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DIMENSION SPLITTING METHOD FOR 3D ROTATING COMPRESSIBLE NAVIER-STOKES EQUATIONS IN THE TURBOMACHINERY

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Dedicated to Professor Roland Glowinski on the occasion of his 70th birthday

Abstract. In this paper, we propose a dimension splitting method for Navier-Stokes equations(NSEs). The main idea is as follows. The domain of flow in 3D is decomposed into several thin layers. In each layer, The 3D NSEs can be represented as the sum of a membrane operator and a normal (bending) operator on the boundary of layer. And The Euler central difference is used to approximate the bending operator. When restricting the 3D NSEs on the boundary in each layer, we obtain a series of two dimensional-three components NSEs (called as 2D-3C NSEs). Then we construct an approximate solution of 3D NSES by solutions of those 2D-3C NSEs.

Key Words. 2D Manifold, Semi-Geodesic Coordinate Navier-Stokes Equations, Dimension Splitting Method.

1. Introduction

In [1, 2], the authors studied two dimensional flow on the stream surface, derived a nonlinear boundary value problem satisfied by stream function defined on the stream surface, and studied its finite element approximation. In [3, 4], Kaitai Li propose a dimensional splitting method for the linearly elastic shell based on differential geometry and tensor analysis. In this paper we will use classical tensor calculation to propose a new method , called "dimensional splitting method" for 3D rotating NSEs (compressible or incompressible).

The main idea is that, a 3D flow domain Ω bounded by four 2D-surfaces is decomposed into several thin layers Ω^i_{i-1} bounded by 2D surfaces $\Im_i,\ i=1,\ 2,\ \cdots,\ m$. 3D rotating Navier-Stokes operators in thin layer $\Omega^i_{i-1}\cup\Omega^{i+1}_i$ under local semi-geodesic coordinate based on the surface $\Im_i$ can be represented into the sum of a membrane operator on $\Im_i$ and a normal (bending) operator to $\Im$, then applying Euler central difference approximate bending operator. Then we obtain a restriction of 3D rotating NSEs on the $\Im_i$, that is a three components-two dimensional NSEs (called 2D-3C NSEs). Solving 2D-3C NSEs on $\Im_i,\ i=1,\ \cdots,\ m$ by parallel algorithms and reiterating until convergence , we can obtain approximate solution of 3D rotating NSEs. It is obvious that the method is different from the classical domain decomposition method because we only solve a two dimensional problem in each sub-domain(stream surface layer), instead of solving a 3D problem, and the 3D domain is decomposed into sub-domains by two dimensional manifold instead

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of flat plane. In addition , this paper provide three methods to solve 2D-3C NSE, those are artificial viscous method, streamline-FEM and stream functions methods.

The contents are organized as following : provide the mathematical description of the blade's surface in section 1; a domain partition's method and rotating NSEs in semi-geodesic coordinate based on two dimensional manifold $\Im$ in section 2; a 2D-3C NSEs-a restriction of 3D rotating Navier-Stokes equation to $\Im$ in section 4; provide a Korn's inequality on the $\Im$ in section 5; prove the existence of solution to corresponding variational formulation in section 6.

2. Geometry of the Channel in the Impeller and Navier-Stokes Equations

Let us consider the geometry of the channel Ω_ε bounded by two blade's surfaces $\Gamma_s^+,\ \Gamma_s^-$ and top- and bottom- surfaces $\Gamma_t,\ \Gamma_b$ in a impeller. Let $D\subset\Re^2$ simply-connected open subset of $\Re^2$, $\mathbf{E}$ denotes a three-dimensional Euclidean space. The surface of blade is a two dimensional manifold $\Im$ which is a smooth injective immersion $\vec{R}\in\ \mathbf{C}^3(D;\mathbf{E}^3)$:

$$D=\{(z,\ r)\}\subset R^2\Rightarrow R^3,\ \vec{R}(z,\ r)=r\vec{e}_r+r\Theta(z,\ r)\vec{e}_\theta+z\vec{k}, \tag{2. 1}$$

where $(\vec{e}_r,\ \vec{e}_\theta,\ \vec{k})$ are base vectors of cylindrical coordinate system rotating with the impeller and $(x^1=z,\ x^2=r)$ are the parameters describing the surface $\Im$ of blade as a submanifold embedding into $\mathbf{E}^3$, are also usually called Gaussian coordinate system on $\Im$.

In this case the Riemannian metric tensors of manifold $\Im$ are given by

$$\begin{cases} a_{\alpha\beta}=\frac{\partial\vec{R}}{\partial x^\alpha}\frac{\partial\vec{R}}{\partial x^\beta}=\frac{\partial r}{\partial x^\alpha}\frac{\partial r}{\partial x^\beta}+r^2\Theta_\alpha\Theta_\beta+\frac{\partial z}{\partial x^\alpha}\frac{\partial z}{\partial x^\beta}=\delta_{\alpha\beta}+r^2\Theta_\alpha\Theta_\beta,\\ a=\det(a_{\alpha\beta})=1+r^2(\Theta_1^2+\Theta_2^2),\end{cases} \tag{2. 2}$$

where

$$\Theta_\alpha=\frac{\partial\Theta}{\partial x^\alpha},$$

$b_{\alpha\beta}$ second fundamental form of the surface $\Im$

$$b_{\alpha\beta}=\frac{\partial^2\vec{R}}{\partial x^\alpha x^\beta}\left(\frac{\partial\vec{R}}{\partial x^1}\times\frac{\partial\vec{R}}{\partial x^2}\right)/\sqrt{a}=\frac{1}{\sqrt{a}}\begin{vmatrix} x_{\alpha\beta} & y_{\alpha\beta} & z_{\alpha\beta}\\ x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\end{vmatrix},$$

where $(x,\ y,\ z)$ denote Cartan coordinate, and $x_\alpha=\frac{\partial x}{\partial x^\alpha},\ y_\alpha=\frac{\partial y}{\partial x^\alpha},\ x_{\alpha\beta}=\frac{\partial^2 x}{\partial x^\alpha x^\beta},\ \cdots,$ Therefore

$$\begin{cases} b_{11}=\frac{1}{\sqrt{a}}(x^2\Theta_{11}+\Theta_2(a-1)),\\ b_{12}=\frac{1}{\sqrt{a}}(x^2\Theta_{12}+\Theta_1 a)=b_{21},\\ b_{22}=\frac{1}{\sqrt{a}}(x^2\Theta_{22}+\Theta_2(a+1)),\\ b=\det(b_{\alpha\beta})=b_{11}b_{22}-b_{12}^2.\end{cases} \tag{2. 3}$$

The mean curvature H and Gaussian curvature K are given by

$$2H=a^{\alpha\beta}b_{\alpha\beta}=\frac{1}{\sqrt{a}}\big(a_{11}b_{22}-2a_{12}b_{12}+a_{22}b_{22}\big),\qquad K=\frac{b}{a}. \tag{2. 4}$$

It is clear that

$$(a_{\alpha\beta})\in\mathbf{C}^2(D;\mathcal{S}^2_>),\quad (b_{\alpha\beta})\in\ \mathbf{C}^2(D;\mathcal{S}^2)$$

are two matrix fields where $\mathcal{S}^2$ and $\mathcal{S}^2_>$ denote the sets of all symmetric matrices of order two , and of all symmetric, positive definite matrices. $(a_{\alpha\beta}):D\to\mathcal{S}^2_>$ and $(b_{\alpha\beta}):D\to\mathcal{S}^2$ are the covariant components of the first and second fundamental forms of the surface $\Im$.

As well known that the geometry of $\Im$ is completely determined by $(a_{\alpha\beta})$, $(b_{\alpha\beta})$ in the following meaning. We recall that $\mathcal{O}^3$ denotes the set of all orthogonal matrices Q order three and that $\mathcal{O}^3_+ = \{Q \in \mathcal{O}^3; \det(Q) = 1\}$ denotes the set of all proper orthogonal matrices of order three. $\mathbf{J}_+(x) = \mathbf{c} + Qo\mathbf{x}$ is a proper isometry of $\mathbf{E}^3$: $\mathbf{E}^3 \to \mathbf{E}^3$ with $\mathbf{c} \in \mathbf{E}^3,\ Q \in \mathcal{O}^3_+$.

Theorem 2. 1([9]) Two immersions $\vec{R} \in C^1(D; \mathbf{E}^3)$ and $\widetilde{\vec{R}} \in C^1(D; \mathbf{E}^3)$ share the same fundamental forms $(a_{\alpha\beta})$ and $(b_{\alpha\beta})$ over an open connected subset D of $\Re^3$ if only if

$$\widetilde{\vec{R}} = \mathbf{J}_+ o\vec{R}, \quad \text{where} \quad \mathbf{J}_+ \quad \text{is a proper isometry of} \quad \mathbf{E}^3, \tag{2. 5}$$

Furthermore, If two matrices fields $(a_{\alpha\beta}) \in C^2(D; \mathcal{S}^2_>)$ and $(b_{\alpha\beta}) \in C^2(D; \mathcal{S}^2)$ satisfy Gauss and Godazzi equations in D

$$\begin{array}{l}
\partial_\beta \Gamma_{\alpha\sigma,\ \tau} - \partial_\sigma \Gamma_{\alpha\beta,\ \tau} + \Gamma^\mu_{\alpha\beta}\Gamma_{\sigma\tau,\ \mu} - \Gamma^\mu_{\alpha\sigma}\Gamma_{\beta\tau,\ \mu} = b_{\alpha\sigma}b_{\beta\tau} - b_{\alpha\beta}b_{\sigma\tau},\\
\partial_\beta b_{\alpha\sigma} - \partial_\sigma b_{\alpha\beta} + \Gamma^\mu_{\alpha\sigma} b_{\beta\mu} - \Gamma^\mu_{\alpha\beta} b_{\sigma\mu} = 0,
\end{array}$$

where

$$\begin{array}{l}
\Gamma_{\alpha\beta,\ \tau} = \frac{1}{2}(\partial_\alpha a_{\alpha\tau} + \partial_\alpha a_{\beta\tau} - \partial_\tau a_{\alpha\beta}),\\
\Gamma^\sigma_{\alpha\beta} = a^{\sigma\tau}\Gamma_{\alpha\beta,\ \tau}, \quad \text{where} \quad (a^{\alpha\beta}) = (a_{\alpha\beta})^{-1},
\end{array}$$

Then there exist an immersion $\vec{R} \in\ C^3(D; \mathbf{E}^3)$ such that

$$a_{\alpha\beta} = \partial_\alpha \vec{R}\partial_\beta \vec{R}, \quad b_{\alpha\beta} = \partial^2_{\alpha\beta}\vec{R} \cdot \{\frac{\partial_1 \vec{R} \times \partial_2 \vec{R}}{|\partial_1 \vec{R} \times \partial_2 \vec{R}|}\}.$$

Lemma 2. 1([2]) Third fundamental tensor is not independent of first and second fundamental tensors $a_{\alpha\beta},\ b_{\alpha\beta}$ they have following relationships

$$\left\{\begin{array}{l}
\varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}b_{\alpha\beta}b_{\lambda\sigma} = 2K, \quad b_{\alpha\beta}b_{\lambda\sigma} - b_{\alpha\lambda}b_{\beta\sigma} = K\varepsilon_{\alpha\sigma}\varepsilon_{\beta\lambda},\\
Ka_{\alpha\beta} - 2Hb_{\alpha\beta} + c_{\alpha\beta} = 0, \quad Ka^{\alpha\beta} - 2Hb^{\alpha\beta} + c^{\alpha\beta} = 0,\\
a^{\alpha\beta} - 2Hb^{\hat{\alpha\beta}} + Kc^{\hat{\alpha\beta}} = 0,
\end{array}\right. \tag{2. 6}$$

$$\left\{\begin{array}{l}
K\hat{b}^{\alpha\beta} = 2Ha^{\alpha\beta} - b^{\alpha\beta}, \quad K^2\hat{c}^{\alpha\beta} = (4H^2 - K)a^{\alpha\beta} - 2Hb^{\alpha\beta},\\
K\hat{b}^{\alpha\beta} = \varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}b_{\lambda\sigma}, \quad K^2\hat{c}^{\alpha\beta} = \varepsilon^{\alpha\lambda}\varepsilon^{\beta\sigma}c_{\lambda\sigma},
\end{array}\right. \tag{2. 7}$$

where $\hat{b}^{\alpha\beta}, \hat{c}^{\alpha\beta}$ are inverse matrixes of $b_{\alpha\beta}, c_{\alpha\beta}$, respectively. Following formulae are useful throughout this paper

$$\left\{\begin{array}{l}
\widehat{b}^{\alpha\beta} = \widehat{c}^{\alpha\lambda}b^\beta_\lambda, \quad \widehat{c}^{\alpha\beta} = \widehat{b}^{\alpha\lambda}\widehat{b}^\beta_\lambda, \quad b^\alpha_\beta = \widehat{b}^{\alpha\lambda}c_{\beta\lambda},\\
c_{\alpha\lambda}b^\lambda_\beta = -2HKa_{\alpha\beta} + (4H^2 - K)b_{\alpha\beta},\\
c_{\alpha\lambda}c^\lambda_\beta = -K(4H^2 - K)a_{\alpha\beta} + 2H(4H^2 - 2K)b_{\alpha\beta},\\
c^\alpha_\alpha = a^{\alpha\beta}c_{\alpha\beta} = b^{\alpha\beta}b_{\alpha\beta} = 4H^2 - 2K;\\
b^{\alpha\beta}c_{\alpha\beta} = 8H^3 - 6HK; \quad c^{\alpha\beta}c_{\alpha\beta} = 16H^4 - 16H^2K + 2K^2.
\end{array}\right. \tag{2. 8}$$

Assume that there are number N blades of an impeller . Then expansion angular of the channel between two successively blades is $2\varepsilon = \frac{2\pi}{N}$. The channel between two blade's is denoted by

$$\left\{\begin{array}{l}
\Omega_\varepsilon = \{(x^1 = z,\ x^2 = r) \in\ D,\ -\varepsilon + \Theta(x^1,\ x^2) \leq \theta \leq \varepsilon + \Theta(x^1,\ x^2)\},\\
\vec{R}(x^1,\ x^2,\ s) = x^2\vec{e}_r + x^2(\varepsilon s + \Theta(x^1,\ x^2))\vec{e}_\theta + x^1\vec{k} \in \Omega_\varepsilon,\\
\quad \forall (x^1,\ x^2) \in D,\ s \in [-1,\ 1].
\end{array}\right. \tag{2. 9}$$

Let us make variable transformation

$$r = x^2, \quad \theta = \varepsilon s + \Theta, \quad z = x^1, \quad -1 \leq s \leq 1, \tag{2. 10}$$

