INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 6, Number 3, Pages 505–519

ANALYSIS OF A STABILIZED FINITE VOLUME METHOD FOR THE TRANSIENT STOKES EQUATIONS

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This paper is dedicated to the special occasion of Professor Roland Glowinski's 70th birthdate.

Abstract. This paper is concerned with the development and study of a stabilized finite volume method for the transient Stokes problem in two and three dimensions. The stabilization is based on two local Gauss integrals and is parameter-free. The analysis is based on a relationship between this new finite volume method and a stabilized finite element method using the lowest equal-order pair (i.e., the $P_1 - P_1$ pair). An error estimate of optimal order in the H^1-norm for velocity and an estimate in the L^2-norm for pressure are obtained. An optimal error estimate in the L^2-norm for the velocity is derived under an additional assumption on the body force.

Key words. Transient Stokes equations, stabilized finite volume method, *inf-sup* condition, local Gauss integrals, optimal error estimate, stability.

1. Introduction

Finite difference, finite element, and finite volume methods are three major numerical methods for solving engineering and science problems. The finite differences are easy to implement and locally conservative but not flexible to handle complex geometry. The finite elements have this flexibility but do not locally conserve mass. The finite volumes lie somewhere between the finite differences and the finite elements. They have the flexibility to handle complicated geometry, and their implementation capability is comparable to that of the finite differences. Moreover, their numerical solutions usually have certain conservation features that are desirable in many engineering and science applications.

The finite volume method has a variety of names: the control volume, covolume, and first-order generalized difference methods [3, 5, 7, 9, 12, 14, 22, 23, 24, 25, 29]. Compared to the finite element method, this method is harder to analyze; particularly, its stability and convergence for multidimensional partial differential equations is more difficult to establish. There exist some preliminary error estimates for second-order elliptic and parabolic partial differential problems. However, for more complex problems such as the Stokes problem under consideration, a fundamental stability and convergence theory for the finite volume method is limited.

Recently, a new stabilized finite element method based on two local Gauss integrals was developed for the stationary Stokes equations [18, 20]. This new method stabilizes the lowest equal-order (i.e., P_1-P_1) elements by the residual of these local integrals on each triangular element. It is free of stabilization parameters, does not require any calculation of high-order derivatives or edge-based data structures, and can be implemented at the element level. Optimal error estimates were obtained using the technique of the standard finite element method [20]. More recently, this stabilized finite element method was extended to the finite volume method for the stationary Stokes equations [19]. After a relationship between this method and a stabilized finite element method was established, an error estimate of optimal order

Received by the editors August 30, 2008.
2000 *Mathematics Subject Classification.* 35Q10, 65N30, 76D05.

in the L^2- and H^1-norms for velocity and an estimate in the L^2-norm for pressure were obtained.

In this paper, we extend the definition and analysis of the stabilized finite volume method to the transient Stokes equations. The crucial argument in the analysis is how to use the relationship between the finite element and finite volume methods developed for the stationary problems to establish the desirable optimal error estimates for the transient problems. This crucial argument will be developed in detail here. This new finite volume method will be applied to porous media flow [6, 8].

This paper is organized as follows: In the next section, we introduce some notation, the transient Stokes equations, and their finite element discretizations. Then, in the third section, a stabilized finite volume method for the transient Stokes equations is developed, and a relationship between this method and a finite element method is considered. Stability and optimal order estimates for the finite volume method are obtained in the last three sections.

2. Preliminary

We focus on two dimensions; a generalization to three dimensions is straightforward. Let Ω be a bounded domain in $\Re^2$, with a Lipschitz-continuous boundary Γ, satisfying a further condition stated in (A1) below. The transient Stokes equations are

$$u_t - \nu\Delta u + \nabla p = f,\ \text{div } u = 0, \quad (x,t) \in \Omega \times (0,T], \tag{2.1}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \quad u(x,t)|_\Gamma = 0, \quad t \in [0,T], \tag{2.2}$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t))$ represents the velocity vector, $p = p(x,t)$ the pressure, $f = f(x,t)$ the prescribed body force, $\nu > 0$ the viscosity, $T > 0$ the final time of interest, and $u_t = \partial u/\partial t$.

To introduce a variational formulation, set

$$X = (H_0^1(\Omega))^2,\ Y = (L^2(\Omega))^2,\ M = L_0^2(\Omega) = \left\{ q \in L^2(\Omega); \int_\Omega q\ dx = 0 \right\},$$

$$V = \{v \in X : \text{div } v = 0\},\ D(A) = (H^2(\Omega))^2 \cap V.$$

As noted, a further assumption on Ω is needed:

(A1) Assume that Ω is regular in the sense that the unique solution $(v,q) \in (X,M)$ of the steady Stokes problem

$$-\Delta v + \nabla q = g, \quad \text{div } v = 0 \quad \text{in } \Omega,\ v|_{\partial\Omega} = 0$$

for a prescribed $g \in Y$ exists and satisfies

$$\|v\|_2 + \|q\|_1 \le c\|g\|_0,$$

where $c > 0$ is a constant depending only on Ω and $\|\cdot\|_i$ denotes the usual norm of the Sobolev space $H^i(\Omega)$ or $(H^i(\Omega))^2$ for $i = 0,1,2$. Below the constant $c > 0$ will depend at most on the data (ν, T, u_0, Ω).

We denote by $(\cdot,\cdot)$ and $\|\cdot\|_0$ the inner product and norm on $L^2(\Omega)$ or $(L^2(\Omega))^2$, as appropriate. The spaces $H_0^1(\Omega)$ and X are equipped with their usual scalar product and norm

$$((u,v)) = (\nabla u, \nabla v),\ \|u\|_1 = ((u,u))^{1/2}.$$

Due to the norm equivalence between $\|u\|_1$ and $\|\nabla u\|_0$ on $H_0^1(\Omega)$, we are using the same notation for them: It is well known that for each $v \in X$ the following inequality holds:

$$\|v\|_0 \le \gamma \|\nabla v\|_0, \tag{2.3}$$

where γ is a positive constant depending only on Ω.

(A2) The initial velocity $u_0 \in D(A)$ and the body force $f(x,t) \in L^2(0,T;Y)$ are assumed to satisfy

$$\|u_0\|_2 + \left(\int_0^T \left(\|f\|_0^2 + \|f\|_1^2 + \|f_t\|_0^2 \right) \, dt \right)^{1/2} \le c.$$

The continuous bilinear forms $a(\cdot,\cdot)$ on $X \times X$ and $d(\cdot,\cdot)$ on $X \times M$ are, respectively, defined by

$$a(u,v) = \nu((u,v)) \quad \forall u,v \in X, \quad d(v,q) = -(v,\nabla q) = (q, \operatorname{div} v) \quad \forall v \in X, \; q \in M,$$

and the generalized bilinear form on $(X,M) \times (X,M)$ is given by

$$B((u,p);(v,q)) = a(u,v) - d(v,p) + d(u,q).$$

Then the following estimates for the bilinear term $B((\cdot,\cdot);(\cdot,\cdot))$ hold [2, 15]:

$$|B((u,p);(u,p)) = \nu \|u\|_1^2, \tag{2.4}$$

$$|B((u,p);(v,q))| \le c(\|u\|_1 + \|p\|_0)(\|v\|_1 + \|q\|_0), \tag{2.5}$$

$$\beta_0(\|u\|_1 + \|p\|_0) \le \sup_{(v,q)\in(X,M)} \frac{|B((u,p);(v,q))|}{\|v\|_1 + \|q\|_0}, \tag{2.6}$$

for all $(u,p), \; (v,q) \in (X,M)$, where $\beta_0 > 0$.

