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NONCONFORMING MIXED FINITE ELEMENT METHOD FOR THE STATIONARY CONDUCTION-CONVECTION PROBLEM

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Abstract. In this paper, a new stable nonconforming mixed finite element scheme is proposed for the stationary conduction-convection problem, in which a new low order Crouzeix-Raviart type nonconforming rectangular element is taken as approximation space for the velocity, the piecewise constant element for the pressure and the bilinear element for the temperature, respectively. The convergence analysis is presented and the optimal error estimates in a broken H^1-norm for the velocity, L^2-norm for the pressure and H^1-seminorm for the temperature are derived.

Key Words. stationary conduction-convection problem, nonconforming mixed finite element, the optimal error estimates.

1. Introduction

We consider the following stationary conduction-convection problem (cf. [1-4]):
Problem (I) Find $u = (u^1, u^2)$, p and T such that

$$\left\{\begin{array}{ll} -\mu\triangle u + (u\cdot\nabla)u + \nabla p = \lambda jT, & \text{in} \quad \Omega, \\ \text{div}u = 0, & \text{in} \quad \Omega, \\ -\triangle T + \lambda u\cdot\nabla T = 0, & \text{in} \quad \Omega, \\ u = 0, \quad T = T_0, & \text{on} \quad \partial\Omega, \end{array}\right. \tag{1.1}$$

where $\Omega \subset R^2$ is a bounded domain with boundary $\partial\Omega$, u denotes the fluid velocity vector field, p the pressure field, T the temperature field, $\mu > 0$ the coefficient of the kinematic viscosity, $\lambda > 0$ the Groshoff number, $j = (0, 1)$ the two-dimensional vector and T_0 the given initial scale function.

The stationary conduction-convection problem (I) is the coupled equations governing steady viscous incompressible flow and heat transfer process, where incompressible fluid is the Boussinnesq's approximation of the Navier-Stokes equations. In atmospheric dynamics it is an important compelling dissipative nonlinear system, which contains not only the velocity vector field and the pressure field but also the temperature field. From the thermodynamics point of view, we know that the movement of the fluid must have viscosity which will produce quantity of heat. Thus, the movement of the fluid must be companied with mutual transformation of temperature, speed and pressure. Therefore, it is very universal to investigate this nonlinear system.

So far some numerical methods have been studied on the conduction-convection problem (cf. [1-2,5-8]), but much less attention is paid to the theoretical error

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analysis of the mixed finite element methods. Shen [9] firstly analyzed the existence and uniqueness of approximation solution, and gave first order accuracy with Bernadi-Raugel element [25] in terms of the small Groshoff number λ that appears in Problem (I) (refer to [10] for the detailed proof). However, the analysis in [9-10] is about the conforming finite elements. Indeed, it seems that there are few studies focusing on the approximation of Problem (I) with the nonconforming finite elements. Recently, these elements have attracted increasing attention from scientists and engineers in various areas since they have some practical advantages. On the one hand, they are usually much easier to be constructed to satisfy the discrete inf-sup condition than the conforming ones, which is usually required in the mixed finite element analysis. On the other hand, from the domain decomposition methods point of view, since the unknowns are associated with the element faces, each degree of freedom belongs to at most two elements, the use of the nonconforming finite elements facilitates the exchange of information across each subdomain and provides spectral radius estimates for the iterative domain decomposition operator [23].

The main aim of this paper is to construct a new low order Crouzeix-Raviart type nonconforming rectangular element and apply it to Problem (I) with a mixed finite element scheme. By virtue of the element's special properties, the convergence analysis is presented and the optimal error estimates are obtained. The remainder of this paper is organized as follows. In section 2, we introduce the variational formulation of Problem (I) and briefly recite the existence and uniqueness of its solution proved in [9-10]. In section 3, we first give the construction of Crouzeix-Raviart type nonconforming mixed finite element scheme and then prove that the approximation spaces of the velocity and the pressure satisfy the discrete inf-sup condition, which yields the existence and uniqueness of approximation solution. In the last section, some important lemmas, the convergence analysis and the optimal error estimates are obtained.

We will employ standard definitions for the Sobolev spaces $W^{k,p}(\Omega)$ with norm $\|\cdot\|_{k,p,\Omega}$, and $H^k(\Omega)=W^{k,2}(\Omega)$ with norm $\|\cdot\|_k$, respectively (cf. [17]). Throughout the paper, C indicates a positive constant, possibly different at different occurrences, which is independent of the mesh parameter h, but may depend on Ω and other parameters introduced in this paper. Notations not especially explained are used with their usual meanings.

2. The existence and uniqueness of the solution to the variational formulation

The variational formulation of Problem (I) is written as:
Problem (I^*) Find $(u,p,T)\in X\times M\times W$, such that $T|_{\partial\Omega}=T_0$

$$
\left\{\begin{array}{ll} a(u,v)+a_1(u;u,v)-b(p,v)=\lambda(jT,v), & \forall\ v\in X,\\ b(q,u)=0, & \forall\ q\in M,\\ d(T,\varphi)+\lambda\bar{a}_1(u;T,\varphi)=0, & \forall\ \varphi\in W_0, \end{array}\right. \tag{2.1}
$$

where

$$
X=H_0^1(\Omega)^2, M=L_0^2(\Omega)=\left\{q,q\in L^2(\Omega),\int_\Omega q dx=0\right\}, W=H^1(\Omega), W_0=H_0^1(\Omega),
$$

$(\cdot,\cdot)$ means the inner product in $L^2(\Omega)^2$ or in $L^2(\Omega)$ according to the context, $a(u,v)=\mu(\nabla u,\nabla v), b(p,v)=(p,\mathrm{div}v), d(T,\varphi)=(\nabla T,\nabla\varphi)$,

$$a_1(u;v,w)=\frac{1}{2}\int_\Omega\sum_{i,j=1}^{2}\Big(u^i\frac{\partial v^j}{\partial x_i}w^j-u^i\frac{\partial w^j}{\partial x_i}v^j\Big)dx, \tag{2.2}$$

and

$$\bar{a}_1(u;T,\varphi)=\frac{1}{2}\int_\Omega\sum_{i=1}^{2}\Big(u^i\frac{\partial T}{\partial x_i}\varphi-u^i\frac{\partial \varphi}{\partial x_i}T\Big)dx. \tag{2.3}$$

It has been shown in [11-15] that there exist positive constants C_i and $C_i^{'}(i=1,2)$ such that

$$\|v\|_0\le C_1\|\nabla v\|_0,\ \ \forall\ v\in H_0^1(\Omega)^2,\ \ \|v\|_0\le C_1^{'}\|\nabla v\|_0,\ \ \forall\ v\in H_0^1(\Omega), \tag{2.4}$$

$$\|v\|_{0,4,\Omega}\le C_2\|\nabla v\|_0,\ \ \forall\ v\in H_0^1(\Omega)^2,\ \ \|v\|_{0,4,\Omega}\le C_2^{'}\|v\|_1,\ \ \forall\ v\in H^1(\Omega). \tag{2.5}$$

Besides, the trilinear forms $a_1(\cdot;\cdot,\cdot)$ and $\bar{a}_1(\cdot;\cdot,\cdot)$ satisfy

$$a_1(u;v,v)=0,\quad a_1(u;v,w)=-a_1(u;w,v),\ \ \forall\ u,v,w\in X, \tag{2.6}$$

$$\bar{a}_1(u;T,T)=0,\ \ \bar{a}_1(u;T,\varphi)=-\bar{a}_1(u;\varphi,T),\ \ \forall\ u\in X,T,\varphi\in W. \tag{2.7}$$

Let $V=\{v\in X,\mathrm{div}v=0\}$ denote the divergence-free subspace of X. Using Green's formula, (2.2) and (2.3) can be rewritten as

$$\begin{aligned}a_1(u;v,w)&=\frac{1}{2}\int_\Omega\sum_{i,j=1}^{2}\Big(u^i\frac{\partial v^j}{\partial x_i}w^j-u^i\frac{\partial w^j}{\partial x_i}v^j\Big)dx\\&=\int_\Omega\sum_{i,j=1}^{2}u^i\frac{\partial v^j}{\partial x_i}w^jdx,\ \ \forall\ u\in V,v\in H^1(\Omega)^2,w\in X,\end{aligned}$$

and

$$\begin{aligned}\bar{a}_1(u;T,\varphi)&=\frac{1}{2}\int_\Omega\sum_{i=1}^{2}\Big(u^i\frac{\partial T}{\partial x_i}\varphi-u^i\frac{\partial \varphi}{\partial x_i}T\Big)dx\\&=\int_\Omega\sum_{i=1}^{2}u^i\frac{\partial T}{\partial x_i}\varphi dx,\ \ \forall\ u\in V,T\in W,\varphi\in W_0,\end{aligned}$$

respectively.

We obtain in view of $H^1(\Omega)\hookrightarrow L^4(\Omega)$

$$|a_1(u;v,w)|\le N\|\nabla u\|_0\|\nabla v\|_0\|\nabla w\|_0,\ \ \forall\ u\in V,v\in H^1(\Omega)^2,w\in X, \tag{2.8}$$

$$|\bar{a}_1(u;T,\varphi)|\le \bar{N}\|\nabla u\|_0\|\nabla T\|_0\|\nabla \varphi\|_0,\ \ \forall\ u\in V,T\in W,\varphi\in W_0, \tag{2.9}$$

where

$$N=\sup_{u\in V,v\in H^1(\Omega)^2,w\in X}\frac{|a_1(u;v,w)|}{\|\nabla u\|_0\|\nabla v\|_0\|\nabla w\|_0},\ \ \bar{N}=\sup_{u\in V,T\in W,\varphi\in W_0}\frac{|\bar{a}_1(u;T,\varphi)|}{\|\nabla u\|_0\|\nabla T\|_0\|\nabla \varphi\|_0}.$$

In order to prove the existence and uniqueness of the solution to Problem (I^*), the following assumption is required as [14].