$\forall\ s =$ constant means that it represents a 2D-manifold $\Im$, its geometric position is reached by angle εs of rotation. Take $(x^1,\ x^2,\ s)$ as new coordinates system: $x^1 = z, \quad x^2 = r, \quad s = \varepsilon^{-1}(\theta - \Theta)$. The channel Ω_ε becomes a cylindrical body $\Omega = \left\{(x^1,\ x^2) \in D,\ -1 \le s \le 1\right\} \subset R^3$. Jacobi determinate of the transformation is given by $J\left(\frac{\partial(r,\ \theta,\ z)}{\partial(x^1,\ x^2,\ s)}\right) = \varepsilon,$. It is clear that it is nonsingular

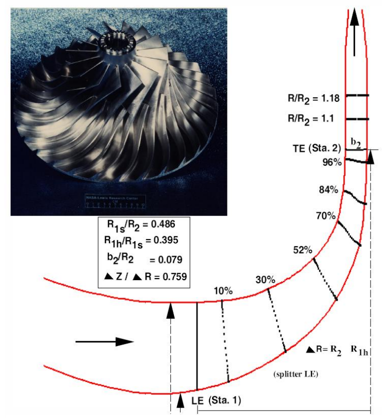


Fig. 1 and Fig 2. Blade and Channel Ω_ε and boundaries of projection at meridian plane

where $D = (x^1,\ x^2) \in \Re^2$:

$$\partial D = \gamma_0 \cup \gamma_1, \qquad \gamma_0 = \widehat{AB} \cup \widehat{CD}, \qquad \gamma_1 = \widehat{CB} \cup \widehat{DA},$$

there are four positive functions $\gamma_0(z),\ \widetilde{\gamma}_0(z),\ \gamma_1(z),\ \widetilde{\gamma}_1(z)$ such that

$$(2.\ 11) \qquad \begin{cases} r := x^2 = \gamma_0(x^1) = \gamma_0(z) \quad \text{on } \widehat{AB}, \quad x^2 = \widetilde{\gamma}_0(x^1) \quad \text{on } \widehat{CD} \\ r := x^2 = \gamma_1(x^1) = \gamma_1(z) \quad \text{on } \widehat{DA}, \quad x^2 = \widetilde{\gamma}_1(x^1) \quad \text{on } \widehat{BC}, \\ r_0 \le \gamma_0(z) \le r_1 \quad \text{on } \widehat{AB}, \quad r_1 < \widetilde{r}_0 \le \widetilde{\gamma}_0(z) \le \widetilde{r}_1 \quad \text{on } \widehat{CD}, \\ r_0 \le \gamma_1(z) \le r_1,\ \forall z_a \le z \le z_d, \quad \text{on } \widehat{DA}, \\ r_0 \le \widetilde{\gamma}_1(z) \le r_1,\ \forall z_b \le z \le z_c \quad \text{on } \widehat{BC}. \end{cases}$$

Assume that turbo-machinery flow in the impeller is stationary flow. We employ rotating coordinate system with same angular velocity ω as impeller. The governing equations are Compressible Navier-Stokes equations

$$(2.\ 12) \qquad \begin{cases} \text{Continuous Equation} \quad \operatorname{div}(\rho w) = 0, \\ \text{Dynamical Equations} \quad -\nabla \cdot \sigma + \operatorname{div}(\rho w w) + 2\rho\omega \times w \\ \qquad\qquad = \rho\omega \times (\omega \times R) + f, \\ \text{Energy Equation} \quad \operatorname{div}(\rho E w) + p\operatorname{div} w - \operatorname{div}(\kappa_0 \operatorname{grad} T) - \Phi = 0, \\ \text{State Equation} \quad p = p(\rho,\ T), \end{cases}$$

where w is relative velocity of fluid, ω angular velocity of the rotator, ρ density of the fluid, p pressure, $E = C_v T$ inner energy in a unite volume, C_v specific heat at constant volume, and μ viscosity, T temperature, κ_0 the coefficient of heat conductivity, $2\omega \times w$ Coriolis force, $F = \frac{f}{\varrho} + \omega \times (\omega \times R)$ volume force including centrifugal force, stress tensor σ and dissipative function Φ are given by

$$(2.\ 13) \qquad \begin{cases} \sigma^{ij}(w) = (-p + \frac{2}{3}\mu \operatorname{div} w) g^{ij} + 2\mu e^{ij}(w) = -g^{ij} p + A^{ijkm} e_{km}(w), \\ \Phi = 2\mu e^{ij}(w) e_{ij}(w) + \frac{2}{3}\mu(\operatorname{div} w)^2, \quad e^{ij}(w) = \frac{1}{2}(\nabla^i w^j + \nabla^j w^i), \end{cases}$$

where

$$\begin{aligned}&\nabla_i w^j = \tfrac{\partial w^j}{\partial x^i} + \Gamma^j_{im} w^m, \qquad \nabla^i w^j = g^{ik}\nabla_k w^j,\\ &A^{ijkm} = 2\mu g^{ik} g^{jm} + \tfrac{\mu}{3} g^{ij} g^{km}.\end{aligned} \tag{2. 14}$$

are covariant derivative and contravariante derivative and viscosity tensor. Γ^i_{jk} is Christoffel symbolism in coordinates x in $\Re^3$. The $e_{ij}(w)$ is the deformation rate tensor of the velocity w.

$$e_{ij}(w) = \frac{1}{2}(\nabla_i w_j + \nabla_j w_i) = \frac{1}{2}(g_{jk}\nabla_i w^k + g_{ik}\nabla_j w^k). \tag{2. 15}$$

In sequence we employ entropy equation in stead of energy equation (for the polytropic gas state equation)

$$\begin{cases} w^i \nabla_i S - \frac{1}{WT}(\frac{\kappa}{\varrho}\Delta T + \Phi/\varrho) = 0,\\ W = g_{ij} w^i w^j \quad \text{module of velocity[2]},\\ S = R\log(T^{\frac{\gamma}{\gamma-1}}/p), \qquad p = A\rho^\gamma, \qquad , \end{cases} \tag{2. 16}$$

where S is the entropy, $1 \le \gamma \le 5/3$ is heat specific radio, A is a constant.

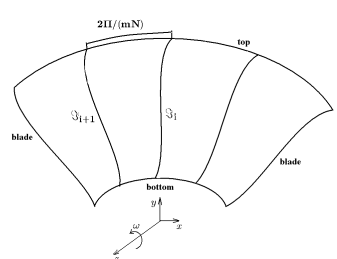
Let Γ_{in} entrance boundary, Γ_{out} exit boundary, $\Gamma_s = \Gamma_s^+ \cup \Gamma_s^-$ positive and negative surfaces of the blade, Γ_t top boundary , Γ_b bottom boundary:

$$\partial\Omega_\varepsilon = \Gamma = \Gamma_1 \cup \Gamma_0, \qquad \Gamma_1 = \Gamma_{in} \cup \Gamma_{out}, \qquad \Gamma_0 = \Gamma_s^+ \cup \Gamma_s^- \cup \Gamma_t \cup \Gamma_b.$$

Then boundary conditions are

$$\begin{cases} w|_{\Gamma_s} = 0, \qquad w|_{\Gamma_b} = 0, \qquad w|_{\Gamma_t} = 0,\\ \sigma\cdot n|_{\Gamma_{in}} = \vec{g}_{in}, \qquad \sigma\cdot n|_{\Gamma_{out}} = \vec{g}_{out},\\ \dfrac{\partial T}{\partial n} + \lambda(T - T_0) = 0, \quad \text{on} \quad \Gamma_t \cup \Gamma_b \cup \Gamma_S \cup \Gamma_{out},\\ T|_{\Gamma_{in}} = T_{in}, \qquad \text{where} \quad \lambda \ge 0 \end{cases} \tag{2. 17}$$

3. Domain Partition and Navier-Stokes Equations in Semi-geodesic coordinate



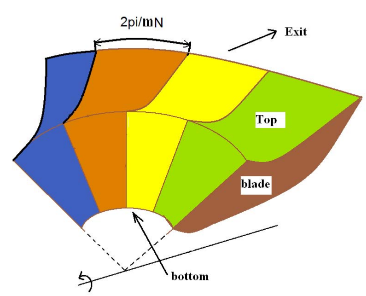


Fig. 3 Section of the Channel and Angular Expansion, Fig. 4 Domain Decomposition

Let consider domain $\Omega = \{(x^1,\ x^2) \in\ D,\ -1 \le s \le 1\}$ decomposition. Making partition on

$$[-1,\ 1] = \{s_0 = -1,\ s_{i+1} = s_i + \triangle s,\ i = 0,\ 1,\ \cdots,\ m,\ s_m = 1\}$$

it is obvious that each $s = s_i$ corresponds a 2D manifold $\Im_i$. By theorem 2. 1 they have the same geometry as the surface of blade. $\Im_i,\ i = 0,\ 1,\ 2,\ \cdots,\ m$ decompose Ω_ε into m sub-domain (is called thin flow's layer and denote byΩ^i_{i-1}, see Fig. 4). Flow layer Ω^{i+1}_i is bounded by $\Im_i,\ \Im_{i+1}$ and $\partial\Omega_\varepsilon$. Assume that Ω_ε consists of the flow layers of number m. In the neighborhood $\{\Omega^i_{i-1} \cup \Omega^{i+1}_i\}$ of $\Im_i$ we establish semi-geodesic coordinate (the abbreviations S-coordinates) $(x^\alpha,\ \xi)$ based on $\Im_i$ and

$g_{\alpha\beta}(x,\ \xi)$ denote metric tensor of $\mathbf{E}^3$ in this coordinate. Then we find relationship between g_{ij} and $a_{\alpha\beta}$(see [2]):

$$\left\{\begin{array}{l} g_{\alpha\beta}(x,\ \xi)=a_{\alpha\beta}(x)-2\xi b_{\alpha\beta}(x)+\xi^2 c_{\alpha\beta}(x);\\ g_{\alpha 3}(x,\ \xi)=g_{3\alpha}(x,\ \xi)=0,\quad g_{33}(x,\ \xi)=1,\\ g^{\alpha\beta}(x,\ \xi)=\kappa^{-2}(a^{\alpha\beta}(x)-2K\hat{b}^{\alpha\beta}(x)\xi+K^2\xi^2\hat{c}^{\alpha\beta}(x)),\\ g^{3\alpha}(x,\ \xi)=g^{\alpha 3}(x,\ \xi)=0,\quad g^{33}(x,\ \xi)=1,\\ g(x,\ \xi)=\det(g_{ij})=\kappa^2(\xi)a(x);\quad \kappa(\xi)=1-2H\xi+K\xi^2. \end{array}\right. \tag{3.1}$$

where the third fundamental form is given by $(c_{\alpha\beta}=a^{\lambda\sigma}b_{\alpha\lambda}b_{\beta\sigma})$ and $(\hat{b}^{\alpha\beta})=(b_{\alpha\beta})^{-1}$, and $(\hat{c}^{\alpha\beta})=(c_{\alpha\beta})^{-1}$.

Let $h:=\Delta\xi$ denotes the distance along the normal from $\Im_i$ to $\Im_{i+1}$, then

$$h=\triangle\xi=\triangle s\ \cdot r\varepsilon\sqrt{a}=(s_{i+1}-s_i)r\varepsilon\sqrt{a}. \tag{3. 2}$$

Let Γ^i_{jk}, ∇_i, and $\overset{*}{\Gamma}{}^{\alpha}{}_{\beta\gamma}$, $\overset{*}{\nabla}_\alpha$ denote Christoffel symbols and covariant derivative in E^3 and on $\Im$ respectively,

$$\left\{\begin{array}{l} \Gamma_{ij,\ k}=\frac{1}{2}(\frac{\partial g_{ik}}{\partial x^j}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^k}),\quad \Gamma^m_{ij}=g^{mk}\Gamma_{ij,\ k},\\ \overset{*}{\Gamma}_{\alpha\beta,\ \lambda}=\frac{1}{2}(\frac{\partial a_{\alpha\lambda}}{\partial x^\beta}+\frac{\partial a_{\beta\lambda}}{\partial x^\alpha}-\frac{\partial a_{\alpha\beta}}{\partial x^\lambda}),\quad \overset{*}{\Gamma}{}^{\lambda}{}_{\alpha\beta}=a^{\lambda\sigma}\ \overset{*}{\Gamma}_{\alpha\beta,\ \sigma}, \end{array}\right.$$

For the tensors of two and one order, covariant derivatives are given by

$$\left\{\begin{array}{l} \nabla_i u^j=\frac{\partial u^j}{\partial x^i}+\Gamma^j_{ik}u^k,\quad \overset{*}{\nabla}_\alpha\ u^\beta=\frac{\partial u^\beta}{\partial x^\alpha}+\overset{*}{\Gamma}{}^{\beta}{}_{\alpha\lambda}\ u^\lambda,\\ \mathrm{div}u=\nabla_i u^i,\quad \overset{*}{\mathrm{div}}\ u=\overset{*}{\nabla}_\alpha\ u^\alpha,\\ \nabla_k e^{ij}=\frac{\partial e^{ij}}{\partial x^k}+\Gamma^i_{km}e^{mj}+\Gamma^j_{km}e^{im},\\ \overset{*}{\nabla}_\lambda\ e^{\alpha\beta}=\frac{\partial e^{\alpha\beta}}{\partial x^\lambda}+\overset{*}{\Gamma}{}^{\alpha}{}_{\lambda\sigma}\ e^{\sigma\beta}+\overset{*}{\Gamma}{}^{\beta}{}_{\lambda\sigma}\ e^{\alpha\sigma}, \end{array}\right. \tag{3. 3}$$