The mixed variational form of (2.1) and (2.2) is to seek $(u,p) \in (X,M)$, $t > 0$, such that, for all $(v,q) \in (X,M)$,

$$(u_t, v) + B((u,p);(v,q)) = (f,v), \tag{2.7}$$

$$u(x,0) = u_0(x). \tag{2.8}$$

For convenience, we recall the Gronwall Lemma that will be frequently used.

Lemma 2.1. ([26]). *Let $g(t)$, $\ell(t)$, and $\xi(t)$ be three nonnegative functions satisfying, for $t \in [0,T]$,*

$$\xi(t) + G(t) \le c + \int_0^t \ell \, ds + \int_0^t g\xi \, ds,$$

where $G(t)$ is a nonnegative function on $[0,T]$. Then

$$\xi(t) + G(t) \le \left(c + \int_0^t \ell \, ds \right) \exp\left(\int_0^t g \, ds \right). \tag{2.9}$$

The following result concerning existence, uniqueness, and regularity of a global strong solution to the Stokes equations is presented under the assumptions (A1) and (A2).

Lemma 2.2. ([16]). *Assume that* (A1) *and* (A2) *hold. Then, for any given $T > 0$ there exists a unique solution (u,p) satisfying the following regularities:*

$$\sup_{0<t\le T} (\|u(t)\|_2^2 + \|p(t)\|_1^2 + \|u_t(t)\|_0^2) \le c,$$

$$\sup_{0<t\le T} \tau(t)\|u_t\|_1^2 + \int_0^T \tau(t) \left(\|u_t\|_2^2 + \|p_t\|_1^2 + \|u_{tt}\|_0^2 \right) \, dt \le c,$$

where $\tau(t) = min\{1, \; t\}$.

For $h > 0$, we introduce finite-dimensional subspaces $(X_h, M_h) \subset (X, M)$, which are associated with K_h, a triangulation of Ω into triangles, assumed to be regular and quasi uniform in the usual sense [4, 13]. We assume that for the finite element spaces (X_h, M_h), the following approximation properties hold: for $v \in (W^{k,r}(\Omega))^2$, $1 \le r,\ 1 \le k \le 2$, and $q \in H^1(\Omega) \cap M$,

$$\|v - I_h v\|_{i,r} \le ch^{k-i}|v|_{k,r}, \quad \|q - J_h q\|_0 \le ch|q|_1, \quad i = 0,\ 1, \tag{2.10}$$

where $\|\cdot\|_{i,r}$ and $|\cdot|_{i,r}$ are the usual norm and semi-norm of the Sobolev space $W^{k,r}(\Omega)$. Particularly, the interpolation operator I_h satisfies

$$\|I_h v\|_1 \le c\|v\|_1. \tag{2.11}$$

Due to the quasi-uniformness of the triangulation K_h, the inverse inequality holds

$$\|v_h\|_1 \le ch^{-1}\|v_h\|_0 \qquad \forall v_h \in X_h. \tag{2.12}$$

Note that the generic positive constant c depends only on Ω.

We consider the finite element spaces

$$X_h = \{v = (v_1, v_2) \in X : v_i|_K \in P_1(K),\ i = 1, 2,\ \forall K \in K_h\},$$
$$M_h = \{q \in M : q|_K \in P_1(K),\ \forall K \in K_h\},$$

where $P_1(K)$ represents the space of linear functions on set K.

It is well known that the lowest equal-order pair of conforming finite elements does not satisfy the discrete *inf-sup* condition

$$\sup_{0 \ne v_h \in X_h} \frac{d(v_h, q_h)}{\|v_h\|_1} \ge \beta\|q_h\|_0,$$

where $\beta > 0$ is independent of h. A technique was used in [20] by adding the residual of two local Gauss integrals on each $K \in K_h$ for the pressure space to enforce this condition. Specifically, we define

$$G(p_h, q_h) = \sum_{K \in K_h} \left\{ \int_{K,2} p_h q_h \, dx - \int_{K,1} p_h q_h \, dx \right\}, \qquad p_h,\ q_h \in M_h,$$

where $\int_{K,i} g(x)\, dx$ indicates an appropriate Gauss integral over K that is exact for polynomials of degree i, $i = 1, 2$, and $g(x) = p_h q_h$ is a polynomial of degree not greater than two. Thus, for all test functions $q_h \in M_h$, the trial function $p_h \in M_h$ must be piecewise constant when $i = 1$. Consequently, we define the L^2-projection operator $\Pi_h :\ L^2(\Omega) \to W_h$

$$(p, q_h) = (\Pi_h p, q_h) \qquad \forall p \in L^2(\Omega),\ q_h \in W_h, \tag{2.13}$$

where $W_h \subset L^2(\Omega)$ denotes the piecewise constant space associated with K_h. The projection operator Π_h has the following properties:

$$\|\Pi_h p\|_0 \le c\|p\|_0 \qquad \forall p \in L^2(\Omega), \tag{2.14}$$

$$\|p - \Pi_h p\|_0 \le ch\|p\|_1 \qquad \forall p \in H^1(\Omega). \tag{2.15}$$

Now, using the definition of Π_h, we can define the bilinear form $G(\cdot,\cdot)$ as follows:

$$G(p_h, q_h) = (p_h - \Pi_h p_h, q_h) = (p_h - \Pi_h p_h, q_h - \Pi_h q_h). \tag{2.16}$$

Then the bilinear form of the finite element method on $(X_h, M_h) \times (X_h, M_h)$ is

$$\mathcal{B}((u_h, p_h), (v_h, q_h)) = a(u_h, v_h) - d(v_h, p_h) + d(u_h, q_h) + G(p_h, q_h).$$

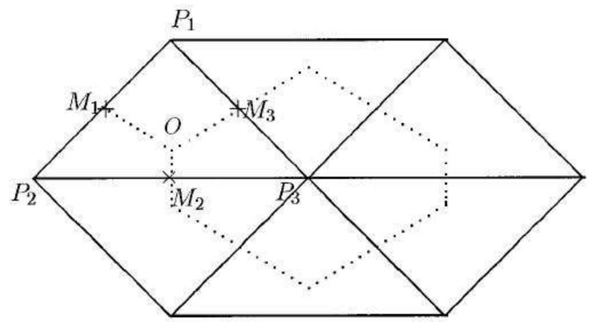

FIGURE 1. Control volumes associated with triangles.