(A_1) Assume that $T_0\in C^{1,\alpha}(\partial\Omega)(0<\alpha\le 1)$, and there exists a prolongation of $T_0\in C_0^{1,\alpha}(R^2)$ (still marked as T_0) such that

$$\|T_0\|_1\le\varepsilon,$$

where ε is a sufficiently small positive constant.

Then the following important conclusion can be found in [9-10].

Theorem 2.1. Under the assumption of (A_1), let $A = 2\mu^{-1}\lambda C_1(1+3C_1^{'})\|T_0\|_1$ and $B = \left(5+\dfrac{1}{C_1^{'}}\right)\|T_0\|_1$. Assume that there exist positive constants δ_1 and δ_2 such that

$$(2.10)\quad \mu^{-1}NA \leq 1-\delta_1 \ (0<\delta_1\leq 1);\quad \delta_1^{-1}\mu^{-1}\lambda^2 C_1 C_1^{'}\bar{N}B \leq 1-\delta_2 \ (0<\delta_2\leq 1).$$

Then Problem (I^*) has a unique solution $(u,p,T)\in X\times M\times W$, satisfying

$$\|\nabla u\|_0 \leq A, \quad \|\nabla T\|_0 \leq B. \tag{2.11}$$

3. Construction of nonconforming mixed finite element scheme

Let $\hat{K}=[-1,1]\times[-1,1]$ be the reference element on $\xi-\eta$ plane, the four vertices of $\hat{K}$ are $\hat{d}_1=(-1,-1)$, $\hat{d}_2=(1,-1)$, $\hat{d}_3=(1,1)$ and $\hat{d}_4=(-1,1)$, the four edges are $\hat{l}_1=\overline{\hat{d}_1\hat{d}_2}$, $\hat{l}_2=\overline{\hat{d}_2\hat{d}_3}$, $\hat{l}_3=\overline{\hat{d}_3\hat{d}_4}$ and $\hat{l}_4=\overline{\hat{d}_4\hat{d}_1}$.

For any $\hat{v}^1,\hat{v}^2\in H^1(\hat{K})$, we define the finite elements $(\hat{K},\hat{P}^i,\hat{\Sigma}^i)$, $i=1,2$, on $\hat{K}$ as follows:

$$\hat{\Sigma}^1=\{\hat{v}_1^1,\hat{v}_2^1,\hat{v}_3^1,\hat{v}_4^1\},\quad \hat{P}^1=\operatorname{span}\{1,\xi,\eta,\eta^2\}, \tag{3.1}$$

$$\hat{\Sigma}^2=\{\hat{v}_1^2,\hat{v}_2^2,\hat{v}_3^2,\hat{v}_4^2\},\quad \hat{P}^2=\operatorname{span}\{1,\xi,\eta,\xi^2\}, \tag{3.2}$$

where $\hat{v}_i^j=\dfrac{1}{|\hat{l}_i|}\displaystyle\int_{\hat{l}_i}\hat{v}^j d\hat{s},\ i=1,2,3,4,\ \ j=1,2.$

It can be easily checked that the interpolations defined above are well-posed and the interpolation functions $\hat{\Pi}^i\hat{v}^i$ (i=1,2) can be expressed as:

(3.3)

$$\hat{\Pi}^1\hat{v}^1=\frac{3}{4}(\hat{v}_2^1+\hat{v}_4^1)-\frac{1}{4}(\hat{v}_1^1+\hat{v}_3^1)+\frac{1}{2}(\hat{v}_2^1-\hat{v}_4^1)\xi+\frac{1}{2}(\hat{v}_3^1-\hat{v}_1^1)\eta+\frac{3}{4}(\hat{v}_1^1-\hat{v}_2^1+\hat{v}_3^1-\hat{v}_4^1)\eta^2,$$

and

(3.4)

$$\hat{\Pi}^2\hat{v}^2=\frac{3}{4}(\hat{v}_1^2+\hat{v}_3^2)-\frac{1}{4}(\hat{v}_2^2+\hat{v}_4^2)+\frac{1}{2}(\hat{v}_2^2-\hat{v}_4^2)\xi+\frac{1}{2}(\hat{v}_3^2-\hat{v}_1^2)\eta+\frac{3}{4}(-\hat{v}_1^2+\hat{v}_2^2-\hat{v}_3^2+\hat{v}_4^2)\xi^2,$$

respectively.

For the sake of convenience, like [9-10] let $\Omega\subset R^2$ be a polygon with boundaries parallel to the axes, $\Im_h$ be an axis parallel rectangular meshes of Ω, where $\Im_h$ satisfies the usual regularity assumption and quasi-uniform assumption (cf. [16]).

For any $K\in\Im_h$, let

$$K=[x_{1K}-h_{x_1},x_{1K}+h_{x_1}]\times[x_{2K}-h_{x_2},x_{2K}+h_{x_2}],$$
$$h_K=\max_{K\in\Im_h}\{h_{x_1},h_{x_2}\} \text{ and } h=\max_{K\in\Im_h}h_K.$$

Define the affine mapping $F_K:\hat{K}\to K$ as follows:

$$\begin{cases} x_1=x_{1K}+h_{x_1}\xi,\\ x_2=x_{2K}+h_{x_2}\eta. \end{cases} \tag{3.5}$$

Then the associated finite element spaces X_h, M_h, W_h and W_{0h} can be defined as:

$$\begin{cases} X_h=\Big\{v=(v^1,v^2),\hat{v}^i|_{\hat{K}}=v^i|_K\circ F_K\in\hat{P}^i,\ \forall\ K\in\Im_h,\\ \qquad\qquad\qquad\qquad \displaystyle\int_F[v^i]ds=0, F\subset\partial K, i=1,2\Big\},\\ M_h=\{\varphi\in M,\varphi|_K\in Q_{0,0}(K),\ \ \forall\ K\in\Im_h\},\\ W_h=\{p\in W,p|_K\in Q_{1,1}(K),\ \ \forall\ K\in\Im_h\},\\ W_{0h}=W_h\cap H_0^1(\Omega), \end{cases} \tag{3.6}$$

where $[v^i]$ $(i=1,2)$ stand for the jump of v^i across the edge F if F is an internal edge, and they are equal to v^i itself if F belongs to $\partial\Omega$, $Q_{j,j}(K)$ is a space of polynomials whose degrees for x_1 and x_2 are equal to or less than j, respectively.

Obviously, $X_h \not\subset H^1(\Omega)^2, M_h \subset M, W_h \subset W$, so this is a nonconforming mixed finite element scheme.

Define the interpolation operator $\Pi_h : v = (v^1, v^2) \in H^1(\Omega)^2 \mapsto \Pi_h v \in X_h$ as:

$$\Pi_h v = (\Pi_h^1 v^1, \Pi_h^2 v^2),\ \Pi_h^i\,|_K = \Pi_K^i,\ \Pi_K^i v^i = \hat{\Pi}^i \hat{v}^i \circ F_K^{-1}, i = 1, 2.$$

Let $\hat{\Pi}\hat{v} = (\hat{\Pi}^1 \hat{v}^1, \hat{\Pi}^2 \hat{v}^2)$ and $\Pi_K v = (\Pi_K^1 v^1, \Pi_K^2 v^2)$, then

$$\int_{l_i} (v - \Pi_K v) ds = 0,\ i = 1, 2, 3, 4, \forall\ v \in H^1(\Omega)^2, \tag{3.7}$$

where $l_i = \hat{l}_i \circ F_K^{-1} (i = 1, 2, 3, 4)$ are the four edges of K, respectively.

For all $v_h = (v_h^1, v_h^2) \in X_h$, we define

$$\|v_h\|_h \;=\; \Big(\sum_{K\in\Im_h} \int_K \nabla v_h \cdot \nabla v_h dx \Big)^{\frac{1}{2}} = \Big(\sum_{K\in\Im_h} (|v_h^1|_{1,K}^2 + |v_h^2|_{1,K}^2) \Big)^{\frac{1}{2}}.$$

Then $\|\cdot\|_h$ is the norm over X_h.

We introduce the bilinear forms $a^h(\cdot,\cdot)$ and $b^h(\cdot,\cdot)$ and the trilinear form $a_1^h(\cdot;\cdot,\cdot)$ as:

$$a^h(u_h, v_h) = \mu \sum_{K\in\Im_h} \int_K \nabla u_h \cdot \nabla v_h dx,\ \ \forall\ u_h, v_h \in X_h, \tag{3.8}$$

$$b^h(p_h, v_h) = \sum_{K\in\Im_h} \int_K p_h \mathrm{div} v_h dx,\ \ \forall\ v_h \in X_h, p_h \in M_h, \tag{3.9}$$

and

$$a_1^h(u_h; v_h, w_h) = \frac{1}{2} \sum_{K\in\Im_h} \int_K \sum_{i,j=1}^{2} \Big(u_h^i \frac{\partial v_h^j}{\partial x_i} w_h^j - u_h^i \frac{\partial w_h^j}{\partial x_i} v_h^j \Big) dx,\ \ \forall\ u_h, v_h, w_h \in X_h, \tag{3.10}$$

respectively.