Then we have

Lemma 3. 1([2]) Under S-coordinate system, Christoffel symbols $(\Gamma^k_{ij},\ \Gamma_{ij,\ k})$ in E^3 can be expressed in means of Christoffel symbols of $\Im$ $(\overset{*}{\Gamma}{}^{\alpha}{}_{\beta\lambda},\ \overset{*}{\Gamma}_{\alpha\beta,\ \lambda})$

$$\left\{\begin{array}{l} \Gamma_{\alpha\beta,\ \lambda}=g_{\lambda\sigma}\ \overset{*}{\Gamma}{}^{\sigma}{}_{\alpha\beta}\ +\xi(H\xi-1)\ \overset{*}{\nabla}_\lambda\ b_{\alpha\beta}\\ \qquad +2\xi(H\xi-1)[\overset{*}{\Gamma}{}^{\sigma}{}_{\beta\lambda}\ b_{\sigma\alpha}-b_{\lambda\sigma}\ \overset{*}{\Gamma}{}^{\sigma}{}_{\alpha\beta}],\\ \Gamma_{\alpha\beta,\ 3}=-J_{\alpha\beta}(\xi),\quad \Gamma_{\alpha 3,\ \beta}=\Gamma_{3\alpha,\ \beta}=J_{\alpha\beta}(\xi),\\ \Gamma_{ij,\ k}=0,\ \text{other case}, \end{array}\right. \tag{3. 4}$$

$$\left\{\begin{array}{l} \Gamma^\lambda_{\alpha\beta}=\overset{*}{\Gamma}{}^{\lambda}{}_{\alpha\beta}\ +\theta^{-1}R^\lambda_{\alpha\beta},\quad \Gamma^\alpha_{\beta 3}=\Gamma^\alpha_{3\beta}=\theta^{-1}I^\alpha_\beta,\quad \Gamma^3_{\alpha\beta}=J_{\alpha\beta}\\ \Gamma^3_{33}=\Gamma^3_{3\beta}=\Gamma^3_{\beta 3}=\Gamma^\alpha_{33}=0 \end{array}\right. \tag{3. 5}$$

and where

$$\begin{array}{l} R^\alpha_{\beta\lambda}=(2H\xi^2-\xi)\ \overset{*}{\nabla}_\lambda\ b^\alpha_\beta-\xi^2 b^\alpha_\mu\ \overset{*}{\nabla}_\lambda\ b^\mu_\beta,\\ I^\alpha_\beta=-b^\alpha_\beta+K\xi\delta^\alpha_\beta,\quad J_{\alpha\beta}=b_{\alpha\beta}-\xi c_{\alpha\beta} \end{array} \tag{3. 6}$$

Lemma 3. 2([2]) Under S-coordinate system covariant derivative of a vector $\vec{u}$ in E^3 can be expressed by covariant derivative of its components on the tangent space at $\Im$. Furthermore it is a rational function of transversal variable ξ

$$\text{(3. 7)}\quad \begin{cases} \nabla_\alpha u^\beta = \overset{*}{\nabla}_\alpha u^\beta + \theta^{-1}(I^\beta_\alpha u^3 + R^\beta_{\alpha\lambda} u^\lambda), \quad \nabla_3 u^3 = \frac{\partial u^3}{\partial \xi}; \\ \nabla_3 u^\beta = \frac{\partial u^\beta}{\partial \xi} + \theta^{-1} I^\beta_\lambda u^\lambda; \quad \nabla_\alpha u^3 = \overset{*}{\nabla}_\alpha u^3 + J_{\alpha\lambda} u^\lambda; \\ \operatorname{div} u = \overset{*}{\operatorname{div}}\, u + \frac{\partial u^3}{\partial \xi} \\ \qquad + \kappa^{-1}[-2Hu^3 + (2Ku^3 - 2u^\alpha \overset{*}{\nabla}_\alpha H)\xi + u^\alpha \overset{*}{\nabla}_\alpha K\xi^2], \end{cases}$$

where and in sequence we consider third component u^3 of $\vec{u}$ as scale function on the 2D manifold $\Im$.

Taking into account of

$$\text{(3. 8)}\quad \nabla_i g_{jk} = 0, \quad \overset{*}{\nabla}_\alpha a_{\beta\sigma} = 0,$$

which will frequently be used throughout this paper and $u^k = g^{kj} u_j$, by using contravariant component of vector u instead of covariant component of vector, the strain rate tensor of velocity on $\Im$ is defined by

$$\text{(3. 9)}\quad \begin{array}{l} \overset{*}{e}_{\alpha\beta}(u) = \frac{1}{2}(\overset{*}{\nabla}_\alpha u_\beta + \overset{*}{\nabla}_\beta u_\alpha) = \frac{1}{2}(a_{\beta\lambda} \overset{*}{\nabla}_\alpha u^\lambda + a_{\alpha\lambda} \overset{*}{\nabla}_\beta u^\lambda), \\ \overset{*}{e}{}^{\alpha\beta}(u) = a^{\alpha\lambda} a^{\beta\sigma} \overset{*}{e}_{\lambda\sigma}(u) \end{array}$$

Lemma 3. 3([2]) Under S-coordinate system the deformation rate tensor of the velocity u are the polynomials of two degree with respect to ξ

$$\text{(3. 10)}\quad e_{ij}(u) = \gamma_{ij}(u) + \overset{1}{\gamma}_{ij}(u)\xi + \overset{2}{\gamma}_{ij}(u)\xi^2,$$

where

$$\text{(3. 11)}\quad \begin{cases} \gamma_{\alpha\beta}(u) = \overset{*}{e}_{\alpha\beta}(u) - b_{\alpha\beta} u^3, \quad \overset{1}{\gamma}_{\alpha\beta}(u) = \overset{1}{e}_{\alpha\beta}(u) + c_{\alpha\beta} u^3 - \overset{*}{\nabla}_\lambda b_{\alpha\beta} u^\lambda, \\ \overset{2}{\gamma}_{\alpha\beta}(u) = \overset{2}{e}_{\alpha\beta}(u) + \frac{1}{2} \overset{*}{\nabla}_\lambda c_{\alpha\beta} u^\lambda, \quad \gamma_{3\alpha}(u) = \frac{1}{2}(a_{\alpha\beta} \frac{\partial u^\beta}{\partial \xi} + \overset{*}{\nabla}_\alpha u^3), \\ \overset{1}{\gamma}_{\alpha 3}(u) = -b_{\alpha\beta} \frac{\partial u^\beta}{\partial \xi}, \quad \overset{2}{\gamma}_{\alpha 3}(u) = \frac{1}{2} c_{\alpha\beta} \frac{\partial u^\beta}{\partial \xi}, \\ \gamma_{33}(u) = \frac{\partial u^3}{\partial \xi}, \quad \overset{1}{\gamma}_{33}(u) = \overset{2}{\gamma}_{33}(u) = 0. \end{cases}$$

where the strain rate tensors on the two-dimensional manifold $\Im$ are given as :

$$\text{(3. 12)}\quad \begin{cases} \overset{*}{e}_{\alpha\beta}(u) = \frac{1}{2}(a_{\alpha\lambda}\delta^\sigma_\beta + a_{\beta\lambda}\delta^\sigma_\alpha) \overset{*}{\nabla}_\sigma u^\lambda; \\ \overset{1}{e}_{\alpha\beta}(u) = -(b_{\alpha\lambda}\delta^\sigma_\beta + b_{\beta\lambda}\delta^\sigma_\alpha) \overset{*}{\nabla}_\sigma u^\lambda; \\ \overset{2}{e}_{\alpha\beta}(u) = \frac{1}{2}(c_{\alpha\sigma}\delta^\lambda_\beta + c_{\beta\sigma}\delta^\lambda_\sigma) \overset{*}{\nabla}_\lambda u^\sigma; \end{cases}$$

Lemma 3. 4 The divergence of the strain rate tensor $e(w)$ of the velocity in S-coordinate is given by

$$\operatorname{div}(e(w)) = \{\nabla_j e^{ij}(w),\ i = 1,\ 2,\ 3\},$$

$$\text{(3. 13)}\quad \begin{aligned} \nabla_j e^{\alpha j}(w) = \ & g^{\alpha\beta} g^{\lambda\sigma} \overset{*}{\nabla}_\lambda e_{\beta\sigma}(w) + [\overset{*}{\nabla}_\lambda (g^{\alpha\beta} g^{\lambda\sigma}) \\ & + \kappa^{-1}(R^\alpha_{\lambda\nu}\delta^\lambda_\mu + R^\lambda_{\lambda\nu}\delta^\alpha_\mu) g^{\nu\beta} g^{\mu\sigma}] e_{\beta\sigma}(w) \\ & + \frac{1}{2}(\kappa^{-1}(I^\alpha_\nu g^{\nu\sigma} + I^\lambda_\lambda g^{\alpha\sigma}) + \partial_\xi g^{\alpha\sigma}) \overset{*}{\nabla}_\sigma w^3 \\ & + \frac{1}{2}[(\kappa^{-1}(I^\alpha_\nu g^{\nu\sigma} + I^\lambda_\lambda g^{\alpha\sigma}) + \partial_\xi (g^{\alpha\sigma} g_{\sigma\beta})] \frac{\partial w^\beta}{\partial \xi} \\ & + \frac{1}{2} g^{\alpha\beta} \overset{*}{\nabla}_\beta \frac{\partial w^3}{\partial \xi} + \frac{1}{2} \frac{\partial^2 w^\alpha}{\partial \xi^2}, \end{aligned}$$

$$\nabla_j e^{3j}(w) = \begin{array}{l} \frac{1}{2}g^{\lambda\sigma}\overset{*}{\nabla}_\lambda\overset{*}{\nabla}_\sigma w^3 + \frac{1}{2}(\overset{*}{\nabla}_\beta g^{\beta\sigma} + \kappa^{-1}R^\beta_{\beta\lambda}g^{\lambda\sigma})\overset{*}{\nabla}_\sigma w^3 \\ +g^{\lambda\nu}g^{\sigma\mu}J_{\lambda\sigma}e_{\nu\mu}(w) + \frac{\partial^2 w^3}{\partial\xi^2} + \kappa^{-1}I^\beta_\beta\frac{\partial w^3}{\partial\xi} \\ +\frac{1}{2}\partial_\xi \overset{*}{\operatorname{div}}\, w + \frac{1}{2}\kappa^{-1}R^\beta_{\beta\lambda}\frac{\partial w^\lambda}{\partial\xi}. \end{array} \tag{3. 14}$$

Proof the proof is omitted here.

In order to compute Coriolis force and centrifugal force we have to introduce permutation tensor in Euclid space E^3 and on 2D manifold $\Im$

$$\varepsilon_{ijk} = \begin{cases} \sqrt{g}, \\ -\sqrt{g}, \\ 0, \end{cases} \quad \varepsilon_{ijk} = \begin{cases} \frac{1}{\sqrt{g}}, & (i,\ j,\ k) \text{ is even permutation of } (1,\ 2,\ 3), \\ -\frac{1}{\sqrt{g}}, & (i,\ j,\ k) \text{ is odd permutation of } (1,\ 2,\ 3), \\ 0, & \text{otherwise}, \end{cases}$$

where $g = \det(g_{ij})$, g_{ij} is metric tensor of $\Re^3$. Similarly

$$\varepsilon_{\alpha\beta} = \begin{cases} \sqrt{a}, \\ -\sqrt{a}, \\ 0, \end{cases} \quad \varepsilon_{\alpha\beta} = \begin{cases} \frac{1}{\sqrt{a}}, & (\alpha,\ \beta) \text{ is even permutation of } (1,\ 2), \\ -\frac{1}{\sqrt{a}}, & (\alpha,\ \beta) \text{ is odd permutation of } (1,\ 2), \\ 0, & \text{otherwise}, \end{cases}$$

Since $\sqrt{g} = \kappa\sqrt{a}$ it is clear that

$$\varepsilon_{3\alpha\beta} = \kappa\varepsilon_{\alpha\beta}, \quad \varepsilon^{3\alpha\beta} = \kappa^{-1}\varepsilon^{\alpha\beta}. \tag{3. 15}$$

Let $\vec{R} = R^\alpha\vec{e}_\alpha + R^3\vec{n}$ denote the radius vector of the point $(x^\alpha,\ \xi)$.

Lemma 3. 5 Coriolis force, centrifugal force and angular velocity vector in semi-geodesic coordinate can be expressed as

$$\begin{cases} \vec{C}(\xi) = 2\vec{\omega}\times\vec{w} = C^\alpha(\xi)\vec{e}_\alpha + C^3(\xi)\vec{n}, \\ \vec{f}_c(\xi) = \vec{\omega}\times(\vec{\omega}\times\vec{R}) = f^\alpha_c(\xi)\vec{e}_\alpha + f^3_c(\xi)\vec{n}, \\ \vec{\omega}(\xi) = \omega\vec{k} = \omega^\alpha(\xi)\vec{e}_\alpha + \omega^3(\xi)\vec{n}, \quad \vec{R} = R^\alpha(\xi)\vec{e}_\alpha + R^3(\xi)\vec{n}, \end{cases} \tag{3. 16}$$

where

$$\begin{cases} \omega^\alpha(\xi) = \omega\vec{k}\vec{e^\alpha} = \omega\kappa^{-1}\vec{k}(a^{\alpha\sigma} - \xi K\widehat{b}^{\alpha\sigma})\vec{r}_\sigma = \omega\kappa^{-1}(a^{\alpha 1} - \xi K\widehat{b}^{\alpha 1}), \\ \omega^3(\xi) = -\frac{x^1\omega}{\sqrt{a}}\kappa\Theta_1(2+\Theta^2), \\ R^\alpha(\xi) = \kappa^{-1}(a^{\alpha\sigma} - \xi K\widehat{b}^{\alpha\sigma})(x^1\delta_{\alpha 1} + x^2(1+\Theta^2)\delta_{\sigma 2} + \Theta\Theta_\sigma(x^2)^2), \\ R^3(\xi) = \xi - \frac{\kappa}{\sqrt{g}}(2+\Theta^2)x^2x^\lambda\Theta_\lambda, \end{cases} \tag{3. 17}$$

$$\begin{cases} C^i(\xi) = 2\varepsilon^{ijk}\omega_j w_k = 2g^{ij}\varepsilon_{jkl}\omega^k w^l, \\ C^\alpha(\xi) = 2\kappa g^{\alpha\beta}\varepsilon_{\beta\lambda}(\omega^\lambda w^3 - \omega^3 w^\lambda) = C^\alpha_\beta(\xi)w^\beta + C^\alpha_3(\xi)w^3, \\ C^3(\xi) = 2\kappa\varepsilon_{\lambda\sigma}\omega^\lambda w^\sigma = C^3_\beta(\xi)w^\beta + C^3_3(\xi)w^3, \\ C^\alpha_\beta(\xi) = -2\kappa g^{\alpha\lambda}\varepsilon_{\lambda\beta}\omega^3; \quad C^\alpha_3(\xi) = 2\kappa g^{\alpha\lambda}\varepsilon_{\lambda\beta}\omega^\beta; \\ C^3_\beta(\xi) = 2\kappa\varepsilon_{\lambda\beta}\omega^\lambda, \quad C^3_3(\xi) = 0. \end{cases} \tag{3. 18}$$

$$\begin{cases} f^i_c(\xi) = \varepsilon^{ijk}g_{jl}\varepsilon_{kpq}\omega^l\omega^p R^q, \\ f^\alpha_c(\xi) = \varepsilon^{\alpha\beta}\varepsilon_{\lambda\sigma}g_{\beta\gamma}\omega^\gamma\omega^\lambda R^\sigma + \omega^3(\omega^\alpha R^3 - \omega^3 R^\alpha), \\ f^3_c(\xi) = \varepsilon^{3jk}\varepsilon_{kpq}g_{jl}\omega^l\omega^p R^q = g_{\alpha\lambda}\omega^\lambda(\omega^3 R^\alpha - \omega^\alpha R^3), \end{cases} \tag{3. 19}$$

