INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 6, Number 2, Pages 232–255

MULTISCALE ASYMPTOTIC METHOD FOR HEAT TRANSFER EQUATIONS IN LATTICE-TYPE STRUCTURES

FANG-MAN ZHAI AND LI-QUN CAO

Abstract. In this paper, we discuss the initial-boundary value problem for the heat transfer equation in lattice-type structures that arises from the aerospace industry and the structural engineering. The main results obtained in this paper are the convergence theorems by using the homogenization method and the multiscale asymptotic method (see Theorems 2.1 and 2.2). Some numerical examples are given for three types of lattice structures. These numerical results suggest that the first-order multiscale method should be a better choice compared with the homogenization method and the second-order multiscale method for solving the heat transfer equations in lattice-type structures.

Key Words. homogenization, multiscale asymptotic expansion, parabolic equation, lattice-type structure, multiscale finite element method.

1. Introduction

In this paper, we consider the initial-boundary value problem for second order parabolic equations with rapidly oscillating coefficients as follows

$$
(1)\qquad \begin{cases} \dfrac{\partial u^{\varepsilon\delta}(x,t)}{\partial t} - \dfrac{\partial}{\partial x_i}\Big(a_{ij}(\dfrac{x}{\varepsilon},t)\dfrac{\partial u^{\varepsilon\delta}(x,t)}{\partial x_j}\Big) = f(x,t),\ (x,t)\in\Omega_{\varepsilon\delta}\times(0,T) \\ u^{\varepsilon\delta}(x,t) = g(x,t), \quad (x,t)\in\partial\Omega\times(0,T) \\ \nu_i a_{ij}(\dfrac{x}{\varepsilon},t)\dfrac{\partial u^{\varepsilon\delta}}{\partial x_j} = 0, \quad (x,t)\in\partial T_{\varepsilon\delta}\times(0,T) \\ u^{\varepsilon\delta}(x,0) = \bar{u}_0(x), \end{cases}
$$

where $f(x,t)$, $g(x,t)$, $\bar{u}_0(x)$ are some known functions. We follow Cioranescu's notation (see [6], p.74) by denoting $\Omega_{\varepsilon\delta} = \Omega\setminus\overline{T}_{\varepsilon\delta}$ the perforated domain, where Ω is a bounded domain of $R^n, n\geq 2$, $T_{\varepsilon\delta} = \tau(\varepsilon T_\delta)$ is the set of all translated images of $\varepsilon\overline{T_\delta}$ of the form $\varepsilon(z+\overline{T_\delta})$, $z=(z_1,\cdots,z_n)\in Z^n$, $T_\delta = Y\setminus Y_\delta$, $Y=(0,1)^n$. Here Y_δ is a lattice structure as shown in Figs. 1-3. The boundaries of Ω and $T_{\varepsilon\delta}$ are respectively $\partial\Omega$ and $\partial T_{\varepsilon\delta}$ and $\vec{\nu}=(\nu_1,\cdots,\nu_n)$ is the unit outer normal to $\partial T_{\varepsilon\delta}$.

In order to apply the extension theorem (see, e.g. Theorem 2.10 of [6], p. 28), we make the following assumption

($\mathbf{H_1}$) The holes do not intersect the boundary $\partial\Omega$.

This assumption restricts the geometry of the open set Ω. For example, Ω can be a finite union of the periodic cells.

Set $\xi=\varepsilon^{-1}x$, and suppose that

($\mathbf{A_1}$) the coefficients $a_{ij}(\xi,t)$ are 1-periodic in ξ;

($\mathbf{A_2}$) $\gamma_0|\eta|^2 \leq a_{ij}(\xi,t)\eta_i\eta_j \leq \gamma_1|\eta|^2, \quad \forall(\eta_1,\cdots,\eta_n)\in R^n$,
γ_0, γ_1 are positive constants independent of ε;

2000 *Mathematics Subject Classification.* 65F10, 65W05.

This work is supported by National Natural Science Foundation of China (grant # 90405016), and Special Funds for Major State Basic Research Projects(grant # 2006CB605102).

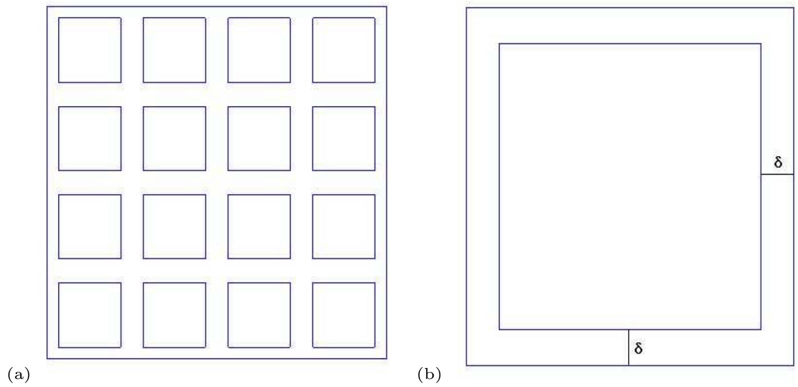


FIGURE 1. (a) lattice-type structure: Type I; (b) the unit cell Y_δ.

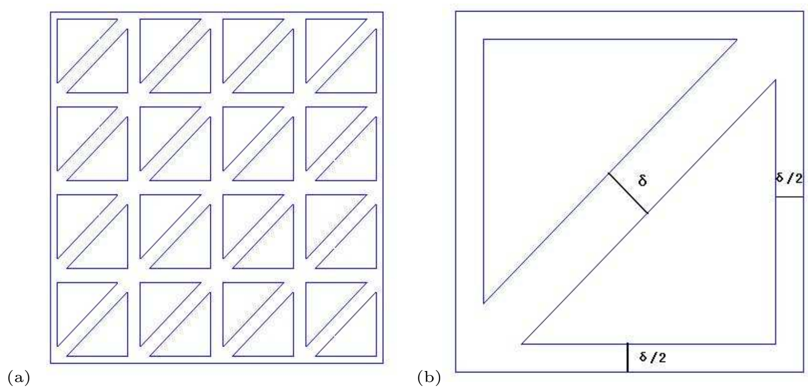


FIGURE 2. (a) lattice-type structure: Type II; (b) the unit cell Y_δ.

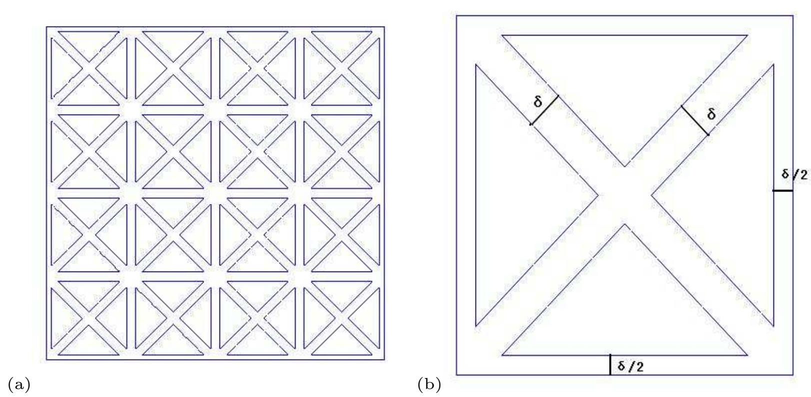


FIGURE 3. (a) lattice-type structure: Type III; (b) the unit cell Y_δ.

($\mathbf{A_3}$) $f \in L^2(0,T;L^2(\Omega)), g \in L^\infty(0,T;H^{1/2}(\partial\Omega)), \partial_t g \in L^2(0,T;H^{1/2}(\partial\Omega))$, $\bar{u}_0 \in H^1(\Omega), g(x,0) = \bar{u}_0(x)$.

Lattice-type structures are characterized by two properties: periodicity and small thickness of the material. Such structures have a wide range of applications in the aerospace industry and the structural engineering. Two small parameters are essential in thermal problem for lattice-type structures. One is periodic parameter ε and the other is thickness parameter δ of the domain. They make direct numerical computation of the solution difficult because it would require a very fine mesh.

The periodic distribution of lattice-type structures suggests that we can use the homogenization method for the perforated domain when $\varepsilon \to 0$ and $\delta \to 0$. The basic idea of the homogenization method is to give the overall behavior of the structures. The homogenization method for lattice-type structures was first studied by Panasenko [2].

We would like to state that, if ε, δ are not sufficiently small, the numerical accuracy of the standard homogenization method may not be satisfactory(see, e.g. [4],[5]). Therefore it is necessary to seek the multiscale asymptotic methods.

For the problems of lattice-type structures, Bakhvalov and Panasenko [2] obtained formal asymptotic expansions for the limit solution when ε and δ are small enough. They justify the first terms of these expansions by proving sharp error estimates. Cioranescu and Saint Jean Paulin ([6],[7],[8]) studied the same types of structures in a different way. They used variational methods to establish convergence theorems, and developed a general method for treating structures with very complicated geometry.

In this paper, we consider parabolic equations in lattice-type structures, and obtain the convergence results for the multiscale asymptotic method. The main difficulty is how to deal with the multiscale asymptotic solution near the boundary $\partial\Omega$. Allegretto, Cao and Lin [1] studied parabolic equations with rapidly oscillating coefficients, and gave the explicit convergence rates for the multiscale asymptotic method in a general domain. There was only a small parameter ε in [1]. But now there are two parameters ε, δ for the lattice-type structures. In such case, we obtain the new convergence rate for the approximate solutions and numerical approximations techniques. In this paper, we focus on the three types of lattice structures, see Figs.1-3.

The remainder of this paper is organized as follows. In Section 2, we derive the convergence theorems for the homogenization method and the multiscale asymptotic method for solving the heat transfer equation in the lattice-type structures. An algorithm based on the multiscale finite element method is proposed in Section 3. Finally, we do some numerical experiments for three types of lattice structures, which support the convergence results of this paper. These numerical results suggest that the first-order multiscale method should be a better choice compared with the homogenization method and the second-order multiscale method for solving the heat transfer equations in lattice-type structures.

Throughout the paper the Einstein summation convention on repeated indices is adopted. By C we shall denote a positive constant independent of ε, δ.

2. The Main Convergence Theorems

In this section, we first present a formal multiscale asymptotic expansion of the solution for problem (1). Then we obtain the main convergence theorems in this paper. For simplicity, we discuss problem (1) only for a type of lattice structure: Type I (see Fig.1) in the two-dimensional case. The others can be treated similarly.

We can take the following particular cell as a representative cell (see [6], p.76), thanks to the periodicity,

$$Y = (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{2}, \frac{1}{2}).$$

Denote the part occupied by the material by

$$Y_\delta = \{y \in Y|\ |y_1| < \frac{\delta}{2} \quad \text{or} \quad |y_2| < \frac{\delta}{2}\},$$

and set $T_\delta = Y \setminus Y_\delta$.

We set formally

$$u_s^{\varepsilon\delta}(x,t) = \sum_{l=0}^{s} \varepsilon^l \sum_{\alpha_1,\cdots,\alpha_l=1}^{n} N^\delta_{\alpha_1\cdots\alpha_l}(\xi,t)\frac{\partial^l u^\delta(x,t)}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}}, \quad 1 \le s \le 2, \tag{2}$$

with $\xi = \varepsilon^{-1}x$.

