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AN OPTIMAL-ORDER ERROR ESTIMATE TO THE MODIFIED METHOD OF CHARACTERISTICS FOR A DEGENERATE CONVECTION-DIFFUSION EQUATION

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Abstract. We prove *a priori* error estimates in a weighted energy norm to the modified method of characteristics (MMOC) for time-dependent convection-diffusion equations with degenerate diffusion. The convergence rates are independent of the lower bound of the diffusion. In other words, these estimates hold uniformly with respect to the degenerate diffusion.

Key Words. Convergence analysis, degenerate convection-diffusion equations, Eulerian-Lagrangian methods, interpolation of spaces, optimal-order estimates

1. Introduction

Time-dependent advection-diffusion equations arise in mathematical models of porous medium flow and transport processes, including petroleum reservoir simulation, environmental modeling, and other applications. In such applications as immiscible displacement of oil by water in a secondary oil recovery process in petroleum industry or a groundwater transport process involving a non-aqueous phase liquid (NAPL), the corresponding governing equation is a degenerate time-dependent nonlinear advection-diffusion equation for the saturation of the invading phase. The diffusion in the saturation equation is due to capillary pressure effect, which could vanish or exhibit significant effect depending on whether the wetting phase or the nonwetting phase occupies the pore space [3, 5, 16, 19]. On the other hand, subsurface geological formations often consist of layered media, in which the diffusion parameters could vary by several orders of magnitude. In all of these applications, the governing equations could be convection-dominated in part of the domain while diffusion-dominated in the other part. Consequently, these problems admit solutions with moving fronts and complex structures and present serious mathematical and numerical difficulties.

Classical finite difference or finite element methods tend to generate numerical solutions with nonphysical oscillations, while classical upwind methods often produce numerical solutions with excessive numerical diffusion that smears out the fronts and generates spurious effects related to grid orientation [9, 14, 16]. Eulerian-Lagrangian methods provide an alternative approach for numerically solving time-dependent advection-diffusion equations. These methods combine the advection and capacity terms in the governing equations to carry out the temporal discretization in the Lagrangian coordinates, and discretize the diffusion term on a fixed mesh in the Eulerian coordinates [6, 11, 13, 24]. They symmetrize the governing equation and stabilize their numerical approximations. Moreover, they generate accurate numerical solutions and significantly reduce the numerical diffusion and

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grid-orientation effect present in upwind methods, even if large time steps and coarse spatial meshes are used. Eulerian-Lagrangian methods were shown to be very competitive in terms of accuracy and efficiency [11, 26, 28].

The modified method of characteristics was one of the pioneering methods in the class of Eulerian-Lagrangian methods and was proposed and analyzed in early 1980s [13]. Since then the MMOC has been successfully applied to the numerical simulation of coupled systems in miscible displacement and immiscible displacement in petroleum industry [9, 14, 17]. Subsequently, various improvements of the MMOC were developed, including the modified method of characteristics with adjusted advection (MMOCAA) [11, 12], the Eulerian-Lagrangian localized adjoint method (ELLAM) [1, 6, 24, 25, 28], the characteristic mixed finite element method (CMFEM) [2, 33], and the Eulerian-Lagrangian discontinuous Galerkin method (ELDG) [32] in the context of linear advection-diffusion equations, single-phase miscible displacement processes, immiscible two-phase flow, and multiphase multicomponent flow and transport processes in porous media.

Extensive research has been carried out on the convergence analysis and error estimates for the MMOC [13], the MMOCAA [12], the CMFEM [2], the ELLAM [22, 23, 27], and the ELDG [32]. However, the generic constants in these estimates depend inversely on the vanishing parameter ε and so will blow up as ε tends to zero. These estimates fail to reflect the uniformly optimal-order convergence rates of the Eulerian-Lagrangian methods observed numerically. An ε-uniform suboptimal-order error estimate was proved for the MMOC scheme for a time-dependent advection-diffusion equation with an incompressible velocity field $\mathbf{v}$ and a nonstandard boundary condition $\mathbf{v} = \mathbf{0}$ and the diffusion of the form $\varepsilon \Delta u$ [4]. Subsequently, ε-uniform estimates for the MMOC, the MMOCAA, the ELLAM, and the ELDG schemes for time-dependent advection-diffusion equations with the diffusion of the form $\varepsilon D(x,t)$ and with a periodic boundary condition or a general flux boundary condition [29, 30, 31] were obtained. However, all of these ε-uniform estimates depend heavily on the lower bound D_{min} and upper bound D_{max} of the diffusion coefficient $D(x,t)$, although they are ε-independent. A suboptimal error estimate was proved for the MMOC scheme for a degenerate time-dependent advection-diffusion equation [10].