In particular, on 2D manifold $\Im$, i. e. $\xi = 0$,

$$\begin{cases} \omega^\alpha(0) = \omega a^{\alpha 1}, \quad \omega^3(0) = -\frac{x^1\omega}{\sqrt{a}}\Theta_1(2+\Theta^2), \\ R^\alpha(0) = (x^1 a^{1\alpha} + x^2(1+\Theta^2)a^{2\alpha} + a^{\alpha\sigma}\Theta\Theta_\sigma(x^2)^2), \\ R^3(0) = -\frac{1}{\sqrt{a}}(2+\Theta^2)x^2x^\alpha\Theta_\alpha, \end{cases} \tag{3. 20}$$

$$
(3.\ 21)\quad \begin{cases} C^\alpha_\lambda = C^\alpha_\beta(0) = \frac{2\omega}{\sqrt{a}} z\Theta_1(2+\Theta^2)a^{\alpha\beta}\varepsilon_{\beta\lambda}, \quad C^\alpha_3 = C^\alpha_3(0) = -\frac{2\omega}{\sqrt{a}}\delta^\alpha_2, \\ C^3_\sigma = C^3_\sigma(0) = 2\omega a^{1\lambda}\varepsilon_{\lambda\sigma}, \quad C^3_3 = 0, \\ f^\alpha_c(0) = a^{\alpha\beta}\varepsilon_{\beta\lambda}\varepsilon_{\nu\sigma}((\omega^\lambda(0)\omega^\nu(0) \\ \qquad + a^{\lambda\nu}\omega^3(0)\omega^3(0))R^\sigma(0) + a^{\lambda\sigma}\omega^3(0)\omega^\nu(0)R^3(0)), \\ f^3_c(0) = a^{\sigma\nu}\varepsilon_{\lambda\sigma}\varepsilon_{\nu\mu}\omega^\lambda(0)(\omega^\mu(0)R^3(0) - \omega^3(0)R^\mu(0)), \end{cases}
$$

Lemma 3. 6 ([2]) The compressible viscous rotating Navier-Stokes (2. 8) in semi-geodesic coordinates represent

$$
(3.\ 22)\quad \begin{aligned} &-2\mu\Big[g^{\alpha\beta}g^{\lambda\sigma}\overset{*}{\nabla}_\lambda e_{\beta\sigma}(w) + \big[\overset{*}{\nabla}_\lambda (g^{\alpha\beta}g^{\lambda\sigma}) + \tfrac{1}{\kappa}(R^\alpha_{\lambda\nu}\delta^\lambda_\mu + R^\lambda_{\lambda\nu}\delta^\alpha_\mu)g^{\nu\beta}g^{\mu\sigma}\big] \\ &\quad \times e_{\beta\sigma}(w) + \tfrac12\big(\tfrac{1}{\kappa}(I^\alpha_\nu g^{\nu\sigma} + I^\lambda_\lambda g^{\alpha\sigma}) + \partial_\xi g^{\alpha\sigma}\big)\overset{*}{\nabla}_\sigma w^3 \\ &\quad + \tfrac12\big[\big(\tfrac{1}{\kappa}(I^\alpha_\nu g^{\nu\sigma} + I^\lambda_\lambda g^{\alpha\sigma}) + \partial_\xi(g^{\alpha\sigma}g_{\sigma\beta})\big]\tfrac{\partial w^\beta}{\partial\xi} + \tfrac12 g^{\alpha\beta}\overset{*}{\nabla}_\beta \tfrac{\partial w^3}{\partial\xi} \\ &\quad + \tfrac12\tfrac{\partial^2 w^\alpha}{\partial\xi^2}\Big] + \overset{*}{\operatorname{div}}\ (\varrho w w^\alpha) + \tfrac{\partial\varrho w^3 w^\alpha}{\partial\xi} + \varrho w^\alpha w^\beta \overset{*}{\nabla}_\beta \ln\kappa \\ &\quad + \varrho w^\alpha w^3 \tfrac{\partial\ln(\kappa\sqrt{a})}{\partial\xi} + g^{\alpha\beta}\overset{*}{\nabla}_\beta\Big[p - \tfrac23\mu(\overset{*}{\operatorname{div}}\ (w) + \tfrac{\partial w^3}{\partial\xi} + w^\beta\overset{*}{\nabla}_\beta\ln\kappa \\ &\quad + w^3\tfrac{\partial\ln(\kappa\sqrt{a})}{\partial\xi})\Big] + \rho C^\alpha(\xi) = \rho f^\alpha_c, \end{aligned}
$$

$$
(3.\ 23)\quad \begin{aligned} &-2\mu\Big[\tfrac12 g^{\lambda\sigma}\overset{*}{\nabla}_\lambda\overset{*}{\nabla}_\sigma w^3 + \tfrac12(\overset{*}{\nabla}_\beta g^{\beta\sigma} + \kappa^{-1}R^\beta_{\beta\lambda}g^{\lambda\sigma})\overset{*}{\nabla}_\sigma w^3 \\ &\quad + g^{\lambda\nu}g^{\sigma\mu}J_{\lambda\sigma}e_{\nu\mu}(w) \quad + \tfrac{\partial^2 w^3}{\partial\xi^2} + \kappa^{-1}I^\beta_\beta\tfrac{\partial w^3}{\partial\xi} + \tfrac12\partial_\xi\overset{*}{\operatorname{div}} w \\ &\quad + \tfrac12\kappa^{-1}R^\beta_{\beta\lambda}\tfrac{\partial w^\lambda}{\partial\xi}\Big] + \tfrac{\partial}{\partial\xi}\Big[p - \tfrac{\mu}{3}(\overset{*}{\operatorname{div}} w + \tfrac{\partial w^3}{\partial\xi} \\ &\quad + w^\alpha\overset{*}{\nabla}_\alpha\ln\kappa + w^3\tfrac{\partial\ln(\kappa\sqrt{a})}{\partial\xi})\Big] \\ &\quad + \overset{*}{\operatorname{div}}(\rho w w^3) + \tfrac{\partial}{\partial\xi}(\rho w^3 w^3) + \rho w^3 w^\beta\overset{*}{\nabla}_\beta\ln\kappa \\ &\quad + \rho w^3 w^3\tfrac{\partial}{\partial\xi}\ln(\kappa\sqrt{a}) + \rho C^3(\xi) = \rho f^3_c, \end{aligned}
$$

$$
(3.\ 24)\quad \begin{cases} \operatorname{div}\ (\varrho w) = \overset{*}{\operatorname{div}}\ (\varrho w) + \frac{\partial\varrho w^3}{\partial\xi} + \varrho w^\alpha\overset{*}{\nabla}_\alpha\ln\kappa + \varrho w^3\frac{\partial\ln(\kappa\sqrt{a})}{\partial\xi} = 0, \\ w^3\frac{\partial S}{\partial\xi} + w^\alpha\overset{*}{\nabla}_\alpha S - \frac{1}{WT}(\frac{k}{\rho}\overset{*}{\Delta} T + \frac{\Phi}{\rho}) = 0. \end{cases}
$$

On the surface $\Im$, i.e., $\xi = 0$, we have

$$
(3.\ 25)\quad \begin{cases} e_{ij}(w)|_{\xi=0} = \gamma_{ij}(w_0), \\ g^{\alpha\beta}|_{\xi=0} = a^{\alpha\beta}, \quad I^\alpha_\beta|_{\xi=0} = -b^\alpha_\beta, \quad J_{\alpha\beta}|_{\xi=0} = b_{\alpha\beta}, \quad R^\alpha_{\beta\lambda}|_{\xi=0} = 0, \\ \overset{*}{\Delta} w^\alpha = a^{\beta\sigma}\overset{*}{\nabla}_\beta\overset{*}{\nabla}_\sigma w^\alpha, \\ w^\alpha\overset{*}{\nabla}_\alpha\ln\kappa|_{\xi=0} = 0, \quad w^3\partial_\xi\ln(\kappa\sqrt{a})|_{\xi=0} = -2Hw^3, \\ b^\lambda_\sigma\overset{*}{\nabla}_\lambda w^\sigma = b^{\lambda\sigma}\overset{*}{\nabla}_\lambda w_\sigma = b^{\lambda\sigma}\overset{*}{e}_{\lambda\sigma}(w) \\ \qquad = b^{\lambda\sigma}(\gamma_{\lambda\sigma}(w) + b_{\lambda\sigma}w^3) = \beta_0(w) + (4H^2 - K)w^3. \end{cases}
$$

where

$$
(3.\ 26).\qquad b^{\alpha\beta}b_{\alpha\beta} = c^\alpha_\alpha = 4H^2 - K = k_1^2 + k_2^2, \quad \beta_0(w) = b^{\alpha\beta}\gamma_{\alpha\beta}(w).
$$

Substituting (3. 26) into (3. 22-3. 24) leads to

Lemma 3. 7 The restriction of 3D rotating Navier-Stokes equations (3. 22-3. 24) on $\Im$, i. e., $\xi = 0$, are given by

$$
(3.\ 27)\quad \begin{cases}
-\mu(\frac{\partial^2 w^\alpha}{\partial \xi^2} - B^\alpha_\beta \frac{\partial w^\beta}{\partial \xi})|_{\xi=0} - \frac{5\mu}{3} a^{\alpha\beta} \overset{*}{\nabla}_\beta \frac{\partial w^3}{\partial \xi}|_{\xi=0} + \frac{\partial}{\partial \xi}(\rho w^3 w^\alpha)|_{\xi=0} \\
\quad -2\mu[\overset{*}{\nabla}_\beta \gamma^{\alpha\beta}(w_0) + H a^{\alpha\beta} \overset{*}{\nabla}_\beta w_0^3] + a^{\alpha\beta} \overset{*}{\nabla}_\beta (p_0 - \frac{2\mu}{3}(\overset{*}{\operatorname{div}} w_0 \\
\quad -2H w_0^3)) + \overset{*}{\operatorname{div}} (\rho_0 w_0 w_0^\alpha) + 2\rho_0 H w_0^3 w_0^\alpha \\
\quad +\rho_0 (C^\alpha_\beta w_0^\beta + C^\alpha_3 w_0^3) = \rho_0 f_c^\alpha, \\
-\frac{8}{3}\mu(\frac{\partial^2 w^3}{\partial \xi^2} - 2H \frac{\partial w^3}{\partial \xi})|_{\xi=0} + [\frac{\partial}{\partial \xi} p - \frac{5}{3}\mu \partial_\xi \overset{*}{\operatorname{div}} w + \partial_\xi(\rho w^3 w^3)]|_{\xi=0} \\
\quad -\mu \overset{*}{\Delta} w_0^3 - 2\mu \beta_0(w_0) + \frac{4}{3} w^\alpha \overset{*}{\nabla}_\alpha H - \frac{4}{3}(K - 2H^2) w_0^3 \\
\quad + \overset{*}{\operatorname{div}} (\rho_0 w_0 w_0^3) - 2H \rho_0 w_0^3 w_0^3 + \rho_0 C^3_\beta w_0^\beta = \rho_0 f_c, \\
\frac{\partial \varrho w^3}{\partial \xi}|_{\xi=0} + \overset{*}{\operatorname{div}} \ (\varrho_0 w_0) - 2H \varrho_0 w_0^3 = 0, \\
w^3 \frac{\partial S}{\partial \xi}|_{\xi=0} + w_0^\alpha \overset{*}{\nabla}_\alpha S_0 - \frac{1}{W_0 T_0}\left(\frac{k}{\rho} \overset{*}{\Delta} T_0 + \frac{\Phi_0}{\rho_0}\right) = 0,
\end{cases}
$$

where

$$
B^\alpha_\beta := 2(b^\alpha_\beta + H\delta^\alpha_\beta), \quad \gamma^{\alpha\beta}(w_0) = a^{\alpha\lambda} a^{\beta\sigma} \gamma_{\lambda\sigma}(w_0), \quad \beta_0(w) = b^{\alpha\beta}\gamma_{\alpha\beta}(w).
$$

Next, the differential operators along normal to the surface $\Im_i$ are approximated by Euler central difference operators

$$
(3.\ 28)\quad \begin{cases}
\frac{\partial w}{\partial \xi}|_{\xi=0} = \frac{w_1 - w_{-1}}{2h}, \quad \frac{\partial^2 w}{\partial \xi^2}|_{\xi=0} = \frac{w_1 - 2w_0 + w_{-1}}{2h^2}, \\
w_0(x) = w|_{\xi=0} = w(x,\ 0), \quad w_{-1}(x) = w|_{s_{i-1}}, \quad w_1 = w|_{s_{i+1}},
\end{cases}
$$

and denote

$$
(3.\ 29)\quad \begin{cases}
F_h^\alpha := & \frac{1}{2}\mu h^{-2}[(\delta^\alpha_\beta - B^\alpha_\beta h) w_1^\beta + (\delta^\alpha_\beta + B^\alpha_\beta h) w_{-1}^\beta] \\
& + \frac{5}{6} a^{\alpha\beta} \overset{*}{\nabla}_\beta \frac{w_1^3 - w_{-1}^3}{h} - \frac{1}{2} h^{-1}(\rho_1 w_1^3 w_1^\alpha - \rho_{-1} w_{-1}^3 w_{-1}^\alpha), \\
F_h^3 := & \frac{8}{3}\mu[h^{-2}(w_1^3 + w_{-1}^3) - 2H h^{-1}(w_1^3 - w_{-1}^3) - \frac{1}{2} h^{-1}(p_1 - p_{-1})] \\
& + \frac{5}{3}\mu \overset{*}{\operatorname{div}} \frac{w_1 - w_{-1}}{h} - \frac{1}{2} h^{-1}(\rho_1 w_1^3 w_1^3 - \rho_{-1} w_{-1}^3 w_{-1}^3).
\end{cases}
$$