Then the approximation of Problem (I^*) reads as follows:

Problem(I^h) Find $(u_h, p_h, T_h) \in X_h \times M_h \times W_h$, such that $T_h|_{\partial\Omega} = T_0$

$$\left\{ \begin{array}{ll} a^h(u_h, v_h) + a_1^h(u_h; u_h, v_h) - b^h(p_h, v_h) = \lambda(jT_h, v_h), & \forall\ v_h \in X_h, \\ b^h(q_h, u_h) = 0, & \forall\ q_h \in M_h, \\ d(T_h, \varphi_h) + \lambda \bar{a}_1(u_h; T_h, \varphi_h) = 0, & \forall\ \varphi_h \in W_{0h}. \end{array} \right. \tag{3.11}$$

Using the similar technique as in [18] we have the following discrete embedding inequality over X_h,

$$\|v_h\|_{0,2k,\Omega} \le C(k) \|v_h\|_h, \quad \forall\ v_h \in X_h, \quad k = 1, 2, \cdots,\ n = 2, \tag{3.12}$$

especially, when $k = 1$ we denote the constant $C(1)$ by C_3, i.e.,

$$\|v_h\|_0 \le C_3 \|v_h\|_h, \quad \forall\ v_h \in X_h. \tag{3.13}$$

It can be checked that the trilinear forms $a_1^h(\cdot;\cdot,\cdot)$ and $\bar{a}_1(\cdot;\cdot,\cdot)$ satisfy the following properties (cf. [18]):

$$a^h(v_h, v_h) = \mu \|v_h\|_h^2, \quad \forall\ v_h \in X_h, \tag{3.14}$$

$$a_1^h(u_h; v_h, v_h) = 0, \quad a_1^h(u_h; v_h, w_h) = -a_1^h(u_h; w_h, v_h),\ \ \forall\ u_h, v_h, w_h \in X_h, \tag{3.15}$$

(3.16)
$$\bar{a}_1(u_h;T_h,T_h)=0,\quad \bar{a}_1(u_h;T_h,\varphi_h)=-\bar{a}_1(u_h;\varphi_h,T_h),\quad \forall\ u_h\in X_h,\quad T_h,\varphi_h\in W_h,$$

$$|a_1^h(u_h;v_h,w_h)|\le N_h\|u_h\|_h\|v_h\|_h\|w_h\|_h,\quad \forall\ u_h,v_h,w_h\in H^1(\Omega)^2\cup X_h, \tag{3.17}$$

(3.18)
$$|\bar{a}_1(u_h;T_h,\varphi_h)|\ \le \bar{N}_h\|u_h\|_h\|\nabla T_h\|_0\|\nabla\varphi_h\|_0,\ \ \forall\, u_h\in H^1(\Omega)^2\cup X_h,\ \ T_h\in W_h,\varphi_h\in W_{0h},$$

where

$$N_h=\sup_{u_h,v_h,w_h\in X_h}\frac{|a_1^h(u_h;v_h,w_h)|}{\|u_h\|_h\|v_h\|_h\|w_h\|_h},\quad \bar{N}_h=\sup_{u_h\in X_h,T_h\in W_h,\varphi_h\in W_{0h}}\frac{|\bar{a}_1(u_h;T_h,\varphi_h)|}{\|u_h\|_h\|\nabla T_h\|_0\|\nabla\varphi_h\|_0},$$

which are norms for the trilinear forms $a_1^h(\cdot;\cdot,\cdot)$ and $\bar{a}_1(\cdot;\cdot,\cdot)$,respectively.

In view of (3.12), we know that there exist two positive constants $N_0>1$ and $\bar{N}_0>1$, such that

$$N_h\le N_0,\qquad \bar{N}_h\le\bar{N}_0,\quad \forall\ 0<h\le 1. \tag{3.19}$$

Therefore

$$|a_1^h(u_h;v_h,w_h)|\le N_0\|u_h\|_h\|v_h\|_h\|w_h\|_h,\quad \forall\ u_h,v_h,w_h\in H^1(\Omega)^2\cup X_h, \tag{3.20}$$

(3.21)
$$|\bar{a}_1(u_h;T_h,\varphi_h)|\le \bar{N}_0\|u_h\|_h\|\nabla T_h\|_0\|\nabla\varphi_h\|_0,\ \ \forall\, u_h\in H^1(\Omega)^2\cup X_h,\ \ T_h\in W_h,\varphi_h\in W_{0h}.$$

Lemma 3.1. The spaces X_h and M_h satisfy the discrete inf-sup condition (cf. [14]), i.e.,

$$\sup_{v_h\in X_h}\frac{b^h(q_h,v_h)}{\|v_h\|_h}\ge\beta\|q_h\|_0,\quad \forall\ q_h\in M_h, \tag{3.22}$$

where β is a positive constant independent of h.

Proof. By (3.3) and (3.4), let

$$\alpha_1=\frac{1}{2}(\hat{v}_2^1-\hat{v}_4^1),\quad \alpha_2=\frac{1}{2}(\hat{v}_3^1-\hat{v}_1^1),\quad \alpha_3=\frac{3}{4}(\hat{v}_1^1-\hat{v}_2^1+\hat{v}_3^1-\hat{v}_4^1),$$

$$\beta_1=\frac{1}{2}(\hat{v}_2^2-\hat{v}_4^2),\quad \beta_2=\frac{1}{2}(\hat{v}_3^2-\hat{v}_1^2),\quad \beta_3=\frac{3}{4}(\hat{v}_2^2-\hat{v}_1^2+\hat{v}_4^2-\hat{v}_3^2).$$

It can be checked that the following inequalities hold

$$|\alpha_i|\le C|\hat{v}^1|_{1,\hat{K}},\quad |\beta_i|\le C|\hat{v}^2|_{1,\hat{K}},\ \ i=1,2,3. \tag{3.23}$$

In fact

$$\begin{aligned}\alpha_1 &= \frac{1}{2}(\hat{v}_2^1-\hat{v}_4^1)=\frac{1}{4}\Big[\int_{\hat{l}_2}\hat{v}^1(1,\eta)d\eta-\int_{\hat{l}_4}\hat{v}^1(-1,\eta)d\eta\Big]\\ &= \frac{1}{4}\int_{\hat{K}}\frac{\partial\hat{v}^1}{\partial\xi}d\xi d\eta\le\frac{1}{2}\parallel\frac{\partial\hat{v}^1}{\partial\xi}\parallel_{0,\hat{K}}\le C|\hat{v}^1|_{1,\hat{K}},\end{aligned}$$

$$\begin{aligned}\alpha_2 &= \frac{1}{2}(\hat{v}_3^1-\hat{v}_1^1)=\frac{1}{4}\Big[\int_{\hat{l}_3}\hat{v}^1(\xi,1)d\xi-\int_{\hat{l}_1}\hat{v}^1(\xi,-1)d\xi\Big]\\ &= \frac{1}{4}\int_{\hat{K}}\frac{\partial\hat{v}^1}{\partial\eta}d\xi d\eta\le\frac{1}{2}\parallel\frac{\partial\hat{v}^1}{\partial\eta}\parallel_{0,\hat{K}}\le C|\hat{v}^1|_{1,\hat{K}},\end{aligned}$$

$$\begin{aligned}
\alpha_3 &= \frac{3}{4}(\hat{v}_1^1-\hat{v}_2^1+\hat{v}_3^1-\hat{v}_4^1)\\
&= \frac{3}{4}\Big[\int_{\hat{K}}\hat{v}^1(\xi,\eta)d\xi d\eta-\int_{\hat{l}_2}\hat{v}^1(1,\eta)d\eta-\int_{\hat{l}_4}\hat{v}^1(-1,\eta)d\eta\\
&\quad+\int_{\hat{l}_3}\hat{v}^1(\xi,1)d\xi+\int_{\hat{l}_1}\hat{v}^1(\xi,-1)d\xi-\int_{\hat{K}}\hat{v}^1(\xi,\eta)d\xi d\eta\Big]\\
&= \frac{3}{4}\Big[\int_{\hat{K}}\eta\frac{\partial\hat{v}^1}{\partial\eta}d\xi d\eta-\int_{\hat{K}}\xi\frac{\partial\hat{v}^1}{\partial\xi}d\xi d\eta\Big]\\
&\le C|\hat{v}^1|_{1,\hat{K}}.
\end{aligned}$$

Similarly

$$\beta_i\le C|\hat{v}^2|_{1,\hat{K}},\ i=1,2,3. \tag{3.24}$$

From (3.23) we can prove

$$\|\Pi_h v\|_h\le C|v|_1,\quad \forall\ v\in X. \tag{3.25}$$

By the definition of interpolation (3.7) and Green's formula, we get for all $v\in X$

$$\begin{aligned}
b^h(q_h,v-\Pi_h v) &= \sum_{K\in\Im_h}\int_K q_h\mathrm{div}(v-\Pi_K v)dx\\
&= \sum_{K\in\Im_h} q_h|_K\int_K \mathrm{div}(v-\Pi_K v)dx\\
&= \sum_{K\in\Im_h} q_h|_K\int_{\partial K}(v-\Pi_K v)\cdot nds\\
&= 0.
\end{aligned}\tag{3.26}$$

The fact that q_h belongs to a piece-wise constant space is used. Here and later n denotes the outward unit normal vector to ∂K. Since the spaces X and M satisfy the inf-sup condition, there exists a constant $\beta_0>0$ such that

$$\sup_{v\in X}\frac{b(q,v)}{\|v\|_1}\ge\beta_0\|q\|_0,\quad\forall\ q\in M. \tag{3.27}$$

Therefore, by (3.25), (3.26) and (3.27), we have

$$\begin{aligned}
\sup_{v_h\in X_h}\frac{b^h(q_h,v_h)}{\|v_h\|_h} &\ge \sup_{v\in X}\frac{b^h(q_h,\Pi_h v)}{\|\Pi_h v\|_h}=\sup_{v\in X}\frac{b^h(q_h,v)}{\|\Pi_h v\|_h}\\
&\ge \frac{1}{C}\sup_{v\in X}\frac{b(q_h,v)}{\|v\|_1}\ge\frac{\beta_0}{C}\|q_h\|_0,
\end{aligned}\tag{3.28}$$

which, by setting $\beta=\dfrac{\beta_0}{C}$ yields the desired result.