In this paper we prove *a priori* optimal-order error estimates in a weighted energy norm to the MMOC scheme for degenerate time-dependent convection-diffusion equations with a degenerate diffusion. The convergence rates are independent of the lower bound of the diffusion, and they do not require the upper bound of the diffusion to tend to zero at the same rate as in the case of vanishing diffusion coefficient ε. The rest of this paper is organized as follows. In §2 we recall preliminary results on Sobolev spaces and interpolation of spaces. In §3 we revisit the MMOC scheme. In §4 we prove optimal-order error estimates in a weighted energy norm to the MMOC scheme. In §5 we prove auxiliary lemmas, which were used in the proof of the main theorem in §4. §6 contains concluding remarks.

2. Model Problem and Preliminaries

We present a model problem and auxiliary results in this section.

2.1. Mathematical Model. We consider a time-dependent linear advection-diffusion equation with a degenerate diffusion in one space dimension

$$
\begin{aligned}
u_t + V(x,t)u_x - (D(x,t)u_x)_x &= f(x,t), \quad (x,t) \in (a,b) \times (0,T), \\
u(x,0) &= u_o(x), \qquad x \in [a,b].
\end{aligned} \tag{1}
$$

Here $V(x,t)$ is a velocity field, $f(x,t)$ accounts for external sources and sinks, $u_o(x)$ is a prescribed initial data, and $u(x,t)$ is the unknown solute concentration of the dissolved substance. $D(x,t)$ is a diffusion coefficient with

$$0 \le D(x,t) \le D_{max} < +\infty \qquad \forall (x,t) \in [a,b] \times [0,T]. \tag{2}$$

We emphasize that the assumption (2) allows the diffusion coefficient to vanish in multiple space-time regions rather than just at separate points. Problem (1) can be viewed as a linearized version of the well-known Buckley-Leverett equation with the capillary pressure effect included [19, 20]. It also describes the mass transport equation in layered geological formations. In particular, the assumption includes the case of the (uniformly) vanishing diffusion case considered in hyperbolic equations [20] as a special case.

Problem (1) needs to be closed with boundary conditions at $x = a$ and $x = b$. However, the degenerate diffusion further complicates the problem. For example, if the diffusion $D(x,t)$ does not vanish within a neighborhood of the outflow boundary, a boundary condition of Dirichlet or Neumann or Robin type needs to be imposed at the outflow boundary. On the other hand, if $D(x,t)$ degenerates in such a neighborhood, no outflow boundary condition should be imposed at the outflow boundary. Since we do not want to impose any restrictions on the behavior of the degenerate diffusion $D(x,t)$, we close problem (1) with periodic boundary conditions at $x = a$ and $x = b$.

2.2. Sobolev Spaces and Approximation Properties. Let $W_p^k(a,b)$ consist of functions whose weak derivatives up to order k are pth Lebesgue integrable in (a,b). Let $H^k(a,b) := W_2^k(a,b)$ and $H_E^1(a,b)$ be a subspace of $H^1(a,b)$ with period $b-a$. For any Banach space X, we introduce Sobolev spaces involving time [15]

$$W_p^k(t_1,t_2;X) := \Big\{ f(x,t) \ : \Big\| \frac{\partial^\alpha f}{\partial t^\alpha}(\cdot,t) \Big\|_X \in L^p(t_1,t_2), \ 0 \le \alpha \le k, \ 1 \le p \le \infty \Big\},$$

$$\|f\|_{W_p^k(t_1,t_2;X)} := \begin{cases} \Big(\displaystyle\sum_{\alpha=0}^{k} \int_{t_1}^{t_2} \Big\| \frac{\partial^\alpha f}{\partial t^\alpha}(\cdot,t) \Big\|_X^p dt \Big)^{1/p}, & 1 \le p < \infty, \\ \displaystyle\max_{0 \le \alpha \le k} \operatorname*{esssup}_{(t_1,t_2)} \Big\| \frac{\partial^\alpha f}{\partial t^\alpha}(\cdot,t) \Big\|_X, & p = \infty. \end{cases}$$