Since

$$
(3.\ 30)\quad \overset{*}{\operatorname{div}} w_0 - 2H w_0^3 = a^{\lambda\sigma}(\overset{*}{e}_{\lambda\sigma} (w_0) - b_{\lambda\sigma} w_0^3) = a^{\lambda\sigma}\gamma_{\lambda\sigma}(w_0) := \gamma_0(w_0)
$$

and vanish covariant derivatives for the metric tensor (3. 8), we claim

$$
(3.\ 31)\quad \begin{cases}
-2\mu[\overset{*}{\nabla}_\beta \gamma^{\alpha\beta}(w_0)] - a^{\alpha\beta} \overset{*}{\nabla}_\beta (\frac{2\mu}{3}(\overset{*}{\operatorname{div}} w_0 - 2H w_0^3)) \\
\qquad = -2\mu a^{\alpha\lambda} a^{\beta\sigma} \overset{*}{\nabla}_\beta \gamma_{\lambda\sigma}(w_0) - \frac{2}{3}\mu a^{\alpha\beta} a^{\lambda\sigma} \overset{*}{\nabla}_\beta \gamma_{\lambda\sigma}(w_0) \\
\qquad = -(2\mu a^{\alpha\lambda} a^{\beta\sigma} + \frac{2}{3}\mu a^{\alpha\beta} a^{\lambda\sigma}) \overset{*}{\nabla}_\beta \gamma_{\lambda\sigma}(w_0) \\
\qquad = -a^{\alpha\beta\lambda\sigma} \overset{*}{\nabla}_\beta \gamma_{\lambda\sigma}(w_0), \\
a^{\alpha\beta\lambda\sigma} = 2\mu a^{\alpha\lambda} a^{\beta\sigma} + \frac{2}{3}\mu a^{\alpha\beta} a^{\lambda\sigma}
\end{cases}
$$

To sum up we assert that the restriction of 3D rotating Navier-Stokes equations on the 2D manifold $\Im_i$:

Theorem 3. 1 The restriction of 3D rotating Naver-Stokes equations (3. 8) on $\Im$(i. e. $\xi = 0$) are given by

$$(3.\ 32)\quad \begin{cases} -a^{\alpha\beta\lambda\sigma} \overset{*}{\nabla}_\beta \gamma_{\lambda\sigma}(w_0) + \mu h^{-2} w_0^\alpha + \overset{*}{\operatorname{div}} (\rho_0 w_0 w_0^\alpha) + 2\rho_0 H w_0^3 w_0^\alpha \\ \qquad + a^{\alpha\beta} \overset{*}{\nabla}_\beta p_0 + l^\alpha(\rho_0,\ w_0) = \rho_0 f_c^\alpha + F_h^\alpha, \\ -\mu \overset{*}{\Delta} w_0^3 + \mu h^{-2} w_0^3 + \overset{*}{\operatorname{div}} (\rho_0 w_0 w_0^3) - 2H\rho_0 w_0^3 w_0^3 + l^3(\rho_0,\ w_0) \\ \qquad = \rho_0 f_c^3 + F_h^3, \\ \overset{*}{\operatorname{div}} (\varrho_0 w_0) - 2H\varrho_0 w_0^3 + d_0 = 0, \\ w_0^\alpha \overset{*}{\nabla}_\alpha S_0 + \frac{w_0^3}{2h}(S_1 - S_{-1}) - \frac{1}{W_0 T_0}\left(\frac{k}{\rho} \overset{*}{\Delta} T_0 + \frac{\Phi_0}{\rho_0} \right) = 0, \end{cases}$$

where

$$(3.\ 33)\quad \begin{cases} l^\alpha(\rho_0,\ w_0) = -2\mu H a^{\alpha\beta} \overset{*}{\nabla}_\beta w_0^3 + \rho_0 (C_\beta^\alpha w_0^\beta + C_3^\alpha w_0^3), \\ l^3 = -2\mu\beta_0(w_0) + \mu(\frac{4}{3}(2H^2 - K)) w_0^3 + (\frac{4}{3}\mu \overset{*}{\nabla}_\beta H) w_0^\beta + \rho_0 C_\beta^3 w_0^\beta, \\ d_0 := \frac{1}{2h}((\rho w^3)_1 - (\rho w^3)_{-1}), \end{cases}$$

Let boundary of $\Im$

$$(3.\ 34)\quad \gamma_s = \Gamma_S \cap \Im_i,\ \gamma_{in} = \Gamma_{in} \cap \Im_i,\ \gamma_{out} = \Gamma_{out} \cap \Im_i, \quad \gamma_0 = \gamma_{in} \cup \gamma_{out},$$

Taking (3. 9) into account the boundary condition (2. 11) become

$$(3.\ 35)\quad \begin{cases} w_0|_{\gamma_s} = 0, \\ \left(\frac{2}{3}\mu a^{\alpha\lambda} \gamma_{\lambda\beta}(w_0) n^\beta - (p_0) n^\alpha\right)|_{\gamma_{in}} = g_{in}^\alpha, \\ (\frac{2}{3}\mu \overset{*}{\nabla}_\beta w_0^3 n^\beta - (p_0) n^3)|_{\gamma_{in}} = \widetilde{g}_{in}^3, \\ (\frac{2}{3}\mu a^{\alpha\lambda} \gamma_{\lambda\beta}(w_0) n^\beta - (p_0) n^\alpha)|_{\gamma_{out}} = g_{out}^\alpha, \\ (\frac{2}{3}\mu \overset{*}{\nabla}_\beta w_0^3 n^\beta - (p_0) n^3)|_{\gamma_{out}} = \widetilde{g}_{out}^3, \end{cases}$$

$$(3.\ 36)\quad \begin{array}{l} \widetilde{g}_{in}^3 = g_{in}^3 - \frac{2}{3}\mu a_{\alpha\beta}(w_1^\alpha - w_{-1}^\alpha) n^\beta, \quad \text{in } \gamma_{out}, \\ \widetilde{g}_{out}^3 = g_{out}^3 - \frac{2}{3}\mu a_{\alpha\beta}(w_1^\alpha - w_{-1}^\alpha) n^\beta, \quad \text{in } \gamma_{out} \end{array}$$

4. The Navier-Stokes Equations on the Surface $\Im$

In sequence we only discuss isentropic ideal gases, in particular for the polytropic gas: $p = A\rho^\gamma$ where A is constant and $\frac{5}{3} \geq \gamma \geq 1$ is the specific heat radio. Hence we omit energy equation. Taking (3. 28) into account, the equations (3. 29) on 2D-manifolds become

$$(4.\ 1)\quad \begin{cases} -a^{\alpha\beta\lambda\sigma} \overset{*}{\nabla}_\beta \gamma_{\lambda\sigma}(w_0) + a^{\alpha\beta} \overset{*}{\nabla}_\beta (A\rho^\gamma) + \overset{*}{\operatorname{div}} (\rho_0 w_0 w_0^\alpha) - 2H\rho_0 w_0^\alpha w_0^3 \\ \qquad + l^\alpha(\rho_0,\ w_0) = F_0^\alpha, \\ -\mu \overset{*}{\Delta} w_0^3 + \overset{*}{\operatorname{div}} (\rho_0 w_0 w_0^3) - 2H\rho_0 w_0^3 w_0^3 + l^3(\rho_0,\ w_0) = F_0^3, \\ \overset{*}{\operatorname{div}} (\rho_0 w_0) - 2H\rho_0 w_0^3 + d_0 = 0, \end{cases}$$

where $l^\alpha,\ l^3$ are defined by (3. 35).

Unless there is a statement to the contrary, the Einstein summation convention, i. e. , repeated indices indicate summation, is used, and a “”' denotes transposition. For the simplicity, we use the abbreviations

$$(4.\ 2)\quad \begin{cases} \|\cdot\|_{0,\ D} = \|\cdot\|_{L^2(D)},\ \|\cdot\|_{0,\ p,\ D} = \|\cdot\|_{L^p(D)}, \\ \|\cdot\|_{m,\ p,\ D} = \|\cdot\|_{H^{m,\ p}(D)},\ \|\cdot\|_{m,\ D} = \|\cdot\|_{m,\ 2,\ D}, \\ V(D) = \{w | w \in\ H^1(D)^3,\ w|_{\gamma_s} = 0,\ \}. \end{cases}$$

Noting that

$$(4.\ 3)\qquad \begin{cases} \gamma_{\alpha\beta}(w_0) = \overset{*}{e}_{\alpha\beta}(w_0) - b_{\alpha\beta}w^3, \\ \gamma_0(w_0) = a^{\alpha\beta}\gamma_{\alpha\beta}(w_0) = \overset{*}{\operatorname{div}}\, w_0 - 2Hw_0^3, \\ a^{\alpha\beta\lambda\sigma}\gamma_{\lambda\sigma}(w_0)b_{\alpha\beta} = 2\mu\beta_0(w_0) + \frac{4}{3}\mu H\gamma_0(w_0), \\ a^{\alpha\beta\lambda\sigma}\gamma_{\lambda\sigma}(w_0)\overset{*}{e}_{\alpha\beta}(v) = a^{\alpha\beta\lambda\sigma}\gamma_{\lambda\sigma}(w_0)\gamma_{\alpha\beta}(v) \\ \qquad +(2\mu\beta_0(w_0) + \frac{4}{3}\mu H\gamma_0(w_0))v^3. \end{cases}$$

$\forall\, v \in\ V(D)$. Since $\overset{*}{\nabla}_\gamma a^{\alpha\beta\lambda\sigma} = 0$, Green formula shows

$$(4.\ 4)\qquad \begin{aligned} &\int_D[-a^{\alpha\beta\lambda\sigma}\overset{*}{\nabla}_\beta \gamma_{\lambda\sigma}(w_0)v_\alpha]\sqrt{a}dx \\ &\quad = -\int_{\partial D} a^{\alpha\beta\lambda\sigma}\gamma_{\lambda\sigma}(w_0)n_\beta v_\alpha d\gamma + \int_D a^{\alpha\beta\lambda\sigma}\gamma_{\lambda\sigma}(w_0)\overset{*}{\nabla}_\beta v_\alpha\sqrt{a}dx \\ &\quad = -\int_{\gamma_0} a^{\alpha\beta\lambda\sigma}\gamma_{\lambda\sigma}(w_0)n_\beta v_\alpha d\gamma + \int_D a^{\alpha\beta\lambda\sigma}\gamma_{\lambda\sigma}(w_0)\overset{*}{e}_{\alpha\beta}(v)\sqrt{a}dx \\ &\quad = -\int_{\gamma_0} a^{\alpha\beta\lambda\sigma}\gamma_{\lambda\sigma}(w_0)n_\beta v_\alpha d\gamma \\ &\quad + \int_D[a^{\alpha\beta\lambda\sigma}\gamma_{\lambda\sigma}(w_0)\gamma_{\alpha\beta}(v) + (2\mu\beta_0(w_0) + \frac{4}{3}\mu H\gamma_0(w_0))v^3]\sqrt{a}dx, \end{aligned}$$

where we used the symmetry of index. Similarly,

$$(4.\ 5)\qquad \begin{cases} (a^{\alpha\beta}\overset{*}{\nabla}_\beta (A\rho^\gamma),\ v_\alpha) = \int_{\gamma_0} A\rho^\gamma a_{\alpha\beta}n^\alpha v^\beta d\gamma - (A\rho^\gamma,\ \overset{*}{\operatorname{div}}(v)), \\ (\overset{*}{\operatorname{div}}(\rho_0 w_0 w_0^\alpha),\ v_\alpha) = \int_{\gamma_0}\rho_0(w_0^\beta n_\beta)(w_0^\alpha v_\alpha)d\gamma - (\rho_0 w_0^\alpha w_0^\beta,\ \overset{*}{e}_{\alpha\beta}(v)), \\ (\overset{*}{\operatorname{div}}(\rho_0 w_0 w_0^3),\ v^3) = \int_{\gamma_0}\rho_0(w_0^\beta n_\beta)(w_0^3 v^3)d\gamma - (\rho_0 w_0^\alpha w_0^3,\ \overset{*}{\nabla}_\alpha v^3), \\ (-\mu\overset{*}{\Delta} w_0^3,\ v^3) = -\int_{\gamma_0}\mu a^{\alpha\beta}\overset{*}{\nabla}_\alpha w_0^3 n_\beta v^3 d\gamma + (\mu\overset{*}{\nabla} w_0^3,\ \overset{*}{\nabla} v^3) \\ \qquad = (\mu\overset{*}{\nabla} w_0^3,\ \overset{*}{\nabla} v^3) - \int_{\gamma_0}\mu\frac{\partial w_0^3}{\partial n}v^3 d\gamma, \end{cases}$$

Multiplying v_α with both sides of the first of (4. 1) , v^3 with both sides of second of (4. 1), adding and taking (4. 4), (4. 5) and (3. 35) into account, and applying

$$\frac{4\mu}{3}H\gamma_0(w_0) + \frac{4\mu}{3}w_0^\beta\overset{*}{\nabla}_\beta H = \frac{4\mu}{3}\gamma_0(Hw_0),$$

the variational formulation for (4. 1) and (3. 42) reads

$$(4.\ 6)\qquad \begin{cases} \text{Find } (w_0,\ \rho_0) \in\ V(D)\times L^\gamma(D), \text{such that } \forall\,(v,\ q) \in\ V(D)\times L^2(D), \\ a_0(w_0,\ v) - (A\rho^\gamma,\ \overset{*}{\operatorname{div}}\, v) + b_0(\rho_0; w_0,\ w_0,\ v) + (l(\rho_0, w_0),\ v) \\ \qquad\qquad =< G,\ v >, \\ (\overset{*}{\operatorname{div}}(\rho_0 w_0) - 2H\rho_0 w_0^3 + d_0,\ q) = 0, \end{cases}$$

where

$$(4.\ 7)\qquad \begin{cases} a_0(w_0,\ v) = (a^{\alpha\beta\lambda\sigma}\gamma_{\lambda\sigma}(w_0),\ \gamma_{\alpha\beta}(v)) + (\mu\overset{*}{\nabla} w_0^3,\ \overset{*}{\nabla} v^3) \\ \quad +(\mu h^{-2}a_{\alpha\beta}w_0^\alpha,\ v^\beta) + (\mu h^{-2}w_0^3,\ v^3), \\ b_0(\rho_0; w_0,\ w_0,\ v) = -(\rho_0 w_0^\alpha w_0^\beta,\ \overset{*}{e}_{\alpha\beta}(v)) - (2H\rho_0 w_0^\alpha w_0^3,\ a_{\alpha\beta}v^\beta) \\ \quad -(\rho_0 w_0^\alpha w_0^3,\ \overset{*}{\nabla}_\alpha v^3) - (2H\rho_0 w_0^3 w_0^3),\ v^3), \\ (l(\rho_0,\ w_0),\ v) = -(2\mu H\overset{*}{\nabla}_\beta w^3,\ v^\beta) + (\frac{4}{3}\mu\gamma_0(Hw_0),\ v^3) \\ \quad +(\rho_0 C_\beta^3 w_0^\beta, v^3) + (\frac{4}{3}\mu(2H^2 - K)w_0^3, v^3) + (\rho_0 C_{\alpha\beta}w_0^\beta + C_{\alpha 3}w_0^3, v^\alpha), \\ < G,\ v >=< F_0,\ v > + \int_{\gamma_0}[a_{\alpha\beta}(A\rho_0^\gamma - \frac{4}{3}\mu\gamma_0(w_0)) - 2\mu\gamma_{\alpha\beta}(w_0)]n^\alpha v^\beta \\ \quad -\int_{\gamma_0}\mu\frac{\partial w_0^3}{\partial n}v^3 d\gamma + \int_{\gamma_0}\rho_0(w_0^\lambda n_\lambda)(a_{\alpha\beta}w_0^\alpha v^\beta + w_0^3 v^3)d\gamma, \\ F_0 = \rho_0 f_c + F_h, \quad d_0 = \frac{1}{2h}((\rho_0 w_0^3)_1 - (\rho_0 w_0^3)_{-1}). \end{cases}$$

