdots & \ddots & \\
\widetilde{A}_{m1}(\tilde{u}) & \cdots & \widetilde{A}_{m\,m-1}(\tilde{u}) & \widetilde{\Lambda}^{(m)}(\tilde{u})
\end{pmatrix}, \tag{4.4}
$$

in which $\widetilde{\Lambda}^{(a)}(\tilde{u})$ $(a = 1, \cdots, m)$ are diagonal matrices, the entries of which are given by $\tilde{\lambda}_i(\tilde{u}) = \lambda_i(u(\tilde{u}))$ $(i = 1, \cdots, n)$, respectively.

If this diffeomorphism is only valid in a local domain, system (1.1) is called to be m-step locally completely reducible.

If there is no such diffeomorphism even in the local sense, system (1,1)is non-completely reducible.

Thus, the standard form of 2-step completely reducible system is

$$\begin{cases} \frac{\partial u^{(1)}}{\partial t} + \Lambda^{(1)}(u)\frac{\partial u^{(1)}}{\partial x} = 0, \\ \frac{\partial u^{(2)}}{\partial t} + \Lambda^{(2)}(u)\frac{\partial u^{(2)}}{\partial x} + A_{21}(u)\frac{\partial u^{(1)}}{\partial x} = 0, \end{cases} \tag{4.5}$$

where

$$u^{(1)} = (u_1, \cdots, u_k)^T, \quad u^{(2)} = (u_{k+1}, \cdots, u_n)^T, \tag{4.6}$$

$$\Lambda^{(1)}(u) = \text{diag}\{\lambda_1(u), \cdots, \lambda_k(u)\},\ \Lambda^{(2)}(u) = \text{diag}\{\lambda_{k+1}(u), \cdots, \lambda_n(u)\}. \tag{4.7}$$

The corresponding matrix composed of left eigenvectors is then of the form

$$L(u) = \begin{pmatrix} I_k & 0 \\ L_{21}(u) & I_{n-k} \end{pmatrix}, \tag{4.8}$$

where I_k and I_{n-k} are $k \times k$ and $(n-k) \times (n-k)$ identity matrices.

Similarly, the standard form of 3-step completely reducible system is

$$\begin{cases} \frac{\partial u^{(1)}}{\partial t} + \Lambda^{(1)}(u)\frac{\partial u^{(1)}}{\partial x} = 0, \\ \frac{\partial u^{(2)}}{\partial t} + \Lambda^{(2)}(u)\frac{\partial u^{(2)}}{\partial x} + A_{21}(u)\frac{\partial u^{(1)}}{\partial x} = 0, \\ \frac{\partial u^{(3)}}{\partial t} + \Lambda^{(3)}(u)\frac{\partial u^{(3)}}{\partial x} + A_{31}(u)\frac{\partial u^{(1)}}{\partial x} + +A_{32}(u)\frac{\partial u^{(2)}}{\partial x} = 0, \end{cases} \tag{4.9}$$

where

$$u^{(1)} = (u_1, \cdots, u_k)^T,\ u^{(2)} = (u_{k+1}, \cdots, u_{k+h})^T,\ u^{(3)} = (u_{k+h+1}, \cdots, u_n)^T, \tag{4.10}$$

$$\begin{aligned} \Lambda^{(1)}(u) &= \text{diag}\{\lambda_1(u), \cdots, \lambda_k(u)\}, \\ \Lambda^{(2)}(u) &= \text{diag}\{\lambda_{k+1}(u), \cdots, \lambda_{k+h}(u)\}, \\ \Lambda^{(3)}(u) &= \text{diag}\{\lambda_{k+h+1}(u), \cdots, \lambda_n(u)\}. \end{aligned} \tag{4.11}$$

The matrix of left eigenvectors is then

$$L(u) = \begin{pmatrix} I_k & 0 & 0 \\ L_{21}(u) & I_h & 0 \\ L_{31}(u) & L_{32}(u) & I_{n-k-h} \end{pmatrix}. \tag{4.12}$$

In order to study the mechanism of the formation of singularities caused only by the dependence of eigenvectors on u in a pure situation, we now assume that all the eigenvalues are constants:

$$\lambda_1, \lambda_2, \cdots, \lambda_n. \tag{4.13}$$

In this case, all the characteristics of the same kind are parallel, then never form any envelope. Hence, it is impossible to have the geometric singularity. Correspondingly, we guess that the solution $u = u(t, x)$ itself can not remain bounded as t tends to the lifespan $\widetilde{T}$, namely, the solution $u = u(t, x)$ and its first order derivatives Du should become unbounded simultaneously at the starting point of singularities. This kind of singularity is then of so called ODE type, as in the case of Riccati's equation. We think it should be the character of singularities caused by eigenvectors.

