NONCONFORMING MIXED FINITE ELEMENT METHOD FOR THE STATIONARY CONDUCTION-CONVECTION PROBLEM

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(Communicated by Xuecheng Tai)

Abstract. In this paper, a new stable nonconforming mixed finite element scheme is proposed for the stationary conduction-convection problem, in which a new low order Crouzeix-Raviart type nonconforming rectangular element is taken as approximation space for the velocity, the piecewise constant element for the pressure and the bilinear element for the temperature, respectively. The convergence analysis is presented and the optimal error estimates in a broken \(H^1\)-norm for the velocity, \(L^2\)-norm for the pressure and \(H^1\)-seminorm for the temperature are derived.

Key Words. stationary conduction-convection problem, nonconforming mixed finite element, the optimal error estimates.

1. Introduction

We consider the following stationary conduction-convection problem (cf. [1-4]):

Problem (I) Find \(u = (u^1, u^2), p\) and \(T\) such that

\[
\begin{align*}
-\mu \Delta u + (u \cdot \nabla)u + \nabla p &= \lambda j T, & \text{in } \Omega, \\
\operatorname{div} u &= 0, & \text{in } \Omega, \\
-\Delta T + \lambda u \cdot \nabla T &= 0, & \text{in } \Omega, \\
u &= 0, \quad T = T_0, & \text{on } \partial \Omega,
\end{align*}
\]

(1.1)

where \(\Omega \subset \mathbb{R}^2\) is a bounded domain with boundary \(\partial \Omega\), \(u\) denotes the fluid velocity vector field, \(p\) the pressure field, \(T\) the temperature field, \(\mu > 0\) the coefficient of the kinematic viscosity, \(\lambda > 0\) the Groshoff number, \(j = (0, 1)\) the two-dimensional vector and \(T_0\) the given initial scale function.

The stationary conduction-convection problem (I) is the coupled equations governing steady viscous incompressible flow and heat transfer process, where incompressible fluid is the Boussinesq’s approximation of the Navier-Stokes equations. In atmospheric dynamics it is an important compelling dissipative nonlinear system, which contains not only the velocity vector field and the pressure field but also the temperature field. From the thermodynamics point of view, we know that the movement of the fluid must have viscosity which will produce quantity of heat. Thus, the movement of the fluid must be companied with mutual transformation of temperature, speed and pressure. Therefore, it is very universal to investigate this nonlinear system.

So far some numerical methods have been studied on the conduction-convection problem (cf. [1-2,5-8]), but much less attention is paid to the theoretical error.
analysis of the mixed finite element methods. Shen [9] firstly analyzed the existence and uniqueness of approximation solution, and gave first order accuracy with Bernadi-Raugel element [25] in terms of the small Groshoff number \( \lambda \) that appears in Problem (I) (refer to [10] for the detailed proof). However, the analysis in [9-10] is about the conforming finite elements. Indeed, it seems that there are few studies focusing on the approximation of Problem (I) with the nonconforming finite elements. Recently, these elements have attracted increasing attention from scientists and engineers in various areas since they have some practical advantages. On the one hand, they are usually much easier to be constructed to satisfy the discrete inf-sup condition than the conforming ones, which is usually required in the mixed finite element analysis. On the other hand, from the domain decomposition methods point of view, since the unknowns are associated with the element faces, each degree of freedom belongs to at most two elements, the use of the nonconforming finite elements facilitates the exchange of information across each subdomain and provides spectral radius estimates for the iterative domain decomposition operator [23].

The main aim of this paper is to construct a new low order Crouzeix-Raviart type nonconforming rectangular element and apply it to Problem (I) with a mixed finite element scheme. By virtue of the element’s special properties, the convergence analysis is presented and the optimal error estimates are obtained. The remainder of this paper is organized as follows. In section 2, we introduce the variational formulation of Problem (I) and briefly recite the existence and uniqueness of its solution proved in [9-10]. In section 3, we first give the construction of Crouzeix-Raviart type nonconforming mixed finite element scheme and then prove that the approximation spaces of the velocity and the pressure satisfy the discrete inf-sup condition, which yields the existence and uniqueness of approximation solution. In the last section, some important lemmas, the convergence analysis and the optimal error estimates are obtained.

We will employ standard definitions for the Sobolev spaces \( W^{k,p}(\Omega) \) with norm \( \| \cdot \|_{k,p,\Omega} \), and \( H^k(\Omega) = W^{k,2}(\Omega) \) with norm \( \| \cdot \|_k \), respectively (cf. [17]). Throughout the paper, \( C \) indicates a positive constant, possibly different at different occurrences, which is independent of the mesh parameter \( h \), but may depend on \( \Omega \) and other parameters introduced in this paper. Notations not especially explained are used with their usual meanings.

2. The existence and uniqueness of the solution to the variational formulation

The variational formulation of Problem (I) is written as:

Problem (I*) Find \((u, p, T) \in X \times M \times W\), such that \( T|_{\partial \Omega} = T_0 \)

\[
\begin{align*}
&\left\{ \begin{array}{l}
  a(u, v) + a_1(u; u, v) - b(p, v) = \lambda (jT, v), \quad \forall \, v \in X, \\
  b(q, u) = 0, \quad \forall \, q \in M, \\
  d(T, \varphi) + \lambda a_1(u; T, \varphi) = 0, \quad \forall \, \varphi \in W_0,
\end{array} \right.
\end{align*}
\]

where

\[
X = H^1_0(\Omega)^2, M = L^2_0(\Omega) = \left\{ q, q \in L^2(\Omega), \int_{\Omega} q dx = 0 \right\}, W = H^1(\Omega), W_0 = H^1_0(\Omega),
\]
\((\cdot, \cdot)\) means the inner product in \(L^2(\Omega)^2\) or in \(L^2(\Omega)\) according to the context, \(a(u, v) = \mu(\nabla u, \nabla v), b(p, v) = (p, \text{div} v), d(T, \phi) = (\nabla T, \nabla \phi)\),

\[
a_1(u; v, w) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^2 \left( u^i \frac{\partial v^j}{\partial x_i} w^j - u^i \frac{\partial w^j}{\partial x_i} v^j \right) dx,
\]

and

\[
\bar{a}_1(u; T, \phi) = \frac{1}{2} \int_\Omega \sum_{i=1}^2 \left( u^i \frac{\partial T}{\partial x_i} \phi - u^i \frac{\partial \phi}{\partial x_i} T \right) dx.
\]

It has been shown in [11-15] that there exist positive constants \(C_i\) and \(C'_i\) \((i = 1, 2)\) such that

\[
\|v\|_0 \leq C_1 \|\nabla v\|_0, \quad \forall v \in H^1_0(\Omega)^2, \quad \|v\|_0 \leq C'_1 \|\nabla v\|_0, \quad \forall v \in H^1_0(\Omega),
\]

\[
\|v\|_{0, \Omega} \leq C_2 \|\nabla v\|_0, \quad \forall v \in H^1(\Omega)^2, \quad \|v\|_{0, \Omega} \leq C'_2 \|v\|_1, \quad \forall v \in H^1(\Omega).
\]

Besides, the trilinear forms \(a_1(\cdot, \cdot, \cdot)\) and \(\bar{a}_1(\cdot, \cdot, \cdot)\) satisfy

\[
a_1(u; v, v) = 0, \quad a_1(u; v, w) = -a_1(u; w, v), \quad \forall u, v, w \in X,
\]

\[
\bar{a}_1(u; T, \phi) = 0, \quad \bar{a}_1(u; T, \varphi) = -\bar{a}_1(u; \varphi, T), \quad \forall u \in X, T, \varphi \in W.
\]

Let \(V = \{v \in X, \text{div} v = 0\}\) denote the divergence-free subspace of \(X\). Using Green’s formula, (2.2) and (2.3) can be rewritten as

\[
a_1(u; v, w) = \frac{1}{2} \int_\Omega \sum_{i,j=1}^2 \left( u^i \frac{\partial v^j}{\partial x_i} w^j - u^i \frac{\partial w^j}{\partial x_i} v^j \right) dx
\]

\[
= \int_\Omega \sum_{i,j=1}^2 u^i \frac{\partial v^j}{\partial x_i} w^j dx, \quad \forall u, v, w \in H^1(\Omega)^2, w \in X,
\]

and

\[
\bar{a}_1(u; T, \varphi) = \frac{1}{2} \int_\Omega \sum_{i=1}^2 \left( u^i \frac{\partial T}{\partial x_i} \varphi - u^i \frac{\partial \varphi}{\partial x_i} T \right) dx
\]

\[
= \int_\Omega \sum_{i=1}^2 u^i \frac{\partial T}{\partial x_i} \varphi dx, \quad \forall u, T \in W, \varphi \in W_0,
\]

respectively.