Remark 4. 1

$$(4.8)\quad \begin{array}{ll}(i) & 2(\omega\times w)v|_{\Im}=(\rho_0 C_{\alpha\beta}w_0^\beta+C_{\alpha 3}w_0^3)v^\alpha+\rho_0 C_\beta^3 w_0^\beta v^3,\\ (ii) & 2H^2-K=\frac{1}{2}(k_1^2+k_2^2),\qquad k_\alpha-\text{Principle curvatures of }\Im\end{array}$$

Therefore

$$(4.9)\quad \begin{array}{l}(l(\rho_0,\ w_0),\ v)=-(2\mu H\overset{*}{\nabla}_\beta w_0^3,\ v^\beta)+(\frac{4\mu}{3}\gamma_0(Hw_0),\ v^3)\\ \quad +(\frac{2\mu}{3}(k_1^2+k_2^2)w_0^3,\ v^3)+(2(\omega\times w)|_{\Im},\ v),\end{array}$$

5. Korn's Inequality on the Surface $\Im$

In the sequel, the constant $C(\Theta,\ D)$ may be different from line to line but should be independent of the vector field w. The inner product on the tangent bundle $T\Im$ induces norms on all tensor space , for example , point-wise norm and Sobolev norms

$$(5.1)\quad \begin{cases}|w|^2=a_{\alpha\beta}w^\alpha w^\beta=a^{\alpha\beta}w_\alpha w_\beta,\qquad w^\alpha=a^{\alpha\beta}w_\beta,\quad w_\alpha=a_{\alpha\beta}w^\beta,\\ \|w\|_{0,\ D}^2=\int_D|w|^2\sqrt{a}dx,\\ |e(w)|^2=a^{\alpha\lambda}a^{\beta\sigma}\overset{*}{e}_{\alpha\beta}(w)\overset{*}{e}_{\lambda\sigma}(w),\\ \|e(w)\|_{0,\ D}^2=\int_D|e(w)|^2\sqrt{a}dx,\\ |\overset{*}{\nabla}w|^2=a^{\alpha\lambda}a^{\beta\sigma}\overset{*}{\nabla}_\alpha w_\beta\overset{*}{\nabla}_\lambda w_\sigma=a^{\alpha\beta}a_{\lambda\sigma}\overset{*}{\nabla}_\alpha w^\lambda\overset{*}{\nabla}_\beta w^\sigma,\\ \|\overset{*}{\nabla}w\|_{0,\ D}^2=\int_D|\overset{*}{\nabla}w|^2\sqrt{a}dx,\\ |r(w)|^2=a^{\alpha\lambda}a^{\beta\sigma}r_{\alpha\beta}(w)r_{\lambda\sigma}(w),\qquad \|r(w)\|^2=\int_D|r(w)|^2\sqrt{a}dx,\\ \|\overset{*}{\nabla}_\alpha w^\beta\|_{0,\ D}^2=\int\limits_D|\overset{*}{\nabla}_\alpha w^\beta|^2\sqrt{a}dx,\\ \|\overset{*}{e}_{\alpha\beta}(w)\|_{0,\ D}^2=\int\limits_D|\overset{*}{e}_{\alpha\beta}(w)|^2\sqrt{a}dx.\\ |\gamma(w)|^2=a^{\alpha\lambda}a^{\beta\sigma}\gamma_{\alpha\beta}(w)\gamma_{\lambda\sigma}(w),\qquad \|\gamma(w)\|_{0,\ D}^2=\int\limits_D|\gamma(w)|^2\sqrt{a}dx.\end{cases}$$

What follows that we will frequently used equalities

$$(5.2)\qquad \overset{*}{\nabla}_\sigma a^{\alpha\beta}=0,\qquad \overset{*}{\nabla}_\sigma a_{\alpha\beta}=0.$$

and notation

$$\begin{array}{l}\overset{*}{e}_{\alpha\beta}(w)=\frac{1}{2}(\overset{*}{\nabla}_\alpha w_\beta+\overset{*}{\nabla}_\beta w_\alpha)=\frac{1}{2}(a_{\beta\lambda}\overset{*}{\nabla}_\alpha w^\lambda+a_{\alpha\lambda}\overset{*}{\nabla}_\beta w^\lambda),\\ r_{\alpha\beta}(w)=\frac{1}{2}(\overset{*}{\nabla}_\alpha w_\beta-\overset{*}{\nabla}_\beta w_\alpha)=\frac{1}{2}(a_{\beta\lambda}\overset{*}{\nabla}_\alpha w^\lambda-a_{\alpha\lambda}\overset{*}{\nabla}_\beta w^\lambda),\\ \gamma_{\alpha\beta}(w)=\overset{*}{e}_{\alpha\beta}(w)-b_{\alpha\beta}w^3,\qquad \beta_0(w)=b^{\alpha\beta}\gamma_{\alpha\beta}(w),\qquad \gamma_0(w)=a^{\alpha\beta}\gamma_{\alpha\beta}(w).\end{array}$$

In this section, we consider the Korn's inequality on the surface $\Im$(a two dimensional Riemannian manifold) which can be found in [8, 9, 16]. For example,

Theorem 5. 1(Th. 2. 7-1, [8])(Korn's inequality "without boundary conditions" on the surface) Let D be a domain in $\Re^2$ and let $\Theta\in C^2(D)$ be an injective mapping such that the two vectors $\vec{e}_\alpha=\partial_\alpha\vec{R}$($\vec{R}$ is defined by (5. 1)) are linearly independent at all points of D. Given $w=(w^\alpha,\ w^3)\in H^1(D)\times H^1(D)\times L^2(D)$, let

$$\gamma_{\alpha\beta}(w):=\overset{*}{e}_{\alpha\beta}(w)-b_{\alpha\beta}w^3\in L^2(D).$$

Then there exists a constant $c_0=c_0(D,\ \Theta)$ such that

$$(5.3)\quad \begin{array}{l}\sum\limits_\alpha\|w^\alpha\|_{1,\ D}^2+\|w^3\|_{0,\ D}^2\le\\ \qquad c_0\{\sum\limits_\alpha\|w^\alpha\|_{0,\ D}^2+\|w^3\|_{0,\ D}^2+\sum\limits_{\alpha,\ \beta}\|\gamma_{\alpha\beta}(w)\|_{0,\ D}^2\},\\ \qquad\forall\ w\in H^1(D)\times H^1(D)\times L^2(D),\end{array}$$

Then Riemann version of Korn's Inequality is given by [15]:

Theorem 5. 2 Let $(\mathcal{M},\ a)$ be an oriented Riemann Manifold and $T\mathcal{M}$ the tangent bundle. Assume $\Omega\subset\mathcal{M}$ be an open set with boundary $\partial\Omega$ of $C^{1,\ 1}$, v be a vector field on the Riemann manifold $\mathcal{M}$. Then there is a positive constant c such that

$$\|\overset{*}{\nabla}v\|^2_{0,\ \Omega}\le C\{\||v|\|^2_{0,\ \Omega}+\||e(v)|\|^2_{0,\ \Omega}\}, \tag{5. 4}$$

where

$$|v|^2=a_{\alpha\beta}v^\alpha v^\beta,\quad |e(v)|^2=a^{\alpha\beta}a_{\lambda\sigma}\overset{*}{\nabla}_\alpha v^\lambda\overset{*}{\nabla}_\beta v^\sigma, \tag{5. 5}$$

Furthermore, if $\gamma\subset\partial\Omega$ with Hausdorff dimension $dim_H(\gamma)>n-2$ and Ω is convex set, then there exists positive constant c such that

$$\|\overset{*}{\nabla}v\|^2_{0,\ \Omega}\le C\||e(v)|\|^2_{0,\ \Omega}, \tag{5. 6}$$

for any vector $v\in H^2(\Omega,\ T\Omega)\cap\{v|_\gamma=0\}$.

Theorem 5. 3(Th. 2. 7-3, [8])(Korn's inequality on the ellipc surface) Assumptions in theorem 5. 1 are satisfy. furthermore, the surface is elliptic, i. e. the curvature tensor(the coefficients of second fundamental form) $b_{\alpha\beta}$ of the surface is positive, or negative, definite at all points in D, or equivalently if there exists a constant c such that

$$\sum_\alpha|\xi^\alpha|^2\le c|b_{\alpha\beta}\xi^\alpha\xi^\beta|,\ \forall\ (\xi^\alpha)\in\Re^2$$

or equivalently if the Gaussian curvature of the surface is everywhere strictly positive $K>0$. Then there exists a constant c_M such that

$$\begin{aligned}&\sum_\alpha\|w_0^\alpha\|^2_{1,\ D}+\|w_0^3\|^2_{0,\ D}\le c_M\sum_{\alpha,\ \beta}\|\gamma_{\alpha\beta}(w_0)\|^2_{0,\ D},\\ &\qquad\forall\ w_0\in\ H^1_0(D)\times H^1_0(D)\times L^2(D),\end{aligned} \tag{5. 7}$$

Inversely , ellipticity of the surface is also necessary condition for the Korn's inequality: If (5. 7) is valid for all vectors in space

$$\{w_0|w_0\in\ H^1(D)\times H^1(D)\times L^2(D),\ w_0^\alpha|_{\gamma_0}=0,\ \gamma_0\subset\partial D\}$$

Then $\gamma_0=\gamma:=\partial D$ and the surface is elliptic.

Remark 5. 1 (5. 6) shows if $\overset{*}{e}_{\alpha\beta}(v)=0$ on the manifold then $\overset{*}{\nabla}_\alpha v^\beta=0$. It is well know[15] that if a vectors v satisfy $\overset{*}{e}_{\alpha\beta}(v)=0$ on the manifolds then the vectors v are called Killing vector field and let $\mathcal{M}$ be a compact Riemannian manifold, then the vector space of Killing field on $\mathcal{M}$ is finite dimensional. In addition, if the vector v of Killing space satisfies $v|_\gamma=0,\ \gamma\in\partial\Omega$ then v vanishes identically on the set Ω. "

Lemma 5. 1 There exist following relationships

$$\begin{cases}|\overset{*}{\nabla}w_0|^2=|e(w_0)|^2+|r(w_0)|^2,\\ \|\overset{*}{\nabla}w_0\|^2_{0,\ D}=\|e(w_0)\|^2_{0,\ D}+\|r(w_0)\|^2_{0,\ D},\\ |\overset{*}{\nabla}w_0|^2+|\overset{*}{\operatorname{div}}w_0|^2+\overset{*}{\operatorname{div}}((w_0\overset{*}{\nabla})w_0-w_0\overset{*}{\operatorname{div}}w_0)=2|e(w_0)|^2,\\ \|\overset{*}{\nabla}w_0\|^2_{0,\ D}+\|\overset{*}{\operatorname{div}}w_0\|^2_{0,\ D}+\int_{\partial D}(w_0\overset{*}{\nabla})w_0-w_0\overset{*}{\operatorname{div}}w_0)\cdot ndl\\ \qquad=2\|e(w_0)\|^2_{0,\ D},\\ |\overset{*}{\nabla}w_0|^2-\overset{*}{\operatorname{div}}((w_0\overset{*}{\nabla})w_0-w_0\overset{*}{\operatorname{div}}w_0)=|\overset{*}{\operatorname{div}}w_0|^2+2|r(w_0)|^2,\\ \|\overset{*}{\nabla}w_0\|^2_{0,\ D}-\int_{\partial D}((w_0\overset{*}{\nabla})w_0-w_0\overset{*}{\operatorname{div}}w_0)ndl\\ \qquad=\|\overset{*}{\operatorname{div}}w_0\|^2_{0,\ D}+2\|r(w_0)\|^2_{0,\ D},\end{cases} \tag{5. 16}$$

$$(5.\ 17)\qquad \begin{cases} \frac{1}{2}(|\overset{*}{\nabla} w_0|^2 + |\overset{*}{\operatorname{div}} w_0|^2 + \overset{*}{\operatorname{div}}((w_0 \overset{*}{\nabla})w_0 - w_0 \overset{*}{\operatorname{div}} w_0)) \\ \qquad = |\gamma(w_0)|^2 + 2a^{\alpha\lambda}a^{\beta\sigma}\gamma_{\lambda\sigma}(w_0)b_{\alpha\beta}w_0^3 + b^{\alpha\beta}b_{\alpha\beta}w_0^3w_0^3, \\ |\overset{*}{\nabla} w_0|^2 + |\overset{*}{\operatorname{div}} w_0|^2 + \overset{*}{\operatorname{div}}((w_0 \overset{*}{\nabla})w_0 - w_0 \overset{*}{\operatorname{div}} w_0) \\ \qquad \leq 4(|\gamma(w_0)|^2 + (k_1^2 + k_2^2)w_0^3w_0^3). \end{cases}$$