We define the cell functions in turn.

$$N_0^\delta(\xi,t) \equiv 1, \quad (\xi,t) \in Y_\delta \times (0,T), \tag{3}$$

$$\begin{cases} \dfrac{\partial}{\partial \xi_i}\Big(a_{ij}(\xi,t)\dfrac{\partial N^\delta_{\alpha_1}(\xi,t)}{\partial \xi_j}\Big) = -\dfrac{\partial}{\partial \xi_i}(a_{i\alpha_1}(\xi,t)), \quad (\xi,t) \in Y_\delta \times (0,T) \\ \nu_i a_{ij}(\xi,t)\dfrac{\partial N^\delta_{\alpha_1}(\xi,t)}{\partial \xi_j} = -\nu_i a_{i\alpha_1}(\xi,t), \quad (\xi,t) \in \partial T_\delta \times (0,T) \\ N^\delta_{\alpha_1}(\xi,t) \quad \text{is 1-periodic in} \quad \xi, \quad \int_{Y_\delta} N^\delta_{\alpha_1}(\xi,t)d\xi = 0, \end{cases} \tag{4}$$

where t plays the role of a parameter, and $\vec{\nu} = (\nu_1,\cdots,\nu_n)$ is the unit outer normal to the boundary ∂T_δ.

$$\begin{cases} \dfrac{\partial}{\partial \xi_i}\Big(a_{ij}(\xi,t)\dfrac{\partial N^\delta_{\alpha_1\alpha_2}(\xi,t)}{\partial \xi_j}\Big) = -\dfrac{\partial}{\partial \xi_i}\Big(a_{i\alpha_1}(\xi,t)N^\delta_{\alpha_2}(\xi,t)\Big) \\ \quad -a_{\alpha_1 j}(\xi,t)\dfrac{\partial N^\delta_{\alpha_2}(\xi,t)}{\partial \xi_j} - a_{\alpha_1\alpha_2}(\xi,t) + \hat{a}^\delta_{\alpha_1\alpha_2}(t), \quad (\xi,t) \in Y_\delta \times (0,T) \\ \nu_i a_{ij}(\xi,t)\dfrac{\partial N^\delta_{\alpha_1\alpha_2}(\xi,t)}{\partial \xi_j} = -\nu_i a_{i\alpha_1}(\xi,t)N^\delta_{\alpha_2}(\xi,t), \quad (\xi,t) \in \partial T_\delta \times (0,T) \\ N^\delta_{\alpha_1\alpha_2}(\xi,t) \quad \text{is 1-periodic in} \quad \xi, \quad \int_{Y_\delta} N^\delta_{\alpha_1\alpha_2}(\xi,t)d\xi = 0, \end{cases} \tag{5}$$

where

$$\hat{a}^\delta_{ij}(t) = \frac{1}{|Y_\delta|}\int_{Y_\delta} \Big[a_{ij}(\xi,t) + a_{ik}(\xi,t)\frac{\partial N^\delta_j(\xi,t)}{\partial \xi_k}\Big]d\xi, \tag{6}$$

and $|Y_\delta|$ denotes the Lebesgue measure of Y_δ.

For any fixed $\delta > 0$, we get the homogenized equation of equation (1) given by

$$\begin{cases} \dfrac{\partial u^\delta(x,t)}{\partial t} - \dfrac{\partial}{\partial x_i}\Big(\hat{a}^\delta_{ij}(t)\dfrac{\partial u^\delta(x,t)}{\partial x_j}\Big) = f(x,t), \quad (x,t) \in \Omega \times (0,T) \\ u^\delta(x,t) = g(x,t), \quad (x,t) \in \partial\Omega \times (0,T) \\ u^\delta(x,0) = \bar{u}_0(x), \end{cases} \tag{7}$$

where the homogenized coefficients matrix $(\hat{a}^\delta_{ij}(t))$ is given as in (6).

Remark 2.1. Note that we add the index δ to the homogenized solution and to the homogenized coefficients because the geometry of the material in the periodic of reference depends on δ. Indeed, the cell functions $N^\delta_{\alpha_1}$ are solutions of a system posed in Y_δ, while the coefficients $\hat{a}^\delta_{ij}$ are integrals computed on Y_δ and containing $N^\delta_{\alpha_1}$. Finally, the solution u^δ depends on δ via $\hat{a}^\delta_{ij}(t)$.

Next we would like to study the limit behavior of the solution $u^\delta(x,t)$ as $\delta \to 0$.

Theorem 2.1. If we assume that the coefficients of problem (1) only depend on time variable t, under the assumptions of $(H_1), (A_1) - (A_3)$, we have

$$
\begin{aligned}
&u^\delta \to u^* \quad \text{strongly in } L^2(0,T;H^1(\Omega)),\\
&u^\delta \to u^* \quad \text{strongly in } L^\infty(0,T;L^2(\Omega)),\\
&\frac{\partial u^\delta}{\partial t} \to \frac{\partial u^*}{\partial t} \quad \text{strongly in } L^2(0,T;H^{-1}(\Omega)), \quad \text{as} \quad \delta \to 0,
\end{aligned} \tag{8}
$$

where $u^*(x,t)$ is the solution of the following equation

$$
\begin{cases}
\dfrac{\partial u^*(x,t)}{\partial t} - \dfrac{\partial}{\partial x_i}\Big(a^*_{ij}(t)\dfrac{\partial u^*(x,t)}{\partial x_j}\Big) = f(x,t), \quad (x,t) \in \Omega \times (0,T)\\
u^*(x,t) = g(x,t), \quad (x,t) \in \partial\Omega \times (0,T)\\
u^*(x,0) = \bar{u}_0(x),
\end{cases} \tag{9}
$$

with

$$
a^*_{ij}(t) = \frac{1}{2}\Big[2a_{ij}(t) - \frac{a_{ik}(t)a_{kj}(t)}{a_{kk}(t)}\Big]. \tag{10}
$$

It is obvious that $(a^*_{ij}(t))$ is a positive-definite matrix for any fixed $t \in (0,T)$. So equation (9) has exactly one solution.

Proof. If we assume that coefficients a_{ij} of problem (1) only depend on time t, a_{ij} are constants for any fixed $t \in (0,T)$. Following the lines of the proof of Theorem 1.1 of ([6], p.75), we have

$$
a^\delta_{ij} \to a^*_{ij}, \quad \text{as} \quad \delta \to 0. \tag{11}
$$

The positive definiteness of the matrix $(a^*_{ij}(t))$ comes from that of the matrix $(a_{ij}(t))$, which is equivalent to the ellipticity condition (A_2). In other words, there are two positive constants γ^*_0, γ^*_1 such that

$$
\gamma^*_0|\eta|^2 \le a^*_{ij}(t)\eta_i\eta_j \le \gamma^*_1|\eta|^2, \quad a.e. \quad t \in (0,T). \tag{12}
$$

In order to obtain (8), we first prove

$$
\begin{aligned}
&u^\delta \rightharpoonup u^* \quad \text{weakly in } L^2(0,T;H^1(\Omega)),\\
&u^\delta \rightharpoonup u^* \quad \text{weakly in } L^\infty(0,T;L^2(\Omega)),\\
&\frac{\partial u^\delta}{\partial t} \rightharpoonup \frac{\partial u^*}{\partial t} \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega)), \quad \text{as} \quad \delta \to 0.
\end{aligned} \tag{13}
$$

We rewrite $\hat{a}^\delta_{ij}(t)$ as

$$
\hat{a}^\delta_{ij}(t) = a^*_{ij}(t) + S^\delta_{ij}(t), \tag{14}
$$

where $S^\delta_{ij} \to 0$ as $\delta \to 0$, thanks to (11).

From (7), we get $(u^\delta - g) \in L^2(0,T;H^1_0(\Omega))$ and the integral identity

$$
\begin{aligned}
&\int_0^t\int_\Omega \frac{\partial u^\delta}{\partial t} v\,dxdt + \int_0^t\int_\Omega \hat{a}^\delta_{ij}\frac{\partial u^\delta}{\partial x_j}\frac{\partial v}{\partial x_i}dxdt\\
&= \int_0^t\int_\Omega fv\,dxdt,
\end{aligned}
$$

holds for any $v \in L^2(0,T;H^1_0(\Omega))$.

Setting $v(x,t) = (u^\delta(x,t) - g(x,t))$, and using the decomposition (14) in (7), we have

$$\begin{aligned}
&\int_0^t\int_\Omega \frac{\partial u^\delta}{\partial t} u^\delta dxdt - \int_0^t\int_\Omega \frac{\partial u^\delta}{\partial t} g dxdt \\
&+ \int_0^t\int_\Omega (a^*_{ij} + S^\delta_{ij}) \frac{\partial u^\delta}{\partial x_j}\frac{\partial u^\delta}{\partial x_i} dxdt \\
&- \int_0^t\int_\Omega (a^*_{ij} + S^\delta_{ij}) \frac{\partial u^\delta}{\partial x_j}\frac{\partial g}{\partial x_i} dxdt \\
&= \int_0^t\int_\Omega f u^\delta dxdt - \int_0^t\int_\Omega f g dxdt
\end{aligned}$$

It is obvious that

$$\int_0^t\int_\Omega \frac{\partial u^\delta}{\partial t} u^\delta dxdt = \frac{1}{2}\int_\Omega\int_0^t \frac{\partial}{\partial t}\Big[(u^\delta)^2\Big] dtdx = \frac{1}{2}\int_\Omega \Big[(u^\delta(x,t))^2 - (\bar{u}_0(x))^2\Big] dx,$$

and

$$-\int_0^t\int_\Omega \frac{\partial u^\delta}{\partial t} g dxdt = -\int_\Omega \Big[(u^\delta g)(x,t) - (\bar{u}_0(x))^2\Big] dx + \int_0^t\int_\Omega u^\delta \frac{\partial g}{\partial t} dxdt.$$

Due to ellipticity of (12), using Cauchy's inequality, we thus get

$$\begin{aligned}
&\frac{1}{2}\|u^\delta(x,t)\|^2_{L^2(\Omega)} + \gamma^*_0 \int_0^t \|\nabla u^\delta\|^2_{L^2(\Omega)} dt \\
&\le \int_0^t |S^\delta_{ij}| \|\nabla u^\delta\|^2_{L^2(\Omega)} dt + \int_0^t |a^*_{ij} + S^\delta_{ij}| \|\nabla u^\delta\|_{L^2(\Omega)} \|g\|_{H^1(\Omega)} dt \\
&+ \int_0^t \|u^\delta\|_{L^2(\Omega)} \|\partial_t g\|_{L^2(\Omega)} dt + \|u^\delta\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)} \\
&+ \int_0^t \|f\|_{L^2(\Omega)} \|u^\delta\|_{L^2(\Omega)} dt + \int_0^t \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)} dt + \frac{1}{2}\|\bar{u}_0\|^2_{L^2(\Omega)}.
\end{aligned}$$

Using Young's inequality, we get

$$\begin{aligned}
&\frac{1}{2}\|u^\delta(x,t)\|^2_{L^2(\Omega)} + \gamma_0^* \int_0^t \|\nabla u^\delta\|^2_{L^2(\Omega)} dt \\
&\le \int_0^t |S^\delta_{ij}| \|\nabla u^\delta\|^2_{L^2(\Omega)} dt + C\gamma \int_0^t \|\nabla u^\delta\|^2_{L^2(\Omega)} dt \\
&+ C\frac{1}{4\gamma} \int_0^t \|g\|^2_{H^1(\Omega)} dt + \mu \|u^\delta(x,t)\|^2_{L^2(\Omega)} + \frac{1}{4\mu}\|g(x,t)\|^2_{L^2(\Omega)} \\
&+ \int_0^t \|u^\delta\|^2_{L^2(\Omega)} dt + \frac{1}{2}\int_0^t \|\partial_t g\|^2_{L^2(\Omega)} dt \\
&+ \int_0^t \|f\|^2_{L^2(\Omega)} dt + \frac{1}{2}\int_0^t \|g\|^2_{L^2(\Omega)} dt + \frac{1}{2}\|\bar{u}_0\|^2_{L^2(\Omega)}.
\end{aligned}$$

We recall $S^\delta_{ij}(t) \to 0$ as $\delta \to 0$ for any fixed $t \in (0,T)$. Therefore, if we take $\delta > 0$ sufficiently small, we have $|S^\delta_{ij}| \le \frac{\gamma_0^*}{4}$. On the other hand, we choose $C\gamma < \frac{\gamma_0^*}{4}$, $\mu < \frac{1}{4}$, and derive

$$\begin{aligned}
&\frac{1}{4}\|u^\delta(x,t)\|^2_{L^2(\Omega)} + \frac{\gamma_0^*}{2} \int_0^t \|\nabla u^\delta\|^2_{L^2(\Omega)} dt \\
&\le C\frac{1}{4\gamma} \int_0^t \|g\|^2_{H^1(\Omega)} dt + \frac{1}{4\mu}\|g(x,t)\|^2_{L^2(\Omega)} \\
&+ \int_0^t \|u^\delta\|^2_{L^2(\Omega)} dt + \frac{1}{2}\int_0^t \|\partial_t g\|^2_{L^2(\Omega)} dt \\
&+ \int_0^t \|f\|^2_{L^2(\Omega)} dt + \frac{1}{2}\int_0^t \|g\|^2_{L^2(\Omega)} dt + \frac{1}{2}\|\bar{u}_0\|^2_{L^2(\Omega)}.
\end{aligned} \tag{15}$$