In order to prove the existence and uniqueness of the solution to Problem (I^h), we first consider the following equations:
Problem $(I^{h'})$ Find $(u_h,T_h)\in V_h\times W_h$, such that $T_h|_{\partial\Omega}=T_0$

$$\begin{cases} a^h(u_h,v_h)+a_1^h(u_h;u_h,v_h)=\lambda(jT_h,v_h), & \forall\ v_h\in V_h,\\ d(T_h,\varphi_h)+\lambda\bar{a}_1(u_h;T_h,\varphi_h)=0, & \forall\ \varphi_h\in W_{0h}, \end{cases}\tag{3.29}$$

where

$$V_h=\{v_h\in X_h; b^h(p_h,v_h)=0,\ \forall\ p_h\in M_h\}. \tag{3.30}$$

It is easy to see that

$$V_h=\{v_h\in X_h;\mathrm{div}v_h=0\}. \tag{3.31}$$

Theorem 3.1. Under the assumption of (A_1), let $A^{'} = 2\mu^{-1}\lambda C_3(1+3C_1^{'})\|T_0\|_1$ and $B = \Big(5+\dfrac{1}{C_1^{'}}\Big)\|T_0\|_1$. Assume that there exist positive constants δ_3 and δ_4 such that

$$(3.32)\quad \mu^{-1}N_0A^{'} \le 1-\delta_3\ (0<\delta_3\le 1);\ \ \delta_3^{-1}\mu^{-1}\lambda^2C_1^{'}C_3\bar{N}_0B \le 1-\delta_4\ (0<\delta_4\le 1).$$

Then Problem (I^h) has a unique solution $(u_h, p_h, T_h) \in X_h \times M_h \times W_h$, satisfying

$$(3.33)\qquad \|u_h\|_h \le A^{'}, \quad \|\nabla T\|_0 \le B.$$

Proof. We use Banach's fixed point theorem (cf. [27]) to prove our theorem. The proof proceeds in the following three steps.

Step 1. To prove the existence of the solution to Problem $(I^{h'})$.

For a given $\tilde{u}_h \in V_h$, we consider the following equation:

$$(3.34)\qquad d(T_h,\psi_h) + \lambda\bar{a}_1(\tilde{u}_h;T_h,\psi_h) = 0, \quad \forall\ \psi_h \in W_{0h}.$$

Let $T_h = w_h + T_0$, $w_h \in W_{0h}$. Substituting it into (3.34), we have

$$(3.35)\quad d(w_h,\psi_h)+\lambda\bar{a}_1(\tilde{u}_h;w_h,\psi_h) = -d(T_0,\psi_h)-\lambda\bar{a}_1(\tilde{u}_h;T_0,\psi_h), \quad \forall\ \psi_h\in W_{0h}.$$

Let

$$(3.36)\qquad D(w_h,\psi_h) = d(w_h,\psi_h)+\lambda\bar{a}_1(\tilde{u}_h;w_h,\psi_h), \quad \forall\ \psi_h \in W_{0h}.$$

Choosing $\psi_h = w_h$ in (3.36), we deduce that

$$(3.37)\qquad D(w_h,w_h) = d(w_h,w_h)+\lambda\bar{a}_1(\tilde{u}_h;w_h,w_h) = d(w_h,w_h) = \|\nabla w_h\|_0^2.$$

So $D(\cdot,\cdot)$ is continuous and coercive over W_{0h}. By the Lax-Milgram Theorem (cf. [14]), we know that (3.35) has a unique solution $T_h \in W_h$, satisfying $T_h|_{\partial\Omega} = T_0$. For this T_h, we consider the following equations:

$$(3.38)\qquad a^h(u_h^*,v_h)+a_1^h(u_h^*;u_h^*,v_h)-b^h(p_h,v_h) = \lambda(jT_h,v_h), \qquad \forall\ v_h\in V_h,$$

and

$$(3.39)\qquad b^h(\varphi_h,u_h^*) = 0, \qquad \forall\ \varphi_h \in M_h,$$

which approximate the stationary Navier-Stokes equations. By (3.14) and Lemma 3.1 we know that there exists a unique solution $(u_h^*,p_h)\in V_h\times M_h$ (cf. [18,26]). Thus (3.34), (3.38) and (3.39) determine a mapping $l_h : \tilde{u}_h \in V_h \to u_h^* \in V_h$, i.e., $u_h^* = l_h\tilde{u}_h$.

Firstly, we estimate $\|\nabla T_h\|_0$.

Choosing $T_h = w_h+T_0, \psi_h = w_h$ in (3.34), we arrive at

$$(3.40)\qquad d(w_h,w_h) = -\lambda\bar{a}_1(\tilde{u}_h;T_0,w_h)-d(T_0,w_h).$$

By (3.21), we have

$$(3.41)\qquad \|\nabla w_h\|_0^2 \le \lambda\bar{N}_0\|\tilde{u}_h\|_h\|\nabla T_0\|_0\|\nabla w_h\|_0+\|\nabla T_0\|_0\|\nabla w_h\|_0.$$

By (A_1), we get

$$(3.42)\qquad \|\nabla w_h\|_0 \le \lambda\bar{N}_0\varepsilon\|\tilde{u}_h\|_h+\|\nabla T_0\|_0.$$

Hence

$$(3.43)\qquad \|\nabla T_h\|_0 \le \|\nabla w_h\|_0+\|\nabla T_0\|_0 \le \lambda\bar{N}_0\varepsilon\|\tilde{u}_h\|_h+2\|T_0\|_1 \equiv \tilde{B}(\|\tilde{u}_h\|_h).$$

Secondly, we estimate $\|u_h^*\|_h$.

Choosing $v_h = u_h^*$ in (3.38), by (3.13) and (3.39), we obtain

$$(3.44)\qquad \mu\|u_h^*\|_h^2 = |\lambda(jT_h,u_h^*)| \le \lambda\|T_h\|_0\|u_h^*\|_0 \le \lambda C_3\|T_h\|_0\|u_h^*\|_h.$$

Applying (2.4) and (3.43), we deduce that

$$
\begin{aligned}
\|u_h^*\|_h &\leq \mu^{-1}\lambda C_3\|T_h\|_0 \leq \mu^{-1}\lambda C_3(\|w_h\|_0+\|T_0\|_0)\\
&\leq \mu^{-1}\lambda C_3(C_1^{'}\|\nabla w_h\|_0+\|T_0\|_0)\\
&\leq \mu^{-1}\lambda C_3[C_1^{'}\|\nabla T_h\|_0+(1+C_1^{'})\|T_0\|_1]\\
&\leq \mu^{-1}\lambda C_3(1+3C_1^{'})\|T_0\|_1+\mu^{-1}\lambda^2 C_1^{'}C_3\bar{N}_0\varepsilon\|\tilde{u}_h\|_h\\
&\equiv \tilde{A}(\|\tilde{u}_h\|_h).
\end{aligned}
\tag{3.45}
$$

Thirdly, we prove l_h is continuous.