Proof

$$\begin{aligned} 4|e(w_0)|^2 = &\ a^{\alpha\lambda}a^{\beta\sigma}(\overset{*}{\nabla}_\alpha w_{0\beta} + \overset{*}{\nabla}_\beta w_{0\alpha})(\overset{*}{\nabla}_\lambda w_{0\sigma} + \overset{*}{\nabla}_\sigma w_{0\lambda}) \\ = &\ 2\overset{*}{\nabla}{}^\alpha w_0^\beta \overset{*}{\nabla}_\alpha w_{0\beta} + 2\overset{*}{\nabla}_\lambda w_0^\alpha \overset{*}{\nabla}_\alpha w_0^\lambda, \\ |\overset{*}{\nabla} w_0|^2 = &\ a^{\alpha\lambda}a^{\beta\sigma}\overset{*}{\nabla}_\alpha w_{0\beta}\overset{*}{\nabla}_\lambda w_{0\sigma} = \overset{*}{\nabla}{}^\alpha w_0^\lambda \overset{*}{\nabla}_\alpha w_{0\lambda}, \\ |\overset{*}{\operatorname{div}} w_0|^2 = &\ \overset{*}{\nabla}_\alpha w_0^\alpha \overset{*}{\nabla}_\lambda w_0^\lambda, \\ |\overset{*}{\nabla} w_0|^2 + |\overset{*}{\operatorname{div}} w_0|^2 = &\ 2|e(w_0)|^2 - \overset{*}{\nabla}_\alpha w_0^\lambda \overset{*}{\nabla}_\lambda w_0^\alpha + \overset{*}{\nabla}_\alpha w_0^\alpha \overset{*}{\nabla}_\lambda w_0^\lambda \\ = &\ 2|e(w_0)|^2 - \overset{*}{\nabla}_\alpha (w_0^\lambda \overset{*}{\nabla}_\lambda w_0^\alpha - w_0^\alpha \overset{*}{\nabla}_\lambda w_0^\lambda) \\ = &\ 2|e(w_0)|^2 - \overset{*}{\operatorname{div}}((w_0 \overset{*}{\nabla} w_0 - w_0 \overset{*}{\operatorname{div}} w_0). \end{aligned}$$

This is the first of (5. 16). Since

$$\overset{*}{\nabla}_\alpha w_{0\beta} = \overset{*}{e}_{\alpha\beta}(w_0) + r_{\alpha\beta}(w_0)$$

hence

$$(5.\ 18)\qquad \begin{aligned} |\overset{*}{\nabla} w_0|^2 &= a^{\alpha\lambda}a^{\beta\sigma}\overset{*}{\nabla}_\alpha w_{0\beta}\overset{*}{\nabla}_\lambda w_{0\sigma} \\ &= a^{\alpha\lambda}a^{\beta\sigma}[\overset{*}{e}_{\alpha\beta}(w_0) + r_{\alpha\beta}(w_0)][\overset{*}{e}_{\lambda\sigma}(w_0) + r_{\lambda\sigma}(w_0)] \\ &= |e(w_0)|^2 + |r(w_0)|^2 + a^{\alpha\lambda}a^{\beta\sigma}(\overset{*}{e}_{\alpha\beta}(w_0)r_{\lambda\sigma}(w_0) \\ &\quad + \overset{*}{e}_{\lambda\sigma}(w_0)r_{\alpha\beta}(w_0)), \end{aligned}$$

Owing to anti-symmetry of index for rotation tensor $r_{\alpha\beta}(w_0)$ and symmetry of index for the strain tensor $\overset{*}{e}_{\alpha\beta}(w_0)$

$$r_{\alpha\beta}(w_0) = -r_{\beta\alpha}(w_0), \quad \overset{*}{e}_{\alpha\beta}(w_0) = \overset{*}{e}_{\beta\alpha}(w_0)$$

we claim

$$a^{\alpha\lambda}a^{\beta\sigma}\overset{*}{e}_{\lambda\sigma}(w_0)r_{\alpha\beta}(w_0) = -a^{\sigma\beta}a^{\lambda\alpha}\overset{*}{e}_{\sigma\lambda}(w_0)r_{\beta\alpha}(w_0) = a^{\alpha\lambda}a^{\beta\sigma}\overset{*}{e}_{\alpha\beta}(w_0)r_{\lambda\sigma}(w_0)$$

Returning to (5. 18) it deduces to (5. 16).

In addition, in similar manner, by virtue of(3. 26)and

$$(5.\ 19)\qquad \left\{ \gamma_{\alpha\beta}(w_0) = \overset{*}{e}_{\lambda\sigma}(w_0) - b_{\alpha\beta}w^3, \quad b^{\alpha\beta} = a^{\alpha\lambda}a^{\beta\sigma}b_{\lambda\sigma}, \right.$$

and (5. 16), then

$$\begin{aligned} |\gamma(w_0)|^2 &= |e(w_0)|^2 - 2a^{\alpha\lambda}a^{\beta\sigma}\overset{*}{e}_{\lambda\sigma}(w_0)b_{\alpha\beta}w_0^3 + a^{\alpha\lambda}a^{\beta\sigma}b_{\lambda\sigma}b_{\alpha\beta}w_0^3w_0^3 \\ &= \tfrac{1}{2}(|\overset{*}{\nabla} w_0|^2 + |\overset{*}{\operatorname{div}} w_0|^2 + \overset{*}{\operatorname{div}}((w_0 \overset{*}{\nabla})w_0 - w_0 \overset{*}{\operatorname{div}} w_0)) \\ &\quad -2a^{\alpha\lambda}a^{\beta\sigma}\gamma_{\lambda\sigma}(w_0)b_{\alpha\beta}w_0^3 - b^{\alpha\beta}b_{\alpha\beta}w_0^3w_0^3, \\ |\gamma(w_0)|^2 &+ 2a^{\alpha\lambda}a^{\beta\sigma}\gamma_{\lambda\sigma}(w_0)b_{\alpha\beta}w_0^3 + b^{\alpha\beta}b_{\alpha\beta}w_0^3w_0^3 \\ &= \tfrac{1}{2}(|\overset{*}{\nabla} w_0|^2 + |\overset{*}{\operatorname{div}} w_0|^2 + \overset{*}{\operatorname{div}}((w_0 \overset{*}{\nabla})w_0 - w_0 \overset{*}{\operatorname{div}} w_0)). \end{aligned}$$

Owing to

$$|2a^{\alpha\lambda}a^{\beta\sigma}\gamma_{\lambda\sigma}(w_0)b_{\alpha\beta}w_0^3| \leq a^{\alpha\lambda}a^{\beta\sigma}\gamma_{\alpha\beta}(w_0)\gamma_{\lambda\sigma}(w_0) + b^{\alpha\beta}b_{\alpha\beta}w_0^3w_0^3,$$

we infer

$$|\overset{*}{\nabla} w_0|^2 + |\overset{*}{\operatorname{div}} w_0|^2 + \overset{*}{\operatorname{div}}((w_0 \overset{*}{\nabla})w_0 - w_0 \overset{*}{\operatorname{div}} w_0) \leq 4(|\gamma(w_0)|^2 + b^{\alpha\beta}b_{\alpha\beta}w_0^3w_0^3)$$

From this and (5. 19) it deduces (5. 17). The proof is complete. ♮

Lemma 5. 2 There exist positive constants $\lambda,\ \Lambda$ such that

$$\lambda|\xi|^2 \le a_{\alpha\beta}\xi^\alpha\xi^\beta \le \Lambda|\xi|^2, \quad \forall\xi\in\ E^2 \tag{5. 20}$$

$$\begin{cases} \Lambda\sum\limits_\alpha |w_0^\alpha|^2 \ge |w_0|^2 \ge \lambda\sum\limits_\alpha |w_0^\alpha|^2, \\ \Lambda\sum\limits_{\alpha,\ \beta} |\overset{*}{\nabla}_\beta\ w_0^\alpha|^2 \ge |\overset{*}{\nabla}\ w_0|^2 \ge \lambda\sum\limits_{\alpha,\ \beta} |\overset{*}{\nabla}_\beta\ w_0^\alpha|^2, \\ \Lambda\sum\limits_\alpha \int_D |w_0^\alpha|^2\sqrt{a}dx \ge \|w_0\|^2_{0,\ D} \ge \lambda\sum\limits_\alpha\int_D |w_0^\alpha|^2\sqrt{a}dx, \\ \Lambda|w_0|^2_{1,\ D} \ge \|\overset{*}{\nabla}\ w_0\|^2_{0,\ D} \ge \lambda|w_0|^2_{1,\ D}, \end{cases} \tag{5. 21}$$

and

$$\begin{cases} \Lambda\sum\limits_{\alpha,\ \beta} |\overset{*}{e}_{\alpha\beta}\ (w_0)|^2 \ge |\overset{*}{e}\ (w_0)|^2 \ge \lambda\sum\limits_{\alpha,\ \beta} |\overset{*}{e}_{\alpha\beta}\ (w_0)|^2, \\ \Lambda\sum\limits_{\alpha,\ \beta} |\gamma_{\alpha\beta}(w_0)|^2 \ge |\gamma(w_0)|^2 \ge \lambda\sum\limits_{\alpha,\ \beta}|\gamma_{\alpha\beta}(w_0)|^2, \end{cases}$$

where

$$|w_0|^2_{1,\ D} = \sum_{\alpha,\ \beta}\int_D |\overset{*}{\nabla}_\beta\ w_0^\alpha|^2\sqrt{a}dx = \sum_{\alpha,\ \beta}\|\overset{*}{\nabla}_\beta\ w_0^\alpha\|^2_{0,\ D}$$

denote a semi-norm in $H^1(D)\times H^1(D)$

Proof Since the positive definition of metric tensor $a_{\alpha\beta}$, it is obvious that (5. 21-5. 23) are valid. ♯

Remark 5. 2 It is obvious that $|\ \overset{*}{\nabla}\ \cdot|_{0,\ D}$ is an equivalent semi-norm in $H^1(D)\times H^1(D)$. Hence, by virtue of (5. 21), we assert that

$$\sum_{\alpha,\ \beta}\|\ \overset{*}{e}_{\alpha\beta}\ (w_0)\|_{0,\ D} \le \sqrt{\frac{\Lambda}{\lambda}}|w_0|_{1,\ D}, \quad \forall w_0\in\ H^1(D)\times H^1(D) \tag{5. 22}$$

Lemma 5. 3

$$\begin{cases} |r(w_0)|^2 + (\overset{*}{\text{div}}\ w_0)^2 = |e(w_0)|^2 \\ \qquad - \overset{*}{\nabla}_\alpha\ (w^\beta\ \overset{*}{\nabla}_\beta\ w_0^\alpha - w_0^\alpha\ \overset{*}{\text{div}}\ w_0) + K|w_0|^2, \\ \|r(w_0)\|^2_{0,\ D} + \|\ \overset{*}{\text{div}}\ w_0)\|^2_{0,\ D} = \|e(w_0)\|^2_{0,\ D} + \int_D K|w_0|^2\sqrt{a}dx \\ \qquad - \int_{\partial D}(w^\beta\ \overset{*}{\nabla}_\beta\ w_0 - w_0\ \overset{*}{\text{div}}\ w_0)\cdot ndl. \end{cases} \tag{5. 23}$$

Proof By similar manner,

$$\begin{aligned} |r(w_0)|^2 &= \tfrac14 a^{\alpha\lambda}a^{\beta\sigma}(\overset{*}{\nabla}_\alpha\ w_{0\beta} - \overset{*}{\nabla}_\beta\ w_{0\alpha})(\overset{*}{\nabla}_\lambda\ w_{0\sigma} - \overset{*}{\nabla}_\sigma\ w_{0\lambda}) \\ &= \tfrac14 a^{\alpha\lambda}a^{\beta\sigma}(2\ \overset{*}{e}_{\alpha\beta}\ (w_0) - 2\ \overset{*}{\nabla}_\beta\ w_{0\alpha})(2\ \overset{*}{e}_{\lambda\sigma}\ (w_0) - 2\ \overset{*}{\nabla}_\sigma\ w_{0\lambda}) \\ &= |e(w_0)|^2 - \tfrac12 a^{\alpha\lambda}a^{\beta\sigma}[\overset{*}{e}_{\alpha\beta}\ (w_0)\ \overset{*}{\nabla}_\sigma\ w_{0\lambda} + \overset{*}{e}_{\lambda\sigma}\ (w_0)\ \overset{*}{\nabla}_\beta\ w_{0\alpha} - 2\ \overset{*}{\nabla}_\beta\ w_{0\alpha}\ \overset{*}{\nabla}_\sigma\ w_{0\lambda}] \\ &= |e(w_0)|^2 - \tfrac12 a^{\alpha\lambda}a^{\beta\sigma}[(\overset{*}{e}_{\alpha\beta}\ (w_0) - \overset{*}{\nabla}_\beta\ w_{0\alpha})\ \overset{*}{\nabla}_\sigma\ w_{0\lambda} + (\overset{*}{e}_{\lambda\sigma}\ (w_0) - \overset{*}{\nabla}_\sigma\ w_{0\lambda})\ \overset{*}{\nabla}_\beta\ w_{0\alpha}] \\ &= |e(w_0)|^2 - \tfrac12 a^{\alpha\lambda}a^{\beta\sigma}[\overset{*}{\nabla}_\alpha\ w_{0\beta}\ \overset{*}{\nabla}_\sigma\ w_{0\lambda} + \overset{*}{\nabla}_\beta\ w_{0\alpha}\ \overset{*}{\nabla}_\lambda\ w_{0\sigma}] \\ &= |e(w_0)|^2 - \overset{*}{\nabla}_\alpha\ w_{0\beta}\ \overset{*}{\nabla}^\beta\ w_0^\alpha, \end{aligned}$$

Since (5. 3),

$$|r(w_0)|^2 = |e(w_0)|^2 - \overset{*}{\nabla}_\alpha\ w_0^\lambda\ \overset{*}{\nabla}_\lambda\ w_0^\alpha, \tag{5. 24}$$

Second term of (5. 24) shows

$$\overset{*}{\nabla}_\alpha w_0^\beta \overset{*}{\nabla}_\beta w_0^\alpha = \overset{*}{\nabla}_\alpha (w_0^\beta \overset{*}{\nabla}_\beta w_0^\alpha) - w_0^\beta \overset{*}{\nabla}_\alpha \overset{*}{\nabla}_\beta w_0^\alpha = \overset{*}{\nabla}_\alpha (w_0^\beta \overset{*}{\nabla}_\beta w_0^\alpha) - w_0^\beta \overset{*}{\nabla}_\alpha \overset{*}{\nabla}_\beta w_0^\alpha, \tag{5. 25}$$

Furthermore, by virtue of Ricci formula and Ricci curvature tensor formula for 2D Riemann manifold[1]

$$\overset{*}{\nabla}_\alpha \overset{*}{\nabla}_\beta w_0^\lambda = \overset{*}{\nabla}_\beta \overset{*}{\nabla}_\alpha w_0^\lambda + \overset{*}{R}{}^\lambda{}_{\sigma\alpha\beta} w_0^\sigma, \quad \overset{*}{R}_{\alpha\beta} = K a_{\alpha\beta},$$

we deduce

$$\begin{aligned} w_0^\beta \overset{*}{\nabla}_\alpha \overset{*}{\nabla}_\beta w_0^\alpha &= w_0^\beta \overset{*}{\nabla}_\beta \overset{*}{\nabla}_\alpha w_0^\alpha + w_0^\beta \overset{*}{R}{}^\alpha{}_{\lambda\alpha\beta} w_0^\lambda = w_0^\beta \overset{*}{\nabla}_\beta \overset{*}{\mathrm{div}}\, w_0 + w_0^\beta \overset{*}{R}_{\lambda\beta} w_0^\lambda \\ &= \overset{*}{\nabla}_\beta (w_0^\beta \overset{*}{\mathrm{div}}\, w_0) - w_0^\beta \overset{*}{\nabla}_\beta \overset{*}{\mathrm{div}}\, w_0 - K a_{\lambda\beta} w_0^\beta w_0^\lambda \\ &= \overset{*}{\nabla}_\beta (w_0^\beta \overset{*}{\mathrm{div}}\, w_0) - \overset{*}{\nabla}_\beta (w_0^\beta \overset{*}{\mathrm{div}}\, w_0) + (\overset{*}{\mathrm{div}}\, w_0)^2 - K|w_0|^2 \end{aligned}$$

i. e.