This bilinear form satisfies the continuity and weak coercivity [20], with (u, p), $(v, q) \in (X, M)$:

$$|\mathcal{B}((u,p),(v,q))| \leq c\,(\|u\|_1 + \|p\|_0)\,(\|v\|_1 + \|q\|_0)\,, \tag{2.17}$$

$$\sup_{(v_h,q_h)\in(X_h,M_h)} \frac{|\mathcal{B}((u_h,p_h),(v_h,q_h))|}{\|v_h\|_1 + \|q_h\|_0} \geq \beta\,(\|u_h\|_1 + \|p_h\|_0)\,, \tag{2.18}$$

where β is independent of h.

The corresponding discrete variational formulation for the Stokes equations is recast:

$$(u_{ht}, v_h) + \mathcal{B}((u_h,p_h),(v_h,q_h)) = (f, v_h) \qquad \forall\ (v_h,q_h) \in (X_h, M_h). \tag{2.19}$$

Because of (2.17), (2.18), and Lemma 2.2, system (2.19) has a unique solution. Moreover, the error estimate (optimal for u) for the finite element solution (u_h, p_h) holds [21], $0 < t \leq T$,

$$\|u - u_h\|_0 + h\,(\|u - u_h\|_1 + \|p - p_h\|_0) \leq c\tau^{-1/2}(t)h^2\,(\|u\|_2 + \|p\|_1)\,. \tag{2.20}$$

3. Finite Volume Method

Let $\mathcal{P}$ be the set containing all the nodes associated with the triangulation K_h, and N be the total number of these nodes. To define the finite volume method, a dual mesh $\tilde{K_h}$ is introduced based on K_h; the elements in $\tilde{K_h}$ are called control volumes. The dual mesh can be constructed by the following rule: For each element $K \in K_h$ with vertices P_j, $j = 1, 2, 3$, select its barycenter O and the midpoint M_j on each of the edges of K, and construct the control volumes in $\tilde{K_h}$ by connecting O to M_j, as illustrated in Fig. 1.

Now, the dual finite element space is defined as

$$\tilde{X_h} = \left\{ v \in \left(L^2(\Omega)\right)^2 : v|_{\tilde{K}} \in P_0(\tilde{K}) \quad \forall \tilde{K} \in \tilde{K_h}; \right. \\ \left. v|_{\tilde{K}} = 0 \text{ on any boundary dual element } \tilde{K} \right\}.$$

Obviously, the dimensions of X_h and $\tilde{X_h}$ are the same. Furthermore, there exists an invertible linear mapping $\Gamma_h :\ X_h \to \tilde{X_h}$ such that for

$$v_h(x) = \sum_{j=1}^{N} v_h(P_j)\phi_j(x), \qquad x \in \Omega,\ v_h \in X_h,\ P_j \in \mathcal{P},$$

we have

$$\Gamma_h v_h(x) = \sum_{j=1}^{N} v_h(P_j)\chi_j(x), \qquad x \in \Omega,\ v_h \in X_h,\ P_j \in \mathcal{P},$$

where $\{\phi_j\}$ indicates the basis for the finite element space X_h, and $\{\chi_j\}$ denotes the basis for the finite volume space $\tilde{X_h}$ that are the characteristic functions associated with the dual partition $\tilde{K_h}$:

$$\chi_j(x) = \begin{cases} 1 & \text{if } x \in \tilde{K_j} \in \tilde{K_h}, \\ 0 & \text{otherwise.} \end{cases}$$

The above idea of connecting the trial and test spaces in the Petrov-Galerkin method through the mapping Γ_h was first introduced in [1, 22] in the context of elliptic problems. Furthermore, the mapping Γ_h satisfies the properties [28]:

Lemma 3.1. *Let $K \in K_h$. If $v_h \in X_h$ and $1 \le r \le \infty$, then*

$$\int_K (v_h - \Gamma_h v_h)dx = 0, \tag{3.1}$$

$$\|v_h - \Gamma_h v_h\|_{L^r(K)} \le ch_K \|v_h\|_{W^{1,r}(K)}, \tag{3.2}$$

where h_K is the diameter of the element K.

Multiplying equation (2.1) by $\Gamma_h v_h \in \tilde{X_h}$ and integrating over the dual elements $\tilde{K} \in \tilde{K_h}$, equation (2.2) by $q_h \in M_h$ and over the primal elements $K \in K_h$, and applying Green's formula, we define the following bilinear forms for the finite volume method:

$$A(u_h, \Gamma_h v_h) = -\sum_{j=1}^{N} v_h(P_j) \cdot \int_{\partial\tilde{K}_j} \frac{\partial u_h}{\partial n} ds, \quad u_h,\ v_h \in X_h,$$

$$D(\Gamma_h v_h, p_h) = \sum_{j=1}^{N} v_h(P_j) \cdot \int_{\partial\tilde{K}_j} p_h n\ ds, \quad p_h \in M_h,$$

$$(f, \Gamma_h v_h) = \sum_{j=1}^{N} v_h(P_j) \cdot \int_{\tilde{K}_j} f\ dx, \quad v_h \in X_h,$$

where n is the unit normal outward to $\partial\tilde{K}_j$.

Now, the new stabilized finite volume method is defined for the solution $(\tilde{u_h}, \tilde{p_h}) \in (X_h, M_h)$ as follows:

$$\begin{aligned}(\tilde{u}_{ht}, v_h) + A(\tilde{u_h}, \Gamma_h v_h) + D(\Gamma_h v_h, \tilde{p_h}) + d(\tilde{u_h}, q_h) + G(\tilde{p_h}, q_h)\\ = (f, \Gamma_h v_h) \quad \forall (v_h, q_h) \in (X_h, M_h).\end{aligned} \tag{3.3}$$

Note that we use v_h in the first term of the above equation instead of $\Gamma_h v_h$. With the latter, the convergence analysis is still open. The next lemma holds [19].

Lemma 3.2. *It holds that*

$$A(u_h, \Gamma_h v_h) = a(u_h, v_h) \qquad \forall u_h,\ v_h \in X_h, \tag{3.4}$$

with the following properties:

$$A(u_h, \Gamma_h v_h) = A(v_h, \Gamma_h u_h), \tag{3.5}$$

$$|A(u_h, \Gamma_h v_h)| \le c\|u_h\|_1 \|v_h\|_1, \tag{3.6}$$

$$|A(v_h, \Gamma_h v_h)| \ge c\|v_h\|_1^2. \tag{3.7}$$

Moreover, the bilinear form $D(\cdot,\cdot)$ satisfies

$$D(\Gamma_h v_h, q_h) = -d(v_h, q_h) \qquad \forall (v_h, q_h) \in (X_h, M_h). \tag{3.8}$$

4. Stability

Detailed results on existence, uniqueness, and regularity of the solution for the continuous problems (2.1) and (2.2) can be found in [4, 15, 27]. For the finite volume method (3.3), we define the bilinear form $\mathcal{C}(\cdot,\cdot)$ on $(X_h, M_h) \times (X_h, M_h)$:

$$\mathcal{C}((\tilde{u_h}, \tilde{p_h}), (v_h, q_h)) = A(\tilde{u_h}, \Gamma_h v_h) + D(\Gamma_h v_h, \tilde{p_h}) + d(\tilde{u_h}, q_h) + G(\tilde{p_h}, q_h). \tag{4.1}$$