We obtain in view of \(H^1(\Omega) \hookrightarrow L^4(\Omega)\)

\[
|a_1(u; v, w)| \leq N \|\nabla u\|_0 \|\nabla v\|_0 \|\nabla w\|_0, \quad \forall u, v, w \in H^1(\Omega)^2, w \in X,
\]

\[
|\bar{a}_1(u; T, \varphi)| \leq \bar{N} \|\nabla u\|_0 \|\nabla T\|_0 \|\nabla \varphi\|_0, \quad \forall u, T \in W, \varphi \in W_0,
\]

where

\[
N = \sup_{u, v \in H^1(\Omega)^2, w \in X} \frac{|a_1(u; v, w)|}{\|\nabla u\|_0 \|\nabla v\|_0 \|\nabla w\|_0}, \quad \bar{N} = \sup_{u, T \in W, \varphi \in W_0} \frac{|\bar{a}_1(u; T, \varphi)|}{\|\nabla u\|_0 \|\nabla T\|_0 \|\nabla \varphi\|_0}.
\]

In order to prove the existence and uniqueness of the solution to Problem \((\star)\), the following assumption is required as [14].

\((A_1)\) Assume that \(T_0 \in C^{1, \alpha}(\partial \Omega)(0 < \alpha \leq 1)\), and there exists a prolongation of \(T_0 \in C^{1, \alpha}(R^2)\) (still marked as \(T_0\)) such that

\[
|T_0|_1 \leq \varepsilon,
\]

where \(\varepsilon\) is a sufficiently small positive constant.
Then the following important conclusion can be found in [9-10].

**Theorem 2.1.** Under the assumption of (A₁), let \( A = 2\mu^{-1}\lambda C₁(1 + 3C₁)\|T₀\|₁ \) and \( B = \left(5 + \frac{1}{C₁}\right)\|T₀\|₁ \). Assume that there exist positive constants \( δ₁ \) and \( δ₂ \) such that

\[
(2.10) ~ ~ \mu^{-1}NA ≤ 1 - δ₁ \left(0 < δ₁ ≤ 1\right); \quad δ₁^{-1}\mu^{-1}\lambda²C₁C₂²N_B ≤ 1 - δ₂ \left(0 < δ₂ ≤ 1\right).
\]

Then Problem \((I^*)\) has a unique solution \((u, p, T) \in X \times M \times W\), satisfying

\[
(2.11) \quad \|\nabla u\|₀ ≤ A, \quad \|\nabla T\|₀ ≤ B.
\]

3. Construction of nonconforming mixed finite element scheme

Let \( \hat{K} = [-1, 1] \times [-1, 1] \) be the reference element on \( \xi - η \) plane, the four vertices of \( \hat{K} \) are \( \hat{d}_₁ = (-1, -1), \hat{d}_₂ = (1, -1), \hat{d}_₃ = (1, 1) \) and \( \hat{d}_₄ = (-1, 1) \), the four edges are \( \hat{l}_₁ = \hat{d}_₁\hat{d}_₂, \hat{l}_₂ = \hat{d}_₂\hat{d}_₃, \hat{l}_₃ = \hat{d}_₃\hat{d}_₄ \) and \( \hat{l}_₄ = \hat{d}_₄\hat{d}_₁ \).

For any \( \hat{v}^ᵢ, \hat{v}^₂ ∈ H^1(\hat{K}) \), we define the finite elements \((\hat{K}, \hat{P}^₁, \hat{S}^₁), i = 1, 2, \) on \( \hat{K} \) as follows:

\[
(3.1) \quad \hat{S}^₁ = \{\hat{v}^₁, \hat{v}^₂, \hat{v}^₃, \hat{v}^₄\}, \quad \hat{P}^₁ = \text{span}\{1, \xi, \eta, \eta^2\},
\]

\[
(3.2) \quad \hat{S}^₂ = \{\hat{v}^₁, \hat{v}^₂, \hat{v}^₃, \hat{v}^₄\}, \quad \hat{P}^₂ = \text{span}\{1, \xi, \eta, \xi^2\},
\]

where \( \hat{v}^ᵢ = \frac{1}{|\hat{l}_ᵢ|} \int_{\hat{l}_ᵢ} \hat{v}^ds, \ i = 1, 2, 3, 4, \ j = 1, 2. \)

It can be easily checked that the interpolations defined above are well-posed and the interpolation functions \( \hat{P}^ᵢ \hat{v}^ᵢ = \{1, i = 1, 2\} \) can be expressed as:

\[
(3.3) \quad \hat{P}^₁ \hat{v}^₁ = \frac{3}{4} (\hat{v}^₁ + \hat{v}^₄) - \frac{1}{4} (\hat{v}^₁ + \hat{v}^₃) + \frac{1}{2} (\hat{v}^₂ - \hat{v}^₃) \xi + \frac{1}{2} (\hat{v}^₃ - \hat{v}^₁) \eta + \frac{3}{4} (\hat{v}^₁ - \hat{v}^₂ + \hat{v}^₃ - \hat{v}^₄) \eta^2,
\]

and

\[
(3.4) \quad \hat{P}^₂ \hat{v}^₂ = \frac{3}{4} (\hat{v}^₁ + \hat{v}^₄) - \frac{1}{4} (\hat{v}^₁ + \hat{v}^₃) + \frac{1}{2} (\hat{v}^₂ - \hat{v}^₃) \xi + \frac{1}{2} (\hat{v}^₃ - \hat{v}^₁) \eta + \frac{3}{4} (\hat{v}^₁ + \hat{v}^₂ - \hat{v}^₃ + \hat{v}^₄) \xi^2,
\]

respectively.

For the sake of convenience, like [9-10] let \( \Omega \subset R^2 \) be a polygon with boundaries parallel to the axes, \( Ω \) be an axis parallel rectangular meshes of \( Ω \), where \( Ω \) satisfies the usual regularity assumption and quasi-uniform assumption (cf. [16]).

For any \( K ∈ Ω \), let

\[
\hat{K} = [x₁K - hₓ₁, x₁K + hₓ₁] × [x₂K - hₓ₂, x₂K + hₓ₂],
\]

\[ h_K = \max\{hₓ₁, hₓ₂\} \text{ and } h = \max_{K ∈ Ω} h_K. \]

Define the affine mapping \( F_K : \hat{K} → K \) as follows:

\[
(3.5) \quad \begin{cases} x₁ = x₁K + hₓ₁\xi, \\ x₂ = x₂K + hₓ₂\eta. \end{cases}
\]

Then the associated finite element spaces \( X_h, M_h, W_h \) and \( W₀h \) can be defined as:

\[
(3.6) \quad \begin{cases} X_h = \{ v = (v^₁, v^₂), \hat{v}^ᵢ|_{\hat{K}} = v^ᵢ|_{K} \circ F_K ∈ \hat{P}^ᵢ, \forall K ∈ Ω \}, \\ M_h = \{ φ ∈ M, φ|_{K} ∈ Q₀₀(K), \forall K ∈ Ω \}, \\ W_h = \{ p ∈ W, p|_{K} ∈ Q₁₁(K), \forall K ∈ Ω \}, \\ W₀h = W_h ∩ H₀₀(Ω), \end{cases}
\]
where \([v^i](i = 1, 2)\) stand for the jump of \(v^i\) across the edge \(F\) if \(F\) is an internal edge, and they are equal to \(v^i\) itself if \(F\) belongs to \(\partial \Omega\). \(Q_{j,j}(K)\) is a space of polynomials whose degree for \(x_1\) and \(x_2\) are equal to or less than \(j\), respectively.

Obviously, \(X_h \not\subset H^1(\Omega)^2, M_h \subset M, W_h \subset W\), so this is a nonconforming mixed finite element scheme.

Define the interpolation operator \(\Pi_h : v = (v^1, v^2) \in H^1(\Omega)^2 \mapsto \Pi_h v \in X_h\) as:

\[
\Pi_h v = (\Pi_h^1 v^1, \Pi_h^2 v^2), \quad \Pi_h|_{K} = \Pi_h^K, \quad \Pi_h v^i = \Pi_h^i \circ F_K^{-1}, i = 1, 2.
\]

Let \(\hat{\Pi} \hat{v} = (\hat{\Pi}^1 \hat{v}^1, \hat{\Pi}^2 \hat{v}^2)\) and \(\Pi_K v = (\Pi_K^1 v^1, \Pi_K^2 v^2)\), then

\[
(3.7) \quad \int_{I_i} (v - \Pi_K v) ds = 0, \quad i = 1, 2, 3, 4, \quad \forall \, v \in H^1(\Omega)^2,
\]

where \(I_i = \hat{I}_i \circ F_K^{-1}(i = 1, 2, 3, 4)\) are the four edges of \(K\), respectively.

For all \(v_h = (v_h^1, v_h^2) \in X_h\), we define

\[
\|v_h\|_h = \left( \sum_{K \in \mathcal{H}_h} \int_K \nabla v_h \cdot \nabla v_h dx \right)^{\frac{1}{2}} = \left( \sum_{K \in \mathcal{H}_h} (|v_h^1|^2_{1,K} + |v_h^2|^2_{1,K}) \right)^{\frac{1}{2}}.
\]

Then \(\| \cdot \|_h\) is the norm over \(X_h\).