For arbitrarily given $\tilde{u}_h^1, \tilde{u}_h^2 \in V_h$, and according to (3.34), (3.38) and (3.39), there exist (u_h^{*1}, p_h^1, T_h^1), $(u_h^{*2}, p_h^2, T_h^2) \in V_h \times M_h \times W_h$, satisfying $T_h^1|_{\partial\Omega} = T_h^2|_{\partial\Omega} = T_0$. By (3.34), we have

$$
d(T_h^1-T_h^2,\psi_h)+\lambda\bar{a}_1(\tilde{u}_h^1-\tilde{u}_h^2;T_h^1,\psi_h)+\lambda\bar{a}_1(\tilde{u}_h^2;T_h^1-T_h^2,\psi_h)=0, \quad \forall\ \psi_h\in W_{0h}.
\tag{3.46}
$$

Choosing $\psi_h = T_h^1 - T_h^2$ in (3.46) and using (3.16) and (3.21), we obtain

$$
\begin{aligned}
\|\nabla(T_h^1-T_h^2)\|_0^2 &= |\lambda\bar{a}_1(\tilde{u}_h^1-\tilde{u}_h^2;T_h^1,T_h^1-T_h^2)|\\
&\leq \lambda\bar{N}_0\|\tilde{u}_h^1-\tilde{u}_h^2\|_h\|\nabla T_h^1\|_0\|\nabla(T_h^1-T_h^2)\|_0.
\end{aligned}
\tag{3.47}
$$

Hence

$$
\|\nabla(T_h^1-T_h^2)\|_0 \leq \lambda\bar{N}_0\|\tilde{u}_h^1-\tilde{u}_h^2\|_h\|\nabla T_h^1\|_0.
\tag{3.48}
$$

Owing to (3.38), we deduce that

$$
\begin{aligned}
&a^h(u_h^{*1}-u_h^{*2},v_h)+a_1^h(u_h^{*1}-u_h^{*2};u_h^{*1},v_h)+a_1^h(u_h^{*2};u_h^{*1}-u_h^{*2},v_h)\\
&-b^h(p_h^1-p_h^2,v_h)=\lambda(j(T_h^1-T_h^2),v_h), \quad \forall\ v_h\in V_h.
\end{aligned}
\tag{3.49}
$$

Choosing $v_h = u_h^{*1} - u_h^{*2}$ in (3.49) and using (2.4), (3.13), (3.20) and (3.39), we derive

$$
\begin{aligned}
&\mu\|u_h^{*1}-u_h^{*2}\|_h^2\\
&=|\lambda(j(T_h^1-T_h^2),u_h^{*1}-u_h^{*2})-a_1^h(u_h^{*1}-u_h^{*2};u_h^{*1},u_h^{*1}-u_h^{*2})|\\
&\leq \lambda C_1^{'}C_3\|\nabla(T_h^1-T_h^2)\|_0\|u_h^{*1}-u_h^{*2}\|_h+N_0\|u_h^{*1}-u_h^{*2}\|_h^2\|u_h^{*1}\|_h.
\end{aligned}
\tag{3.50}
$$

Therefore

$$
\|u_h^{*1}-u_h^{*2}\|_h \leq \mu^{-1}\lambda C_1^{'}C_3\|\nabla(T_h^1-T_h^2)\|_0+\mu^{-1}N_0\|u_h^{*1}-u_h^{*2}\|_h\|u_h^{*1}\|_h.
\tag{3.51}
$$

Substituting (3.48) into (3.51) yields

$$
\begin{aligned}
&\|u_h^{*1}-u_h^{*2}\|_h\\
&\leq \mu^{-1}\lambda^2 C_1^{'}C_3\bar{N}_0\|\tilde{u}_h^1-\tilde{u}_h^2\|_h\|\nabla T_h^1\|_0+\mu^{-1}N_0\|u_h^{*1}-u_h^{*2}\|_h\|u_h^{*1}\|_h.
\end{aligned}
\tag{3.52}
$$

From (3.43) and (3.45) we know that $\|u_h^{*i}\|_h \leq \tilde{A}(\|\tilde{u}_h^i\|_h)$ and $\|\nabla T_h^i\|_0 \leq \tilde{B}(\|\tilde{u}_h\|_h^i)$ (i=1,2) are bounded if $\|\tilde{u}_h^i\|_h$ (i=1,2) are bounded. By (3.52) and the assumption that $\mu^{-1}N_0\tilde{A} \leq 1-\delta_3$ $(0<\delta_3\leq 1)$, we arrive at

$$
\|u_h^{*1}-u_h^{*2}\|_h \leq \delta_3^{-1}\mu^{-1}\lambda^2 C_1^{'}C_3\bar{N}_0\tilde{B}\|\tilde{u}_h^1-\tilde{u}_h^2\|_h.
\tag{3.53}
$$

Therefore u_h^* is continuously dependent on $\tilde{u}_h$.

Finally, we consider the following equation:

$$
u_h = s l_h u_h,\ 0\leq s\leq 1.
$$

Obviously, $s = 0$ is a trivial case. So we only need to consider $0 < s \le 1$. Substituting $s^{-1}u_h = l_h u_h$ into (3.45), we obtain

$$s^{-1}\|u_h\|_h \le \mu^{-1}\lambda C_3(1+3C_1^{'})\|T_0\|_1 + \mu^{-1}\lambda^2 C_1^{'} C_3 \bar{N}_0 \varepsilon \|u_h\|_h. \tag{3.54}$$

Noting that $0 < s \le 1$, then $s(2-s)^{-1} \le 1$. Assume that

$$\varepsilon = (2\mu^{-1}\lambda^2 C_1^{'} C_3 \bar{N}_0)^{-1}. \tag{3.55}$$

By (3.54), we have

$$\|u_h\|_h \le 2\mu^{-1}\lambda C_3(1+3C_1^{'})\|T_0\|_1 \equiv A^{'}. \tag{3.56}$$

Substituting (3.56) into (3.43) yields

$$\|\nabla T_h\|_0 \le 2\|T_0\|_1 + \lambda \bar{N}_0 (2\mu^{-1}\lambda^2 C_1^{'} C_3 \bar{N}_0)^{-1}\|u_h\|_h = \Big(5 + \frac{1}{C_1^{'}}\Big)\|T_0\|_1 \equiv B. \tag{3.57}$$

Observe that $\delta_3^{-1}\mu^{-1}\lambda^2 C_1^{'} C_3 \bar{N}_0 B \le 1 - \delta_4$ $(0 < \delta_4 \le 1)$, we know, by (3.53), that l_h is a contractive operator. Using Banach's fixed point theorem, Problem $(I^{h'})$ has a unique solution $(u_h, T_h) \in V_h \times W_h$, satisfying $T_h|_{\partial\Omega} = T_0$.

Step 2. To prove the existence of the solution to Problem (I^h).

Clearly, the solution $(u_h, T_h) \in V_h \times W_h$ to Problem $(I^{h'})$ satisfies the last two equations of Problem (I^h). For the given (u_h, T_h), we define

$$f(v_h) = a^h(u_h, v_h) + a_1^h(u_h; u_h, v_h) - \lambda(jT_h, v_h), \qquad \forall\ v_h \in X_h. \tag{3.58}$$

Noting that $f(v_h)$ is a bounded linear functional over X_h and $f(v_h) = 0$ on V_h, then there exists a $p_h \in M_h$ such that

$$f(v_h) = (p_h, \mathrm{div} v_h), \qquad \forall\ v_h \in X_h, \tag{3.59}$$

(cf. [15]). Therefore, Problem (I^h) has a solution $(u_h, p_h, T_h) \in X_h \times M_h \times W_h$, satisfying $T_h|_{\partial\Omega} = T_0$.

Step 3. To prove the uniqueness of the solution to Problem (I^h).

Suppose that Problem (I^h) has another solution $(u_h^1, p_h^1, T_h^1) \in X_h \times M_h \times W_h$, satisfying $T_h^1|_{\partial\Omega} = T_0$. By Problem (I^h), we deduce that

$$\begin{aligned} &a^h(u_h - u_h^1, v_h) + a_1^h(u_h - u_h^1; u_h, v_h) + a_1^h(u_h^1; u_h - u_h^1, v_h) \\ &-b^h(p_h - p_h^1, v_h) = \lambda(j(T_h - T_h^1), v_h),\ \forall\ v_h \in X_h, \end{aligned} \tag{3.60}$$

$$b^h(\varphi_h, u_h - u_h^1) = 0, \qquad \forall\ \varphi_h \in M_h, \tag{3.61}$$

and

$$d(T_h - T_h^1, \psi_h) + \lambda \bar{a}_1(u_h - u_h^1; T_h, \psi_h) + \lambda \bar{a}_1(u_h^1; T_h - T_h^1, \psi_h) = 0, \qquad \forall\ \psi_h \in W_{0h}. \tag{3.62}$$

Taking $v_h = u_h - u_h^1$ in (3.60) and applying (2.4), (3.13), (3.20) and (3.56) yield

$$\begin{aligned} \mu\|u_h - u_h^1\|_h^2 &= |\lambda(j(T_h - T_h^1), u_h - u_h^1) - a_1^h(u_h - u_h^1; u_h, u_h - u_h^1)| \\ &\le \lambda C_1^{'} C_3 \|\nabla(T_h - T_h^1)\|_0 \|u_h - u_h^1\|_h + N_0 A^{'} \|u_h - u_h^1\|_h^2. \end{aligned} \tag{3.63}$$

Noting that $\mu^{-1}N_0 A^{'} \le 1 - \delta_3$ $(0 < \delta_3 \le 1)$, we have

$$\|u_h - u_h^1\|_h \le \delta_3^{-1}\mu^{-1}\lambda C_1^{'} C_3 \|\nabla(T_h - T_h^1)\|_0. \tag{3.64}$$

Choosing $\psi_h = T_h - T_h^1$ in (3.62) and using (3.16) and (3.21), we can write

$$\begin{aligned} \|\nabla(T_h - T_h^1)\|_0^2 &= |-\lambda \bar{a}_1(u_h - u_h^1; T_h, T_h - T_h^1)| \\ &\le \lambda \bar{N}_0 \|u_h - u_h^1\|_h \|\nabla T_h\|_0 \|\nabla(T_h - T_h^1)\|_0. \end{aligned} \tag{3.65}$$

Then it stems from (3.57) that

$$\|\nabla(T_h - T_h^1)\|_0 \le \lambda \bar{N}_0 B \|u_h - u_h^1\|_h. \tag{3.66}$$

Substituting (3.64) into (3.66) yields

$$\|\nabla(T_h - T_h^1)\|_0 \le \delta_3^{-1}\mu^{-1}\lambda^2 B C_1^{'} C_3 \bar{N}_0 \|\nabla(T_h - T_h^1)\|_0. \tag{3.67}$$

On the one hand, noting that $\delta_3^{-1}\mu^{-1}\lambda^2 C_1^{'} C_3 \bar{N}_0 B \le 1 - \delta_4$ $(0 < \delta_4 \le 1)$, we have $\|\nabla(T_h - T_h^1)\|_0 = 0$, hence $T_h = T_h^1$. Together with (3.64) we know that $\|u_h - u_h^1\|_h = 0$, i.e., $u_h = u_h^1$.
On the other hand, by (3.60), we have

$$b^h(p_h - p_h^1, v_h) = 0, \quad \forall\ v_h \in X_h. \tag{3.68}$$

Thanks to Lemma 3.1, we get

$$\beta\|p_h - p_h^1\|_0 \le \sup_{v_h \in X_h} \frac{b^h(p_h - p_h^1, v_h)}{\|v_h\|_h} = 0. \tag{3.69}$$

That is $p_h = p_h^1$. We complete the proof.