The following result establishes its continuity and weak coercivity [19]:

Theorem 4.1. *It holds that*

$$\begin{aligned} |\mathcal{C}((\tilde{u_h}, \tilde{p_h}), (v_h, q_h))| \leq c\,(\|\tilde{u_h}\|_1 + \|\tilde{p_h}\|_0)\,(\|v_h\|_1 + \|q_h\|_0) \\ \forall (\tilde{u_h}, \tilde{p_h}),\ (v_h, q_h) \in (X_h, M_h). \end{aligned} \tag{4.2}$$

Moreover,

$$\begin{aligned} \sup_{(v_h,q_h)\in(X_h,M_h)} \frac{|\mathcal{C}((\tilde{u_h}, \tilde{p_h}), (v_h, q_h))|}{\|v_h\|_1 + \|q_h\|_0} \geq \beta\,(\|\tilde{u_h}\|_1 + \|\tilde{p_h}\|_0) \\ \forall (\tilde{u_h}, \tilde{p_h}) \in (X_h, M_h), \end{aligned} \tag{4.3}$$

where β is independent of h.

It follows from this theorem that the stabilized finite volume system (3.3) has a unique solution $(\tilde{u_h}, \tilde{p_h}) \in (X_h, M_h)$.

5. Error Analysis

To obtain error estimates for the finite volume solution $(\tilde{u_h}, \tilde{p_h})$, we also define the projection operator $(R_h, Q_h):\ (X, M) \to (X_h, M_h)$ by

$$\begin{aligned} &\mathcal{C}((R_h(v,q), Q_h(v,q)); (v_h, q_h)) = B((v,q); (v_h, q_h)) \\ &\qquad\qquad \forall (v,q) \in (X, M),\ (v_h, q_h) \in (X_h, M_h), \end{aligned} \tag{5.1}$$

which are well defined and satisfy the following approximation properties:

Lemma 5.1. *The projection operator (R_h, Q_h) satisfies*

$$\|v - R_h(v,q)\|_1 + \|q - Q_h(v,q)\|_0 \leq c(\|v\|_1 + \|q\|_0), \tag{5.2}$$

for all $(v,q) \in (X, M)$ and

$$\|v - R_h(v,q)\|_0 + h\,(\|v - R_h(v,q)\|_1 + \|q - Q_h(v,q)\|_0) \leq ch^2(\|v\|_2 + \|q\|_1), \tag{5.3}$$

for all $(v,q) \in (D(A), H^1(\Omega) \cap M)$.

Proof. First, using the triangle inequality, (2.5), (4.3), and (5.1) gives

$$\begin{aligned} \|v - R_h(v,q)\|_1 + \|q - Q_h(v,q)\|_0 &\leq \|v\|_1 + \|q\|_0 + \|R_h(v,q)\|_1 + \|Q_h(v,q)\|_0 \\ &\leq \|v\|_1 + \|q\|_0 + \frac{1}{\beta} \sup_{(v_h,q_h)\in(X_h,M_h)} \frac{\mathcal{C}((R_h(v,q), Q_h(v,q)); (v_h, q_h))}{\|v_h\|_1 + \|q_h\|_0} \\ &= \|v\|_1 + \|q\|_0 + \frac{1}{\beta} \sup_{(v_h,q_h)\in(X_h,M_h)} \frac{B((v,q); (v_h, q_h))}{\|v_h\|_1 + \|q_h\|_0} \\ &\leq c(\|v\|_1 + \|q\|_0). \end{aligned} \tag{5.4}$$

Next, we see from the definition of (R_h, Q_h), the triangle inequality, (2.14)–(2.16), and (4.2) that

$$\begin{aligned}
&\|v - R_h(v,q)\|_1 + \|q - Q_h(v,q)\|_0 \\
&\quad \le \|v - I_h v\|_1 + \|q - \rho_h q\|_0 + \|I_h v - R_h(v,q)\|_1 + \|\rho_h q - Q_h(v,q)\|_0 \\
&\quad \le \|v - I_h v\|_1 + \|q - \rho_h q\|_0 \\
&\qquad + \frac{1}{\beta} \sup_{(v_h,q_h)\in(X_h,M_h)} \frac{|\mathcal{C}((I_h v - R_h(v,q), \rho_h q - Q_h(v,q)); (v_h,q_h))|}{\|v_h\|_1 + \|q_h\|_0} \\
&\quad \le \|v - I_h v\|_1 + \|q - \rho_h q\|_0 \\
&\qquad + \frac{1}{\beta} \sup_{(v_h,q_h)\in(X_h,M_h)} \frac{|\mathcal{C}((I_h v - v, \rho_h q - q); (v_h,q_h))| + |G(q,q_h)|}{\|v_h\|_1 + \|q_h\|_0} \\
&\quad \le c(\|v - I_h v\|_1 + \|q - \rho_h q\|_0) + \frac{1}{\beta} \sup_{(v_h,q_h)\in(X_h,M_h)} \frac{|G(q,q_h)|}{\|v_h\|_1 + \|q_h\|_0} \\
&\quad \le ch(\|v\|_2 + \|q\|_1). \qquad (5.5)
\end{aligned}$$

Finally, to establish the estimate in the L^2-norm, we consider the dual problem for $(\Phi, \Psi) \in X \times M$ satisfying

$$B((w,r);(\Phi,\Psi)) = (w, v - R_h(v,q)) \qquad \forall (w,r) \in X \times M, \tag{5.6}$$

which satisfies

$$\|\Phi\|_2 + \|\Psi\|_1 \le c\|v - R_h(v,q)\|_0. \tag{5.7}$$

Obviously, using (2.10) and (5.5) and setting $(w,r) = (e,\eta) = (v - R_h(v,q), q - Q_h(v,q))$ in (5.6) and $(v_h, q_h) = (I_h\Phi, \rho_h\Psi)$ in (5.1), respectively, we see that

$$\begin{aligned}
\|e\|_0^2 =& \mathcal{C}((e,\eta);(\Phi - I_h\Phi, \Psi - \rho_h\Psi)) + G(q, \rho_h\Psi) - G(\eta,\Psi) \\
\le& c\left(\|e\|_1 + \|\eta\|_0\right)\left(\|\Phi - I_h\Phi\|_1 + \|\Psi - \rho_h\Psi\|_0\right) + G(q, \rho_h\Psi - \Psi) \\
& + G(q,\Psi) - G(\eta,\Psi) \\
\le& ch\left\{\left(\|e\|_1 + \|\eta\|_0\right)\left(\|\Phi\|_2 + \|\Psi\|_1\right) + h\|q\|_1\|\Psi\|_1\right\} \\
\le& ch\left(\|e\|_1 + \|\eta\|_0 + h\|q\|_1\right)\left(\|\Phi\|_2 + \|\Psi\|_1\right). \qquad (5.8)
\end{aligned}$$

Thus, by combining (5.8) with (5.7) and using (5.5), we have

$$\|v - R_h(v,q)\|_0 \le ch^2(\|v\|_2 + \|q\|_1),$$

which, together with (5.5), yields (5.3). #

Because of $u_0 \in D(A)$, we can define $p_0 \in H^1(\Omega) \cap M$ [16]. Now, we define $(u_{0h}, p_{0h}) = (R_h(u_0,p_0), Q_h(u_0,p_0))$.