We define a uniform space-time partition on $[a,b] \times [0,T]$: $x_i := a + ih$ for $0 \le i \le I$ with $h := (b-a)/I$ and $t_n := n\Delta t$ for $0 \le n \le N$ with $\Delta t := T/N$. Let $x_{i-\frac{1}{2}}$ be the center of the interval $[x_{i-1}, x_i]$. We define the following (discrete) energy semi-norms

$$\begin{aligned}
|f(\cdot,t_n)|_{H_D^1(a,b)} &:= \Big(\int_a^b D(x,t_n) f_x^2(x,t_n) dx \Big)^{\frac{1}{2}}, \\
|f(\cdot,t_n)|_{\hat{H}_D^1(a,b)} &:= \Big(\sum_{i=1}^{I} D(x_{i-\frac{1}{2}},t_n) f_x^2(x_{i-\frac{1}{2}},t_n) h \Big)^{\frac{1}{2}}, \\
\|f\|_{L^\infty(0,T;L^2(a,b))} &:= \max_{0 \le n \le N} \|f(\cdot,t_n)\|_{L^2(a,b)}, \\
|f|_{L^2(0,T;H_D^1(a,b))} &:= \Big(\sum_{n=0}^{N} |f(\cdot,t_n)|_{H_D^1(a,b)}^2 \Delta t \Big)^{\frac{1}{2}}, \\
|f|_{L^2(0,T;\hat{H}_D^1(a,b))} &:= \Big(\sum_{n=0}^{N} |f(\cdot,t_n)|_{\hat{H}_D^1(a,b)}^2 \Delta t \Big)^{\frac{1}{2}}.
\end{aligned}$$

Let $S_h(a,b) \subset H_E^1(a,b)$ be the finite element space that consists of continuous and piecewise-linear functions with respect to the spatial partition in $[a,b]$. We let

$\Pi_h v \in S_h(a,b)$ be the piecewise-linear interpolant of v for any $v \in H^1_E(a,b)$. The following estimates hold [7]

$$
\begin{aligned}
\|\Pi_h v - v\|_{H^k(a,b)} &\le C_1 h^{2-k}\, \|v\|_{H^2(a,b)} && \forall v \in H^2(a,b),\ k = 0,1,\\
\|v_h\|_{H^1(a,b)} &\le C_2 h^{-1} \|v_h\|_{L^2(a,b)} && \forall v_h \in S_h(a,b),\\
\|v_h\|_{L^\infty(a,b)} &\le C_2 h^{-1/2} \|v_h\|_{L^2(a,b)} && \forall v_h \in S_h(a,b).
\end{aligned} \tag{3}
$$

The generic constant C may assume different values at different occurances.

3. Revisit of MMOC Method

We present the modified method of characteristics as a time-stepping procedure for Eq. (1). Let $y = r(\theta; \bar{x}, \bar{t})$ denote the characteristics passing through $\bar{x}$ at time $\bar{t}$ which is defined by

$$
\frac{dr}{dt} = V(r,t), \quad r(t;\bar{x},\bar{t})\Big|_{t=\bar{t}} = \bar{x}. \tag{4}
$$

This problem can not be solved analytically for a general velocity field. In this paper, we use an Euler quadrature to define the approximate characteristic curve

$$
r_h(t;\bar{x},\bar{t}) = \bar{x} - V(\bar{x},\bar{t})(\bar{t} - t).
$$

In the MMOC formulation the capacity and convection terms in Eq. (1) are combined to form a material derivative at time step t_n, which is approximated by a backward difference quotient along the approximate characteristic $r_h(t;x,t_n)$ in the time stepping procedure [13]

$$
\begin{aligned}
& u_t(x,t_n) + V(x,t_n) u_x(x,t_n) \\
&= \sqrt{1 + V(x,t_n)^2}\frac{du}{dt}(x,t_n) \\
&= \frac{u(x,t_n) - u(x^*,t_{n-1})}{\Delta t} \\
&\quad + \frac{1}{\Delta t}\int_{t_{n-1}}^{t_n} \sqrt{(r_h(t;x,t_n) - x^*)^2 + (t - t_{n-1})^2}\frac{d^2u}{dt^2}(r_h(t;x,t_n),t)dt.
\end{aligned} \tag{5}
$$

Here $x^* = r_h(t_{n-1};x,t_n)$.