4. Error estimates

This section is devoted to establishing convergence results for the discrete velocity, pressure and temperature spaces. First of all, we state the following lemma which can be found in [14].
Lemma 4.1. There exists an operator $r_h : W \to W_h$, such that for all $\varphi \in W$

$$(\nabla(\varphi - r_h\varphi), \nabla\varphi_h) = 0, \quad \forall\ \varphi_h \in W_h,$$

$$\int_\Omega (\varphi - r_h\varphi)dx = 0, \quad \|\nabla r_h\varphi\|_0 \le \|\nabla\varphi\|_0,$$

and

$$\|\varphi - r_h\varphi\|_1 \le Ch|\varphi|_2, \quad \forall\ \varphi \in H^2(\Omega).$$

Next we prove the following two important lemmas.
Lemma 4.2. Assume that $u \in H^2(\Omega)^2$ and $p \in H^1(\Omega)$, then we have

$$\Big|\sum_{K\in\Im_h} \int_{\partial K} \frac{\partial u}{\partial n} \cdot v_h ds\Big| \le Ch|u|_2\|v_h\|_h, \quad \forall\ v_h \in X_h, \tag{4.1}$$

and

$$\Big|\sum_{K\in\Im_h} \int_{\partial K} p v_h \cdot n ds\Big| \le Ch|p|_1\|v_h\|_h, \quad \forall\ v_h \in X_h. \tag{4.2}$$

Proof. Note that

$$\begin{aligned}
&\sum_{K\in\Im_h} \int_{\partial K} \frac{\partial u}{\partial n} \cdot v_h ds \\
&= \sum_{K\in\Im_h} \Big[-\int_{l_1} \Big(\frac{\partial u^1}{\partial x_2} v_h^1 + \frac{\partial u^2}{\partial x_2} v_h^2\Big) dx_1 + \int_{l_2} \Big(\frac{\partial u^1}{\partial x_1} v_h^1 + \frac{\partial u^2}{\partial x_1} v_h^2\Big) dx_2 \\
&+ \int_{l_3} \Big(\frac{\partial u^1}{\partial x_2} v_h^1 + \frac{\partial u^2}{\partial x_2} v_h^2\Big) dx_1 - \int_{l_4} \Big(\frac{\partial u^1}{\partial x_1} v_h^1 + \frac{\partial u^2}{\partial x_1} v_h^2\Big) dx_2\Big].
\end{aligned} \tag{4.3}$$

For any $K \in \Im_h$ and $v \in H^1(K)(H^1(K)^2)$, we define the following operators:

$$P_{0l_i} v = \frac{1}{2h_{x_1}} \int_{l_i} v dx_1, \quad i = 1, 3, \quad P_{0l_i} v = \frac{1}{2h_{x_2}} \int_{l_i} v dx_2, \quad i = 2, 4,$$

$$P_0 v = \frac{1}{|K|}\int_K v dx.$$

It is easy to see that the above operators are affine equivalent. Let the corresponding ones onto the reference element $\hat{K}$ be denoted by $\hat{P}_{0\hat{l}_i}(i=1,2,3,4)$ and $\hat{P}_0$. By (4.3) we get

(4.4)

$$\begin{aligned}
&\sum_{K\in\Im_h}\int_{\partial K}\frac{\partial u}{\partial n}\cdot v_h ds\\
&=\sum_{K\in\Im_h}\Big\{-\int_{l_1}\Big[\Big(\frac{\partial u^1}{\partial x_2}-P_0\frac{\partial u^1}{\partial x_2}\Big)(v_h^1-P_{0l_1}v_h^1)+\Big(\frac{\partial u^2}{\partial x_2}-P_0\frac{\partial u^2}{\partial x_2}\Big)(v_h^2-P_{0l_1}v_h^2)\Big]dx_1\\
&+\int_{l_2}\Big[\Big(\frac{\partial u^1}{\partial x_1}-P_0\frac{\partial u^1}{\partial x_1}\Big)(v_h^1-P_{0l_2}v_h^1)+\Big(\frac{\partial u^2}{\partial x_1}-P_0\frac{\partial u^2}{\partial x_1}\Big)(v_h^2-P_{0l_2}v_h^2)\Big]dx_2\\
&+\int_{l_3}\Big[\Big(\frac{\partial u^1}{\partial x_2}-P_0\frac{\partial u^1}{\partial x_2}\Big)(v_h^1-P_{0l_3}v_h^1)+\Big(\frac{\partial u^2}{\partial x_2}-P_0\frac{\partial u^2}{\partial x_2}\Big)(v_h^2-P_{0l_3}v_h^2)\Big]dx_1\\
&-\int_{l_4}\Big[\Big(\frac{\partial u^1}{\partial x_1}-P_0\frac{\partial u^1}{\partial x_1}\Big)(v_h^1-P_{0l_4}v_h^1)+\Big(\frac{\partial u^2}{\partial x_1}-P_0\frac{\partial u^2}{\partial x_1}\Big)(v_h^2-P_{0l_4}v_h^2)\Big]dx_2\Big\}\\
&=\sum_{K\in\Im_h}[F_1+F_2+F_3+F_4],
\end{aligned}$$

where

$$F_1=-\int_{l_1}\Big[\Big(\frac{\partial u^1}{\partial x_2}-P_0\frac{\partial u^1}{\partial x_2}\Big)(v_h^1-P_{0l_1}v_h^1)+\Big(\frac{\partial u^2}{\partial x_2}-P_0\frac{\partial u^2}{\partial x_2}\Big)(v_h^2-P_{0l_1}v_h^2)\Big]dx_1,$$

$$F_2=\int_{l_2}\Big[\Big(\frac{\partial u^1}{\partial x_1}-P_0\frac{\partial u^1}{\partial x_1}\Big)(v_h^1-P_{0l_2}v_h^1)+\Big(\frac{\partial u^2}{\partial x_1}-P_0\frac{\partial u^2}{\partial x_1}\Big)(v_h^2-P_{0l_2}v_h^2)\Big]dx_2,$$

$$F_3=\int_{l_3}\Big[\Big(\frac{\partial u^1}{\partial x_2}-P_0\frac{\partial u^1}{\partial x_2}\Big)(v_h^1-P_{0l_3}v_h^1)+\Big(\frac{\partial u^2}{\partial x_2}-P_0\frac{\partial u^2}{\partial x_2}\Big)(v_h^2-P_{0l_3}v_h^2)\Big]dx_1,$$

$$F_4=-\int_{l_4}\Big[\Big(\frac{\partial u^1}{\partial x_1}-P_0\frac{\partial u^1}{\partial x_1}\Big)(v_h^1-P_{0l_4}v_h^1)+\Big(\frac{\partial u^2}{\partial x_1}-P_0\frac{\partial u^2}{\partial x_1}\Big)(v_h^2-P_{0l_4}v_h^2)\Big]dx_2.$$

Using Hölder inequality, trace theorem and interpolation theorem, we have

$$\begin{aligned}
F_i &\le Ch_K(|u^1|_{2,K}|v_h^1|_{1,K}+|u^2|_{2,K}|v_h^2|_{1,K})\\
&\le Ch_K|u|_{2,K}|v_h|_{1,K},\quad i=1,2,3,4.
\end{aligned}\tag{4.5}$$

By using (4.5), we get

$$\Big|\sum_{K\in\Im_h}\int_{\partial K}\frac{\partial u}{\partial n}\cdot v_h ds\Big|\le Ch|u|_2\|v_h\|_h.$$

Similarly, we can get (4.2). Then Lemma 4.2 is proved.

Lemma 4.3. Assume that $u\in H^2(\Omega)^2\cap H_0^1(\Omega)^2$, then there holds

$$\Big|\sum_{K\in\Im_h}\int_{\partial K}(u\cdot n)(u\cdot v_h)ds\Big|\le Ch\|u\|_1\|u\|_2\|v_h\|_h,\qquad \forall\ v_h\in X_h.$$

Proof.