$$w_0^\beta \overset{*}{\nabla}_\alpha \overset{*}{\nabla}_\beta w_0^\alpha = \overset{*}{\nabla}_\sigma (w_0^\beta \overset{*}{\nabla}_\beta w_0^\alpha - w_0^\alpha \overset{*}{\mathrm{div}}\, w_0) + (\overset{*}{\mathrm{div}}\, w_0)^2 - K|w_0|^2, \tag{5. 26}$$

To sum up, (5. 25) and (5. 26) imply (5. 23). The proof is complete. ♮

Theorem 5. 4 Let $\Im$ be a 2D Surface with boundary $\partial\Im$ of $C^{1,\,1}$ defined previously. Let D be a domain in $\Re^2$, $\Theta \in C^3(\overline{D}, E^3)$ be an injective immersion and $w_0 = \{w_0^\alpha, w_0^3\} \in V(D)$:

$$\left\{ \begin{array}{l} V(D) := \{w_0 \in H^1(D) \times H^1(D) \times L^2(D), \quad w_0|_{\partial D} = 0, \quad \text{or} \\ \quad w_0|_{\gamma_s} = 0, \quad (w_0^\beta \overset{*}{\nabla}_\beta w_0^\alpha - w_0^\alpha \overset{*}{\mathrm{div}}\, w_0) n_\alpha|_{\gamma_0} = 0, \ \partial D = \gamma_s \cup \gamma_0\}, \end{array} \right. \tag{5. 27}$$

Furthermore let strain tensor of the vector field w

$$\begin{aligned} &\overset{*}{e}_{\alpha\beta}(w_0) := \tfrac{1}{2}(\overset{*}{\nabla}_\alpha w_{0\beta} + \overset{*}{\nabla}_\beta w_{0\alpha}) = \tfrac{1}{2}(a_{\beta\lambda} \overset{*}{\nabla}_\alpha w_0^\lambda + a_{\alpha\lambda} \overset{*}{\nabla}_\beta w_0^\lambda) \in L^2(D), \\ &\gamma_{\alpha\beta}(w_0) := \overset{*}{e}_{\alpha\beta}(w_0) - b_{\alpha\beta} w_0^3 \in L^2(D). \end{aligned}$$

Then for all $w_0 \in V(D)$,

$$\begin{aligned} &\lambda \sum_{\alpha,\,\beta} \| \overset{*}{\nabla}_\alpha w_0^\beta \|_{0,\,D}^2 \le \| \overset{*}{\nabla} w_0 \|_{0,\,D}^2 \le 2\|e(w_0)\|_{0,\,D}^2 \\ &\quad \le 2\Lambda \sum_{\alpha,\,\beta} \| \overset{*}{e}_{\alpha\beta}(w_0) \|_{0,\,D}^2, \end{aligned} \tag{5. 28}$$

$$\left\{ \begin{array}{l} \{\| \overset{*}{\nabla} w_0 \|_{0,\,D}^2 + \| \overset{*}{\mathrm{div}}\, w_0 \|_{0,\,D}^2 \le 4\|\gamma(w_0)\|_{0,\,D}^2 + K_0 \|w_0^3\|_{0,\,D}^2, \\ \lambda \sum_{\alpha,\,\beta} \| \overset{*}{\nabla}_\alpha w_0^\beta \|_{0,\,D}^2 + \| \overset{*}{\mathrm{div}}\, w_0 \|_{0,\,D}^2 \le \\ \quad 4\Lambda \sum_{\alpha,\,\beta} \|\gamma_{\alpha\beta}(w_0)\|_{0,\,D}^2 + K_0 \|w_0^3\|_{0,\,D}^2, \end{array} \right. \tag{5. 29}$$

$$\left\{ \begin{array}{l} \{\sum_{\alpha,\beta} \|\frac{\partial w_0^\beta}{\partial x^\alpha}\|_{0,D}^2 + \sum_\alpha \|w_0^\alpha\|_{0,D}^2\} \le C(\sum_{\alpha,\beta} \| \overset{*}{e}_{\alpha\beta}(w_0) \|_{0,D}^2 + \sum_\alpha \|w_0^\alpha\|_{0,D}^2), \\ \{\sum_{\alpha,\,\beta} \|\frac{\partial w_0^\beta}{\partial x^\alpha}\|_{0,\,D}^2 + \sum_\alpha \|w_0^\alpha\|_{0,\,D}^2 + \| \overset{*}{\mathrm{div}}\, w_0 \|_{0,\,D}^2\} \\ \qquad \le C(\sum_{\alpha,\,\beta} \|\gamma_{\alpha\beta}(w_0)\|_{0,\,D}^2 + \|w_0\|_{0,\,D}^2). \end{array} \right. \tag{5. 30}$$

where

$$K_0 = 4 \min_D (k_1^2 + k_2^2), \quad \|w_0\|_{0,\,D}^2 = \sum_i \|w_0^i\|_{0,\,D}^2, \tag{5. 31}$$

.

Proof Integrating both sides of the first of (5. 18) and using Gauss theorem and boundary conditions

$$\int_D \overset{*}{\nabla}_\alpha (w^\beta \overset{*}{\nabla}_\beta w^\alpha - w^\alpha \overset{*}{\operatorname{div}} w)\sqrt{a}dx = \int_{\partial D}(w^\beta \overset{*}{\nabla}_\beta w^\alpha - w^\alpha \overset{*}{\operatorname{div}} w)n_\alpha ds = 0$$

we infer (5. 28). By similar manner, from the second of (5. 17) and (5. 20) assert (5. 29).

Next, let consider Sobolev norm. Because the covariant derivative

$$\overset{*}{\nabla}_\alpha w_0^\beta = \frac{\partial w_0^\beta}{\partial x^\alpha} + \overset{*}{\Gamma^\beta}_{\alpha\lambda} w_0^\lambda, \qquad \frac{\partial w_0^\beta}{\partial x^\alpha} = \overset{*}{\nabla}_\alpha w_0^\beta - \overset{*}{\Gamma^\beta}_{\alpha\lambda} w_0^\lambda$$

we claim

$$\sum_{\alpha,\ \beta}\|\frac{\partial w_0^\beta}{\partial x^\alpha}\|_{0,\ D}^2 \le C\{\sum_{\alpha,\ \beta}\| \overset{*}{\nabla}_\alpha w_0^\beta\|_{0,\ D}^2 + \sum_\alpha \|w_0^\alpha\|_{0,\ D}^2\}, \tag{5. 32}$$

To sum-up, (5, 28)(5. 29) and (5. 32) imply (5. 30). The proof is complete. ♮.

6. Existence of Solution of Variational Problem

In this section we study the variational problem (4. 6) on the manifold $\Im$

$$\begin{cases} \text{Find}(w_0, p_0) \in V(D) \times L^2(D), \text{ such that} \forall\, (v, q) \in V(D) \times L^2(D), \\ a_0(w_0,\ v) - (p_0,\ \overset{*}{\operatorname{div}}\, v) + b_0(\rho; w_0,\ w_0,\ v) + (l(\rho_0,\ w_0),\ v) \\ \qquad\qquad = < G,\ v >, \\ (\overset{*}{\operatorname{div}}\,(\rho_0 w_0) - 2H\rho_0 w_0^3 + d_0^3,\ q) = 0, \end{cases} \tag{6. 1}$$

where $p_0 = A\rho^\gamma$ and all terms in (6. 1) are defined by (4. 7). Variational problem is a irregular problem. In order to regularization we introduce an artificial viscosity η such that

$$\begin{cases} \text{Find}(w_0, p_0) \in V(D) \times L^2(D), \text{ such that} \forall\, (v, q) \in V(D) \times L^2(D), \\ a_0(w_0,\ v) - (p_0,\ \overset{*}{\operatorname{div}}\, v) + b_0(\rho; w_0,\ w_0,\ v) + (l(\rho_0,\ w_0),\ v) \\ \qquad\qquad = < G,\ v >, \\ \eta(\overset{*}{\nabla} p_0,\ \overset{*}{\nabla} q) + ((\overset{*}{\operatorname{div}}\,(\rho_0 w_0) - 2H\rho_0 w_0^3 + d_0^3,\ q) = 0, \end{cases} \tag{6. 2}$$

Our primary objective consist in showing that the bilinear form defined by (4. 7) is $V(D)$-elliptic.

Lemma 6. 1 Let there be given a domain D in $\Re^2$ and an injective mapping $\vec{\Theta} \in C^3(\bar{D}; E^3)$ such that the two vectors $\vec{a}_\alpha = \partial_\alpha \vec{\Theta}$ are linearly independent at all points of $\bar{D}$. Let γ_0 be a $d\gamma$-measurable subset of $\gamma = \partial D$ that satisfies length $\gamma_0 > 0$. Then bilinear form $a_0(\cdot,\ \cdot)$ in $V(D)$ defined by (4. 7) is symmetric, continuous and elliptic

$$\begin{cases} (i) & a_0(w,\ v) = a_0(v,\ w),\ \forall\ w,\ v \in\ V(D); \\ (ii) & |a_0(w,\ v)| \le C\|w\|_{1,\ D}\|v\|_{0,\ D},\ \forall\ w,\ v \in\ V(D); \\ (iii) & a_0(w,\ w) \ge C_0\|w\|_{1,\ D}^2,\ \forall\ w\ \in\ V(D). \end{cases} \tag{6. 3}$$

where $V(D)$ is defined by (5, 27) with the Sobolev norm

$$\|w\|_{1,\ D}^2 = \sum_{i,\ j}(\|\partial_i w^j\|_{0,\ D}^2 + \|w^i\|_{0,\ D}^2) = |w|_{0,\ 1}^2 + \|w\|_{0,\ D}^2.$$

where denote $x^3 = \xi$.

Proof Indeed it is enough to prove (iii). Since (3. 32),

$$a^{\alpha\beta\lambda\sigma}\gamma_{\lambda\sigma}(w_0)\gamma_{\alpha\beta}(w_0) = 2\mu\gamma^{\alpha\beta}(w_0)\gamma_{\alpha\beta}(w_0)h + \frac{2}{3}(\overset{*}{\text{div}}\ w_0)^2 \geq 2\mu\gamma^{\alpha\beta}(w_0)\gamma_{\alpha\beta}(w_0),$$

it infer

$$a_0(w_0,\ w_0) \geq \nu(2\|\gamma(w_0)\|^2_{0,\ D} + \|\overset{*}{\nabla}\ w_0^3\|^2_{0,\ D} + h^{-2}\|w\|^2_{0,\ D})$$

Taking (5. 32) into account,

$$\begin{aligned} a_0(w_0,\ w_0) &\geq C(\sum_{\alpha,\ \beta}\|\frac{\partial w_0^\beta}{\partial x^\alpha}\|^2_{0,\ D} + \sum_\alpha \|w_0^\alpha\|^2_{0,\ D}) + \nu|w_0^3|^2_{1,\ D}) \\ &\geq C(|w_0|^2_{1,\ D} + \sum_\alpha \|w_0^\alpha\|^2_{0,\ D}) \end{aligned}$$

Employing Poincare inequality, semi-norm $|w|_{1,\ D}$ is equivalent to the full norm $\|w\|_{1,\ D}$ in the $V(D)$ we assert (iii). To sum up, it concludes our proof. ♯

Lemma 6. 2 Let there be given a domain D in $\Re^2$ and an injective mapping $\vec{\Theta} \in C^3(\bar{D}; E^3)$ such that the two vectors $\vec{a}_\alpha = \partial_\alpha\vec{\Theta}$ are linearly independent at all points of $\bar{D}$. Let γ_0 be a $d\gamma$-measurable subset of $\gamma = \partial D$ that satisfies length $\gamma_0 > 0$. Then the trilinear form $b_0(\cdot,\ \cdot,\ \dot{)}$ defined by (4. 9) is continuous: there exists a constant $M(\Theta,\ D)$ independent of $\rho_0,\ w_0$

$$\begin{aligned} &|b_0(\rho_0,\ w_0,\ w_0,\ v)| \leq M\|w_0\|_{0,\ q}\|\rho_0\|_{0,\ \gamma}\|v\|_{1,\ +\infty}, \\ &\forall\ w_0 \in L^q(D),\ \rho_0 \in L^\gamma(D),\ v \in C^\infty(D), \end{aligned} \tag{6. 4}$$

where $\gamma \geq 1, \quad q = \frac{2\gamma}{\gamma - 1}$.

Proof From (4. 7) and using Hölder inequality it is easy to obtain (6. 4), Proof is complete. ♯

Theorem 6. 1 Let there be given a domain D in $\Re^2$ and an injective mapping $\vec{\Theta} \in C^3(\bar{D}; E^3)$ such that the two vectors $\vec{a}_\alpha = \partial_\alpha\vec{\Theta}$ are linearly independent at all points of $\bar{D}$. Let γ_0 be a $d\gamma$-measurable subset of $\gamma = \partial D$ that satisfies length $\gamma_0 > 0$. For given $(G,\ d_0^3) \in V^*(D) \times H^{-1}(D)$, there exist the positive numbers $C_0,\ \eta$ satisfying

$$C_0 \geq \frac{C}{\chi}\|G\|_{V^*(D)} + 1 + \chi), \qquad \eta \geq \frac{C}{\chi}\|d_0^3\|_{V^*(D)} + 1 + \chi) \tag{6. 5}$$

for any positive number χ and a unique solution $(w_0,\ p_0) : \|w_0\|_{1,\ D} \leq \chi,\ \|p_0\|_{1,\ D} \leq \chi$ of variational problem (6. 2).

Furthermore there exists a sequence $(w_0(\eta_k),\ p_0(\eta_k))$ of solution to (6. 2) with artificial viscosity η_k weakly converging to $(w_0,\ p_0) \in (V(D) \times H_0^1(D))$ which satisfy (6. 1).

The proof is omitted.

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