In order to prove this conjecture, it suffices to show that the boundedness of the C^0 norm of $u = u(t,x)$ implies the boundedness of the C^1 norm $\|u\|_1 = \|u\|_0 + \|u_x\|_0$. More precisely, it is only necessary to show that for any given $T_0 > 0$, if the Cauchy problem under consideration admits a unique C^1 solution $u = u(t,x)$ on $0 \le t \le T$ with $0 < T < T_0$, such that the C^0 norm of $u = u(t,x)$ has a uniform a priori estimate

$$\|u(t,\cdot)\|_0 \le C_0(T_0), \quad \forall t \in [0,T], \tag{4.14}$$

where $C_0(T_0)$ is a positive constant independent of T but possibly depending on T_0, then we have

$$\|u_x(t,\cdot)\|_0 \le C_1(T_0), \quad \forall t \in [0,T], \tag{4.15}$$

where $C_1(T_0)$ is also a positive constant independent of T but possibly depending on T_0.

Thus we need only to establish a uniform a priori estimate on u_x.

For the simplest case that the system is 2-step completely reducible with constant eigenvalues, we have proved the previous conjecture, however, up to now, even for 3-step completely reducible systems with constant eigenvalues, the previous conjecture can be proved only under some additional hypotheses (cf. [7, 9]). As to the non-completely reducible system, for instance, the system given by A. Jeffrey, we have no idea at all. How to prove this conjecture in the general situation, or how to construct a counter example to show that this conjecture is in general not correct, is still an open problem.

5. Is there no shock formation for quasilinear LD hyperbolic systems?

Now we turn to the next problem: What happens if both eigenvalues and eigenvectors depend on u? For small and decaying initial data, we know that the dependence of eigenvectors on u has no influence on the formation of singularities and the singularity must be of the type of shock formation. In particular, there is no singularity then no shock formation for quasilinear LD hyperbolic systems. For arbitrary initial data with bounded C^1 norm, as said by Yann Brenier, "Solutions of linearly degenerate system of hyperbolic conservation laws are in general believed to be smooth or to blow up in Sup norm, not in Lipschitz norm" [1]. Actually, A. Majda has essentially given the following conjecture in his monograph [13] "If the system is totally linearly degenerate, then the system typically has smooth global solutions for any $\varphi(x)$, unless there exists a T_* so that as $t \uparrow T_*,\ u(t,x)$ escapes form every compact subset. In particular, the shock wave formation never happens for any smooth initial data $\varphi(x)$".

Thus, another conjecture is that there is no shock formation for quasilinear LD hyperbolic systems of conservation laws.

This conjecture is true for diagonal systems or for small and decaying C^1 initial data. The question is what happens for arbitrary C^1 initial data with bounded C^1 norm. In order to prove this conjecture, it still suffices to establish the previous uniform a priori estimate for the C^0 norm of u_x under the assumption that there is a uniform a priori estimate for the C^0 norm of u.

For 2-step completely reducible systems of 2 equations

$$\begin{cases} \frac{\partial u_1}{\partial t} + \lambda_1(u)\frac{\partial u_1}{\partial x} = 0, \\ \frac{\partial u_2}{\partial t} + \lambda_2(u)\frac{\partial u_2}{\partial x} + a(u)\frac{\partial u_1}{\partial x} = 0 \end{cases} \tag{5.1}$$

with the LD conditions

$$\begin{cases} \frac{\partial\lambda_1}{\partial u_1} - \frac{a(u)}{\lambda_2(u)-\lambda_1(u)}\frac{\partial\lambda_1}{\partial u_2} \equiv 0, \\ \frac{\partial\lambda_2}{\partial u_2} \equiv 0, \end{cases} \tag{5.2}$$

this conjecture can be proved (see [10]).

More generally, the same result is still valid (see [10]) for 2-step completely reducible systems of the form

$$\begin{cases} \frac{\partial u^{(1)}}{\partial t} + \Lambda^{(1)}(u)\frac{\partial u^{(1)}}{\partial x} = 0, \\ \frac{\partial u_n}{\partial t} + \lambda_n(u)\frac{\partial u_n}{\partial x} + a_1(u)\frac{\partial u_1}{\partial x} + \cdots + a_{n-1}(u)\frac{\partial u_{n-1}}{\partial x} = 0 \end{cases} \tag{5.3}$$

with the corresponding LD conditions, where

$$u^{(1)} = (u_1, \cdots, u_{n-1})^T \tag{5.4}$$

and

$$\Lambda^{(1)}(u) = \mathrm{diag}\{\lambda_1(u), \cdots, \lambda_{n-1}(u)\}. \tag{5.5}$$

Up to now, we have not asked our system to be of conservation laws. If it is the case, some additional conditions should be imposed to the original hyperbolic system so that it may be possible to give us some benefits to prove the previous conjecture.