If we set $E(t) = \int_0^t \int_\Omega (u^\delta(x,t))^2 dx dt$, and

$$\begin{aligned}
F(t) = \quad & C\frac{1}{4\gamma} \int_0^t \|g\|^2_{H^1(\Omega)} dt + \frac{1}{4\mu}\|g(x,t)\|^2_{L^2(\Omega)} \\
&+ \frac{1}{2}\int_0^t \|\partial_t g\|^2_{L^2(\Omega)} dt + \int_0^t \|f\|^2_{L^2(\Omega)} dt + \frac{1}{2}\int_0^t \|g\|^2_{L^2(\Omega)} dt \\
&+ \frac{1}{2}\|\bar{u}_0\|^2_{L^2(\Omega)},
\end{aligned}$$

then we conclude

$$\begin{cases} \dfrac{dE(t)}{dt} \le E(t) + F(t) \\ E(0) = 0. \end{cases}$$

Using Gronwall's inequality, we obtain

$$E(t) \le \quad Ce^t \Big\{ \int_0^t \|g\|^2_{H^1(\Omega)} dt + \|g(x,t)\|^2_{L^2(\Omega)} + \int_0^t \|\partial_t g\|^2_{L^2(\Omega)} dt + \int_0^t \|f\|^2_{L^2(\Omega)} dt + \|\bar{u}_0\|^2_{L^2(\Omega)} \Big\}.$$

Substituting into (15), we have

$$\frac{1}{4}\|u^\delta(x,t)\|^2_{L^2(\Omega)} + \frac{\gamma_0^*}{2} \int_0^t \|\nabla u^\delta\|^2_{L^2(\Omega)} dt$$

$$\le C(1+e^t) \Big\{ \int_0^t \|g\|^2_{H^1(\Omega)} dt + \|g(x,t)\|^2_{L^2(\Omega)}$$

$$+ \int_0^t \|\partial_t g\|^2_{L^2(\Omega)} dt + \int_0^t \|f\|^2_{L^2(\Omega)} dt + \|\bar{u}_0\|^2_{L^2(\Omega)} \Big\}.$$

Therefore, we get

$$\|u^\delta\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u^\delta\|_{L^2(0,T;L^2(\Omega))}$$
$$\le C(T) \Big\{ \|g\|_{L^2(0,T;H^1(\Omega))} + \|g\|_{L^\infty(0,T;L^2(\Omega))}$$
$$+ \|\partial_t g\|_{L^2(0,T;L^2(\Omega))} + \|f\|_{L^2(0,T;L^2(\Omega))} + \|\bar{u}_0\|_{L^2(\Omega)} \Big\},$$

where C is independent of ε, δ but dependent of T.

Hence, up to a subsequence, we have

$$\begin{array}{ll} u^\delta \rightharpoonup u^0 & \text{weakly in } L^2(0,T;H^1(\Omega)), \\ u^\delta \rightharpoonup u^0 & \text{weakly}^* \text{ in } L^\infty(0,T;L^2(\Omega)). \end{array}$$

From (7) and (14), after multiplication by any $v \in L^2(0,T;H_0^1(\Omega))$ and integration by parts, we obtain

$$(\frac{\partial u^\delta}{\partial t}, v)_{L^2(\Omega)} = -\int_\Omega (a^*_{ij} + S^\delta_{ij}) \frac{\partial u^\delta}{\partial x_j} \frac{\partial v}{\partial x_i} dx + \int_\Omega f v dx$$
$$\le |a^*_{ij} + S^\delta_{ij}| \|u^\delta\|_{H^1(\Omega)} \|v\|_{H^1_0(\Omega)} + \|f\|_{L^2(\Omega)} \|v\|_{H^1_0(\Omega)}$$
$$\le (\gamma_1^* + \frac{\gamma_0^*}{4}) \|u^\delta\|_{H^1(\Omega)} \|v\|_{H^1_0(\Omega)} + \|f\|_{L^2(\Omega)} \|v\|_{H^1_0(\Omega)}.$$

We thus have

$$\langle \frac{\partial u^\delta}{\partial t}, v \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} = (\frac{\partial u^\delta}{\partial t}, v)_{L^2(\Omega)}$$
$$\le C(\|u^\delta\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}) \|v\|_{H^1_0(\Omega)},$$

$$\|\frac{\partial u^\delta}{\partial t}\|_{H^{-1}(\Omega)} \le C(\|u^\delta\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}),$$

and

$$\|\frac{\partial u^\delta}{\partial t}\|_{L^2(0,T;H^{-1}(\Omega))} \le C(T)(\|u^\delta\|_{L^2(0,T;H^1(\Omega))} + \|f\|_{L^2((0,T)\times\Omega)})$$
$$\le C(T)\Big\{\|g\|_{L^2(0,T;H^1(\Omega))} + \|g\|_{L^\infty(0,T;L^2(\Omega))}$$
$$+\|\partial_t g\|_{L^2(0,T;L^2(\Omega))} + \|f\|_{L^2(0,T;L^2(\Omega))} + \|\bar{u}_0\|_{L^2(\Omega)}\Big\}.$$

This implies

$$\frac{\partial u^\delta}{\partial t} \rightharpoonup \frac{\partial u^0}{\partial t} \quad \text{weakly in } Ł^2(0,T;H^{-1}(\Omega)).$$

Multiplying both sides of equation (7) by any $v \in L^2(0,T;H_0^1(\Omega))$ and integrating over $(0,T)$ respect to t, we get

$$\int_0^T\int_\Omega \frac{\partial u^\delta}{\partial t} v dx dt + \int_0^T\int_\Omega \hat{a}_{ij}^\delta \frac{\partial u^\delta}{\partial x_j}\frac{\partial v}{\partial x_i} dx dt = \int_0^T\int_\Omega f v dx dt.$$

As $\delta \to 0$, we find that u^0 is the solution of (9) and $u^0 = u^*$ by uniqueness. Actually all sequences u^δ have the same limitation. Therefore, we prove (13).

The next step is to establish the claimed strong convergence result. We take $u^\delta - u^*$ as a test function in (7) and (9), respectively. Using (14), we obtain

$$\begin{aligned}(16)\qquad &\int_0^t\int_\Omega \frac{\partial(u^\delta - u^*)}{\partial t}(u^\delta - u^*)dxdt + \int_0^t\int_\Omega a_{ij}^*\frac{\partial(u^\delta - u^*)}{\partial x_i}\frac{\partial(u^\delta - u^*)}{\partial x_j}dxdt \\ &= -\int_0^t\int_\Omega S_{ij}^\delta \frac{\partial u^\delta}{\partial x_j}\frac{\partial(u^\delta - u^*)}{\partial x_i}dxdt.\end{aligned}$$

It is obvious that

$$\begin{aligned}(17)\qquad &\int_0^t\int_\Omega \frac{\partial(u^\delta - u^*)}{\partial t}(u^\delta - u^*)dxdt = \frac{1}{2}\int_0^t \frac{d}{dt}\|u^\delta - u^*\|_{L^2(\Omega)}^2 dt \\ &= \frac{1}{2}\|u^\delta(x,t) - u^*(x,t)\|_{L^2(\Omega)}^2,\end{aligned}$$

$$(18)\qquad \int_0^t\int_\Omega a_{ij}^*(t)\frac{\partial(u^\delta - u^*)}{\partial x_i}\frac{\partial(u^\delta - u^*)}{\partial x_j}dxdt \ge \gamma_0^*\int_0^t \|\nabla(u^\delta - u^*)\|_{L^2(\Omega)}^2 dt,$$

and

$$\begin{aligned}(19)\qquad &|\int_0^t\int_\Omega S_{ij}^\delta \frac{\partial u^\delta}{\partial x_j}\frac{\partial(u^\delta - u^*)}{\partial x_i}dxdt| \\ &\le |S_{ij}^\delta|\|\nabla u^\delta\|_{L^2((0,T)\times\Omega)}\|u^\delta - u^*\|_{L^2(0,T;H^1(\Omega))}.\end{aligned}$$

Since $S_{ij}^\delta \to 0$ as $\delta \to 0$, from (16) to (19), we complete the proof of (8).

Next we define

$$(20)\quad u_s^{\varepsilon\delta}(x,t) = u^\delta(x,t) + \sum_{l=1}^s \varepsilon^l \sum_{\alpha_1,\cdots,\alpha_l=1}^n N_{\alpha_1\cdots\alpha_l}^\delta(\xi,t)\frac{\partial^l u^\delta(x,t)}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}}, \quad 1 \le s \le 2.$$

In a standard way (see [1]), we can prove that

$$\begin{aligned}(21)\qquad &\sup_{0\le t\le T}\int_{\Omega_{\varepsilon\delta}} (u^{\varepsilon\delta}(x,t) - u_s^{\varepsilon\delta}(x,t))^2 dx + \int_0^T \|u^{\varepsilon\delta} - u_s^{\varepsilon\delta}\|_{H^1(\Omega_{\varepsilon\delta})}^2 dt \\ &\le C(T,\delta)\varepsilon, \quad 1 \le s \le 2,\end{aligned}$$

where $C(T,\delta)$ is a constant independent of ε but dependent on T and δ.

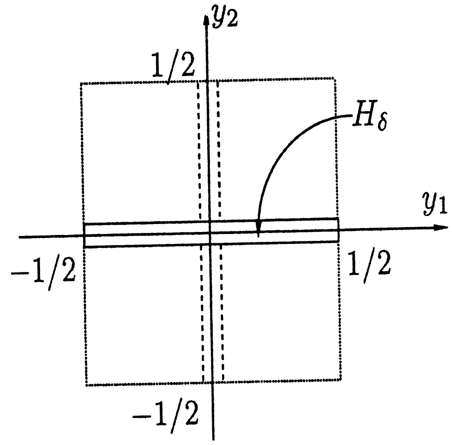


FIGURE 4. The horizontal bar H_δ

Since we do not know how $C(T,\delta)$ depends on $\delta > 0$, we can not estimate the magnitude of the error when comparing $u^{\varepsilon\delta}(x,t)$ with $u^{\delta}(x,t)$. To overcome this difficulty, we define the multiscale asymptotic solution given by

$$(22)\quad U_s^{\varepsilon\delta}(x,t) = u^*(x,t) + \sum_{l=1}^{s} \varepsilon^l \sum_{\alpha_1,\cdots,\alpha_l=1}^{n} N^{\delta}_{\alpha_1\cdots\alpha_l}(\xi,t)\frac{\partial^l u^*(x,t)}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}}, \quad 1 \le s \le 2.$$

where $u^*(x,t)$ is the solution of the homogenized equation (9).

As for the unit cell functions $N^{\delta}_{\alpha_1}(\xi,t)$, $N^{\delta}_{\alpha_1\alpha_2}(\xi,t)$ of the expansion (22), the following estimates hold as proved in ([6], p.77-98).

Lemma 2.1. Let $N^{\delta}_{\alpha_1}(\xi,t)$, $N^{\delta}_{\alpha_1\alpha_2}(\xi,t), \alpha_1,\alpha_2 = 1,\cdots,n$ be the weak solutions of problems (4) and (5), respectively. For any fixed $t \in (0,T)$, we have

$$(23)\quad \begin{aligned} &\|N^{\delta}_{\alpha_1}(\xi,t)\|_{L^2(Y_\delta)} \le C\delta^{\frac{3}{2}},\\ &\|\nabla_\xi N^{\delta}_{\alpha_1}(\xi,t)\|_{L^2(Y_\delta)} \le C\delta^{\frac{1}{2}},\\ &\|N^{\delta}_{\alpha_1\alpha_2}(\xi,t)\|_{L^2(Y_\delta)} \le C\delta^{\frac{5}{2}},\\ &\|\nabla_\xi N^{\delta}_{\alpha_1\alpha_2}(\xi,t)\|_{L^2(Y_\delta)} \le C\delta^{\frac{3}{2}}, \end{aligned}$$

where $C > 0$ is a constant independent of δ.

The following lemma estimates the difference between $\hat{a}^{\delta}_{ij}(t)$ and $a^*_{ij}(t)$.

Lemma 2.2. Let the homogenized coefficients $\hat{a}^{\delta}_{ij}(t)$ and $a^*_{ij}(t)$ be as given in (6) and (10), respectively. If the coefficients $a_{ij}(\xi,t)$ do not depend on ξ, we can prove that

$$(24)\quad |\hat{a}^{\delta}_{ij} - a^*_{ij}| \le C\delta^{\frac{1}{2}},$$

where $C > 0$ is a constant independent of δ.