We introduce the bilinear forms \(a^h(\cdot, \cdot)\) and \(b^h(\cdot, \cdot)\) and the trilinear form \(a^h_1(\cdot; \cdot; \cdot)\) as:

\[
(3.8) \quad a^h(u_h, v_h) = \mu \sum_{K \in \mathcal{H}_h} \int_K \nabla u_h \cdot \nabla v_h dx, \quad \forall \, u_h, v_h \in X_h,
\]

\[
(3.9) \quad b^h(p_h, v_h) = \sum_{K \in \mathcal{H}_h} \int_K p_h \text{div} v_h dx, \quad \forall \, v_h \in X_h, p_h \in M_h,
\]

and

\[
(3.10) \quad a^h_1(u_h; v_h, w_h) = \frac{1}{2} \sum_{K \in \mathcal{H}_h} \int_K \sum_{i,j = 1}^2 \left( \frac{\partial w_h^j}{\partial x_i} u_h^i - u_h^i \frac{\partial w_h^j}{\partial x_i} \right) dx, \quad \forall \, u_h, v_h, w_h \in X_h,
\]

respectively.

Then the approximation of Problem \((I^*)\) reads as follows:

Problem \((I^*)\) \quad Find \((u_h, p_h, T_h) \in X_h \times M_h \times W_h\), such that \(T_h|_{\partial \Omega} = T_0\)

\[
(3.11) \quad \begin{cases} 
    a^h(u_h, v_h) + a^h_1(u_h; v_h, w_h) - b^h(p_h, v_h) = \lambda(j T_h, v_h), & \forall \, v_h \in X_h, \\
    b^h(q_h, u_h) = 0, & \forall \, q_h \in M_h, \\
    d(T_h, \varphi_h) + \lambda \hat{a}_1(u_h; T_h, \varphi_h) = 0, & \forall \, \varphi_h \in W_{\partial h}.
\end{cases}
\]

Using the similar technique as in [18] we have the following discrete embedding inequality over \(X_h\),

\[
(3.12) \quad \|v_h\|_{0,2,\Omega} \leq C(k)\|v_h\|_h, \quad \forall \, v_h \in X_h, \quad k = 1, 2, \ldots, n = 2,
\]

especially, when \(k = 1\) we denote the constant \(C(1)\) by \(C_3\), i.e.,

\[
(3.13) \quad \|v_h\|_0 \leq C_3\|v_h\|_h, \quad \forall \, v_h \in X_h.
\]

It can be checked that the trilinear forms \(a^h_1(\cdot; \cdot; \cdot)\) and \(\hat{a}_1(\cdot; \cdot; \cdot)\) satisfy the following properties (cf. [18]):

\[
(3.14) \quad a^h(v_h, v_h) = \mu\|v_h\|^2_h, \quad \forall \, v_h \in X_h,
\]

\[
(3.15) \quad a^h_1(u_h; v_h, w_h) = 0, \quad a^h_1(u_h; v_h, w_h) = -a^h_1(u_h; w_h, v_h), \quad \forall \, u_h, v_h, w_h \in X_h,
\]
(3.16) \[ \bar{a}_1(u_h; T_h, T_h) = 0, \quad \bar{a}_1(u_h; T_h, \varphi_h) = -\bar{a}_1(u_h; \varphi_h, T_h), \quad \forall \, u_h \in X_h, \quad T_h, \varphi_h \in W_h, \]

(3.17) \[ |a^h_1(u_h; v_h, w_h)| \leq N_h \| u_h \|_h \| v_h \|_h \| w_h \|_h, \quad \forall \, u_h, v_h, w_h \in \mathcal{H}^1(\Omega)^2 \cup X_h, \]

(3.18) \[ |\bar{a}_1(u_h; T_h, \varphi_h)| \leq \bar{N}_h \| u_h \|_h \| \nabla T_h \|_0 \| \nabla \varphi_h \|_0, \quad \forall \, u_h \in \mathcal{H}^1(\Omega)^2 \cup X_h, \quad T_h \in W_h, \varphi_h \in W_{oh}, \]

where

\[ N_h = \sup_{u_h, v_h, w_h \in X_h} \frac{|a^h_1(u_h; v_h, w_h)|}{\| u_h \|_h \| v_h \|_h \| w_h \|_h}, \quad \bar{N}_h = \sup_{u_h \in X_h, T_h \in W_h, \varphi_h \in W_{oh}} \frac{|\bar{a}_1(u_h; T_h, \varphi_h)|}{\| u_h \|_h \| \nabla T_h \|_0 \| \nabla \varphi_h \|_0}, \]

which are norms for the trilinear forms \( a^h_1(\cdot, \cdot, \cdot) \) and \( \bar{a}_1(\cdot, \cdot, \cdot) \), respectively.

In view of (3.12), we know that there exist two positive constants \( N_0 > 1 \) and \( \bar{N}_0 > 1 \), such that

(3.19) \[ N_h \leq N_0, \quad \bar{N}_h \leq \bar{N}_0, \quad \forall \, 0 < h \leq 1. \]

Therefore

(3.20) \[ |a^h_1(u_h; v_h, w_h)| \leq N_0 \| u_h \|_h \| v_h \|_h \| w_h \|_h, \quad \forall \, u_h, v_h, w_h \in \mathcal{H}^1(\Omega)^2 \cup X_h, \]

(3.21) \[ |\bar{a}_1(u_h; T_h, \varphi_h)| \leq \bar{N}_0 \| u_h \|_h \| \nabla T_h \|_0 \| \nabla \varphi_h \|_0, \quad \forall \, u_h \in \mathcal{H}^1(\Omega)^2 \cup X_h, \quad T_h \in W_h, \varphi_h \in W_{oh}. \]

**Lemma 3.1.** The spaces \( X_h \) and \( M_h \) satisfy the discrete inf-sup condition (cf. [14]), i.e.,

(3.22) \[ \sup_{v_h \in X_h} \frac{b^h(q_h, v_h)}{\| v_h \|_h} \geq \beta \| q_h \|_0, \quad \forall \, q_h \in M_h, \]

where \( \beta \) is a positive constant independent of \( h \).

**Proof.** By (3.3) and (3.4), let

\[ \alpha_1 = \frac{1}{2} (\hat{v}_1^2 - \hat{v}_4^2), \quad \alpha_2 = \frac{1}{2} (\hat{v}_3^2 - \hat{v}_1^2), \quad \alpha_3 = \frac{3}{4} (\hat{v}_1^4 - \hat{v}_2^3 + \hat{v}_3^3 - \hat{v}_4^3), \]

\[ \beta_1 = \frac{1}{2} (\hat{v}_2^2 - \hat{v}_4^2), \quad \beta_2 = \frac{1}{2} (\hat{v}_3^2 - \hat{v}_1^2), \quad \beta_3 = \frac{3}{4} (\hat{v}_2^3 - \hat{v}_2^3 + \hat{v}_3^3 - \hat{v}_4^3). \]

It can be checked that the following inequalities hold

(3.23) \[ |\alpha_1| \leq C|\hat{v}_1|_{1, \bar{K}}, \quad |\beta_1| \leq C|\hat{v}_2|_{1, \bar{K}}, \quad i = 1, 2, 3. \]

In fact

\[ \alpha_1 = \frac{1}{4} (\hat{v}_2^2 - \hat{v}_4^2) = \frac{1}{4} \left[ \int_{i_2} \hat{v}_1^4 (\eta - 1) d\eta - \int_{i_4} \hat{v}_1^4 (-1, \eta) d\eta \right] \]

\[ = \frac{1}{4} \left[ \int_{i_3} \int_{\bar{K}} - \frac{\partial \hat{v}_1^4}{\partial \xi} d\xi d\eta \right] \leq \frac{1}{2} \| \frac{\partial \hat{v}_1^4}{\partial \xi} \|_{0, \bar{K}} \leq C|\hat{v}_1|_{1, \bar{K}}, \]

\[ \alpha_2 = \frac{1}{4} (\hat{v}_3^2 - \hat{v}_1^2) = \frac{1}{4} \left[ \int_{i_3} \hat{v}_1^4 (\xi, 1) d\xi - \int_{i_1} \hat{v}_1^4 (\xi, -1) d\xi \right] \]

\[ = \frac{1}{4} \left[ \int_{i_3} \int_{\bar{K}} - \frac{\partial \hat{v}_1^4}{\partial \eta} d\xi d\eta \right] \leq \frac{1}{2} \| \frac{\partial \hat{v}_1^4}{\partial \eta} \|_{0, \bar{K}} \leq C|\hat{v}_1|_{1, \bar{K}}, \]
\[ \alpha_3 = \frac{3}{4} (\hat{v}_1 - \hat{v}_2 + \hat{v}_3 - \hat{v}_4) \]
\[ = \frac{3}{4} \left[ \int_K \hat{v}_1(\xi, \eta)d\xi d\eta - \int_{i_3} \hat{v}_1(1, \eta)d\eta - \int_{i_4} \hat{v}_1(-1, \eta)d\eta + \int_{i_3} \hat{v}_1(\xi, 1)d\xi + \int_{i_4} \hat{v}_1(\xi, -1)d\xi - \int_K \hat{v}_1(\xi, \eta)d\eta \right] \]
\[ \leq C|\hat{v}_1|_{1,K}. \]