Lemma 5.2. *Under the assumptions of Lemma 2.2, we see that, for $t \in [0,T]$,*

$$\|\tilde{u}_h(t)\|_0^2 + \int_0^t \left(\nu\|\tilde{u}_h\|_1^2 + G(\tilde{p}_h,\tilde{p}_h)\right) ds \le c, \tag{5.9}$$

$$\nu\|\tilde{u}_h(t)\|_1^2 + G(\tilde{p}_h(t),\tilde{p}_h(t)) + \int_0^t \|\tilde{u}_{ht}\|_0^2 ds \le c, \tag{5.10}$$

$$\|u(t) - \tilde{u}_h(t)\|_0^2 + \int_0^t \left(\nu\|u - \tilde{u}_h\|_1^2 + G(p - \tilde{p}_h, p - \tilde{p}_h)\right) ds \le ch^2. \tag{5.11}$$

Proof. Choosing $(v,q) = 2(\tilde{u}_h, \tilde{p}_h)$ in (3.3) and using the definition of $\mathcal{C}(\cdot;\cdot)$, we see that

$$\frac{d}{dt}\|\tilde{u}_h\|_0^2 + 2\nu\|\tilde{u}_h\|_1^2 + 2G(\tilde{p}_h, p_h) \le \|f\|_0\|\Gamma_h\tilde{u}_h\|_0 \le \nu\|\tilde{u}_h\|_1^2 + \nu^{-1}\gamma^2\|f\|_0^2.$$

Integrating the above inequality from 0 to t and noting

$$\|\tilde{u}_h(0)\|_0 \le \|u_0\|_0 + \|u_0 - R_h(u_0,p_0)\|_0 \le c(\|u_0\|_1 + \|p_0\|_0),$$

we obtain (5.9).

Subtracting (3.3) from (2.7) with $(v,q) = (v_h,q_h)$, we see that

$$(u_t - \tilde{u}_{ht}, v_h) + \mathcal{C}((u-\tilde{u}_h, p-\tilde{p}_h);(v_h,q_h)) = (f, v_h - \Gamma_h v_h) + G(p,q_h), \tag{5.12}$$

for all $(v_h,q_h) \in (X_h,M_h)$. Set $(v_h,q_h) = 2(e_h,\eta_h)$ in (5.12), where $(e_h,\eta_h) = (R_h(u,p)-\tilde{u}_h, Q_h(u,p)-\tilde{p}_h)$. Also, set $E = u - R_h(u,p)$. Then, using the definition of R_h and Q_h, we obtain

$$\frac{d}{dt}\|u-\tilde{u}_h\|_0^2 + 2\nu\|e_h\|_1^2 + 2G(\eta_h,\eta_h) = 2(f, v_h - \Gamma_h v_h) + 2(u_t - \tilde{u}_{ht}, E). \tag{5.13}$$

Using Lemma 2.2 and (2.3), we see that

$$|(u_t - \tilde{u}_{ht}, E)| \le c\|E\|_0\|u_t - \tilde{u}_{ht}\|_0,$$

$$|(f, e_h - \Gamma_h e_h)| \le ch\|f\|_0\|e_h\|_1 \le ch^2\|f\|_0^2 + \frac{\nu}{4}\|e_h\|_1^2.$$

Noting that

$$\|E\|_0 + h\|E\|_1 \le ch^2(\|u\|_2 + \|p\|_1), \tag{5.14}$$

and combining this inequality with (5.13), we have

$$\begin{aligned}&\frac{d}{dt}\|u-\tilde{u}_h\|_0^2 + \nu\|e_h\|_1^2 + G(\eta_h,\eta_h)\\ &\quad\le ch^2\left(\|f\|_0^2 + \|u_t - \tilde{u}_{ht}\|_0(\|u\|_2 + \|p\|_1)\right).\end{aligned} \tag{5.15}$$

Then, by integrating (5.15) from 0 to t and using the Schwarz inequality, Lemma 2.2, (A2), and the following inequality:

$$\|u_0 - R_h(u_0,p_0)\|_0 \le ch^2(\|u_0\|_2 + \|p_0\|_1),$$

we have

$$\begin{aligned}&\|u(t) - \tilde{u}_h(t)\|_0^2 + \int_0^t \left(\nu\|e_h\|_1^2 + G(\eta_h,\eta_h)\right)\, ds\\ &\quad\le ch^2 + ch^2\left(\int(\|u_t\|_0^2 + \|\tilde{u}_{ht}\|_0^2)ds\right)^{1/2},\end{aligned} \tag{5.16}$$

which, combining with (5.3), (5.14), and Lemma 2.2, yields

$$\begin{aligned}&\|u(t) - \tilde{u}_h(t)\|_0^2 + \int_0^t \left(\nu\|u-\tilde{u}_h\|_1^2 + G(p-\tilde{p}_h, p-\tilde{p}_h)\right) ds\\ &\quad\le ch^2\left\{1 + \left(\int_0^t \|\tilde{u}_{ht}\|_0^2 ds\right)^{1/2}\right\}.\end{aligned} \tag{5.17}$$

To estimate $\int_0^t \|\tilde{u}_{ht}\|_0^2 ds$, we differentiate the term $d(\tilde{u}_h,q_h) + G(\tilde{p}_h,\tilde{q}_h)$ with respect to time t in (3.3) and set $(v_h,q_h) = (\tilde{u}_{ht},\tilde{p}_h)$ to have

$$\|\tilde{u}_{ht}\|_0^2 + \frac{1}{2}\frac{d}{dt}\left(\nu\|\tilde{u}_h\|_1^2 + G(\tilde{p}_h,\tilde{p}_h)\right) = (f, \Gamma_h\tilde{u}_{ht}) \le c\|f\|_0\|\tilde{u}_{ht}\|_0,$$

so

$$\|\tilde{u}_{ht}\|_0^2 + \frac{d}{dt}\left(\nu\|\tilde{u}_h\|_1^2 + G(\tilde{p}_h,\tilde{p}_h)\right) \le c\|f\|_0^2. \tag{5.18}$$