Proof. We first decompose Y_δ into $H_\delta \cup V_\delta \cup K_\delta$ (see Figures 4-6). We define the functions $\Phi^{\delta}_k(\xi,t)$ and $\Psi^{\delta}_k(\xi,t)$ given by

$$\begin{cases} \dfrac{\partial}{\partial\xi_i}\Big(a_{ij}(\xi,t)\dfrac{\partial\Phi^{\delta}_k(\xi,t)}{\partial\xi_j}\Big) = -\dfrac{\partial}{\partial\xi_i}(a_{ik}(\xi,t)), \quad (\xi,t)\in H_\delta\times(0,T)\\ \nu_i a_{ij}(\xi,t)\dfrac{\partial\Phi^{\delta}_k(\xi,t)}{\partial\xi_j} = -\nu_i a_{ik}(\xi,t), \quad (\xi,t)\in[\partial H_\delta\setminus(\partial H_\delta\cap\partial Y)]\times(0,T)\\ \Phi^{\delta}_k(\xi,t) \quad \text{is 1-periodic in} \quad \xi_1, \end{cases}$$

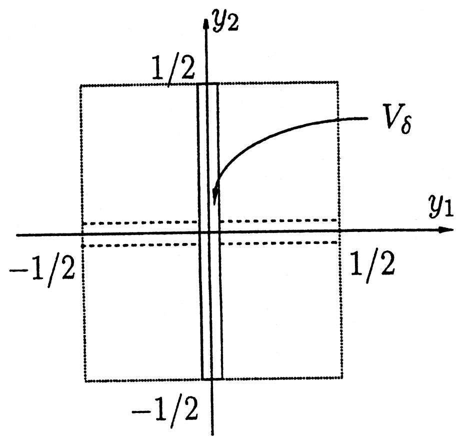


FIGURE 5. The vertical bar V_δ

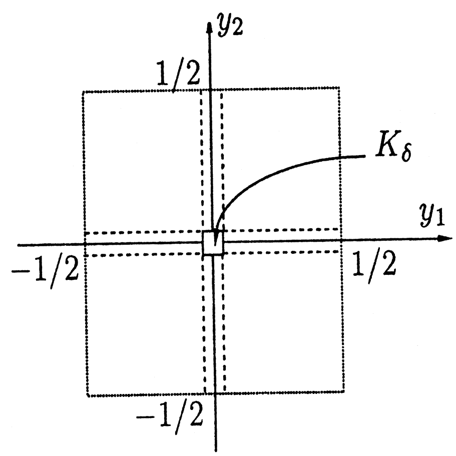


FIGURE 6. The central square K_δ

and

$$\begin{cases} \dfrac{\partial}{\partial \xi_i}\Big(a_{ij}(\xi,t)\dfrac{\partial \Psi_k^\delta(\xi,t)}{\partial \xi_j}\Big) = -\dfrac{\partial}{\partial \xi_i}(a_{ik}(\xi,t)), \quad (\xi,t) \in V_\delta \times (0,T) \\ \nu_i a_{ij}(\xi,t)\dfrac{\partial \Psi_k^\delta(\xi,t)}{\partial \xi_j} = -\nu_i a_{ik}(\xi,t), \quad (\xi,t) \in [\partial V_\delta \setminus (\partial V_\delta \cap \partial Y)] \times (0,T) \\ \Psi_k^\delta(\xi,t) \quad \text{is 1-periodic in} \quad \xi_2, \end{cases}$$

where t plays the role of a parameter, and $\vec{\nu} = (\nu_1, \cdots, \nu_n)$ is the unit outer normal to the associated boundary.

If we assume that the coefficients $a_{ij}(\xi,t)$ do not depend on ξ, a possible choice up to a constant could be

$$\Phi_k^\delta(\xi,t) = -\frac{a_{2k}(t)}{a_{22}(t)}\xi_2 \quad \text{on} \quad H_\delta, \quad \Psi_k^\delta(\xi,t) = -\frac{a_{1k}(t)}{a_{11}(t)}\xi_1 \quad \text{on} \quad V_\delta. \tag{25}$$

We recall (6), and set

$$\hat{a}_{ij}^{\delta}(t) = \frac{1}{|Y_\delta|}\int\limits_{Y_\delta}\Big[a_{ij}(\xi,t) + a_{ik}(\xi,t)\frac{\partial N_j^{\delta}(\xi,t)}{\partial \xi_k}\Big]d\xi \equiv Q_1 + Q_2, \tag{26}$$

where

$$\begin{aligned} Q_1 &= \frac{1}{|Y_\delta|}\Big[-\int\limits_{K_\delta} a_{ik}\frac{\partial N_j^{\delta}}{\partial \xi_k}d\xi + \int\limits_{H_\delta} a_{ik}\frac{\partial (N_j^{\delta}-\Phi_j^{\delta})}{\partial \xi_k}d\xi \\ &+ \int\limits_{V_\delta} a_{ik}\frac{\partial (N_j^{\delta}-\Psi_j^{\delta})}{\partial \xi_k}d\xi\Big], \\ Q_2 &= a_{ij} + \frac{1}{|Y_\delta|}\Big[\int\limits_{H_\delta} a_{ik}\frac{\partial \Phi_j^{\delta}}{\partial \xi_k}d\xi + \int\limits_{V_\delta} a_{ik}\frac{\partial \Psi_j^{\delta}}{\partial \xi_k}d\xi\Big]. \end{aligned}$$

It follows from (25) that

$$\begin{aligned} Q_2 &= a_{ij}(t) + \frac{2}{2-\delta}(a_{ij}^{*}(t) - a_{ij}(t)) \\ &= \frac{2}{2-\delta}a_{ij}^{*}(t) + \frac{-\delta}{2-\delta}a_{ij}(t). \end{aligned} \tag{27}$$

Following the lines of the proof of (1.66) in ([6],p.96), we can derive

$$|Q_1| \le C\delta^{1/2}, \tag{28}$$

where C is a constant independent of δ.

From (26), (27) and (28), we get

$$\hat{a}_{ij}^{\delta}(t) - a_{ij}^{*}(t) = \frac{\delta}{2-\delta}a_{ij}^{*}(t) + \frac{-\delta}{2-\delta}a_{ij}(t) + Q_1.$$

Therefore, we complete the proof of Lemma 2.2.

We next give the main convergence theorem of this paper.

Theorem 2.2. Suppose that $\Omega_{\varepsilon\delta}$ is a lattice-type structure which is the union of entire cells. Let $u^{\varepsilon\delta}$ be the weak solution of equation (1) and let the approximate solution $U_s^{\varepsilon\delta}(x,t)$ be as given in (22). Under assumptions of $(A_1)-(A_3),(H_1)$, if $\partial_t a_{ij} \in L^\infty((0,T)\times Y_\delta)$, $u^* \in L^2(0,T;W^{4,\infty}(\Omega))$, $\partial_t u^* \in L^2(0,T;W^{2,\infty}(\Omega))$, it holds

$$\begin{aligned} &\sup_{0\le t\le T}\int\limits_{\Omega_{\varepsilon\delta}}(u^{\varepsilon\delta}(x,t) - U_s^{\varepsilon\delta}(x,t))^2dx + \int\limits_0^T \|u^{\varepsilon\delta} - U_s^{\varepsilon\delta}\|^2_{H^1(\Omega_{\varepsilon\delta})}dt \\ &\le C(T)\Big\{\varepsilon^2\delta^3 + \delta^2 + \delta^{\lambda}\varepsilon^{2/q}\Big\}, \quad 1 \le s \le 2, \end{aligned} \tag{29}$$

where C is a constant independent of ε and δ , but dependent on T, $q = 2p/(p-2)$, $2 < p < r$, $r >> 2$, $1/p = \lambda/2 + (1-\lambda)/r$, $0 < \lambda < 1$, and λ is close to 1.

Proof. We first prove (29) for $s = 2$.

From (22), (4)-(5), we can obtain

$$\frac{\partial (u^{\varepsilon\delta} - U_2^{\varepsilon\delta})}{\partial t} - \frac{\partial}{\partial x_i}\Big[a_{ij}\frac{\partial (u^{\varepsilon\delta} - U_2^{\varepsilon\delta})}{\partial x_j}\Big] = F_0(x,\xi,t), \tag{30}$$

where

(31)

$$
\begin{aligned}
F_0(x,\xi,t) &= \varepsilon\Big\{-\sum_{l=1}^{2}\varepsilon^{l-1}\sum_{\alpha_1,\cdots,\alpha_l=1}^{n}\frac{\partial N^{\delta}_{\alpha_1\cdots\alpha_l}(\xi,t)}{\partial t}\frac{\partial^l u^*(x,t)}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}}\\
&- \sum_{l=1}^{2}\varepsilon^{l-1}\sum_{\alpha_1,\cdots,\alpha_l=1}^{n}N^{\delta}_{\alpha_1\cdots\alpha_l}(\xi,t)\frac{\partial^{l+1}u^*(x,t)}{\partial t\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}}\\
&+ \sum_{\alpha_1,\alpha_2=1}^{n}a_{ij}(\xi,t)\frac{\partial N^{\delta}_{\alpha_1\alpha_2}(\xi,t)}{\partial\xi_j}\frac{\partial^3 u^*(x,t)}{\partial x_i\partial x_{\alpha_1}\partial x_{\alpha_2}}\\
&+ \sum_{\alpha_1,\alpha_2=1}^{n}\frac{\partial}{\partial\xi_i}(a_{ij}(\xi,t)N^{\delta}_{\alpha_1\alpha_2}(\xi,t))\frac{\partial^3 u^*(x,t)}{\partial x_j\partial x_{\alpha_1}\partial x_{\alpha_2}}\\
&+ \sum_{l=1}^{2}\varepsilon^{l-1}\sum_{\alpha_1,\cdots,\alpha_l=1}^{n}a_{ij}(\xi,t)N^{\delta}_{\alpha_1\cdots\alpha_l}(\xi,t)\frac{\partial^{l+2}u^*(x,t)}{\partial x_i\partial x_j\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}}\Big\}\\
&+ (\hat{a}^{\delta}_{ij}-a^*_{ij})\frac{\partial^2 u^*(x,t)}{\partial x_i\partial x_j}.
\end{aligned}
$$

For $(x,t)\in\partial\Omega\times(0,T)$, we have

$$u^{\varepsilon\delta}(x,t)-U_2^{\varepsilon\delta}(x,t)=\psi_{\varepsilon\delta}(x,\xi,t), \tag{32}$$

where

$$\psi_{\varepsilon\delta}(x,\xi,t)=-\sum_{l=1}^{2}\varepsilon^l\sum_{\alpha_1,\cdots,\alpha_l=1}^{n}N^{\delta}_{\alpha_1\cdots\alpha_l}(\xi,t)\frac{\partial^l u^*(x,t)}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}}.$$

$$u^{\varepsilon\delta}(x,0)-U_2^{\varepsilon\delta}(x,0)=p_{\varepsilon\delta}(x,\xi),\quad x\in\Omega, \tag{33}$$

where

$$p_{\varepsilon\delta}(x,\xi)=-\sum_{l=1}^{2}\varepsilon^l\sum_{\alpha_1,\cdots,\alpha_l=1}^{n}\Big(N^{\delta}_{\alpha_1\cdots\alpha_l}(\xi,t)\frac{\partial^l u^*(x,t)}{\partial x_{\alpha_1}\cdots\partial x_{\alpha_l}}\Big)\Big|_{t=0}. \tag{34}$$

For $(x,t)\in\partial T_{\varepsilon\delta}\times(0,T)$, from (4) and (5), we get

$$\nu_i a_{ij}(\frac{x}{\varepsilon},t)\frac{\partial(u^{\varepsilon\delta}-U_2^{\varepsilon\delta})}{\partial x_j}=S_{\varepsilon\delta}(x,\xi,t), \tag{35}$$

where

$$S_{\varepsilon\delta}(x,\xi,t)=-\varepsilon^2\nu_i a_{ij}(\xi,t)N^{\delta}_{\alpha_1\alpha_2}(\xi,t)\frac{\partial^3 u^*(x,t)}{\partial x_j\partial x_{\alpha_1}\partial x_{\alpha_2}}.$$

Since $t\in(0,T)$ plays the role of a parameter, from (4), we derive

$$
\begin{cases}
\dfrac{\partial}{\partial\xi_i}\Big(a_{ij}(\xi,t)\dfrac{\partial N^{\delta}_{\alpha_1,t}(\xi,t)}{\partial\xi_j}\Big)=-\dfrac{\partial}{\partial\xi_i}(\partial_t a_{i\alpha_1}(\xi,t))\\
\quad -\dfrac{\partial}{\partial\xi_i}\Big(\partial_t a_{ij}(\xi,t)\dfrac{\partial N^{\delta}_{\alpha_1}(\xi,t)}{\partial\xi_j}\Big),\quad (\xi,t)\in Y_\delta\times(0,T)\\
\partial_t(\nu_i a_{ij}(\xi,t)\dfrac{\partial N^{\delta}_{\alpha_1}(\xi,t)}{\partial\xi_j})=-\nu_i\partial_t a_{i\alpha_1}(\xi,t),\quad (\xi,t)\in\partial T_\delta\times(0,T)\\
N^{\delta}_{\alpha_1,t}(\xi,t)\quad\text{is 1-periodic in}\quad\xi,\quad \int\limits_{Y_\delta}N^{\delta}_{\alpha_1,t}(\xi,t)d\xi=0.
\end{cases}
$$

where $N^{\delta}_{\alpha_1,t}(\xi,t)=\partial_t N^{\delta}_{\alpha_1}(\xi,t)$.