Similarly
\[ \beta_i \leq C|\hat{v}_i^2|_{1,K}, \quad i = 1, 2, 3. \]

From (3.23) we can prove
\[ \| \Pi_h v \|_h \leq C|v|_1, \quad \forall \ v \in X. \]

By the definition of interpolation (3.7) and Green's formula, we get for all \( v \in X \)
\[ b^h(q_h, v - \Pi_h v) = \sum_{K \in \mathcal{T}_h} \int_K q_h \text{div}(v - \Pi_K v) dx \]
\[ = \sum_{K \in \mathcal{T}_h} q_h |K| \int_K \text{div}(v - \Pi_K v) dx \]
\[ = \sum_{K \in \mathcal{T}_h} q_h |K| \int_{\partial K} (v - \Pi_K v) \cdot nds \]
\[ = 0. \]

The fact that \( q_h \) belongs to a piece-wise constant space is used. Here and later \( n \)
denotes the outward unit normal vector to \( \partial K \). Since the spaces \( X \) and \( M \) satisfy
the inf-sup condition, there exists a constant \( \beta_0 > 0 \) such that
\[ \sup_{v \in X} \frac{b(q, v)}{\|v\|_1} \geq \beta_0 \|q\|_0, \quad \forall \ q \in M. \]

Therefore, by (3.25), (3.26) and (3.27), we have
\[ \sup_{v \in X} \frac{b^h(q_h, v)}{\|v\|_h} \geq \frac{\sup_{v \in X} \frac{b^h(q_h, v)}{\|v\|_1}}{\|\Pi_h v\|_h} = \sup_{v \in X} \frac{b^h(q_h, v)}{\|\Pi_h v\|_h} \]
\[ \geq \frac{1}{C} \sup_{v \in X} \frac{b(q_h, v)}{\|v\|_1} \geq \frac{\beta_0}{C} \|q_h\|_0, \]

which, by setting \( \beta = \frac{\beta_0}{C} \), yields the desired result.

In order to prove the existence and uniqueness of the solution to Problem \((I^h)\),
we first consider the following equations:

Problem \((I^h)\) Find \((u_h, T_h) \in V_h \times W_h\), such that \( T_h|_{\partial \Omega} = T_0 \)
\[ \left\{ \begin{array}{ll}
\alpha^i_h (u_h, v_h) + a^i_1(u_h; u_h, v_h) = \lambda_j(T_h, v_h), & \forall \ v_h \in V_h, \\
d(T_h, \varphi_h) + \lambda a_1(u_h; T_h, \varphi_h) = 0, & \forall \ \varphi_h \in W_h,
\end{array} \right. \]

where
\[ \left\{ \begin{array}{ll}
V_h = \{ v_h \in X_h; b^h(p_h, v_h) = 0, \forall \ p_h \in M_h \}, \\
\end{array} \right. \]

It is easy to see that
\[ \left\{ \begin{array}{ll}
V_h = \{ v_h \in X_h; \text{div}v_h = 0 \}. \\
\end{array} \right. \]
Theorem 3.1. Under the assumption of \((A_1)\), let \(A' = 2\mu^{-1}\lambda C_3(1 + 3C_1')\|T_0\|_1\) and \(B = \left(5 + \frac{1}{C_1}\right)\|T_0\|_1\). Assume that there exist positive constants \(\delta_3\) and \(\delta_4\) such that
\[
\mu^{-1}N_0 A' \leq 1 - \delta_3 \ (0 < \delta_3 \leq 1); \quad \delta_3^{-1} \mu^{-1} \lambda^2 C_3' C_3 N_0 B \leq 1 - \delta_4 \ (0 < \delta_4 \leq 1).
\]
Then Problem \((I^b)\) has a unique solution \((u_h, p_h, T_h) \in X_h \times M_h \times W_h\), satisfying
\[
\|u_h\|_h \leq A', \quad \|\nabla T\|_0 \leq B.
\]

Proof. We use Banach’s fixed point theorem (cf. [27]) to prove our theorem. The proof proceeds in the following three steps.

Step 1. To prove the existence of the solution to Problem \((I^h')\). For a given \(\hat{u}_h \in V_h\), we consider the following equation:
\[
d(T_h, \psi_h) + \lambda \tilde{a}_1(\hat{u}_h; T_h, \psi_h) = 0, \quad \forall \psi_h \in W_{0h}.
\]
Let \(T_h = w_h + T_0,\ w_h \in W_{0h}\). Substituting it into (3.34), we have
\[
d(w_h, \psi_h) + \lambda \tilde{a}_1(\hat{u}_h; w_h, \psi_h) = -d(T_0, \psi_h) - \lambda \tilde{a}_1(\hat{u}_h; T_0, \psi_h), \quad \forall \psi_h \in W_{0h}.
\]
Let
\[
D(w_h, \psi_h) = d(w_h, \psi_h) + \lambda \tilde{a}_1(\hat{u}_h; w_h, \psi_h), \quad \forall \psi_h \in W_{0h}.
\]
Choosing \(\psi_h = w_h\) in (3.36), we deduce that
\[
D(w_h, w_h) = d(w_h, w_h) + \lambda \tilde{a}_1(\hat{u}_h; w_h, w_h) = d(w_h, w_h) = \|\nabla w_h\|_0^2.
\]
So \(D(\cdot, \cdot)\) is continuous and coercive over \(W_{0h}\). By the Lax-Milgram Theorem (cf. [14]), we know that (3.35) has a unique solution \(T_h \in W_h\), satisfying \(T_h|_{\partial \Omega} = T_0\). For this \(T_h\), we consider the following equations:
\[
a^h(u_h^*, v_h) + a^h_1(u_h^*, u_h^*, v_h) - b^h(p_h, v_h) = \lambda(jT_h, v_h), \quad \forall v_h \in V_h,
\]
and
\[
b^h(\varphi_h, u_h^*) = 0, \quad \forall \varphi_h \in M_h,
\]
which approximate the stationary Navier-Stokes equations. By (3.14) and Lemma 3.1 we know that there exists a unique solution \((u_h^*, p_h) \in V_h \times M_h\) (cf. [18,26]). Thus (3.34), (3.38) and (3.39) determine a mapping \(l_h : \hat{u}_h \in V_h \to u_h^* \in V_h\), i.e., \(u_h^* = l_h \hat{u}_h\).

Firstly, we estimate \(\|\nabla T_h\|_0\).
Choosing \(T_h = w_h + T_0, \psi_h = w_h\) in (3.34), we arrive at
\[
d(w_h, w_h) = -\lambda \tilde{a}_1(\hat{u}_h; T_0, w_h) - d(T_0, w_h).
\]
By (3.21), we have
\[
\|\nabla w_h\|_0^2 \leq \lambda \tilde{N}_0 \|\hat{u}_h\|_h \|\nabla T_0\|_0 \|\nabla w_h\|_0 + \|\nabla T_0\|_0 \|\nabla w_h\|_0.
\]
By \((A_1)\), we get
\[
\|\nabla w_h\|_0 \leq \lambda \tilde{N}_0 \varepsilon \|\hat{u}_h\|_h + \|\nabla T_0\|_0.
\]
Hence
\[
\|\nabla T_h\|_0 \leq \|\nabla w_h\|_0 + \|\nabla T_0\|_0 \leq \lambda \tilde{N}_0 \varepsilon \|\hat{u}_h\|_h + 2\|T_0\|_0 = \tilde{B}(\|\hat{u}_h\|_h).
\]
Secondly, we estimate \(\|u_h^*\|_h\).
Choosing \(v_h = u_h^*\) in (3.38), by (3.13) and (3.39), we obtain
\[
\mu \|u_h^*\|_h^2 = |\lambda(jT_h, u_h^*)| \leq \lambda \|T_h\|_0 \|u_h^*\|_0 \leq \lambda C_3 \|T_h\|_0 \|u_h^*\|_h.
\]
Applying (2.4) and (3.43), we deduce that
\[
\|u_h^i\|_h \leq \mu^{-1}\lambda C_3\|T_h\|_0 \leq \mu^{-1}\lambda C_3(\|w_h\|_0 + \|T_0\|_0) \\
\leq \mu^{-1}\lambda C_3(C'_1\|\nabla w_h\|_0 + \|T_0\|_0) \\
\leq \mu^{-1}\lambda C_3[C'_1\|\nabla T_h\|_0 + (1 + C'_1)\|T_0\|_1] \\
\leq \mu^{-1}\lambda C_3(1 + 3C'_1)\|T_0\|_1 + \mu^{-1}\lambda^2 C'_1 C_3 \tilde{N}_0 \varepsilon \|\tilde{u}_h\|_h \\
= \hat{A}(\|\tilde{u}_h\|_h).
\]

(3.45)