Now, integrating (5.18) from 0 to t, noting that

$$\nu\|\tilde{u}_{0h}\|_1^2 + G(\tilde{p}_{0h},\tilde{p}_{0h}) \le c(\|\tilde{u}_{0h}\|_1^2 + \|\tilde{p}_{0h}\|_0^2) \le c(\|u_0\|_1^2 + \|p_0\|_0^2),$$

and using Lemma 2.2, we see that

$$\int_0^t \|\tilde{u}_{ht}\|_0^2 ds + \nu\|\tilde{u}_h(t)\|_1^2 + G(\tilde{p}_h(t), \tilde{p}_h(t))$$
$$\leq \nu\|\tilde{u}_{0h}\|_1^2 + G(\tilde{p}_{0h}, \tilde{p}_{0h}) + c\int_0^t \|f\|_0^2 ds \leq c, \tag{5.19}$$

which implies (5.10). Finally, combining (5.17) with (5.10) gives (5.11). #

Lemma 5.3. *Under the assumptions of Lemma 2.2, it holds that, for $t \in [0, T]$,*

$$\|\tilde{u}_{ht}(t)\|_0^2 + \int_0^t \left(\nu\|\tilde{u}_{ht}\|_1^2 + G(\tilde{p}_{ht}, \tilde{p}_{ht})\right) ds \leq c, \tag{5.20}$$

$$\tau(t)\left(\nu\|\tilde{u}_{ht}(t)\|_1^2 + G(\tilde{p}_{ht}(t), \tilde{p}_{ht}(t))\right) + \int_0^t \tau(s)\|\tilde{u}_{htt}\|_0^2 ds \leq c, \tag{5.21}$$

$$\tau(t)\|u_t(t) - \tilde{u}_{ht}(t)\|_0^2 + \int_0^t \tau(s)\left(\nu\|e_{ht}\|_1^2 + G(\eta_{ht}, \eta_{ht})\right) ds \leq ch^2. \tag{5.22}$$

Proof. By differentiating (3.3) with respect to time, it follows that

$$(\tilde{u}_{htt}, v_h) + \mathcal{C}((\tilde{u}_{ht}, \tilde{p}_{ht}); (v_h, q_h)) = (f_t, \Gamma_h v_h), \tag{5.23}$$

for $(v_h, q_h) \in (X_h, M_h)$. Taking $(v_h, q_h) = (\tilde{u}_{ht}, \tilde{p}_{ht})$ in equation (5.23), we deduce

$$\frac{1}{2}\frac{d}{dt}\|u_{ht}\|_0^2 + \nu\|\tilde{u}_{ht}\|_1^2 + G(\tilde{p}_{ht}, \tilde{p}_{ht}) \leq \frac{1}{2\nu}\|\tilde{u}_{ht}\|_1^2 + \frac{\nu}{2}\|f_t\|_0^2, \tag{5.24}$$

so

$$\frac{d}{dt}\|\tilde{u}_{ht}\|_0^2 + \nu\|\tilde{u}_{ht}\|_1^2 + G(\tilde{p}_{ht}, \tilde{p}_{ht}) \leq c\|f_t\|_0^2. \tag{5.25}$$

Integrating (5.25) and using assumption (A2) and Lemma 5.2, we obtain (5.20).

Next, differentiating again the term $d(\tilde{u}_{ht}, q_h) + G(\tilde{p}_{ht}, q_h)$ in (5.23) and taking $(v_h, q_h) = (\tilde{u}_{htt}, \tilde{p}_{ht})$, we see that

$$(\tilde{u}_{htt}, \tilde{u}_{htt}) + \frac{d}{dt}(\nu\|\tilde{u}_{ht}\|_1^2 + G(\tilde{p}_{ht}, \tilde{p}_{ht})) = (f_t, \Gamma_h\tilde{u}_{htt})$$
$$\leq \|f_t\|_0\|\tilde{u}_{htt}\|_0 \leq \frac{1}{2}\|f_t\|_0^2 + \frac{1}{2}\|\tilde{u}_{htt}\|_0^2. \tag{5.26}$$

Similarly, multiplying (5.26) by $\tau(s)$ and integrating from 0 to t, we see that

$$\int_0^t \tau(s)\|\tilde{u}_{htt}\|_0^2 ds + \tau(t)(\nu\|\tilde{u}_{ht}\|_1^2 + G(\tilde{p}_{ht}, \tilde{p}_{ht}))$$
$$\leq c\left(\int_0^t \tau(s)\|f_t\|_0^2 ds + \int_0^t (\nu\|\tilde{u}_{ht}\|_1^2 + G(\tilde{p}_{ht}, \tilde{p}_{ht}))ds\right). \tag{5.27}$$

Combining (5.27), Lemma 5.2, and (A2) completes the proof of (5.21).

To show (5.22), differentiating (5.12) with respect to time t gives

$$(u_{tt} - \tilde{u}_{htt}, v_h) + \mathcal{C}((e_{ht}, \eta_{ht}); (v_h, q_h)) = (f_t, v_h - \Gamma_h v_h), \tag{5.28}$$

for all $(v_h, q_h) \in (X_h, M_h)$. Taking $(v_h, q_h) = (e_{ht}, \eta_{ht})$ in (5.28) and using (3.2) and (5.3), we see that

(5.29)
$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u_t - \tilde{u}_{ht}\|_0^2 + \nu\|e_{ht}\|_1^2 + G(\eta_{ht}, \eta_{ht}) &= (f_t, e_{ht} - \Gamma_h e_{ht}) - (u_{tt} - \tilde{u}_{htt}, E_t)\\ &\le c\|f_t\|_0 h\|e_{ht}\|_1 + \|u_{tt} - \tilde{u}_{htt}\|_0\|E_t\|_0\\ &\le \frac{c}{2\nu}h^2\|f_t\|_0^2 + \frac{1}{2}\nu\|e_{ht}\|_1^2 + c(\|u_{tt}\|_0 + \|\tilde{u}_{htt}\|_0)h^2(\|u\|_2 + \|p\|_1).\end{aligned}$$

We multiply (5.29) by $\tau(s)$, integrate from 0 to t, and apply Lemma 2.2 and (5.21) to obtain

$$\begin{aligned}\tau(t)\|u_t - \tilde{u}_{ht}\|_0^2 + \int_0^t \tau(s)(\nu\|e_{ht}\|_1^2 + G(\eta_{ht}, \eta_{ht}))ds\\ \le ch^2\left(\int_0^t \tau(s)\|f_t\|_0^2 ds + \int_0^t \tau(s)(\|u_{tt}\|_0 + \|\tilde{u}_{htt}\|_0)(\|u\|_2 + \|p\|_1)ds\right) \le ch^2,\end{aligned} \tag{5.30}$$

which completes the proof. #

Lemma 5.4. *Under the assumptions of Lemma 2.2, it holds that, for $t \in [0, T]$,*

$$\nu\tau(t)\|u(t) - \tilde{u}_h(t)\|_1^2 + \int_0^t \tau(s)\|u_t - \tilde{u}_{ht}\|_0^2 ds \le ch^2. \tag{5.31}$$

Proof. Differentiating the term $d(u - \tilde{u}_h, q_h) + G(p - \tilde{p}_h, q_h)$ in (5.12), we have

$$\begin{aligned}(u_t - \tilde{u}_{ht}, v_h) + A(u - \tilde{u}_h, \Gamma_h v_h) + D(\Gamma_h v_h, p - \tilde{p}_h) + d(u_t - \tilde{u}_{ht}, q_h)\\ + G(p_t - \tilde{p}_{ht}, q_h) = (f, v_h - \Gamma_h v_h) + G(p_t, q_h).\end{aligned} \tag{5.32}$$