If $\partial_t a_{ij}\in L^\infty((0,T)\times Y_\delta)$, it holds

$$
\begin{aligned}
\gamma_0\|\nabla N^{\delta}_{\alpha_1,t}\|^2_{L^2(Y_\delta)} &\le c_1|Y_\delta|^{1/2}\|\nabla N^{\delta}_{\alpha_1,t}\|_{L^2(Y_\delta)}\\
&+ c_2\|\nabla N^{\delta}_{\alpha_1}\|_{L^2(Y_\delta)}\|\nabla N^{\delta}_{\alpha_1,t}\|_{L^2(Y_\delta)}.
\end{aligned}
$$

Given $|Y_\delta| = 2\delta(1-\frac{\delta}{2})$, it follows from Lemma 2.1 that

$$\|\nabla N^\delta_{\alpha_1,t}\|_{L^2(Y_\delta)} \le C\delta^{1/2}, \tag{36}$$

where C is a constant independent of ε, δ.

Following the lines of proof of (1.72) in ([6], p. 98), we have

$$\|N^\delta_{\alpha_1,t}\|_{L^2(Y_\delta)} \le C\delta^{3/2}. \tag{37}$$

We can check that

$$\|N^\delta_{\alpha_1\alpha_2,t}\|_{L^2(Y_\delta)} \le C\delta^{5/2}, \tag{38}$$

where $N^\delta_{\alpha_1\alpha_2,t} = \partial_t N^\delta_{\alpha_1\alpha_2}$.

Given $|\Omega_{\varepsilon\delta}| \approx \delta$, it follows from (31), Lemmas 2.1 and 2.2 that

$$\begin{aligned}\|F_0\|_{L^2(\Omega_{\varepsilon\delta})} \;&\le\; C\varepsilon\Big\{\delta^{3/2}\|u^*\|_{W^{1,\infty}(\Omega)} + \varepsilon\delta^{5/2}\|u^*\|_{W^{2,\infty}(\Omega)}\\ &+\; \delta^{3/2}\|\partial_t u^*\|_{W^{1,\infty}(\Omega)} + \varepsilon\delta^{5/2}\|\partial_t u^*\|_{W^{2,\infty}(\Omega)}\\ &+\; \delta^{5/2}\|u^*\|_{W^{3,\infty}(\Omega)} + \delta^{3/2}\|u^*\|_{W^{3,\infty}(\Omega)}\\ &+\; \varepsilon\delta^{5/2}\|u^*\|_{W^{4,\infty}(\Omega)}\Big\} + C\delta\|u^*\|_{W^{2,\infty}(\Omega)},\end{aligned}$$

and

(39)

$$\begin{aligned}\|F_0\|_{L^2(0,T;L^2(\Omega_{\varepsilon\delta}))} \;&\le\; C\varepsilon\Big\{\delta^{3/2}\|u^*\|_{L^2(0,T;W^{1,\infty}(\Omega))} + \varepsilon\delta^{5/2}\|u^*\|_{L^2(0,T;W^{2,\infty}(\Omega))}\\ &+\; \delta^{3/2}\|\partial_t u^*\|_{L^2(0,T;W^{1,\infty}(\Omega))} + \varepsilon\delta^{5/2}\|\partial_t u^*\|_{L^2(0,T;W^{2,\infty}(\Omega))}\\ &+\; \delta^{5/2}\|u^*\|_{L^2(0,T;W^{3,\infty}(\Omega))} + \delta^{3/2}\|u^*\|_{L^2(0,T;W^{3,\infty}(\Omega))}\\ &+\; \varepsilon\delta^{5/2}\|u^*\|_{L^2(0,T;W^{4,\infty}(\Omega))}\Big\} + C\delta\|u^*\|_{L^2(0,T;W^{2,\infty}(\Omega))}\\ &\le\; C(T)(\varepsilon\delta^{3/2}+\delta).\end{aligned}$$

where $C(T)$ is a constant independent of ε, δ but dependent of T.

In order to estimate $\|\psi_{\varepsilon\delta}\|_{H^{1/2}(\partial\Omega)}$, it suffices to find a function $\Psi_{\varepsilon\delta} \in H^1(\Omega_{\varepsilon\delta})$ such that

$$\Psi_{\varepsilon\delta} + \varepsilon N^\delta_{\alpha_1}(\xi,t)\frac{\partial u^*(x,t)}{\partial x_{\alpha_1}} + \varepsilon^2 N^\delta_{\alpha_1\alpha_2}(\xi,t)\frac{\partial^2 u^*(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}} \in H^1(\Omega_{\varepsilon\delta},\partial\Omega),$$

where $H^1(\Omega_{\varepsilon\delta},\partial\Omega) = \{v \in H^1(\Omega_{\varepsilon\delta}),\ v|_{\partial\Omega} = 0\}$.

Let φ_ε be a scalar function such that $\varphi_\varepsilon \in \mathcal{D}(\Omega)$, $\varphi_\varepsilon = 1$ if $dist(x,\partial\Omega) \le \varepsilon$, $\varphi_\varepsilon = 0$ if $dist(x,\partial\Omega) \ge 2\varepsilon$, $|\nabla\varphi_\varepsilon| \le c_2\varepsilon^{-1}$. Set

$$\Psi_{\varepsilon\delta} = -\varphi_\varepsilon\Big(\varepsilon N^\delta_{\alpha_1}(\xi,t)\frac{\partial u^*(x,t)}{\partial x_{\alpha_1}} + \varepsilon^2 N^\delta_{\alpha_1\alpha_2}(\xi,t)\frac{\partial^2 u^*(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\Big);$$

$$K_{\varepsilon\delta} = \{x:\ dist(x,\partial\Omega) \le 2\varepsilon\} \cap \Omega_{\varepsilon\delta}.$$

It is easy to see that $\Psi_{\varepsilon\delta} \in H^1(\Omega_{\varepsilon\delta})$ and

$$\begin{aligned}\frac{\partial\Psi_{\varepsilon\delta}}{\partial x_j} \;&=\; -\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x_j}N^\delta_{\alpha_1}\frac{\partial u^*}{\partial x_{\alpha_1}} - \varphi_\varepsilon\frac{\partial N^\delta_{\alpha_1}}{\partial\xi_j}\frac{\partial u^*}{\partial x_{\alpha_1}}\\ &-\; \varepsilon\varphi_\varepsilon N^\delta_{\alpha_1}\frac{\partial^2 u^*}{\partial x_j\partial x_{\alpha_1}} - \varepsilon^2\frac{\partial\varphi_\varepsilon}{\partial x_j}N^\delta_{\alpha_1\alpha_2}\frac{\partial^2 u^*}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\\ &-\; \varepsilon\varphi_\varepsilon\frac{\partial N^\delta_{\alpha_1\alpha_2}}{\partial\xi_j}\frac{\partial^2 u^*}{\partial x_{\alpha_1}\partial x_{\alpha_2}} - \varepsilon^2\varphi_\varepsilon N^\delta_{\alpha_1\alpha_2}\frac{\partial^3 u^*}{\partial x_j\partial x_{\alpha_1}\partial x_{\alpha_2}}.\end{aligned} \tag{40}$$

Using Hölder's inequality, it is easy to check that

$$\|u\|_{L^p(Q)} \le \|u\|^\lambda_{L^2(Q)}\|u\|^{1-\lambda}_{L^r(Q)}, \quad \text{for} \quad u \in L^r(Q), \tag{41}$$

where $2 < p < r$, $1/p = \lambda/2 + (1-\lambda)/r$, $0 < \lambda < 1$, and λ is close to 1.

From (4) and (5), under the assumptions of Theorem 2.2, for any fixed $t \in (0, T)$, we can prove that

$$\|N^{\delta}_{\alpha_1}(\xi, t)\|_{W^{m,\infty}(Y_\delta)} \le C, \quad m = 0, 1, \tag{42}$$

where C is a constant independent of δ.

Setting $Q = Y_\delta$, and using Lemma 2.1, (41)-(42), we get

$$\begin{aligned}
&\|N^{\delta}_{\alpha_1}(\xi, t)\|_{L^p(Y_\delta)} \le C\delta^{\frac{3\lambda}{2}},\\
&\|\nabla_\xi N^{\delta}_{\alpha_1}(\xi, t)\|_{L^p(Y_\delta)} \le C\delta^{\frac{\lambda}{2}},\\
&\|N^{\delta}_{\alpha_1\alpha_2}(\xi, t)\|_{L^p(Y_\delta)} \le C\delta^{\frac{5\lambda}{2}},\\
&\|\nabla_\xi N^{\delta}_{\alpha_1\alpha_2}(\xi, t)\|_{L^p(Y_\delta)} \le C\delta^{\frac{3\lambda}{2}},
\end{aligned}$$

where $C > 0$ is a constant independent of δ, and $p = 2r/((r-2)\lambda + 2)$, $r >> 2$.

Now we consider the first term on the right side of (40). Using the cut-off function φ_ε, we have

$$\begin{aligned}
&\int_{\Omega_{\varepsilon\delta}} |\varepsilon \frac{\partial \varphi_\varepsilon}{\partial x_j} N^{\delta}_{\alpha_1}(\tfrac{x}{\varepsilon}) \frac{\partial u^*(x)}{\partial x_{\alpha_1}}|^2 dx\\
&\quad \le C \int_{K_{\varepsilon\delta}} |N^{\delta}_{\alpha_1}(\tfrac{x}{\varepsilon})|^2 |\frac{\partial u^*(x)}{\partial x_{\alpha_1}}|^2 dx\\
&\quad \le C\Big(\int_{K_{\varepsilon\delta}} |N^{\delta}_{\alpha_1}|^p dx\Big)^{2/p} \Big(\int_{K_{\varepsilon\delta}} |\frac{\partial u^*}{\partial x_{\alpha_1}}|^q dx\Big)^{2/q}\\
&\quad \le C\Big(\sum_{z'} \varepsilon^n \int_{z'+Y_\delta} |N^{\delta}_{\alpha_1}|^p d\xi\Big)^{2/p} \Big(\int_{K_{\varepsilon\delta}} |\frac{\partial u^*}{\partial x_{\alpha_1}}|^q dx\Big)^{2/q}\\
&\quad \le C\delta^{3\lambda} \|u^*\|^2_{W^{1,q}(K_{\varepsilon\delta})},
\end{aligned}$$

where $C > 0$ is a constant independent of δ, $K_{\varepsilon\delta} = \bigcup_{z'} \varepsilon(z' + Y_\delta)$, and $q = 2p/(p-2)$.

The other terms on the right side of (40) can be treated similarly. We thus get

$$\begin{aligned}
\|\Psi_{\varepsilon\delta}\|_{H^1(\Omega_{\varepsilon\delta})} \le\ & C\Big\{\delta^{\frac{3\lambda}{2}} \|u^*\|_{W^{1,q}(K_{\varepsilon\delta})} + \delta^{\frac{\lambda}{2}} \|u^*\|_{W^{1,q}(K_{\varepsilon\delta})}\\
+\ & \varepsilon\delta^{\frac{3\lambda}{2}} \|u^*\|_{W^{2,q}(K_{\varepsilon\delta})} + \varepsilon\delta^{\frac{5\lambda}{2}} \|u^*\|_{W^{2,q}(K_{\varepsilon\delta})}\\
+\ & \varepsilon\delta^{\frac{3\lambda}{2}} \|u^*\|_{W^{2,q}(K_{\varepsilon\delta})} + \varepsilon^2\delta^{\frac{5\lambda}{2}} \|u^*\|_{W^{3,q}(K_{\varepsilon\delta})}\Big\}.
\end{aligned}$$

Following the lines of the proof of Lemma 1.5 of ([12], Chap.I), and using Theorem 1 of ([9], p.258), we can prove that

$$\|u^*\|_{W^{s,q}(K_{\varepsilon\delta})} \le C\varepsilon^{1/q} \|u^*\|_{W^{s+1,q}(\Omega)}, \quad s = 1, 2, 3,$$

and

$$\begin{aligned}
\|\Psi_{\varepsilon\delta}\|_{H^1(\Omega_{\varepsilon\delta})} \le\ & C\Big\{\delta^{\frac{3\lambda}{2}} \varepsilon^{1/q} \|u^*\|_{W^{2,q}(\Omega)} + \delta^{\frac{\lambda}{2}} \varepsilon^{1/q} \|u^*\|_{W^{2,q}(\Omega)}\\
+\ & \varepsilon^{(1+1/q)} \delta^{\frac{3\lambda}{2}} \|u^*\|_{W^{3,q}(\Omega)} + \varepsilon^{(1+1/q)} \delta^{\frac{5\lambda}{2}} \|u^*\|_{W^{3,q}(\Omega)}\\
+\ & \varepsilon^{(2+1/q)} \delta^{\frac{5\lambda}{2}} \|u^*\|_{W^{4,q}(\Omega)}\Big\},
\end{aligned}$$

where $q = 2p/(p-2)$, $2 < p < r$, $r >> 2$, $1/p = \lambda/2 + (1-\lambda)/r$, $0 < \lambda < 1$, and λ is close to 1.