(4.6)

$$\begin{aligned}
&\sum_{K\in\Im_h}\int_{\partial K}(u\cdot n)(u\cdot v_h)ds\\
&=\sum_{K\in\Im_h}\Big(\int_{l_1}-u^2u\cdot v_h dx_1+\int_{l_2}u^1u\cdot v_h dx_2+\int_{l_3}u^2u\cdot v_h dx_1+\int_{l_4}-u^1u\cdot v_h dx_2\Big)\\
&=\sum_{K\in\Im_h}\Big[\int_{l_1}-(u^2u-P_0(u^2u))\cdot(v_h-P_{0l_1}v_h)dx_1+\int_{l_2}(u^1u-P_0(u^1u))\cdot(v_h-P_{0l_2}v_h)dx_2\\
&+\int_{l_3}(u^2u-P_0(u^2u))\cdot(v_h-P_{0l_3}v_h)dx_1+\int_{l_4}-(u^1u-P_0(u^1u))\cdot(v_h-P_{0l_4}v_h)dx_2\Big]\\
&=\sum_{K\in\Im_h}[I_1+I_2+I_3+I_4],
\end{aligned}$$

where

$$\begin{aligned}
I_1&=-\int_{l_1}(u^2u-P_0(u^2u))\cdot(v_h-P_{0l_1}v_h)dx_1,\\
I_2&=\int_{l_2}(u^1u-P_0(u^1u))\cdot(v_h-P_{0l_2}v_h)dx_2,\\
I_3&=\int_{l_3}(u^2u-P_0(u^2u))\cdot(v_h-P_{0l_3}v_h)dx_1,\\
I_4&=-\int_{l_4}(u^1u-P_0(u^1u))\cdot(v_h-P_{0l_4}v_h)dx_2.
\end{aligned}$$

Using Hölder inequality, trace theorem and interpolation theorem, we have

$$\begin{aligned}
I_1 &= -\int_{l_1}(u^2u-P_0(u^2u))\cdot(v_h-P_{0l_1}v_h)dx_1\\
&\le \Big(\int_{l_1}(u^2u-P_0(u^2u))^2dx_1\Big)^{\frac12}\Big(\int_{l_1}(v_h-P_{0l_1}v_h)^2dx_1\Big)^{\frac12}\\
&= \Big(\int_{\hat l_1}(\widehat{u^2u}-\hat P_0\widehat{(u^2u)})^2h_{x_1}d\xi\Big)^{\frac12}\Big(\int_{\hat l_1}(\widehat{v}_h-\hat P_{0\hat l_1}\widehat{v}_h)^2h_{x_1}d\xi\Big)^{\frac12}\\
&= h_{x_1}\|\widehat{u^2u}-\hat P_0\widehat{(u^2u)}\|_{0,\hat l_1}\|\widehat{v}_h-\hat P_{0\hat l_1}\widehat{v}_h\|_{0,\hat l_1}\\
&\le Ch_{x_1}\|\widehat{u^2u}-\hat P_0\widehat{(u^2u)}\|_{1,\hat K}\|\widehat{v}_h-\hat P_{0\hat l_1}\widehat{v}_h\|_{1,\hat K}\\
&\le Ch_{x_1}|\widehat{u^2u}|_{1,\hat K}|\widehat{v}_h|_{1,\hat K}\\
&\le Ch_{x_1}|u^2u|_{1,K}|v_h|_{1,K}.
\end{aligned}\tag{4.7}$$

Similarly

$$I_2\le Ch_{x_2}|u^1u|_{1,K}|v_h|_{1,K},\tag{4.8}$$

$$I_3\le Ch_{x_1}|u^2u|_{1,K}|v_h|_{1,K},\tag{4.9}$$

$$I_4\le Ch_{x_2}|u^1u|_{1,K}|v_h|_{1,K}.\tag{4.10}$$

Combining (4.7), (4.8), (4.9) and (4.10), yields the desired result.

The following theorem is the main result of this section.

Theorem 4.1. Under the assumption of (A_1), let $(u,p,T)\in(H^2(\Omega)^2\cap H_0^1(\Omega)^2)\times H^1(\Omega)\times H^2(\Omega)$ and $(u_h,p_h,T_h)\in X_h\times M_h\times W_h$ be the solutions to Problems (I) and (I^h), respectively, then we have

$$\|u-u_h\|_h+\|p-p_h\|_0+\|\nabla(T-T_h)\|_0\le Ch(|u|_2+|p|_1+|T|_2+\|u\|_1\|u\|_2).$$

Proof. By Problems (I) and (I^h), we get the following equations:

$$
\begin{aligned}
&a^h(u-u_h,v_h)+a_1^h(u-u_h;u,v_h)+a_1^h(u_h;u-u_h,v_h)-b^h(p-p_h,v_h)\\
&-\mu\sum_{K\in\Im_h}\int_{\partial K}\frac{\partial u}{\partial n}\cdot v_h ds+\sum_{K\in\Im_h}\int_{\partial K} pv_h\cdot nds\\
&-\frac{1}{2}\sum_{K\in\Im_h}\int_{\partial K}(u\cdot n)(u\cdot v_h)ds=\lambda(j(T-T_h),v_h),\ \ \forall\ v_h\in X_h,
\end{aligned}\tag{4.11}
$$

$$
b^h(q_h,u-u_h)=0,\quad \forall\ q_h\in M_h,\tag{4.12}
$$

and

$$
d(T-T_h,\varphi_h)+\lambda\bar{a}_1(u-u_h;T,\varphi_h)+\lambda\bar{a}_1(u_h;T-T_h,\varphi_h)=0,\ \ \forall\ \varphi_h\in W_{0h}.\tag{4.13}
$$

Choosing $\varphi_h=r_hT-T_h\in W_{0h}$ in (4.13) and using Lemma 4.1, we have

$$
\begin{aligned}
&d(T-T_h,T-T_h)=d(T-T_h,T-r_hT)+d(T-T_h,r_hT-T_h)\\
&=d(T-r_hT,T-r_hT)-\lambda\bar{a}_1(u-u_h;T,r_hT-T_h)-\lambda\bar{a}_1(u_h;T-T_h,r_hT-T_h).
\end{aligned}\tag{4.14}
$$

By Hölder inequality and Cauchy inequality, it follows easily from (3.21) and Theorem 3.1 that

$$
\begin{aligned}
&\|\nabla(T-T_h)\|_0^2\\
&\le\|\nabla(T-r_hT)\|_0^2+\lambda\bar{N}_0\|u-u_h\|_h\|\nabla T\|_0(\|\nabla(T-r_hT)\|_0+\|\nabla(T-T_h)\|_0)\\
&+\lambda\bar{N}_0\|u_h\|_h\|\nabla(T-T_h)\|_0(\|\nabla(T-r_hT)\|_0+\|\nabla(T-T_h)\|_0)\\
&\le C\|\nabla(T-r_hT)\|_0^2+(\lambda\bar{N}_0B)^2\|u-u_h\|_h^2+\frac{1}{2}\|\nabla(T-T_h)\|_0^2+\frac{\theta_1^2}{2}\|\nabla(u-u_h)\|_0^2,
\end{aligned}\tag{4.15}
$$

where θ_1 is an undetermined positive constant. Thus

$$
\|\nabla(T-T_h)\|_0\le C\|\nabla(T-r_hT)\|_0+(\theta_1+2\lambda\bar{N}_0B)\|u-u_h\|_h.\tag{4.16}
$$

Choosing $v_h=\Pi_hu-u_h$ in (4.11), and by (4.12), (3.20), Hölder inequality, Theorems 2.1 and 3.1, Lemmas 4.2 and 4.3, we have

$$
\begin{aligned}
&\mu\|u-u_h\|_h^2\\
&=|a^h(u-u_h,u-\Pi_hu)+a^h(u-u_h,\Pi_hu-u_h)|\\
&=\Big|a^h(u-u_h,u-\Pi_hu)+\lambda(j(T-T_h),\Pi_hu-u_h)-a_1^h(u-u_h;u,\Pi_hu-u_h)\\
&-a_1^h(u_h;u-u_h,\Pi_hu-u_h)+b^h(p-p_h,\Pi_hu-u_h)+\mu\sum_{K\in\Im_h}\int_{\partial K}\frac{\partial u}{\partial n}\cdot(\Pi_hu-u_h)ds\\
&-\sum_{K\in\Im_h}\int_{\partial K}p(\Pi_hu-u_h)\cdot nds+\frac{1}{2}\sum_{K\in\Im_h}\int_{\partial K}(u\cdot n)(u\cdot(\Pi_hu-u_h))ds\Big|\\
&\le\mu\|u-u_h\|_h\|u-\Pi_hu\|_h+\lambda\|T-T_h\|_0(\|\Pi_hu-u\|_0+\|u-u_h\|_0)\\
&+N_0\|u-u_h\|_h\|\nabla u\|_0(\|\Pi_hu-u\|_h+\|u-u_h\|_h)\\
&+N_0\|u_h\|_h\|u-u_h\|_h(\|\Pi_hu-u\|_h+\|u-u_h\|_h)+\|p-p_h\|_0\|\Pi_hu-u\|_h
\end{aligned}\tag{4.17}
$$

$$+Ch|u|_2\|\Pi_h u-u_h\|_h+Ch|p|_1\|\Pi_h u-u_h\|_h+Ch\|u\|_1\|u\|_2\|\Pi_h u-u_h\|_h$$
$$\le \mu\|u-u_h\|_h\|u-\Pi_h u\|_h+\lambda C_1^{'}C_3\|\nabla(T-T_h)\|_0\|\Pi_h u-u\|_h$$
$$+\lambda C_1^{'}C_3\|\nabla(T-T_h)\|_0\|u-u_h\|_h+2N_0A^{'}\|u-u_h\|_h\|\Pi_h u-u\|_h$$
$$+N_0A^{'}\|u-u_h\|_h^2+\|p-p_h\|_0\|\Pi_h u-u\|_h+Ch|u|_2\|\Pi_h u-u\|_h+Ch|u|_2\|u-u_h\|_h$$
$$+Ch|p|_1\|\Pi_h u-u\|_h+Ch|p|_1\|u-u_h\|_h+Ch\|u\|_1\|u\|_2\|\Pi_h u-u\|_h$$
$$+Ch\|u\|_1\|u\|_2\|u-u_h\|_h.$$