Taking $(v_h, q_h) = (e_{ht}, \eta_h)$ in (5.32) and noting that

$$(u_t - \tilde{u}_{ht}, E_t) \le \frac{1}{2}\|u - \tilde{u}_{ht}\|_0^2 + \frac{1}{2}\|E_t\|_0^2, \tag{5.33}$$

we have

$$\begin{aligned}\|u_t - \tilde{u}_{ht}\|_0^2 + \frac{1}{2}\frac{d}{dt}(\nu\|e_h\|_1^2 + G(\eta_h, \eta_h))\\ \le c\|f\|_0 h\|e_{ht}\|_1 + \frac{1}{2}\|u_t - \tilde{u}_{ht}\|_0^2 + \frac{1}{2}\|E_t\|_0^2.\end{aligned} \tag{5.34}$$

That is,

$$\begin{aligned}\|u_t - \tilde{u}_{ht}\|_0^2 + \frac{d}{dt}(\nu\|e_h\|_1^2 + G(\eta_h, \eta_h))\\ \le c\|f\|_0 h\|e_{ht}\|_1 + \|E_t\|_0^2.\end{aligned} \tag{5.35}$$

Multiplying (5.35) by $\tau(s)$, integrating from 0 to t, and using Lemma 5.1, Lemma 5.2, and (5.22) yields

$$\begin{aligned}&\int_0^t \tau(s)\|u_t - \tilde{u}_{ht}\|_0^2 ds + \tau(t)(\nu\|e_h\|_1^2 + G(\eta_h, \eta_h))\\ \le\, &c\int_0^t \tau(s)\|f\|_0 h\|e_{ht}\|_1 ds + \int_0^t \tau(s)\|E_t\|_0^2 ds + \int_0^t (\nu\|e_h\|_1^2 + G(\eta_h, \eta_h))ds\\ \le\, &ch\left(\int_0^t \|f\|_0^2 ds\right)^{1/2}\left(\int_0^t \tau(s)\|e_{ht}\|_1^2 ds\right)^{1/2} + \int_0^t \tau(s)h^2(\|u_t\|_2^2 + \|p_t\|_1^2)ds\\ &+ \int_0^t (\nu\|e_h\|_1^2 + G(\eta_h, \eta_h))ds \le ch^2.\end{aligned} \tag{5.36}$$

This completes the proof. #

Lemma 5.5. *Under the assumptions of Lemma* 2.2*, it holds that, for* $t \in [0,T]$*,*

$$\tau^{1/2}(t)\|p(t)-\tilde{p}_h(t)\|_0 \le ch. \tag{5.37}$$

Proof. It follows from the *inf-sup* condition (4.3) and (3.3) that

$$\begin{aligned}\|\eta_h(t)\|_0 \le & \beta^{-1} \sup_{(v_h,q_h)\in(X_h,M_h)} \frac{\mathcal{C}((e_h(t),\eta_h(t));(v_h,q_h))}{\|v_h\|_1+\|q_h\|_0}\\ \le & \beta^{-1}\gamma\|u_t(t)-\tilde{u}_{ht}(t)\|_0 + ch\|f\|_0. \end{aligned}\tag{5.38}$$

Using Lemma 5.3 and (A2), we see that

$$\tau^{1/2}(t)\|\eta_h(t)\|_0 \le c\tau^{1/2}(t)\|u_t(t)-\tilde{u}_{ht}(t)\|_0 + ch\tau^{1/2}(t)\|f\|_0 \le ch. \tag{5.39}$$

Thus, using Lemmas 5.1 and 2.2 yields

$$\begin{aligned}\tau^{1/2}(t)\|p(t)-\tilde{p}_h(t)\|_0 \le & \tau^{1/2}(t)\|\eta_h(t)\|_0 + \tau^{1/2}(t)\|p(t)-Q_h(u(t),p(t))\|_0\\ \le & ch + ch(\|u(t)\|_2+\|p(t)\|_1) \le ch,\end{aligned}$$

which is (5.37). #

Theorem 5.6. *Under the assumptions of Lemma* 2.2*, it holds that, for* $t \in [0,T]$*,*

$$\tau^{1/2}(t)\|u(t)-\tilde{u}_h(t)\|_1 + \tau^{1/2}(t)\|p(t)-\tilde{p}_h(t)\|_0 \le ch.$$

This theorem follows from Lemmas 5.4 and 5.5.

6. L^2-Error Estimate

Now, we estimate the error $\|u-u_h\|_0$ using a parabolic duality argument for a backward-in-time Stokes problem [16, 17]. The dual problem is to seek $(\Phi(t),\Psi(t)) \in X \times M$ such that, for $t\in[0,T]$ and $g \in L^2(0,T,Y)$,

$$(v,\Phi_t) - B\big((v,q);(\Phi,\Psi)\big) = (v,g) \tag{6.1}$$

for all $(v,q)\in(X,M)$, with $\Phi(T)=0$. This problem is well-posed and has a unique solution (Φ,Ψ) with [17]

$$\Phi \in C(0,T,V)\cap L^2(0,T,D(A))\cap H^1(0,T,Y),\ \ \Psi\in L^2(0,T,H^1(\Omega)\cap M).$$

We recall the following regularity results [17]:

Lemma 6.1. *The solution* (Φ,Ψ) *of* (6.1) *satisfies*

$$\sup_{0\le t\le T}\|\Phi(t)\|_1^2 + \int_0^T \left(\|\Phi\|_2^2+\|\Psi\|_1^2+\|\Phi_t\|_0^2\right)dt \le c\int_0^T \|g\|_0^2 dt. \tag{6.2}$$

Lemma 6.2. *Under the assumptions of Lemma* 2.2 *and* $f\in L^\infty(0,T,(H^1(\Omega))^2)$*, it holds that*

$$\int_0^T \|u-\tilde{u}_h\|_0^2 ds \le ch^4. \tag{6.3}$$

Proof. We introduce the dual Galerkin projection $(\Phi_h(t),\Psi_h(t))$ of $(\Phi(t),\Psi(t))$:

$$\mathcal{C}((v_h,q_h));(\Phi_h,\Psi_h)) = B((v_h,q_h);(\Phi,\Psi)) \quad \forall\ (v_h,q_h)\in(X_h,M_h),$$

which gives

$$\mathcal{C}((v_h,q_h);(\Phi-\Phi_h,\Psi-\Psi_h)) = G(q_h,\Psi) \quad \forall\ (v_h,q_h)\in(X_h,M_h). \tag{6.4}$$

By using a similar approach to the proof of Lemma 5.1, we can prove

$$\|\Phi-\Phi_h\|_0 + h\|\Phi-\Phi_h\|_1 + h\|\Psi-\Psi_h\|_0 \le ch^2(\|\Phi\|_2+\|\Psi\|_1). \tag{6.5}$$

Taking $(v_h, q_h) = (\Phi_h, \Psi_h)$ in (5.12), we have

$$(e_t, \Phi_h) + \mathcal{C}\big((e,\eta); (\Phi_h, \Psi_h)\big) = (f, \Phi_h - \Gamma_h \Phi_h) + G(p, \Psi_h), \tag{6.6}$$

where $(e, \eta) = (u - \tilde{u}_h, p - \tilde{p}_h)$. Adding (6.6) and (6.1) with $(v, q) = (e, \eta)$ and $g = e$, we see that

$$\begin{aligned} \|e\|_0^2 =& \frac{d}{dt}(e, \Phi) - (e_t, \Phi - \Phi_h) - \mathcal{C}((e,\eta); (\Phi - \Phi_h, \Psi - \Psi_h)) \\ & - (f, \Phi_h - \Gamma_h \Phi_h) + G(\eta, \Psi) - G(p, \Psi_h). \end{aligned} \tag{6.7}$$