We use the trace theorem, and get

$$\begin{aligned}
&\|\psi_{\varepsilon\delta}\|_{L^2(0,T;H^{1/2}(\partial\Omega))} \le C\|\Psi_{\varepsilon\delta}\|_{L^2(0,T;H^1(\Omega_{\varepsilon\delta}))}\\
&\quad\le C\Big\{\delta^{\frac{3\lambda}{2}}\varepsilon^{1/q}\|u^*\|_{L^2(0,T;W^{2,q}(\Omega))}\\
&\quad+\delta^{\frac{\lambda}{2}}\varepsilon^{1/q}\|u^*\|_{L^2(0,T;W^{2,q}(\Omega))}\\
&\quad+\varepsilon^{(1+1/q)}\delta^{\frac{3\lambda}{2}}\|u^*\|_{L^2(0,T;W^{3,q}(\Omega))}\\
&\quad+\varepsilon^{(1+1/q)}\delta^{\frac{5\lambda}{2}}\|u^*\|_{L^2(0,T;W^{3,q}(\Omega))}\\
&\quad+\varepsilon^{(2+1/q)}\delta^{\frac{5\lambda}{2}}\|u^*\|_{L^2(0,T;W^{4,q}(\Omega))}\Big\}\\
&\quad\le C(T)\delta^{\lambda/2}\varepsilon^{1/q}.
\end{aligned} \tag{43}$$

where $C(T)$ is a constant independent of ε, δ but dependent of T.

From (34), it follows from Lemma 2.1 that

$$\|p_{\varepsilon\delta}\|_{L^2(\Omega_{\varepsilon\delta})} \le C\Big[\varepsilon\delta^{3/2}\|\bar{u}_0\|_{W^{1,\infty}(\Omega)} + \varepsilon^2\delta^{5/2}\|\bar{u}_0\|_{W^{2,\infty}(\Omega)}\Big]. \tag{44}$$

From (35), we observe that $\partial T_{\varepsilon\delta} \subset \Omega$, and have

$$\|S_{\varepsilon\delta}\|_{L^2(\partial T_{\varepsilon\delta})} \le C\varepsilon^2\|N^{\delta}_{\alpha_1\alpha_2}\|_{L^2(\partial T_{\varepsilon\delta})}\|u^*\|_{W^{3,\infty}(\Omega)},$$

where

$$\begin{aligned}
\|N^{\delta}_{\alpha_1\alpha_2}\|^2_{L^2(\partial T_{\varepsilon\delta})} &= \int_{\partial T_{\varepsilon\delta}} |N^{\delta}_{\alpha_1\alpha_2}|^2 ds(x)\\
&= C\varepsilon^{n-1}\sum_{i=1}^{N_\varepsilon}\int_{\partial T_\delta} |N^{\delta}_{\alpha_1\alpha_2}|^2 ds(\xi)\\
&\approx C\varepsilon^{n-1}\frac{|\Omega|}{|Y|}\varepsilon^{-n}\int_{\partial T_\delta} |N^{\delta}_{\alpha_1\alpha_2}|^2 ds(\xi).
\end{aligned}$$

Using the trace theorem and Lemma 2.1, we derive

$$\|N^{\delta}_{\alpha_1\alpha_2}\|_{L^2(\partial T_\delta)} \le C\|N^{\delta}_{\alpha_1\alpha_2}\|_{H^1(Y_\delta)} \le C\delta^{3/2},$$

and consequently

$$\|S_{\varepsilon\delta}\|_{L^2(0,T;L^2(\partial T_{\varepsilon\delta}))} \le C\varepsilon^{3/2}\delta^{3/2}\|u^*\|_{L^2(0,T;W^{3,\infty}(\Omega))}. \tag{45}$$

From (30)-(35), combining (39), (43), (44) and (45), and using a priori estimate for parabolic equations, we obtain

$$\begin{aligned}
&\sup_{0\le t\le T}\int_{\Omega_{\varepsilon\delta}} (u^{\varepsilon\delta}(x,t) - U_s^{\varepsilon\delta}(x,t))^2 dx + \int_0^T \|u^{\varepsilon\delta} - U_s^{\varepsilon\delta}\|^2_{H^1(\Omega_{\varepsilon\delta})} dt\\
&\quad\le C\Big\{\|\psi_{\varepsilon\delta}\|^2_{L^2(0,T;H^{\frac{1}{2}}(\partial\Omega))} + \|F_0\|^2_{L^2(0,T;L^2(\Omega))}\\
&\quad+\|S_{\varepsilon\delta}\|^2_{L^2(0,T;L^2(\partial T_{\varepsilon\delta}))} + \|p_{\varepsilon\delta}\|^2_{L^2(\Omega_{\varepsilon\delta})}\Big\}\\
&\quad\le C(T)\Big\{\varepsilon^2\delta^3 + \delta^2 + \delta^{\lambda}\varepsilon^{2/q}\Big\}, \quad s = 2,
\end{aligned}$$

where C is independent of ε, δ but dependent of T, $q = 2p/(p-2)$, $2 < p < r$, $r >> 2$, $1/p = \lambda/2 + (1-\lambda)/r$, $0 < \lambda < 1$, and λ is close to 1.

We next prove (29) for $s = 1$. From (4) and (5), we have

$$
\begin{aligned}
&\frac{\partial(u^{\varepsilon\delta} - U_1^{\varepsilon\delta})}{\partial t} - \frac{\partial}{\partial x_i}\Big[a_{ij}(\frac{x}{\varepsilon},t)\frac{\partial(u^{\varepsilon\delta} - U_1^{\varepsilon\delta})}{\partial x_j}\Big] \\
&= \Big[(a_{ij} + a_{ik}\frac{\partial N_j^\delta}{\partial \xi_k} + \frac{\partial}{\partial \xi_k}(a_{ki}N_j^\delta) - \hat{a}_{ij}^\delta\Big]\frac{\partial^2 u^*(x,t)}{\partial x_i \partial x_j} \\
&\quad +(\hat{a}_{ij}^\delta - a_{ij}^*)\frac{\partial^2 u^*(x,t)}{\partial x_i \partial x_j} + \varepsilon a_{ij} N_{\alpha_1}^\delta \frac{\partial^3 u^*(x,t)}{\partial x_i \partial x_j \partial x_{\alpha_1}} \\
&\quad -\varepsilon \frac{\partial N_{\alpha_1}^\delta(\xi,t)}{\partial t}\frac{\partial u^*(x,t)}{\partial x_{\alpha_1}} - \varepsilon N_{\alpha_1}^\delta(\xi,t)\frac{\partial^2 u^*(x,t)}{\partial t \partial x_{\alpha_1}} \\
&\quad = -\frac{\partial}{\partial \xi_i}\Big(a_{ij}\frac{\partial N_{\alpha_1\alpha_2}^\delta}{\partial \xi_j}\Big)\frac{\partial^2 u^*(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}} \\
&\quad +(\hat{a}_{ij}^\delta - a_{ij}^*)\frac{\partial^2 u^*(x,t)}{\partial x_i \partial x_j} + \varepsilon a_{ij} N_{\alpha_1}^\delta \frac{\partial^3 u^*(x,t)}{\partial x_i \partial x_j \partial x_{\alpha_1}} \\
&\quad -\varepsilon \frac{\partial N_{\alpha_1}^\delta(\xi,t)}{\partial t}\frac{\partial u^*(x,t)}{\partial x_{\alpha_1}} - \varepsilon N_{\alpha_1}^\delta(\xi,t)\frac{\partial^2 u^*(x,t)}{\partial t \partial x_{\alpha_1}} \\
&\quad = -\varepsilon\frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial N_{\alpha_1\alpha_2}^\delta}{\partial \xi_j}\frac{\partial^2 u^*(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}\Big) + F_1(x,\xi,t),
\end{aligned}
$$

where

$$
\begin{aligned}
F_1(x,\xi,t) &= \varepsilon a_{ij}\frac{\partial N_{\alpha_1\alpha_2}^\delta}{\partial \xi_j}\frac{\partial^3 u^*(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}\partial x_i} \\
&+ (\hat{a}_{ij}^\delta - a_{ij}^*)\frac{\partial^2 u^*(x,t)}{\partial x_i \partial x_j} + \varepsilon a_{ij} N_{\alpha_1}^\delta \frac{\partial^3 u^*(x,t)}{\partial x_i \partial x_j \partial x_{\alpha_1}} \\
&- \varepsilon \frac{\partial N_{\alpha_1}^\delta(\xi,t)}{\partial t}\frac{\partial u^*(x,t)}{\partial x_{\alpha_1}} - \varepsilon N_{\alpha_1}^\delta(\xi,t)\frac{\partial^2 u^*(x,t)}{\partial t \partial x_{\alpha_1}}.
\end{aligned}
$$

For $(x,t) \in \partial T_{\varepsilon\delta} \times (0,T)$, from (4) and (5), we get

$$
\begin{aligned}
\nu_i a_{ij}\frac{\partial(u^{\varepsilon\delta} - U_1^{\varepsilon\delta})}{\partial x_j} &= -\varepsilon \nu_i a_{i\alpha_1} N_{\alpha_2}^\delta \frac{\partial^2 u^*(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}} \\
&= \varepsilon \nu_i a_{ij}\frac{\partial N_{\alpha_1\alpha_2}^\delta}{\partial \xi_j}\frac{\partial^2 u^*(x,t)}{\partial x_{\alpha_1}\partial x_{\alpha_2}}.
\end{aligned}
$$

Repeating the process of the proof for $s = 2$, we can prove (29) for $s = 1$. Therefore, the proof of Theorem 2.2 is complete.

Corollary 2.1. Under the assumptions of Theorem 2.2, we have

$$
\sup_{0\le t\le T}\int_{\Omega_{\varepsilon\delta}} (u^{\varepsilon\delta}(x,t) - u^*(x,t))^2 dx \le C(T)\Big\{\varepsilon^2\delta^3 + \delta^2 + \delta^\lambda \varepsilon^{2/q}\Big\}, \tag{46}
$$

where C is a constant independent of ε and δ , but dependent on T, $q = 2p/(p-2)$, $2 < p < r$, $r >> 2$, $1/p = \lambda/2 + (1-\lambda)/r$, $0 < \lambda < 1$, and λ is close to 1.

Remark 2.2. We prove Theorems 2.1 and 2.2 for a kind of lattice structure of Type I. In fact, Theorems 2.1 and 2.2 are valid for other lattice-type structures such as Type II and Type III.

Remark 2.3. If the coefficients $a_{ij}(\frac{x}{\varepsilon},t)$ do not depend on x and t, i.e. the coefficients a_{ij} are constants, we can also prove Theorems 2.1 and 2.2.

3. Multiscale Finite Element Method and Numerical Examples

In this section, we first present the multiscale finite element method for solving heat transfer equation in lattice-type structures. Then we show some numerical results for three kinds of lattice-type structures which have the same volume ratio of a solid material.

From (20) and (22), the multiscale finite element method for solving problem (1) is composed of three steps:

Step 1 Compute the cell functions $N^{\delta}_{\alpha_1}(\xi,t)$, $N^{\delta}_{\alpha_1\alpha_2}(\xi,t)$, $\xi=\varepsilon^{-1}x$ in a reference cell Y_δ.

Step 2 Solve numerically the initial-boundary value problem of the homogenized parabolic equation (9) with constant coefficients over a whole domain $\Omega\times(0,T)$ in a coarse mesh.

Step 3 Calculate the higher-order derivatives $D^\alpha_x u^*(x,t)$ by using the finite difference method. Note that we cannot directly compute higher-order derivatives for the finite element solutions.

We implement the subdivisions for Y_δ and Ω, respectively. h_0 and h denote the sizes of the corresponding meshes.