We incorporate Eq. (5) into Eq. (1) and multiply the equation by any test function $w \in H^1_E(a,b)$. We integrate the resulting equation on the interval (a,b), leading to an MMOC reference equation for problem (1): Find $u(x,t_n) \in H^1_E(a,b)$ for $n = 1,\ldots,N$, such that for any $w(x) \in H^1_E(a,b)$

$$
\begin{aligned}
& \int_a^b \frac{u(x,t_n) - u(x^*,t_{n-1})}{\Delta t} w(x)dx + \int_a^b D(x,t_n)u_x(x,t_n)w_x(x)dx \\
&\quad = \int_a^b f(x,t_n)w(x)dx - \frac{1}{\Delta t}E(u,w).
\end{aligned} \tag{6}
$$

Here $E(u,w)$ is the local truncation error of the MMOC reference equation

$$
E(u,w) = \int_a^b w(x)\int_{t_{n-1}}^{t_n} \sqrt{(r_h(t;x,t_n) - x^*)^2 + (t - t_{n-1})^2}\frac{d^2u}{dt^2}dtdx.
$$

The MMOC scheme reads: Find $u_h(x,t_n) \in S_h(a,b)$ for $n = 1, \ldots, N$, such that for any $w_h(x) \in S_h(a,b)$

$$\begin{aligned}(7)\qquad &\int_a^b \frac{u_h(x,t_n) - u_h(x^*,t_{n-1})}{\Delta t} w_h(x)dx + \int_a^b D(x,t_n) u_{h,x}(x,t_n) w_{h,x}(x)dx \\ &= \int_a^b f(x,t_n) w_h(x)dx.\end{aligned}$$

4. Optimal-Order Error Estimates

Let the Courant number $Cr := \max_{(x,t)\in[a,b]\times[0,T]} |V(x,t)|\Delta t/h$ and $\lambda = 1$ if $Cr < 1$ or 0 otherwise. We prove *a priori* optimal-order error estimates in a weighted energy norm for the MMOC scheme for problem (1). The main result is given in the theorem below.

Theorem 4.1. *Assume $D, V \in L^\infty(0,T;W^2_\infty(a,b))$. Then the following optimal-order and superconvergence error estimate holds for the MMOC scheme*

$$\begin{aligned}(8)\qquad &\|u_h - u\|_{L^\infty(0,T;L^2(a,b))} + |u_h - u|_{L^2(0,T;\hat{H}^1_D(a,b))} \\ &\le C\Delta t \Big\| \frac{d^2 u}{dt^2} \Big\|_{L^2(0,T;L^2)} + C\min\{h,\Delta t\}\|u\|_{L^\infty(0,T;H^2)} \\ &+ Ch^2\|u\|_{L^\infty(0,T;H^3)} + C\lambda\big(\Delta t\|u\|_{L^\infty(0,T;H^2)} + h^2\|u\|_{H^1(0,T;H^2)}\big).\end{aligned}$$

Here the constant C is independent of the mesh parameter h or Δt, the true solution u, or the degeneracy of the diffusion coefficient D.

Proof. We let $e = u_h - u$ and choose the test function $w(x)$ in (6) to be $w_h(x) \in S_h(a,b)$. We then subtract Eq. (7) from the MMOC reference equation (6) to obtain an error equation for any $w_h(x) \in S_h(a,b)$

$$\begin{aligned}(9)\qquad &\int_a^b e(x,t_n) w_h(x)dx + \Delta t \int_a^b D(x,t_n) e_x(x,t_n) w_{hx}(x)dx \\ &= \int_a^b e(x^*,t_{n-1}) w_h(x)dx + E(u,w_h).\end{aligned}$$