Thirdly, we prove \(l_h\) is continuous.
For arbitrarily given \(\tilde{u}_h^1, \tilde{u}_h^2 \in V_h\), and according to (3.34), (3.38) and (3.39), there exist \((u_h^1, p_h^1, T_h^1)\), \((u_h^2, p_h^2, T_h^2)\) \(\in V_h \times M_h \times W_h\), satisfying \(T_h^1|_{\partial \Omega} = T_h^2|_{\partial \Omega} = T_0\). By (3.34), we have
\[
d(T_h^1 - T_h^2, \psi_h) + \lambda \tilde{a}_1(\tilde{u}_h^1 - \tilde{u}_h^2; T_h^1, \psi_h) + \lambda \tilde{a}_1(\tilde{u}_h^2 - \tilde{u}_h^2; T_h^2, \psi_h) = 0, \quad \forall \psi_h \in W_{oh}.
\]
Choosing \(\psi_h = T_h^1 - T_h^2\) in (3.46) and using (3.16) and (3.21), we obtain
\[
\|\nabla(T_h^1 - T_h^2)\|_0^2 = |\lambda \tilde{a}_1(\tilde{u}_h^1 - \tilde{u}_h^2; T_h^1, T_h^1 - T_h^2)| \\
\leq \lambda \tilde{N}_0 \|\tilde{u}_h^1 - \tilde{u}_h^2\|_h \|\nabla T_h^1\|_0 \|\nabla(T_h^1 - T_h^2)\|_0.
\]

Hence
\[
\|\nabla(T_h^1 - T_h^2)\|_0 \leq \lambda \tilde{N}_0 \|\tilde{u}_h^1 - \tilde{u}_h^2\|_h \|\nabla T_h^1\|_0.
\]

(3.48)

Owing to (3.38), we deduce that
\[
a^h(u_h^{1*} - u_h^{2*}, v_h) + a_h^1(u_h^{1*} - u_h^{2*}; u_h^{1*}, v_h) + a_h^1(u_h^{2*}; u_h^{1*} - u_h^{2*}, v_h) \\
- b^h(p_h^1 - p_h^2, v_h) = \lambda(j(T_h^1 - T_h^2), v_h), \quad \forall v_h \in V_h.
\]
Choosing \(v_h = u_h^{1*} - u_h^{2*}\) in (3.49) and using (2.4), (3.13), (3.20) and (3.39), we derive
\[
\mu \|u_h^{1*} - u_h^{2*}\|_h^2 \\
= |\lambda(j(T_h^1 - T_h^2), u_h^{1*} - u_h^{2*}) - a_h^1(u_h^{1*} - u_h^{2*}; u_h^{1*} - u_h^{2*})| \\
\leq \lambda C'_1 C_3 \|\nabla(T_h^1 - T_h^2)\|_0 \|u_h^{1*} - u_h^{2*}\|_h + \tilde{N}_0 \|u_h^{1*} - u_h^{2*}\|_h \|u_h^{1*}\|_h.
\]

Therefore
\[
\|u_h^{1*} - u_h^{2*}\|_h \leq \mu^{-1}\lambda C'_1 C_3 \|\nabla(T_h^1 - T_h^2)\|_0 + \mu^{-1}\tilde{N}_0 \|u_h^{1*} - u_h^{2*}\|_h \|u_h^{1*}\|_h.
\]

(3.51)

Substituting (3.48) into (3.51) yields
\[
\|u_h^{1*} - u_h^{2*}\|_h \\
\leq \mu^{-1}\lambda^2 C'_1 C_3 \tilde{N}_0 \|\tilde{u}_h^1 - \tilde{u}_h^2\|_h \|\nabla T_h^1\|_0 + \mu^{-1}\tilde{N}_0 \|u_h^{1*} - u_h^{2*}\|_h \|u_h^{1*}\|_h.
\]

(3.52)

From (3.43) and (3.45) we know that \(\|u_h^{1*}\|_h \leq \tilde{A}(\|\tilde{u}_h\|_h)\) and \(\|\nabla T_h^1\|_0 \leq \tilde{B}(\|\tilde{u}_h\|_h)\) \((i=1,2)\) are bounded if \(\|\tilde{u}_h\|_h \) \((i=1,2)\) are bounded. By (3.52) and the assumption that \(\mu^{-1}\tilde{N}_0 \tilde{A} \leq 1 - \delta_3 \) \((0 < \delta_3 \leq 1)\), we arrive at
\[
\|u_h^{1*} - u_h^{2*}\|_h \leq \delta_3^{-1}\mu^{-1}\lambda^2 C'_1 C_3 \tilde{N}_0 \tilde{B}(\|\tilde{u}_h\|_h) \|\tilde{u}_h^1 - \tilde{u}_h^2\|_h.
\]

(3.53)

Therefore \(u_h^{*}\) is continuously dependent on \(\tilde{u}_h\).

Finally, we consider the following equation:
\[
u_h = s \lambda \tilde{u}_h, \quad 0 \leq s \leq 1.
\]
Obviously, \( s = 0 \) is a trivial case. So we only need to consider \( 0 < s \leq 1 \).
Substituting \( s^{-1}u_h = l_h u_h \) into (3.45), we obtain
\[
(3.54) \quad s^{-1}||u_h|| \leq \mu^{-1} \lambda C_3 (1 + 3C_1') ||T_0||_1 + \mu^{-1} \lambda^2 C_1' C_3 \bar{N}_0 \varepsilon ||u_h||.
\]
Noting that \( 0 < s \leq 1 \), then \( s(2-s)^{-1} \leq 1 \). Assume that
\[
(3.55) \quad \varepsilon = (2\mu^{-1} \lambda^2 C_1' C_3 \bar{N}_0)^{-1}.
\]
By (3.54), we have
\[
(3.56) \quad ||u_h|| \leq 2\mu^{-1} \lambda C_3 (1 + 3C_1') ||T_0||_1 \equiv A'.
\]
Substituting (3.56) into (3.43) yields
\[
(3.57) \quad ||\nabla T_h||_0 \leq 2 ||T_0||_1 + \lambda \bar{N}_0 (2\mu^{-1} \lambda^2 C_1' C_3 \bar{N}_0)^{-1} ||u_h|| = \left( 5 + \frac{1}{C_1} \right) ||T_0||_1 \equiv B.
\]
Observe that \( \delta_3^{-1} \mu^{-1} \lambda^2 C_1' C_3 \bar{N}_0 \delta_4 \leq 1 - \delta_4 \) \((0 < \delta_4 \leq 1)\), we know, by (3.53), that \( l_h \) is a contractive operator. Using Banach’s fixed point theorem, Problem \((I^h')\) has a unique solution \((u_h, T_h) \in V_h \times W_h\), satisfying \( T_h|_{\partial \Omega} = T_0 \).

**Step 2.** To prove the existence of the solution to Problem \((I^h)\).

Clearly, the solution \((u_h, T_h) \in V_h \times W_h\) to Problem \((I^h')\) satisfies the last two equations of Problem \((I^h)\). For the given \((u_h, T_h)\), we define
\[
(3.58) \quad f(v_h) = a^h(u_h, v_h) + a^h_1(u_h; u_h, v_h) - \lambda (jT_h, v_h), \quad \forall v_h \in X_h.
\]
Noting that \( f(v_h) \) is a bounded linear functional over \( X_h \) and \( f(v_h) = 0 \) on \( V_h \), then there exists a \( p_h \in M_h \) such that
\[
(3.59) \quad f(v_h) = (p_h, \text{div}v_h), \quad \forall v_h \in X_h,
\]
(cf. [15]). Therefore, Problem \((I^h)\) has a solution \((u_h, p_h, T_h) \in X_h \times M_h \times W_h\), satisfying \( T_h|_{\partial \Omega} = T_0 \).