By use of the assumption that $\mu^{-1}N_0A^{'}\le 1-\delta_3$ $(0<\delta_3\le 1)$ and Cauchy inequality, we have

(4.18)
$$\begin{aligned}
&\|u-u_h\|_h\\
&\le \delta_3^{-1}\|u-u_h\|_h\|u-\Pi_h u\|_h+\mu^{-1}\delta_3^{-1}\lambda C_1^{'}C_3\|\nabla(T-T_h)\|_0\|\Pi_h u-u\|_h\\
&+\mu^{-1}\delta_3^{-1}\lambda C_1^{'}C_3\|\nabla(T-T_h)\|_0\|u-u_h\|_h+2\mu^{-1}\delta_3^{-1}N_0A^{'}\|u-u_h\|_h\|\Pi_h u-u\|_h\\
&+\mu^{-1}\delta_3^{-1}\|p-p_h\|_0\|\Pi_h u-u\|_h+Ch|u|_2\|\Pi_h u-u\|_h+Ch|u|_2\|u-u_h\|_h\\
&+Ch|p|_1\|\Pi_h u-u\|_h+Ch|p|_1\|u-u_h\|_h+Ch\|u\|_1\|u\|_2\|\Pi_h u-u\|_h\\
&+Ch\|u\|_1\|u\|_2\|u-u_h\|_h\\
&\le C\|u-\Pi_h u\|_h^2+Ch^2(|u|_2^2+|p|_1^2+\|u\|_1^2\|u\|_2^2)\\
&+2(\mu^{-1}\delta_3^{-1}\lambda C_1^{'}C_3)^2\|\nabla(T-T_h)\|_0^2+\frac{1}{2}\|u-u_h\|_h^2+\frac{\theta_2^2}{2}\|p-p_h\|_0^2,
\end{aligned}$$

where θ_2 is an undetermined positive constant. As a consequence, we have

$$\begin{aligned}
&\|u-u_h\|_h\\
&\le C\|u-\Pi_h u\|_h+Ch(|u|_2+|p|_1+\|u\|_1\|u\|_2)\\
&+2\mu^{-1}\delta_3^{-1}\lambda C_1^{'}C_3\|\nabla(T-T_h)\|_0+\theta_2\|p-p_h\|_0.
\end{aligned}\tag{4.19}$$

By(4.11), (3.20), Hölder inequality, Theorems 2.1 and 3.1, Lemmas 4.2 and 4.3, we have

(4.20)
$$\begin{aligned}
&|b^h(P_0p-p_h,v_h)|\\
&=|b^h(P_0p-p,v_h)+b^h(p-p_h,v_h)|\\
&=\Big|a^h(u-u_h,v_h)+a_1^h(u-u_h;u,v_h)+a_1^h(u_h;u-u_h,v_h)-\lambda(j(T-T_h),v_h)\\
&-\mu\sum_{K\in\Im_h}\int_{\partial K}\frac{\partial u}{\partial n}\cdot v_h ds+\sum_{K\in\Im_h}\int_{\partial K}pv_h\cdot n ds-\frac{1}{2}\sum_{K\in\Im_h}\int_{\partial K}(u\cdot n)(u\cdot v_h)ds\Big|\\
&\le[\mu\|u-u_h\|_h+2N_0A\|u-u_h\|_h+\lambda C_1^{'}C_3\|\nabla(T-T_h)\|_0\\
&+Ch(|u|_2+|p|_1+\|u\|_1\|u\|_2)]\|v_h\|_h,
\end{aligned}$$

where $P_0p|_K=\dfrac{1}{|K|}\displaystyle\int_K p dx$.

According to the discrete inf-sup condition Lemma 3.1, we have

$$\|p-p_h\|_0\le\|p-P_0p\|_0+\frac{1}{\beta}\sup_{v_h\in X_h}\frac{b^h(P_0p-p_h,v_h)}{\|v_h\|_h}.\tag{4.21}$$

In view of (4.20) and (4.21), it follows

(4.22)
$$\begin{aligned}\|p-p_h\|_0 \quad \leq \quad & \|p-P_0p\|_0+\beta^{-1}(\mu+2N_0A)\|u-u_h\|_h+\beta^{-1}\lambda C_1^{'}C_3\|\nabla(T-T_h)\|_0 \\ & +Ch(|u|_2+|p|_1+\|u\|_1\|u\|_2).\end{aligned}$$

Substituting (4.22) into (4.19), we obtain

(4.23)
$$\begin{aligned}&\|u-u_h\|_h \\ &\leq C(\|u-\Pi_hu\|_h+\|p-P_0p\|_0)+Ch(|u|_2+|p|_1+\|u\|_1\|u\|_2) \\ &+(2\mu^{-1}\delta_3^{-1}\lambda C_1^{'}C_3+\theta_2\beta^{-1}\lambda C_1^{'}C_3)\|\nabla(T-T_h)\|_0+\theta_2\beta^{-1}(\mu+2N_0A)\|u-u_h\|_h.\end{aligned}$$

Substituting (4.16) into (4.23) and employing interpolation theorem and Lemma 4.2, yield

(4.24)
$$\begin{aligned}&\|u-u_h\|_h \\ &\leq Ch(|u|_2+|p|_1+|T|_2+\|u\|_1\|u\|_2)+[2\theta_1\mu^{-1}\delta_3^{-1}\lambda C_1^{'}C_3+\theta_1\theta_2\beta^{-1}\lambda C_1^{'}C_3 \\ &+4\mu^{-1}\delta_3^{-1}\lambda^2\bar{N}_0BC_1^{'}C_3+2\theta_2\beta^{-1}\lambda^2\bar{N}_0BC_1^{'}C_3+\theta_2\beta^{-1}(\mu+2N_0A)]\|u-u_h\|_h.\end{aligned}$$

Noting that $\delta_3^{-1}\mu^{-1}\lambda^2C_1^{'}C_3\bar{N}_0B \leq 1-\delta_4$ $(0<\delta_4\leq 1)$. Let $\delta_4=\dfrac{7}{8}$, then $4\delta_3^{-1}\mu^{-1}\lambda^2C_1^{'}C_3\bar{N}_0B\leq\dfrac{1}{2}$. Consequently

(4.25)
$$\begin{aligned}\|u-u_h\|_h \quad \leq \quad & Ch(|u|_2+|p|_1+|T|_2+\|u\|_1\|u\|_2)+2[\theta_1(2\mu^{-1}\delta_3^{-1}\lambda C_1^{'}C_2^{'} \\ & +\theta_2\beta^{-1}\lambda C_1^{'}C_3)+\theta_2\beta^{-1}(2\lambda^2\bar{N}_0BC_1^{'}C_3+\mu+2N_0A)]\|u-u_h\|_h.\end{aligned}$$

Let
$$\theta_2=\frac{\beta}{8(2\lambda^2\bar{N}_0BC_1^{'}C_3+\mu+2N_0A)}\leq\frac{\beta}{16\lambda^2\bar{N}_0BC_1^{'}C_3},$$
then $\theta_2\beta^{-1}\lambda C_1^{'}C_3\leq\dfrac{1}{16\lambda\bar{N}_0B}$, together with $2\mu^{-1}\delta_3^{-1}\lambda C_1^{'}C_3\leq\dfrac{1}{4\lambda\bar{N}_0B}$ and (4.28), we have

(4.26) $$\|u-u_h\|_h\leq Ch(|u|_2+|p|_1+|T|_2+\|u\|_1\|u\|_2)+\Big(\frac{5\theta_1}{8\lambda\bar{N}_0B}+\frac{1}{4}\Big)\|u-u_h\|_h.$$

Let $\theta_1=\dfrac{2\lambda\bar{N}_0B}{5}$, then $\dfrac{5\theta_1}{8\lambda\bar{N}_0B}=\dfrac{1}{4}$. By (4.29), we have

(4.27) $$\|u-u_h\|_h\leq Ch(|u|_2+|p|_1+|T|_2+\|u\|_1\|u\|_2).$$

Substituting (4.27) into (4.16) yields

(4.28) $$\|\nabla(T-T_h)\|_0\leq Ch(|u|_2+|p|_1+|T|_2+\|u\|_1\|u\|_2).$$

Then substituting (4.27) and (4.28) into (4.22), we obtain

(4.29) $$\|p-p_h\|_0\leq Ch(|u|_2+|p|_1+|T|_2+\|u\|_1\|u\|_2).$$

Combining (4.27), (4.28) and (4.29), yields the desired result.

Remark 1: It can be checked that the results obtained in the present work are valid to the rectangular nonconforming element proposed in [19-20] and the rotated Q_1 element discussed in [21,23-24] for the approximation of the velocity in the stationary conduction-convection problem. At the same time, the former element can be applied to the anisotropic meshes.

Remark 2: It should be pointed out that the total degrees of freedom of the nonconforming element of this paper are the same as the conforming one used in [10].

Remark 3: From (2.10) and (3.32) we know that λ and μ must satisfy some relations, especially, the Groshoff number λ is not too big.

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