Let $\theta_h = u_h - \Pi_h u \in S_h(a,b)$, and $\rho = \Pi_h u - u$. The error estimates for ρ are given in (3), so we need only to estimate θ_h. We choose $w_h(x) = \theta_h(x,t_n)$ in Eq. (9) and rewrite the error equation in terms of θ_h and ρ as follows:

$$\begin{aligned}(10)\qquad &\int_a^b \theta_h^2(x,t_n)dx + \Delta t \int_a^b D(x,t_n)\theta_{hx}^2(x,t_n)dx \\ &= \int_a^b \theta_h(x^*,t_{n-1})\theta_h(x,t_n)dx - \Delta t \int_a^b D(x,t_n)\rho_x(x,t_n)\theta_{hx}(x,t_n)dx \\ &+ \int_a^b \rho(x^*,t_{n-1})\theta_h(x,t_n)dx - \int_a^b \rho(x,t_n)\theta_h(x,t_n)dx + E(u,\theta_h).\end{aligned}$$

The left side of the error equation is already in the desired form. So we need only to bound the right side of the error equation term by term. We bound the first term on the right-hand side of Eq. (10) by

$$
\begin{aligned}
&\Big|\int_a^b \theta_h(x^*, t_{n-1})\theta_h(x, t_n)dx\Big| \\
&\leq \frac{1}{2}\int_a^b \theta_h^2(x, t_n)dx + \frac{1}{2}\int_a^b \theta_h^2(x^*, t_{n-1})dx \\
&\leq \frac{1}{2}\int_a^b \theta_h^2(x, t_n)dx + \frac{1 + C\Delta t}{2}\int_{a_h^*}^{b_h^*} \theta_h^2(x^*, t_{n-1})\Big|\frac{dx^*}{dx}\Big|^{-1}dx^* \\
&\leq \frac{1}{2}\int_a^b \theta_h^2(x, t_n)dx + \frac{1 + C\Delta t}{2}\int_{a_h^*}^{b_h^*} \theta_h^2(y, t_{n-1})dy \\
&\leq \frac{1}{2}\|\theta_h(\cdot, t_n)\|_{L^2}^2 + \frac{1 + C\Delta t}{2}\|\theta_h(\cdot, t_{n-1})\|_{L^2}^2.
\end{aligned} \tag{11}
$$

Here the constant C depends on $\|V\|_{L^\infty(0,T;W^1_\infty)}$. We have used the substitution of variables from x to x^*, and changed the limits a and b of the integral to a^* and b^*. We also utilize the periodicity of V to conclude that

$$
\begin{aligned}
r_{hx}(t_{n-1}; x, t_n)^{-1} &= (1 - V_x(x, t_n)\Delta t)^{-1} = 1 + O(\Delta t), \\
b^* - a^* &= (b - a) - (V(b, t_n) - V(a, t_n))\Delta t = b - a.
\end{aligned} \tag{12}
$$

The estimates of the second through the fourth terms on the right side of Eq. (10) presents the major difficulties. For clarity of exposition, the proofs are presented in Lemmas 5.1 and 5.2; there we obtain

$$
\begin{aligned}
&\Big|\int_a^b \rho(x^*, t_{n-1})\theta_h(x, t_n)dx - \int_a^b \rho(x, t_n)\theta_h(x, t_n)dx\Big| \\
&\quad \leq C\Delta t\|\theta_h(\cdot, t_n)\|_{L^2}^2 + C\Delta t \min\{h^2, (\Delta t)^2\}\|u\|_{L^\infty(0,T;H^2)}^2 \\
&\quad\quad + C\lambda\big((\Delta t)^3\|u\|_{L^\infty(0,T;H^2)}^2 + h^4\|u\|_{H^1(t_{n-1},t_n;H^2)}^2 \\
&\quad\quad + \Delta t h^4\|u\|_{L^\infty(0,T;H^3)}^2\big),
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
&\Big|\Delta t\int_a^b D(x, t_n)\rho_x(x, t_n)\theta_{hx}(x, t_n)dx\Big| \\
&\quad \leq C\Delta t\|\theta_h(\cdot, t_n)\|_{L^2}^2 + \frac{\Delta t}{2}\int_a^b D(x, t_n)\theta_{hx}^2(x, t_n)dx \\
&\quad\quad + C\Delta t \min\{h^2, (\Delta t)^2\}\|u\|_{L^\infty(0,T;H^2)}^2 + C\lambda\Delta t h^4\|u\|_{L^\infty(0,T;H^3)}^2.
\end{aligned} \tag{14}
$$