**Step 3.** To prove the uniqueness of the solution to Problem \((I^h)\).

Suppose that Problem \((I^h)\) has another solution \((u^1_h, p^1_h, T^1_h) \in X_h \times M_h \times W_h\), satisfying \( T^1_h|_{\partial \Omega} = T_0 \). By Problem \((I^h)\), we deduce that
\[
(3.60) \quad a^h(u_h - u^1_h, v_h) + a^h_1(u_h - u^1_h; u_h, v_h) + a^h_1(u^1_h; u_h - u^1_h, v_h)
\]
\[
-b^h(p_h - p^1_h, v_h) = \lambda (j(T_h - T^1_h), v_h), \quad \forall v_h \in X_h,
\]
\[
(3.61) \quad b^h(\varphi_h, u_h - u^1_h) = 0, \quad \forall \varphi_h \in M_h,
\]
and
\[
(3.62) \quad d(T_h - T^1_h, \psi_h) + \lambda \bar{a}_1(u_h - u^1_h; T_h, \psi_h) + \lambda \bar{a}_1(u^1_h; T_h - T^1_h, \psi_h) = 0, \quad \forall \psi_h \in W_{oh}.
\]
Taking \( v_h = u_h - u^1_h \) in (3.60) and applying (2.4), (3.13), (3.20) and (3.56) yield
\[
\mu ||u_h - u^1_h|| = |\lambda (j(T_h - T^1_h), u_h - u^1_h)| - a^h_1(u_h - u^1_h; u_h, u_h - u^1_h)|
\]
\[
\leq \lambda C_1 C_3 ||\nabla(T_h - T^1_h)||_0 ||u_h - u^1_h|| + N_0 A' ||u_h - u^1_h||^2.
\]
Noting that \( \mu^{-1} N_0 A' \leq 1 - \delta_3 \) \((0 < \delta_3 \leq 1)\), we have
\[
(3.64) \quad ||u_h - u^1_h||_h \leq \delta_3^{-1} \mu^{-1} \lambda C_1 C_3 ||\nabla(T_h - T^1_h)||_0.
\]
Choosing \( \psi_h = T_h - T^1_h \) in (3.62) and using (3.16) and (3.21), we can write
\[
\|\nabla(T_h - T^1_h)\|_0 = | - \lambda \bar{a}_1(u_h - u^1_h; T_h, T_h - T^1_h)|
\]
\[
\leq \lambda \bar{N}_0 ||u_h - u^1_h||_h ||\nabla T_h||_0 ||\nabla(T_h - T^1_h)||_0.
\]
Then it stems from (3.57) that
\begin{equation}
\|\nabla(T_h - T_h^l)\|_0 \leq \lambda N_B\|u_h - u_h^l\|_h.
\end{equation}
Substituting (3.64) into (3.66) yields
\begin{equation}
\|\nabla(T_h - T_h^l)\|_0 \leq \delta^{-1}_3 \mu^{-1} \lambda^2 B C^4_1 C_3 N_{\bar{B}} \|\nabla(T_h - T_h^l)\|_0.
\end{equation}
On the one hand, noting that \(\delta^{-1}_3 \mu^{-1} \lambda^2 B C^4_1 C_3 N_{\bar{B}} \leq 1 - \delta_4 (0 < \delta_4 \leq 1)\), we have \(\|\nabla(T_h - T_h^l)\|_0 = 0\), hence \(T_h = T_h^l\). Together with (3.64) we know that \(\|u_h - u_h^l\|_h = 0\), i.e., \(u_h = u_h^l\).

On the other hand, by (3.60), we have
\begin{equation}
b^h(p_h - p_h^l, v_h) = 0, \quad \forall \ v_h \in X_h.
\end{equation}
Thanks to Lemma 3.1, we get
\begin{equation}
\beta \|p_h - p_h^l\|_0 \leq \sup_{v_h \in X_h} \frac{b^h(p_h - p_h^l, v_h)}{\|v_h\|_h} = 0.
\end{equation}
That is \(p_h = p_h^l\). We complete the proof.

4. Error estimates

This section is devoted to establishing convergence results for the discrete velocity, pressure and temperature spaces. First of all, we state the following lemma which can be found in [14].

Lemma 4.1. There exists an operator \(r_h : W \rightarrow W_h\), such that for all \(\varphi \in W\)
\[ (\nabla(\varphi - r_h \varphi), \nabla \varphi_h) = 0, \quad \forall \varphi_h \in W_h, \]
\[ \int_\Omega (\varphi - r_h \varphi) dx = 0, \quad \|\nabla r_h \varphi\|_0 \leq \|\nabla \varphi\|_0, \]
and
\[ \|\varphi - r_h \varphi\|_1 \leq C h |\varphi|_2, \quad \forall \varphi \in H^2(\Omega). \]

Next we prove the following two important lemmas.

Lemma 4.2. Assume that \(u \in H^2(\Omega)^2\) and \(p \in H^1(\Omega)\), then we have
\begin{equation}
\left| \sum_{K \in \mathcal{K}_h} \int_{\partial K} \frac{\partial u}{\partial n} \cdot v_h ds \right| \leq C h |u|_2 \|v_h\|_h, \quad \forall \ v_h \in X_h,
\end{equation}
and
\begin{equation}
\left| \sum_{K \in \mathcal{K}_h} \int_{\partial K} p v_h \cdot n ds \right| \leq C h |p|_1 \|v_h\|_h, \quad \forall \ v_h \in X_h.
\end{equation}

Proof. Note that
\begin{equation}
\sum_{K \in \mathcal{K}_h} \int_{\partial K} \frac{\partial u}{\partial n} \cdot v_h ds
= \sum_{K \in \mathcal{K}_h} \left[ - \int_{l_1} \left( \frac{\partial u_1^1}{\partial x_1} v_h^1 + \frac{\partial u_2^1}{\partial x_2} v_h^2 \right) dx_1 + \int_{l_2} \left( \frac{\partial u_1^2}{\partial x_1} v_h^1 + \frac{\partial u_2^2}{\partial x_2} v_h^2 \right) dx_2 \right]
+ \int_{l_3} \left( \frac{\partial u_1^3}{\partial x_2} v_h^1 + \frac{\partial u_2^3}{\partial x_1} v_h^2 \right) dx_1 - \int_{l_4} \left( \frac{\partial u_1^4}{\partial x_1} v_h^1 + \frac{\partial u_2^4}{\partial x_2} v_h^2 \right) dx_2.
\end{equation}

For any \(K \in \mathcal{K}_h\) and \(v \in H^1(K)(H^1(K)^2)\), we define the following operators:
\[ P_{0i} v = \frac{1}{2h_{x_i}} \int_{l_i} v dx_{x_i}, \quad i = 1, 3, \quad P_{0i} v = \frac{1}{2h_{x_i}} \int_{l_i} v dx_{x_i}, \quad i = 2, 4, \]
\[ P_0v = \frac{1}{|K|} \int_K v dx. \]

It is easy to see that the above operators are affine equivalent. Let the corresponding ones onto the reference element \( \hat{K} \) be denoted by \( \hat{P}_i \) \( (i = 1, 2, 3, 4) \) and \( \hat{P}_0 \). By (4.3) we get

\begin{equation}
\sum_{K \in \mathcal{K}_h} \int_{\partial K} \frac{\partial u}{\partial n} \cdot v_h ds = \sum_{K \in \mathcal{K}_h} \left\{ - \int_{l_1} \left[ \left( \frac{\partial u^1}{\partial x_2} - P_0 \frac{\partial u^1}{\partial x_2} \right)(v_h^1 - P_{0l_1}v_h^1) + \left( \frac{\partial u^2}{\partial x_2} - P_0 \frac{\partial u^2}{\partial x_2} \right)(v_h^2 - P_{0l_1}v_h^2) \right] dx_1 \\
+ \int_{l_2} \left[ \left( \frac{\partial u^1}{\partial x_1} - P_0 \frac{\partial u^1}{\partial x_1} \right)(v_h^1 - P_{0l_2}v_h^1) + \left( \frac{\partial u^2}{\partial x_1} - P_0 \frac{\partial u^2}{\partial x_1} \right)(v_h^2 - P_{0l_2}v_h^2) \right] dx_2 \\
+ \int_{l_3} \left[ \left( \frac{\partial u^1}{\partial x_2} - P_0 \frac{\partial u^1}{\partial x_2} \right)(v_h^1 - P_{0l_3}v_h^1) + \left( \frac{\partial u^2}{\partial x_2} - P_0 \frac{\partial u^2}{\partial x_2} \right)(v_h^2 - P_{0l_3}v_h^2) \right] dx_1 \\
- \int_{l_4} \left[ \left( \frac{\partial u^1}{\partial x_1} - P_0 \frac{\partial u^1}{\partial x_1} \right)(v_h^1 - P_{0l_4}v_h^1) + \left( \frac{\partial u^2}{\partial x_1} - P_0 \frac{\partial u^2}{\partial x_1} \right)(v_h^2 - P_{0l_4}v_h^2) \right] dx_2 \right\} \tag{4.4}
\end{equation}

where

\begin{align*}
F_1 &= - \int_{l_1} \left[ \left( \frac{\partial u^1}{\partial x_2} - P_0 \frac{\partial u^1}{\partial x_2} \right)(v_h^1 - P_{0l_1}v_h^1) + \left( \frac{\partial u^2}{\partial x_2} - P_0 \frac{\partial u^2}{\partial x_2} \right)(v_h^2 - P_{0l_1}v_h^2) \right] dx_1, \\
F_2 &= \int_{l_2} \left[ \left( \frac{\partial u^1}{\partial x_1} - P_0 \frac{\partial u^1}{\partial x_1} \right)(v_h^1 - P_{0l_2}v_h^1) + \left( \frac{\partial u^2}{\partial x_1} - P_0 \frac{\partial u^2}{\partial x_1} \right)(v_h^2 - P_{0l_2}v_h^2) \right] dx_2, \\
F_3 &= \int_{l_3} \left[ \left( \frac{\partial u^1}{\partial x_2} - P_0 \frac{\partial u^1}{\partial x_2} \right)(v_h^1 - P_{0l_3}v_h^1) + \left( \frac{\partial u^2}{\partial x_2} - P_0 \frac{\partial u^2}{\partial x_2} \right)(v_h^2 - P_{0l_3}v_h^2) \right] dx_1, \\
F_4 &= - \int_{l_4} \left[ \left( \frac{\partial u^1}{\partial x_1} - P_0 \frac{\partial u^1}{\partial x_1} \right)(v_h^1 - P_{0l_4}v_h^1) + \left( \frac{\partial u^2}{\partial x_1} - P_0 \frac{\partial u^2}{\partial x_1} \right)(v_h^2 - P_{0l_4}v_h^2) \right] dx_2.
\end{align*}

Using Hölder inequality, trace theorem and interpolation theorem, we have

\begin{equation}
F_i \leq Ch_K (||u^1||_{2,K}||v_h^1||_{1,K} + ||u^2||_{2,K}||v_h^2||_{1,K}) \leq Ch_K ||u||_{2,K}||v_h||_{1,K}, \quad i = 1, 2, 3, 4. \tag{4.5}
\end{equation}

By using (4.5), we get

\[ \left| \sum_{K \in \mathcal{K}_h} \int_{\partial K} \frac{\partial u}{\partial n} \cdot v_h ds \right| \leq Ch ||u||_{2,K} ||v_h||_{1,K}. \]

Similarly, we can get (4.2). Then Lemma 4.2 is proved.