Define first-order difference quotients given by

$$\Delta_{x_j}u^*_h(N_p,t_i)=\frac{1}{\tau(N_p)}\sum_{e\in\sigma(N_p)}[\frac{\partial u^*_h}{\partial x_j}]_e(N_p,t_i), \tag{47}$$

where $\sigma(N_p)$ is the set of elements with node N_p; $\tau(N_p)$ is the number of elements of $\sigma(N_p)$, $[\frac{\partial u^*_h}{\partial x_j}]_e(N_p,t_i)$ is the value of the derivative $\frac{\partial u^*_h}{\partial x_j}$ at node N_p associated with element e at time $t=t_i$.

We define second-order difference quotients as follows

$$\Delta^2_{x_l x_m}u^*_h(N_p,t_i)=\frac{1}{\tau(N_p)}\sum_{e\in\sigma(N_p)}[\sum_{j=1}^{d}\Delta_{x_l}u^*_h(P_j,t_i)\frac{\partial\chi_j}{\partial x_m}]_e(N_p,t_i), \tag{48}$$

where d is the number of nodes on e, P_j are the nodes of e, $\chi_j(x)$ are Lagrange's type shape functions.

The multiscale finite element formula is written as

$$\begin{array}{rcl} U^{\varepsilon\delta}_{1,h_0,h}(N_p,t_i) &=& u^*_h(N_p,t_i)+\varepsilon N^{\delta,h_0}_{\alpha_1}\Delta_{x_{\alpha_1}}u^*_h(N_p,t_i),\\ U^{\varepsilon\delta}_{2,h_0,h}(N_p,t_i) &=& u^*_h(N_p,t_i)+\varepsilon N^{\delta,h_0}_{\alpha_1}\Delta_{x_{\alpha_1}}u^*_h(N_p,t_i)\\ &+& \varepsilon^2 N^{\delta,h_0}_{\alpha_1\alpha_2}\Delta^2_{x_{\alpha_1}x_{\alpha_2}}u^*_h(N_p,t_i). \end{array} \tag{49}$$

Next we do some numerical experiments for heat transfer equation for three types of lattice structures as shown in Figs. 1-3 in the two-dimensional case.

$$\begin{cases} \dfrac{\partial u^{\varepsilon\delta}(x,t)}{\partial t}-\dfrac{\partial}{\partial x_i}(a_{ij}(\frac{x}{\varepsilon},t)\dfrac{\partial u^{\varepsilon\delta}(x,t)}{\partial x_j})=f(x,t), \quad (x,t)\in\Omega_{\varepsilon\delta}\times(0,T)\\ u^{\varepsilon\delta}(x,t)=0, \quad (x,t)\in\partial\Omega\times(0,T)\\ \nu_i a_{ij}(\frac{x}{\varepsilon},t)\dfrac{\partial u^{\varepsilon\delta}}{\partial x_j}=0, \quad (x,t)\in\partial T_{\varepsilon\delta}\times(0,T)\\ u^{\varepsilon\delta}(x,0)=0, \quad x\in\Omega_{\varepsilon\delta}. \end{cases} \tag{50}$$

where $\Omega_{\varepsilon\delta}$ is a lattice-type structure which is the union of entire cells, $\partial T_{\varepsilon\delta}$ is the surface of holes and δ is the thickness of solid walls.

We take $\varepsilon=\frac{1}{8}$, and the thicknesses of three types of lattice structures are given as in Table 1.

Table 1. Thicknesses of solid walls for three types of lattice-type structures

	thickness(δ)
Type I	0.0150888
Type II	0.0176776
Type III	0.0125

Example 3.1. Suppose that the coefficients a_{ij} are constants.

Case 1.1: $\Omega_{\varepsilon\delta}$ is a lattice structure of Type I as shown in Fig.1 (a). $f(x,t) = 100(x+y)(2+\sin(20\pi t)),\ a_{ij}(\frac{x}{\varepsilon}) = 100\delta_{ij},\ t_* = 1.0.$

Case 1.2: $\Omega_{\varepsilon\delta}$ is a lattice structure of Type II as shown in Fig.2 (a), $f(x,t) = 100(x+y)(2+\sin(20\pi t)),\ a_{ij}(\frac{x}{\varepsilon}) = 100\delta_{ij},\ t_* = 1.0.$

Case 1.3: $\Omega_{\varepsilon\delta}$ is a lattice structure of Type III as shown in Fig.3 (a), $f(x,t) = 100(x+y)(2+\sin(20\pi t)),\ a_{ij}(\frac{x}{\varepsilon}) = 100\delta_{ij},\ t_* = 1.0.$

Example 3.2. Suppose that the coefficients a_{ij} only depend on t.

Case 2.1: $\Omega_{\varepsilon\delta}$ is a lattice structure of Type I as shown in Fig.1 (a), $f(x,t) = 100,\ a_{ij}(\frac{x}{\varepsilon},t) = 100(2+\sin(20\pi t))\delta_{ij},\ t_* = 1.0.$

Case 2.2: $\Omega_{\varepsilon\delta}$ is a lattice structure of Type II as shown in Fig.2 (a), $f(x,t) = 100,\ a_{ij}(\frac{x}{\varepsilon},t) = 100(2+\sin(20\pi t))\delta_{ij},\ t_* = 1.0.$

Case 2.3: $\Omega_{\varepsilon\delta}$ is a lattice structure of Type III as shown in Fig.3 (a), $f(x,t) = 100,\ a_{ij}(\frac{x}{\varepsilon},t) = 100(2+\sin(20\pi t))\delta_{ij},\ t_* = 1.0$, where δ_{ij} is the Kronecker symbol.

Since it is extremely difficult to find out the exact solution of (50), in order to show the numerical accuracy of our method, we replace $u^{\varepsilon\delta}(x,t)$ with its approximate solution in a very fine mesh. We would like to point out that we do not need to solve the original problem (50) in a very fine mesh in real engineering problems. Here we use the linear Lagrangian element to solve problem (50). Without confusion we continue to use $u^{\varepsilon\delta}(x,t)$ to denote the numerical solution in a fine mesh.

We implement triangle partitions for Y_δ and Ω, respectively, and use the linear Lagrangian elements. The computational cost is listed in Tables 2 and 3.

In Cases 1.1-1.3, the homogenized coefficients a^*_{ij} for three types of lattice structures can be calculated by algebraic expressions(see [6], p.75, p.112, p.127), and $\hat{a}^\delta_{ij}$ defined in (6) can be computed numerically. Some numerical results are as shown in Tables 4 and 5. In Cases 2.1-2.3, the homogenized coefficients $a^*_{ij}(t)$ and $\hat{a}^\delta_{ij}(t)$ can be easily obtained by multiplying the factor $(2+\sin(20\pi t))$.

For simplicity, we denote $u^*(x,t)$ the finite element solution for the homogenized heat transfer equation (9) in a coarse mesh, and $U_1^{\varepsilon\delta}(x,t), U_2^{\varepsilon\delta}(x,t)$ the first-order and the second-order multiscale finite element solutions, respectively.

Set

$$e_0 = u^{\varepsilon\delta}(x,t) - u^*(x,t),\ e_1 = u^{\varepsilon\delta}(x,t) - U_1^{\varepsilon\delta}(x,t),\ e_2 = u^{\varepsilon\delta}(x,t) - U_2^{\varepsilon\delta}(x,t).$$

We introduce some notations:

$\|u\|_{L^2} = \Big(\int\limits_\Omega |u(x,t)|^2 dx\Big)^{1/2},\ \ \|u\|_{H^1} = \Big(\int\limits_\Omega |u(x,t)|^2 + |\nabla u(x,t)|^2 dx\Big)^{1/2},\ \ \|u\|_{(0)} = \Big(\int\limits_0^t \|u\|^2_{L^2} dt\Big)^{1/2}$, and $\|u\|_{(1)} = \Big(\int\limits_0^t \|u\|^2_{H^1} dt\Big)^{1/2}$.

Some numerical results are shown as in Table 6.

Figs.7-9 show some numerical results for solutions $u^{\varepsilon\delta}(x,t)$, $u^*(x,t)$, $U_1^{\varepsilon\delta}(x,t)$, and $U_2^{\varepsilon\delta}(x,t)$ along the diagonal of the square $\Omega_{\varepsilon\delta}$ in Cases 1.1-1.3, 2.1-2.3 at time t_* , which t_* are as given in Cases 1.1-1.3, 2.1-2.3 .

Figs.10-15 clearly show the evolution of the relative errors of approximate solutions with time t in Cases 1.1-1.3, 2.1-2.3, where the horizontal axis denotes time t, the vertical axis is the relative error, where erre0L2, erre1L2, erre2L2, erre0H1, erre1H1, and erre2H1 denote $\frac{\|e_0\|_{L^2}}{\|u^{\varepsilon\delta}\|_{L^2}}$, $\frac{\|e_1\|_{L^2}}{\|u^{\varepsilon\delta}\|_{L^2}}$, $\frac{\|e_2\|_{L^2}}{\|u^{\varepsilon\delta}\|_{L^2}}$, $\frac{\|e_0\|_{H^1}}{\|u^{\varepsilon\delta}\|_{H^1}}$, $\frac{\|e_1\|_{H^1}}{\|u^{\varepsilon\delta}\|_{H^1}}$, and $\frac{\|e_2\|_{H^1}}{\|u^{\varepsilon\delta}\|_{H^1}}$, respectively.

TABLE 2. Comparison of the numbers of nodes

	original equation	cell problem	homogenized equation
case 1.1	12309	684	441
case 1.2	20181	921	441
case 1.3	28257	1173	441

TABLE 3. Comparison of the numbers of elements

	original equation	cell problem	homogenized equation
case 1.1	16664	912	800
case 1.2	27416	1232	800
case 1.3	38704	1576	800

TABLE 4. Homogenized coefficients a^*_{ij} for three types of structures

	a^*_{11}	a^*_{12}	a^*_{21}	a^*_{22}
case 1.1	50.0	0.0	0.0	50.0
case 1.2	50.0	20.71068	20.71068	50.0
case 1.3	50.0	0.0	0.0	50.0

TABLE 5. Homogenized coefficients $\hat{a}^{\delta}_{ij}$ for three types of structures

	$\hat{a}^{\delta}_{11}$	$\hat{a}^{\delta}_{12}$	$\hat{a}^{\delta}_{21}$	$\hat{a}^{\delta}_{22}$
case 1.1	51.37552	-7.98636×10^{-8}	-7.98636×10^{-8}	51.37552
case 1.2	51.02844	20.46541	20.46541	51.02845
case 1.3	51.41881	1.97295×10^{-7}	1.97295×10^{-7}	51.41881

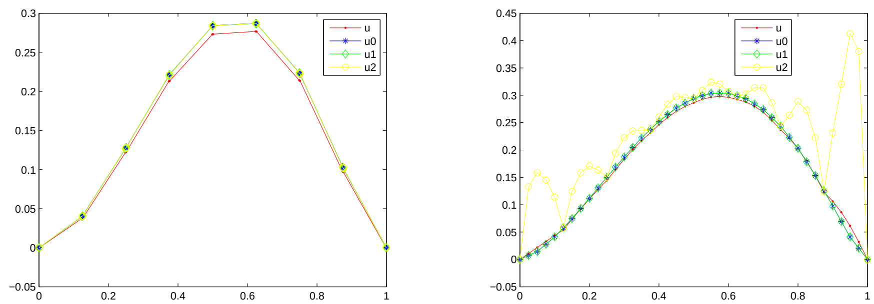
Remark 3.1. Observing the numerical results in Tables 4 and 5, it is obvious that if $\delta > 0$ is very small, $\hat{a}^{\delta}_{ij}$ and a^*_{ij} are very close, which are entirely consistent with (24).

Remark 3.2. Table 6 clearly shows that the first-order multiscale finite element method has the better numerical accuracy. But neither the homogenization method nor the second-order multiscale finite element method has good numerical accuracy in these cases (see Cases 1.2-1,3, 2.2-2.3).

Remark 3.3 Theoretically, the first-order multiscale method yields the same convergent order as the second-order one. Numerically, in the computation of all kinds of physical fields for composites, when the difference between different materials is very large, the first-order multiscale method is insufficient to describe local fluctuation of the solution for considering problems. We need to seek the second-order multiscale method. Numerous numerical results clearly show that the numerical accuracy of the second-order multiscale method is better than that of the homogenization method or the first-order multiscale method. However, for lattice-type structures, since the solid part of the structures is made of only one kind of material and the cell functions are very small (see Lemma 2.1 of this paper), the numerical accuracy of the first-order multiscale method is sufficient good. The numerical results of the second-order multiscale method may be bad because of the numerical errors.