We use the expression of $E(u, \theta_h)$ (below (6)) to bound this term by

$$
\begin{aligned}
E(u, \theta_h) &\leq C(\Delta t)^{3/2}\|\theta_h(\cdot, t_n)\|_{L^2}\Big\|\frac{d^2u}{dt^2}\Big\|_{L^2(t_{n-1},t_n;L^2)} \\
&\leq C\Delta t\|\theta_h(\cdot, t_n)\|_{L^2}^2 + C(\Delta t)^2\Big\|\frac{d^2u}{dt^2}\Big\|_{L^2(t_{n-1},t_n;L^2)}^2.
\end{aligned}
$$

We combine these estimates and the estimates (11)–(14) to get

$$\begin{aligned}
&\|\theta_h(\cdot,t_n)\|_{L^2}^2 + \Delta t\int_a^b D(x,t_n)\theta_{hx}^2(x,t_n)dx \\
&\leq \frac{1+C\Delta t}{2}(\|\theta_h(\cdot,t_n)\|_{L^2}^2 + \|\theta_h(\cdot,t_{n-1})\|_{L^2}^2) \\
&\quad + \frac{\Delta t}{2}\int_a^b D(x,t_n)\theta_{hx}^2(x,t_n)dx + C\Delta t\min\{h^2,(\Delta t)^2\}\|u\|_{L^\infty(0,T;H^2)}^2 \\
&\quad + C(\Delta t)^2\Big\|\frac{d^2u}{dt^2}\Big\|_{L^2(t_{n-1},t_n;L^2)}^2 + C\lambda\big((\Delta t)^3\|u\|_{L^\infty(0,T;H^2)}^2 \\
&\quad + h^4\|u\|_{H^1(t_{n-1},t_n;H^2)}^2 + \Delta t h^4\|u\|_{L^\infty(0,T;H^3)}^2\big).
\end{aligned}$$

We sum the estimate for $n = 1,\ldots,N_1(\leq N)$ and cancel like terms to obtain

$$\begin{aligned}
&\|\theta_h(\cdot,t_{N_1})\|_{L^2}^2 + \Delta t\sum_{n=1}^{N_1}\int_a^b D(x,t_n)\theta_{hx}^2(x,t_n)dx \\
&\leq C\Delta t\sum_{n=1}^{N_1-1}\|\theta_h(\cdot,t_n)\|_{L^2}^2 + C\min\{h^2,(\Delta t)^2\}\|u\|_{L^\infty(0,T;H^2)}^2 \\
&\quad + C(\Delta t)^2\Big\|\frac{d^2u}{dt^2}\Big\|_{L^2(0,t_{N_1};L^2)}^2 + C\lambda\big((\Delta t)^2\|u\|_{L^\infty(0,T;H^2)}^2 \\
&\quad + h^4\|u\|_{H^1(0,t_{N_1};H^2)}^2 + h^4\|u\|_{L^\infty(0,T;H^3)}^2\big).
\end{aligned}$$

We then apply Gronwall inequality to conclude

$$\begin{aligned}
&\|\theta_h\|_{L^\infty(0,T;L^2(a,b))} + |\theta_h|_{L^2(0,T;H_D^1(a,b))} \\
&\leq C\min\{h,\Delta t\}\|u\|_{L^\infty(0,T;H^2)} + C\Delta t\Big\|\frac{d^2u}{dt^2}\Big\|_{L^2(0,T;L^2)} \\
&\quad + C\lambda\big(\Delta t\|u\|_{L^\infty(0,T;H^2)} + h^2\|u\|_{H^1(0,T;H^2)} + h^2\|u\|_{L^\infty(0,T;H^3)}\big).
\end{aligned}$$

We combine this estimate with the known estimate for ρ to finish the proof. □

5. Auxiliary Lemmas

We prove auxiliary lemmas that were used in section 4.

5.1. A Superconvergence Estimate on Interpolation.

We prove the following superconvergence estimate on the interpolation error.