**Lemma 4.3.** Assume that \( u \in H^2(\Omega)^2 \cap H^1_0(\Omega)^2 \), then there holds

\[ \left| \sum_{K \in \mathcal{K}_h} \int_{\partial K} (u \cdot n)(u \cdot v_h) ds \right| \leq Ch ||u||_1 ||u||_2 ||v_h||, \quad \forall v_h \in X_h. \]
Theorem 4

The following theorem is the main result of this section. Combining (4.7), (4.8), (4.9) and (4.10), yields the desired result.

Proof.

\( \sum_{K \in \mathcal{A}_h} \int_{\partial K} (u \cdot n)(u \cdot v_h)ds \)

\( = \sum_{K \in \mathcal{A}_h} \left( \int_{I_1} -u^2 u \cdot v_h dx_1 + \int_{I_2} u^1 u \cdot v_h dx_2 + \int_{I_3} u^2 u \cdot v_h dx_1 + \int_{I_4} -u^1 u \cdot v_h dx_2 \right) \)

\( = \sum_{K \in \mathcal{A}_h} \left[ \int_{I_1} -(u^2 u - P_0(u^2 u)) \cdot (v_h - P_{0l_1} v_h) dx_1 + \int_{I_2} (u^1 u - P_0(u^1 u)) \cdot (v_h - P_{0l_2} v_h) dx_2 + \int_{I_3} (u^2 u - P_0(u^2 u)) \cdot (v_h - P_{0l_3} v_h) dx_1 + \int_{I_4} -(u^1 u - P_0(u^1 u)) \cdot (v_h - P_{0l_4} v_h) dx_2 \right] \)

\( = \sum_{K \in \mathcal{A}_h} [I_1 + I_2 + I_3 + I_4], \)

where

\[ I_1 = -\int_{I_1} (u^2 u - P_0(u^2 u)) \cdot (v_h - P_{0l_1} v_h) dx_1, \]

\[ I_2 = \int_{I_2} (u^1 u - P_0(u^1 u)) \cdot (v_h - P_{0l_2} v_h) dx_2, \]

\[ I_3 = \int_{I_3} (u^2 u - P_0(u^2 u)) \cdot (v_h - P_{0l_3} v_h) dx_1, \]

\[ I_4 = -\int_{I_4} (u^1 u - P_0(u^1 u)) \cdot (v_h - P_{0l_4} v_h) dx_2. \]

Using Hölder inequality, trace theorem and interpolation theorem, we have

\[ I_1 = -\int_{I_1} (u^2 u - P_0(u^2 u)) \cdot (v_h - P_{0l_1} v_h) dx_1 \]

\[ \leq \left( \int_{I_1} (u^2 u - P_0(u^2 u))^2 dx_1 \right)^{\frac{1}{2}} \left( \int_{I_1} (v_h - P_{0l_1} v_h)^2 dx_1 \right)^{\frac{1}{2}} \]

\[ = \left( \int_{I_1} (u^2 u - \hat{P}_0(u^2 u))^2 h_{x_1} d\xi \right)^{\frac{1}{2}} \left( \int_{I_1} (\hat{v}_h - \hat{P}_{0l_1} \hat{v}_h)^2 h_{x_1} d\xi \right)^{\frac{1}{2}} \]

\[ \leq Ch_{x_1} ||u^2 u - \hat{P}_0(u^2 u)||_{1,K} ||\hat{v}_h - \hat{P}_{0l_1} \hat{v}_h||_{1,K} \]

\[ \leq Ch_{x_1} ||u^2 u||_{1,K} ||\hat{v}_h||_{1,K} \]

Similarly

\[ I_2 \leq Ch_{x_2} ||u^1 u||_{1,K} ||v_h||_{1,K}, \]

\[ I_3 \leq Ch_{x_1} ||u^2 u||_{1,K} ||v_h||_{1,K}, \]

\[ I_4 \leq Ch_{x_2} ||u^1 u||_{1,K} ||v_h||_{1,K}. \]

Combining (4.7), (4.8), (4.9) and (4.10), yields the desired result.

The following theorem is the main result of this section.

**Theorem 4.1.** Under the assumption of (A1), let \((u, p, T) \in (H^2(\Omega) \cap H^1_0(\Omega))^2 \times H^1(\Omega) \cap H^1_0(\Omega) \times H^1(\Omega) \cap H^1_0(\Omega)\) and \((u_h, p_h, T_h) \in X_h \times M_h \times W_h\) be the solutions to Problems (I) and (Ih), respectively, then we have

\[ ||u - u_h||_h + ||p - p_h||_0 + ||\nabla(T - T_h)||_0 \leq Ch(||u||_2 + |p|_1 + |T|_2 + ||u||_1 ||u||_2). \]
Proof. By Problems (I) and (I'), we get the following equations:

\[
\begin{align*}
& a^h(u - u_h, v_h) + a_1^h(u - u_h; u, v_h) + a_2^h(u_h; u - u_h, v_h) - b^h(p - p_h, v_h) \\
& - \mu \sum_{K \in \mathcal{S}_h} \int_{\partial K} \frac{\partial u}{\partial n} \cdot v_h ds + \sum_{K \in \mathcal{S}_h} \int_{\partial K} p v_h \cdot nds \\
& - \frac{1}{2} \sum_{K \in \mathcal{S}_h} \int_{\partial K} (u \cdot n)(u \cdot v_h) ds = \lambda(j(T - T_h), v_h), \quad \forall v_h \in X_h,
\end{align*}
\]

(4.11)

\[
b^h(q_h, u - u_h) = 0, \quad \forall q_h \in M_h,
\]

(4.12)

and

\[
d(T - T_h, \varphi_h) + \lambda \bar{a}_1(u - u_h; T, \varphi_h) + \lambda \bar{a}_1(u_h; T - T_h, \varphi_h) = 0, \quad \forall \varphi_h \in W_{0h}.
\]

(4.13)

Choosing \( \varphi_h = r_h T - T_h \in W_{0h} \) in (4.13) and using Lemma 4.1, we have

\[
d(T - T_h, T - T_h) = d(T - T_h, T - r_h T) + d(T - T_h, r_h T - T_h)
\]

(4.14)

\[
= d(T - r_h T, T - r_h T) - \lambda \bar{a}_1(u - u_h; T, r_h T - T_h) - \lambda \bar{a}_1(u_h; T - T_h, r_h T - T_h).
\]

By Hölder inequality and Cauchy inequality, it follows easily from (3.21) and Theorem 3.1 that

\[
\|
\nabla (T - T_h)\|^2_0 \
\leq \|
\nabla (T - T_h)\|^2_0 + \lambda \bar{N}_0 \|u - u_h\|_h \|
\nabla T\|_0 \|
\nabla (T - T_h)\|_0 + \|
\nabla (T - T_h)\|_0 \\
+ \lambda \bar{N}_0 \|u_h\|_h \|
\nabla (T - T_h)\|_0 \|
\nabla (T - T_h)\|_0 + \|
\nabla (T - T_h)\|_0 \\
\leq C\|
\nabla (T - T_h)\|^2_0 + (\lambda \bar{N}_0 B)^2 \|u - u_h\|^2_h + \frac{1}{2} \|
\nabla (T - T_h)\|^2_0 + \frac{\theta_1^2}{2} \|
\nabla (u - u_h)\|^2_0,
\]

(4.15)

where \( \theta_1 \) is an undetermined positive constant. Thus

\[
\|
\nabla (T - T_h)\|_0 \leq C\|
\nabla (T - T_h)\|_0 + (\theta_1 + 2\lambda \bar{N}_0 B)\|u - u_h\|_h.
\]

(4.16)