Applying Lemma 6.1, (2.14), and (3.2), we have

$$\begin{aligned} |(e_t, \Phi - \Phi_h)| \leq& c(\|u_t\|_0 + \|\tilde{u}_{ht}\|_0)\|\Phi - \Phi_h\|_0 \\ \leq& ch^2(\|u_t\|_0 + \|\tilde{u}_{ht}\|_0)(\|\Phi\|_2 + \|\Psi\|_1), \\ (f, \Phi_h - \Gamma_h \Phi_h) =& (f - \Gamma_h f, \Phi_h - \Gamma_h \Phi_h) \leq ch\|f\|_1 h \|\Phi_h\|_1 \\ \leq& ch^2 \|f\|_1(\|\Phi - \Phi_h\|_1 + \|\Phi\|_1), \\ |G(\eta, \Psi)| \leq& chG^{1/2}(\eta, \eta)\|\Psi\|_1, \\ G(p, \Psi_h) =& G(p, \Psi_h - \Psi) + G(p, \Psi) \leq ch\|p\|_1 h \|\Psi\|_1 + ch\|p\|_1 h\|\Psi\|_1. \end{aligned} \tag{6.8}$$

As for the bilinear term, by using (6.4), (5.1), (5.5), (2.14), and (2.15), we have

$$\begin{aligned} & |\mathcal{C}\big((e,\eta); (\Phi - \Phi_h, \Psi - \Psi_h)\big)| \\ & \leq |\mathcal{C}\big((u - R_h(u,p), p - Q_h(u,p)); (\Phi - \Phi_h, \Psi - \Psi_h)\big)| + |G(Q_h(u,p) - \tilde{p}_h, \Psi)| \\ & \leq c\left(\|u - R_h(u,p)\|_1 + \|p - Q_h(u,p)\|_0\right)\left(\|\Phi - \Phi_h\|_1 + \|\Psi - \Psi_h\|_0\right) \\ & \qquad + |G(Q_h(u,p) - p + \eta, \Psi)| \\ & \leq ch^2\left(\|u\|_2 + \|p\|_1\right)\left(\|\Phi\|_2 + \|\Psi\|_1\right) + chG^{1/2}(\eta,\eta)\|\Psi\|_1. \end{aligned}$$

Then, combining the above estimates with (6.7), we see that

$$\begin{aligned} \|e\|_0^2 = & \frac{d}{dt}(e, \Phi) + ch^2(\|u_t\|_0 + \|u_{ht}\|_0)(\|\Phi\|_2 + \|\Psi\|_1) + chG^{1/2}(\eta,\eta)\|\Psi\|_1 \\ & + ch^2\left(\|u\|_2 + \|p\|_1\right)\left(\|\Phi\|_2 + \|\Psi\|_1\right) + ch^2\|f\|_1(\|\Phi\|_1 + h\|\Phi\|_2). \end{aligned} \tag{6.9}$$

Integrating the above equation from 0 to T gives

(6.10)

$$\begin{aligned} & \int_0^T \|e(s)\|_0^2 ds = -(e(0), \Phi(0)) \\ & + ch^2 \left(\int_0^T \left(\|u_t\|_0^2 + \|u_{ht}\|_0^2 + \|u\|_2^2 + \|p\|_1^2\right) ds\right)^{1/2} \left(\int_0^T \left(\|\Phi\|_2^2 + \|\Psi\|_1^2\right) ds\right)^{1/2} \\ & + ch \left(\int_0^T G(\eta,\eta) ds\right)^{1/2} \left(\int_0^T \|\Psi\|_1^2 ds\right)^{1/2} \\ & + ch^2 \left(\int_0^T \|f\|_1^2 ds\right)^{1/2} \left(\int_0^T (\|\Phi\|_1^2 + h^2\|\Psi\|_2^2) ds\right)^{1/2}. \end{aligned}$$

In addition, by the definition of R_h, we have

$$|(e(0), \Phi(0)| = |(u_0 - R_h(u_0, p_0), \Phi(0))| \leq ch^2(\|u_0\|_2 + \|p_0\|_1)\|\Phi(0)\|_1. \tag{6.11}$$

Combining (6.11) with (6.10) and using Lemma 6.1 with $g = e$ complete the proof of (6.3). #

Lemma 6.3. *Under the assumptions of Lemma 6.2, it holds that, for $t \in [0,T]$,*

$$\tau^{1/2}(t)\|u(t) - u_h(t)\|_0 \le ch^2. \tag{6.12}$$

Proof. Taking $(v_h, q_h) = (e_h, \eta_h) = (R_h(u,p) - u_h, Q_h(u,p) - p_h)$ in (5.12) and using (3.2), we see that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|e_h\|_0^2 + \nu\|e_h\|_1^2 + G(\eta_h,\eta_h) &= (f, e_h - \Gamma_h e_h) - (u_t - R_h(u,p)_t, e_h)\\ &= (f - \Gamma_h f, e_h - \Gamma_h e_h) - (u_t - R_h(u,p)_t, e_h)\\ &\le ch^4\frac{1}{2\nu}\|f\|_1^2 + \frac{\nu}{2}\|e_h\|_1^2 + \|E_t\|_0\|e_h\|_0, \end{aligned} \tag{6.13}$$

so

$$\frac{d}{dt}\|e_h\|_0^2 + \nu\|e_h\|_1^2 + G(\eta_h,\eta_h) \le ch^4\|f\|_1^2 + c\|E_t\|_0\|e_h\|_0. \tag{6.14}$$

Multiplying (6.14) by $\tau(t)$, integrating from 0 to t, and using Lemmas 2.2 and 6.2, we obtain

$$\begin{aligned}&\tau(t)\|e_h(t)\|_0^2 + \int_0^t \tau(s)\left(\nu\|e_h\|_1^2 + G(\eta_h,\eta_h)\right)ds\\ &\le c\int_0^t \|e_h\|_0^2 ds + ch^4\int_0^t \tau(s)\|f\|_1^2 ds\\ &\quad + ch^2\left(\int_0^t \tau(s)(\|u_t\|_2^2 + \|p_t\|_1^2)ds\right)^{1/2}\left(\int_0^t \|e_h\|_0^2 ds\right)^{1/2},\end{aligned}$$

which completes the proof. #

The next theorem follows from Lemma 6.3 and Theorem 5.6.

Theorem 6.4. *Under the assumptions of Lemma 6.2, it holds that, for $t \in [0,T]$,*

$$\|u(t) - \tilde{u}_h(t)\|_0 + h\|u(t) - \tilde{u}_h(t)\|_1 + h\|p(t) - \tilde{p}_h(t)\|_0 \le c\tau^{-1/2}(t)h^2. \tag{6.15}$$

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