Lemma 5.1. *Assume $u \in L^\infty(0,T;H^3(a,b)) \cap H^1(0,T;H^2(a,b))$. Then the following superconvergence estimate holds*

$$\begin{aligned}
&\Big|\int_a^b \rho(x^*,t_{n-1})\theta_h(x,t_n)dx - \int_a^b \rho(x,t_n)\theta_h(x,t_n)dx\Big| \\
&\leq C\Delta t\|\theta_h(\cdot,t_n)\|_{L^2}^2 + C\Delta t\min\{h^2,(\Delta t)^2\}\|u\|_{L^\infty(0,T;H^2)}^2 \\
&\quad + C\lambda\big((\Delta t)^3\|u\|_{L^\infty(0,T;H^2)}^2 + h^4\|u\|_{H^1(t_{n-1},t_n;H^2)}^2 \\
&\quad + \Delta t h^4\|u\|_{L^\infty(0,T;H^3)}^2\big).
\end{aligned}$$

Proof. When $Cr \geq 1$ that implies $h \leq C\Delta t$, we use the interpolation property (3) to get the estimate

$$\begin{aligned}
&\Big|\int_a^b \rho(x,t_n)\theta_h(x,t_n)dx - \int_a^b \rho(x^*,t_{n-1})\theta_h(x,t_n)dx\Big| \\
&\leq C\|\theta_h(\cdot,t_n)\|_{L^2}(\|\rho(\cdot,t_n)\|_{L^2} + \|\rho(\cdot,t_{n-1})\|_{L^2}) \\
&\leq Ch^2\|\theta_h(\cdot,t_n)\|_{L^2}\ \|u\|_{L^\infty(0,T;H^2)} \\
&\leq C\Delta t\|\theta_h(\cdot,t_n)\|_{L^2}^2 + C\Delta t\min\{h^2,(\Delta t)^2\}\|u\|_{L^\infty(0,T;H^2)}^2.
\end{aligned} \tag{15}$$

Next we focus on the case $Cr < 1$ and decompose the left side of (13) as follows:

$$\begin{aligned}
&\int_a^b (\rho(x,t_n) - \rho(x^*,t_{n-1}))\theta_h(x,t_n)dx \\
&\quad = \int_a^b \int_{t_{n-1}}^{t_n} \rho_t(x,t)dt\theta_h(x,t_n)dx \\
&\qquad + \int_a^b (\rho(x,t_{n-1}) - \rho(x^*,t_{n-1}))\theta_h(x,t_n)dx.
\end{aligned} \tag{16}$$

The first term on the right-hand side is bounded by

$$\begin{aligned}
&\Big|\int_a^b \int_{t_{n-1}}^{t_n} \rho_t(x,t)dt \ \theta_h(x,t_n)dx\Big| \\
&\quad \le (\Delta t)^{1/2}\|\theta_h(\cdot,t_n)\|_{L^2} \ \|\rho\|_{H^1(t_{n-1},t_n;L^2)} \\
&\quad \le C\Delta t\|\theta_h(\cdot,t_n)\|_{L^2}^2 + Ch^4\|u\|_{H^1(t_{n-1},t_n;H^2)}^2.
\end{aligned} \tag{17}$$

A standard estimate of the second term on the right side of (16) can be bounded as follows:

$$\begin{aligned}
&\Big|\int_a^b (\rho(x,t_{n-1}) - \rho(x^*,t_{n-1}))\theta_h(x,t_n)dx\Big| \\
&\quad = \Big|\int_a^b \int_{x^*}^{x} \rho_y(y,t_{n-1})dy\theta_h(x,t_n)dx\Big| \\
&\quad \le C\Delta t\|\rho(\cdot,t_{n-1})\|_{W^1_\infty(a,b)}\|\theta_h(\cdot,t_n)\|_{L^2(a,b)} \\
&\quad \le C\Delta t\|\theta_h(\cdot,t_n)\|_{L^2(a,b)}^2 + C\Delta t h^2\|u\|_{L^\infty(0,T;W^2_\infty(a,b))}^2,
\end{aligned} \tag{18}$$

which will lead to a suboptimal estimate of order $O(h+\Delta t)$ as in [10]. On the other hand, an integration by parts switches the derivative from ρ to θ_h and leads to an optimal-order error estimate as in [13]. However, such an optimal-order estimate depends inversely on the lower bound of the diffusion $D(x,t)$ and could blow up as the lower bound of the diffusion tends to zero.