Choosing \( v_h = \Pi_h u - u_h \) in (4.11), and by (4.12), (3.20), Hölder inequality, Theorems 2.1 and 3.1, Lemmas 4.2 and 4.3, we have

\[
\mu \|u - u_h\|^2_h \\
= |a^h(u - u_h, u - \Pi_h u) + a^h(u - u_h, \Pi_h u - u_h)| \\
= |a^h(u - u_h, u - \Pi_h u) + \lambda j(T - T_h), \Pi_h u - u_h) - a_1^h(u - u_h; u, \Pi_h u - u_h) \\
- a_1^h(u_h; u - u_h, \Pi_h u - u_h) + b^h(p - p_h, \Pi_h u - u_h) + \mu \sum_{K \in \mathcal{S}_h} \int_{\partial K} \frac{\partial u}{\partial n} \cdot (\Pi_h u - u_h) ds \\
- \sum_{K \in \mathcal{S}_h} \int_{\partial K} p(\Pi_h u - u_h) \cdot nds + \frac{1}{2} \sum_{K \in \mathcal{S}_h} \int_{\partial K} (u \cdot n)(u \cdot (\Pi_h u - u_h)) ds | \\
\leq \mu \|u - u_h\|_h \|u - \Pi_h u\|_h + \lambda \|T - T_h\|_0 (\|\Pi_h u - u\|_0 + \|u - u_h\|_0) \\
+ N_0 \|u - u_h\|_h \|
\nabla u\|_0 (\|\Pi_h u - u\|_h + \|u - u_h\|_h) \\
+ N_0 \|u_h\|_h \|u - u_h\|_h (\|\Pi_h u - u\|_h + \|u - u_h\|_h) + \|p - p_h\|_0 \|\Pi_h u - u\|_h
\]

(4.17)
\[ + Ch|u_2|\|\Pi_h u - u_h\|_h + Ch|p|_1|\Pi_h u - u_h\|_h + Ch|u_1|\|\Pi_h u - u_h\|_h \]
\[ \leq \mu|u - u_h|_h|u - \Pi_h u\|_h + \lambda C_1 C_3\|\nabla(T - T_h)\|_0|\Pi_h u - u\|_h \]
\[ + \lambda C_1 C_3\|\nabla(T - T_h)\|_0|u - u_h|_h + 2N_0 A'|\|u - u_h\|_h|\Pi_h u - u\|_h \]
\[ + N_0 A'|\|u - u_h\|_h^2 + p - p_h|_0|\Pi_h u - u\|_h + Ch|u_2|\|\Pi_h u - u\|_h + Ch|u_2|\|u - u_h\|_h \]
\[ + Ch|p|_1|\Pi_h u - u_h\|_h + Ch|p|_1|u - u_h\|_h + Ch|u_1|\|u_2|_2|\Pi_h u - u\|_h \]
\[ + Ch|u_1|\|u_2\|\|u - u_h\|_h. \]

By use of the assumption that \( \mu^{-1}N_0 A' \leq 1 - \delta_3 \) (0 < \( \delta_3 \) ≤ 1) and Cauchy inequality, we have
\[ \|u - u_h\|_h \leq \delta_3\|u - u_h\|_h + \lambda C_1 C_3\|\nabla(T - T_h)\|_0 + Ch|u_2| + Ch|p|_1 + Ch|u_1| \leq \frac{\theta_2}{2}\|p - p_h\|_0, \]
where \( \theta_2 \) is an undetermined positive constant. As a consequence, we have
\[ \|u - u_h\|_h \leq C\|u - \Pi_h u\|_h + Ch(\|u_2\| + |p|_1 + \|u_1\|) \leq \frac{\theta_2}{2}\|p - p_h\|_0. \]

By (4.11), (3.20), Hölder inequality, Theorems 2.1 and 3.1, Lemmas 4.2 and 4.3, we have
\[ \|b^h(P_0 p - p_h, v_h)\| = \|b^h(P_0 p - p, v_h) + b^h(p - p_h, v_h)\| \]
\[ = a^h(u - u_h, v_h) + a^h(u - u_h; u, v_h) + a^h(u_h; u - u_h, v_h) - \lambda(T - T_h), v_h \]
\[ \leq \mu|u - u_h|_h + 2N_0 A'|\|u - u_h\|_h + \lambda C_1 C_3\|\nabla(T - T_h)\|_0 \]
\[ + Ch(\|u_2\| + |p|_1 + \|u_1\|)\|v_h\|_h, \]
where \( P_0 p|_K = \frac{1}{|K|}\int_K p dx. \)

According to the discrete inf-sup condition Lemma 3.1, we have
\[ \|p - p_h\|_0 \leq \|p - P_0 p\|_0 + \frac{1}{\beta} \sup_{v_h \in X_h} \frac{b^h(P_0 p - p_h, v_h)}{\|v_h\|_h}. \]
In view of (4.20) and (4.21), it follows
\[
\| p - p_h \|_0 \leq \| p - P_0 p \|_0 + \beta^{-1}(\mu + 2N_0 A)\| u - u_h \|_h + \beta^{-1}\lambda C_1 C_3 \| \nabla (T - T_h) \|_0 \\
+ Ch(|u|_2 + |p|_1 + \| u \|_1 \| u \|_2).
\]
Substituting (4.22) into (4.19), we obtain
\[
\| u - u_h \|_h \leq C(\| u - \Pi_h u \|_h + \| p - P_0 p \|_0) + Ch(|u|_2 + |p|_1 + \| u \|_1 \| u \|_2) \\
+(2\mu^{-1}\delta_3^{-1}\lambda C_1 C_3 + \theta_2 \beta^{-1}\lambda C_1 C_3)\| \nabla (T - T_h) \|_0 + \theta_2 \beta^{-1}(\mu + 2N_0 A)\| u - u_h \|_h.
\]
Substituting (4.16) into (4.23) and employing interpolation theorem and Lemma 4.2, yield
\[
\| u - u_h \|_h \leq Ch(|u|_2 + |p|_1 + |T|_2 + \| u \|_1 \| u \|_2) + 2\theta_1 \mu^{-1}\delta_3^{-1}\lambda C_1 C_3 + \theta_1 \theta_2 \beta^{-1}\lambda C_1 C_3 \\
+ 4\mu^{-1}\delta_3^{-1}\bar{N}_0 B C_1 C_3 + 2\theta_2 \beta^{-1}(\lambda C_1 C_3 + \mu + 2N_0 A)\| u - u_h \|_h.
\]
Noting that \( \delta_3^{-1}\mu^{-1}\lambda C_1 C_3 \bar{N}_0 B \leq 1 - \delta_4 \) (0 < \( \delta_4 \) ≤ 1), Let \( \delta_4 = \frac{7}{8} \), then
\[
4\delta_3^{-1}\mu^{-1}\lambda C_1 C_3 \bar{N}_0 B \leq \frac{1}{2}. \text{ Consequently}
\]
\[
\| u - u_h \|_h \leq Ch(|u|_2 + |p|_1 + |T|_2 + \| u \|_1 \| u \|_2) + 2\theta_1 (2\mu^{-1}\delta_3^{-1}\lambda C_1 C_2 \\
+ \theta_2 \beta^{-1}\lambda C_1 C_3) + \theta_2 \beta^{-1}(2\lambda \bar{N}_0 B C_1 C_3 + \mu + 2N_0 A)\| u - u_h \|_h.
\]
Let
\[
\theta_2 = \frac{\beta}{8(2\lambda \bar{N}_0 B C_1 C_3 + \mu + 2N_0 A)} \leq \frac{\beta}{16\lambda \bar{N}_0 B C_1 C_3},
\]
then \( \theta_2 \beta^{-1}\lambda C_1 C_3 \leq \frac{1}{16\lambda \bar{N}_0 B} \), together with \( 2\mu^{-1}\delta_3^{-1}\lambda C_1 C_3 \leq \frac{1}{4\lambda \bar{N}_0 B} \) and (4.28), we have
\[
\| u - u_h \|_h \leq Ch(|u|_2 + |p|_1 + |T|_2 + \| u \|_1 \| u \|_2) + \left( \frac{5\theta_1}{8\lambda \bar{N}_0 B} + \frac{1}{4} \right)\| u - u_h \|_h.
\]
Let \( \theta_1 = \frac{2\lambda \bar{N}_0 B}{5} \), then \( \frac{5\theta_1}{8\lambda \bar{N}_0 B} = \frac{1}{4} \). By (4.29), we have
\[
\| u - u_h \|_h \leq Ch(|u|_2 + |p|_1 + |T|_2 + \| u \|_1 \| u \|_2).
\]
Substituting (4.27) into (4.16) yields
\[
\| \nabla (T - T_h) \|_0 \leq Ch(|u|_2 + |p|_1 + |T|_2 + \| u \|_1 \| u \|_2).
\]
Then substituting (4.27) and (4.28) into (4.22), we obtain
\[
\| p - p_h \|_0 \leq Ch(|u|_2 + |p|_1 + |T|_2 + \| u \|_1 \| u \|_2).
\]
Combining (4.27), (4.28) and (4.29), yields the desired result.

Remark 1: It can be checked that the results obtained in the present work are valid to the rectangular nonconforming element proposed in [19-20] and the rotated \( Q_1 \) element discussed in [21,23-24] for the approximation of the velocity in the stationary conduction-convection problem. At the same time, the former element can be
applied to the anisotropic meshes.

**Remark 2**: It should be pointed out that the total degrees of freedom of the non-conforming element of this paper are the same as the conforming one used in [10].

**Remark 3**: From (2.10) and (3.32) we know that $\lambda$ and $\mu$ must satisfy some relations, especially, the Groshoff number $\lambda$ is not too big.

**Acknowledgment** We would like to thank the anonymous referees for many constructive comments and suggestions which led to an improved presentation of this paper.

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