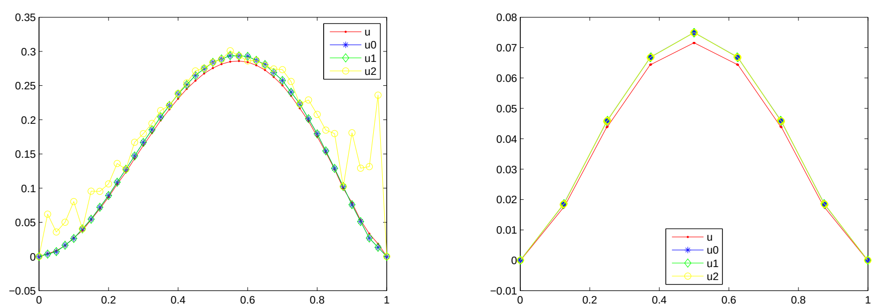
TABLE 6. Comparison of computational results

	$\frac{\|e_0\|_{(0)}}{\|u^{\varepsilon\delta}\|_{(0)}}$	$\frac{\|e_1\|_{(0)}}{\|u^{\varepsilon\delta}\|_{(0)}}$	$\frac{\|e_2\|_{(0)}}{\|u^{\varepsilon\delta}\|_{(0)}}$	$\frac{\|e_0\|_{(1)}}{\|u^{\varepsilon\delta}\|_{(1)}}$	$\frac{\|e_1\|_{(1)}}{\|u^{\varepsilon\delta}\|_{(1)}}$	$\frac{\|e_2\|_{(1)}}{\|u^{\varepsilon\delta}\|_{(1)}}$
Case 1.1	0.039937	0.039845	0.10197	0.97475	0.11316	4.8086
Case 1.2	0.028745	0.028619	0.19844	1.0017	0.13893	6.4708
Case 1.3	0.029338	0.029285	0.14833	0.96451	0.12759	10.980
Case 2.1	0.039772	0.039685	0.099143	0.97188	0.10799	4.8241
Case 2.2	0.029107	0.028990	0.18847	1.0029	0.13521	6.0305
Case 2.3	0.029407	0.029358	0.14177	0.96259	0.12420	10.690



(a) (b)

FIGURE 7. Computational results: (a) case 1.1; (b) case 1.2.

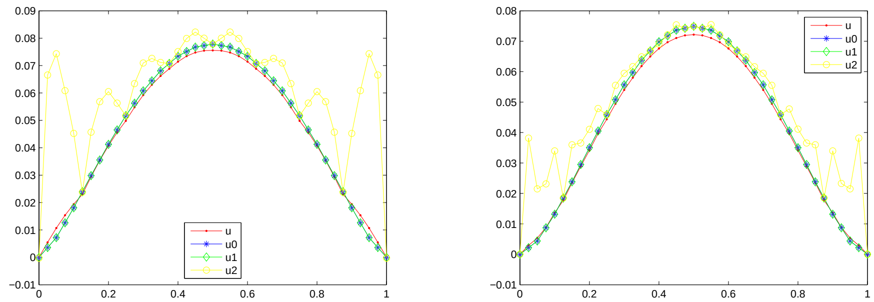


(a) (b)

FIGURE 8. Computational results: (a) case 1.3,; (b) case 2.1.

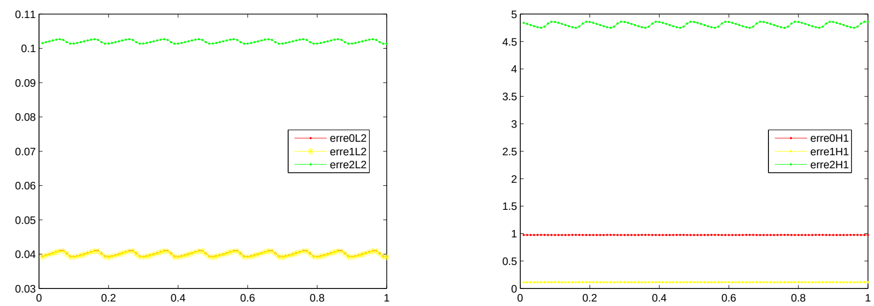
Conclusions

This paper discussed the initial-boundary value problem for the heat transfer equation in lattice-type structures. The new contribution obtained in this paper was the determination of the convergence rate for the approximate solutions by using the homogenization method and the multiscale asymptotic methods. We did some numerical experiments for three types of lattice structures. The numerical results suggested that the first-order multiscale method should be a better choice compared with the homogenization method and the second-order multiscale method for solving the heat transfer equations in lattice-type structures.



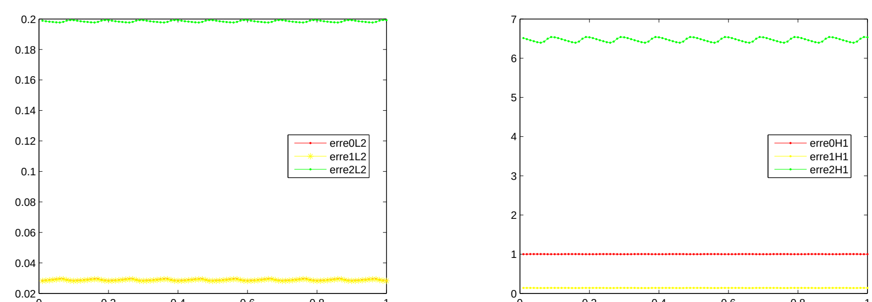
(a) (b)

FIGURE 9. Computational results: (a) case 2.2,; (b) case 2.3.



(a) (b)

FIGURE 10. (a) case 1.1, the evolution of L^2 relative errors with t; (b) case 1.1, the evolution of H^1 relative errors with t

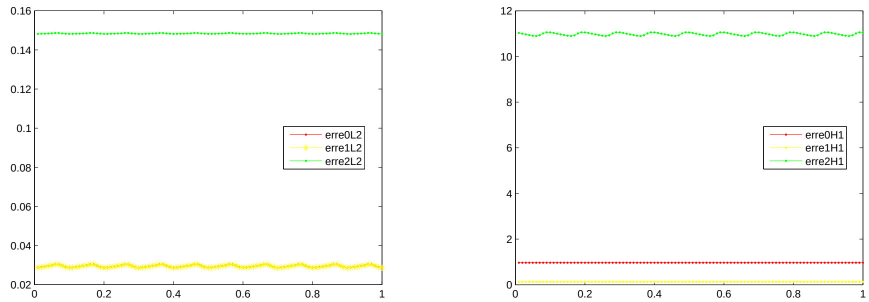


(a) (b)

FIGURE 11. (a) case 1.2, the evolution of L^2 relative errors with t; (b) case 1.2, the evolution of H^1 relative errors with t

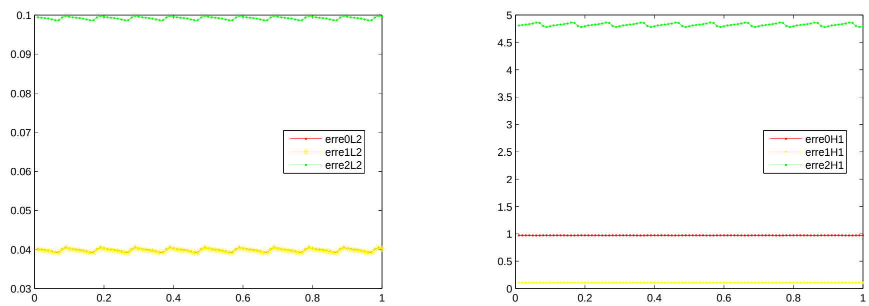
Acknowledgements

The authors express their thanks to the referees, whose careful readings and suggestions led to improvements in the presentation of the results.



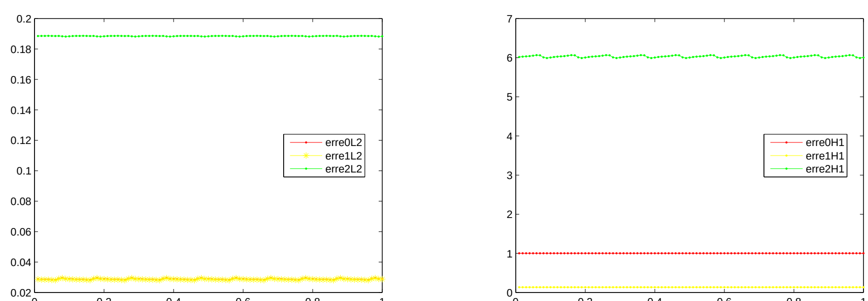
(a) (b)

FIGURE 12. (a) case 1.3, the evolution of L^2 relative errors with t; (b) case 1.3, the evolution of H^1 relative errors with t



(a) (b)

FIGURE 13. (a) case 2.1, the evolution of L^2 relative errors with t; (b) case 2.1, the evolution of H^1 relative errors with t

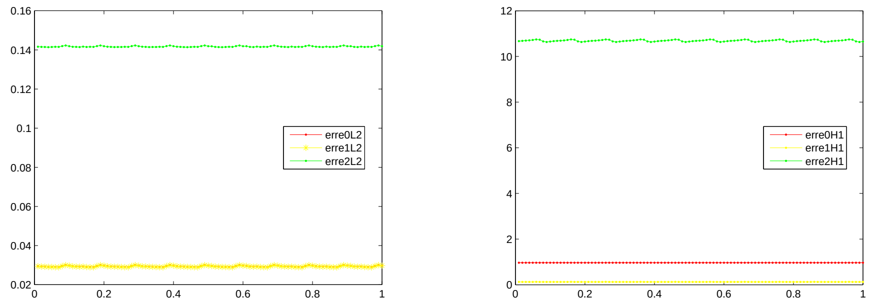


(a) (b)

FIGURE 14. (a) case 2.2, the evolution of L^2 relative errors with t; (b) case 2.2, the evolution of H^1 relative errors with t

References

[1] W. Allegretto, L.Q. Cao and Y.P. Lin, Multiscale asymptotic expansion for second order parabolic equations with rapidly oscillating coefficients, Discrete and Continuous Dynamical Systems, 2008, 20(3): 543-576.



(a) (b)

FIGURE 15. (a) case 2.3, the evolution of L^2 relative errors with t; (b) case 2.3, the evolution of H^1 relative errors with t

[2] N.S. Bakhvalov and G.P. Panasenko, Homogenization: Averaging Processes in Periodic Media, Kluwer, Dordrecht, 1989.
[3] A. Bensoussan, J.L. Lions, and G. Papanicolaou, Asymptotic Analysis of Periodic Structures, North-Holland, Amsterdan, 1978.
[4] L.Q. Cao, Multiscale asymptotic expansion and finite element methods for mixed boundary value problems of second order elliptic equation in perforated domains, Numer. Math., 2006, 103:11-45.
[5] L.Q. Cao and J.Z. Cui, Asymptotic expansions and numerical algorithms of eigenvalues and eigenfunctions of the Dirichlet problem for second order elliptic equations in perforated domains, Numer. Math., 2004, 96: 525-581.
[6] D. Cioranescu and J. Saint Jean Paulin, Homogenization of Reticulated Structures, Springer, New York, 1999.
[7] D. Cioranescu, An Introduction to Homogenization, Oxford University Press, Oxford New York, 1999.
[8] D. Cioranescu and J. Saint Jean Paulin, Homogenization in open sets with holes, J. Math. Anal. and Appl., 1979, 71(2): 590-607.
[9] L.C. Evans, Partial Differential Equations, American Mathematical Society, Providence, Rhode Island, 1998.
[10] J.L. Lions, Some Methods in the Mathematical Analysis of Systems and Their Control, Science Press, Beijing, Gordon and Breach, New York, 1981.
[11] R. Miller, Extension theorems for homogenization on lattice structures, Appl. Math. Lett., 1992, 5: 73-78.
[12] O.A. Oleinik, A.S. Shamaev and G.A. Yosifian, Mathematical Problems in Elasticity and Homogenization, North-Holland, Amsterdam New York, 1992.

State Key Laboratory of Scientific and Engineering Computing , Institute of Computational Mathematics and Science-Engineering Computing, Academy of Mathematics and Systems Science ,Chinese Academy of Sciences, Beijing, 100080, China
E-mail: `zhaifm@lsec.cc.ac.cn`

State Key Laboratory of Scientific and Engineering Computing , Institute of Computational Mathematics and Science-Engineering Computing, Academy of Mathematics and Systems Science ,Chinese Academy of Sciences, Beijing, 100080, China
E-mail: `clq@lsec.cc.ac.cn`