Consider 2-step completely reducible systems

$$\begin{cases} \frac{\partial u^{(1)}}{\partial t} + \Lambda^{(1)}(u)\frac{\partial u^{(1)}}{\partial x} = 0, \\ \frac{\partial u^{(2)}}{\partial t} + \Lambda^{(2)}(u)\frac{\partial u^{(2)}}{\partial x} + A_{21}(u)\frac{\partial u^{(1)}}{\partial x} = 0, \end{cases} \tag{5.6}$$

where

$$u^{(1)} = (u_1, \cdots, u_k), \quad u^{(2)} = (u_{k+1}, \cdots, u_n) \tag{5.7}$$

and

$$\Lambda^{(1)}(u) = \mathrm{diag}\{\lambda_1(u), \cdots, \lambda_k(u)\}, \quad \Lambda^{(2)}(u) = \mathrm{diag}\{\lambda_{k+1}(u), \cdots, \lambda_n(u)\}. \tag{5.8}$$

Suppose that by means of a C^2 diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$:

$$w = w(u), \tag{5.9}$$

the system can be rewritten in a form of conservation laws:

$$\frac{\partial w}{\partial t} + \frac{\partial f(w)}{\partial x} = 0. \tag{5.10}$$

Then, it can be shown that system (5.6) is partly rich, namely, for the second part of system (5.6), Lax's transformation is still valid with respect to variables $u^{(2)} = (u_{k+1}, \cdots, u_n)$. Then, under the hypothesis that the C^0 norm of $u = u(t,x)$ has a uniform a priori bound, we can prove that

$$\|u_x^{(1)}(t,\cdot)\|_0 \leq C_1(T_0), \quad \forall t \in [0,T] \tag{5.11}$$

implies

$$\|u_x^{(2)}(t,\cdot)\|_0 \leq C_2(T_0), \quad \forall t \in [0,T]. \tag{5.12}$$

Hence, if (5.6) is a block successive closed system, namely, if$\Lambda^{(1)}(u)$ depends only on $u^{(1)}$, then the conjuncture can be proved, since in this case we do have (5.11).

This argument is still OK for m-step completely reducible systems. The conclusion is as follows: For any m-step completely reducible system, suppose that it is strictly hyperbolic and LD. Suppose furthermore that it is a block successive closed system, namely, for $i = 1, \cdots, m-1$, the coefficients in any i-step subsystem depend only on the variable $(u^{(1)}, \cdots, u^{(i)})$ so that any i-step subsystem is closed.

Suppose finally that by means of a C^2 diffeomorphism this system can be rewritten in a form of conservation laws. Then there is no shock formation for the Cauchy problem with any C^1 initial data with bounded C^1 norm (see [10]).

Recently, for quasilinear LD hyperbolic systems with source terms

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = F(u), \tag{5.13}$$

this conjecture was also verified in some special cases (see [2, 3, 10]).

However, this conjecture is still open in more general cases.

6. Is there no envelope for LD characteristics of the same family?

Another related conjecture is that if the i-th characteristic $\lambda_i(u)$ is LD, then the i-th family of characteristics never form any envelope at least on the domain $0 \le t \le \widetilde{T}$, where $\widetilde{T}$ is the lifespan of the C^1 solution.

This conjecture is true for diagonal systems or for small and decaying C^1 initial data. But we don't know if it is true or not for general hyperbolic systems with arbitrary C^1 initial data with bounded C^1 norm. It is still open up to now.

Here, we would like to say that in order to show this conjecture, the hypothesis that the initial data have a bounded C^1 norm is essential. To illustrate this, consider the following system

$$\begin{cases} \frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x} = 0, \\ \frac{\partial u_2}{\partial t} + (1+u_1)\frac{\partial u_2}{\partial x} = 0. \end{cases} \tag{6.1}$$

For this system, $\lambda_1 = u_1$ is GN, while $\lambda_2 = 1 + u_1$ is LD. For the following initial data

$$t = 0 : u_1 = \varphi_1(x) \triangleq -2x, \quad u_2 = \varphi_2(x), \tag{6.2}$$

it is easy to see that all the 1-st characteristics passing through the x-axis meet at the point $(t, x) = (\frac{1}{2}, 0)$ and all the 2-nd characteristics passing through the x-axis also meet at this point. Since the lifespan $\widetilde{T}$ is equal to $\frac{1}{2}$, the family of LD characteristics forms an envelope on $t = \widetilde{T}$ in this case with unbounded C^1 initial data.

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