To derive an optimal-order estimate, we adopt a different approach and use the following expressions in the second term on the right side of (16)

$$\begin{aligned}
&\rho(x,t_{n-1}) - \rho(x^*,t_{n-1}) \\
&\quad = \int_0^1 \frac{d}{d\tau}\rho(x^* + \tau(x-x^*),t_{n-1})d\tau \\
&\quad = \int_0^1 \rho_x(x^* + \tau(x-x^*),t_{n-1})(x-x^*)d\tau
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial x}\big(\rho(x^* + \tau(x-x^*),t_{n-1})\big) \\
&\quad = \rho_x(x^* + \tau(x-x^*),t_{n-1})(x^*_x + \tau(1-x^*_x)) \\
&\quad = \rho_x(x^* + \tau(x-x^*),t_{n-1})(1-(1-\tau)V_x(x,t_n)\Delta t)
\end{aligned}$$

and then integrate the resulting term by parts to yield

$$
\begin{aligned}
&\int_a^b (\rho(x,t_{n-1}) - \rho(x^*,t_{n-1}))\theta_h(x,t_n)dx \\
&\quad = \int_a^b \int_0^1 \rho_x(x^* + \tau(x-x^*),t_{n-1})(x-x^*)d\tau\ \theta_h(x,t_n)dx \\
&\quad = \int_a^b \int_0^1 \frac{\partial}{\partial x}\Big(\rho(x^* + \tau(x-x^*),t_{n-1})\Big) \\
&\qquad\qquad (1-(1-\tau)V_x(x,t_n)\Delta t)^{-1}(x-x^*)d\tau\ \theta_h(x,t_n)dx \\
&\quad = \int_a^b \int_0^1 \frac{\partial}{\partial x}\Big(\rho(x^* + \tau(x-x^*),t_{n-1})\Big)(x-x^*)d\tau\ \theta_h(x,t_n)dx \\
&\qquad + \int_a^b \int_0^1 \frac{\partial}{\partial x}\Big(\rho(x^* + \tau(x-x^*),t_{n-1})\Big)O((\Delta t)^2)d\tau\ \theta_h(x,t_n)dx \\
&\quad = -\int_0^1 \int_a^b \rho(x^* + \tau(x-x^*),t_{n-1})(x-x^*)_x\theta_h(x,t_n)dxd\tau \\
&\qquad - \int_0^1 \int_a^b \rho(x^* + \tau(x-x^*),t_{n-1})(x-x^*)\theta_{hx}(x,t_n)dxd\tau \\
&\qquad + \int_a^b \int_0^1 \frac{\partial}{\partial x}\Big(\rho(x^* + \tau(x-x^*),t_{n-1})\Big)O((\Delta t)^2)d\tau\ \theta_h(x,t_n)dx.
\end{aligned} \tag{19}
$$

The boundary term resulting from integration by parts vanishes due to the periodity of problem (1). Let $y = x^* + \tau(x-x^*)$. We bound the first and third terms on the right side by

$$
\begin{aligned}
&\Big|\int_0^1 \int_a^b \rho(x^* + \tau(x-x^*),t_{n-1})(x-x^*)_x\theta_h(x,t_n)dxd\tau \\
&\qquad + \int_a^b \int_0^1 \frac{\partial}{\partial x}\Big(\rho(x^* + \tau(x-x^*),t_{n-1})\Big)O((\Delta t)^2)d\tau\ \theta_h(x,t_n)dx\Big| \\
&\quad \le C\Delta t \int_0^1 \int_a^b \big|V_x(x,t_n)\rho(x^* + \tau(x-x^*),t_{n-1})\theta_h(x,t_n)\big|dxd\tau \\
&\qquad + C(\Delta t)^2 \int_0^1 \int_a^b \big|\rho_x(x^* + \tau(x-x^*),t_{n-1})\theta_h(x,t_n)\big|dxd\tau \\
&\quad \le C\Delta t\|\theta_h(x,t_n)\|_{L^2}\Big[\Big(\int_0^1 \int_a^b \rho^2(x^* + \tau(x-x^*),t_{n-1})dxd\tau\Big)^{1/2} \\
&\qquad + \Delta t\Big(\int_0^1 \int_a^b \rho_x^2(x^* + \tau(x-x^*),t_{n-1})dxd\tau\Big)^{1/2}\Big] \\
&\quad \le C\Delta t\|\theta_h(\cdot,t_n)\|_{L^2}\Big[\Big(\int_0^1 \int_a^b \rho^2(y,t_{n-1})(\tau+(1-\tau)x^*_{h,x})^{-1}dyd\tau\Big)^{1/2} \\
&\qquad\qquad + \Delta t\Big(\int_0^1 \int_a^b \rho_x^2(y,t_{n-1})(\tau+(1-\tau)x^*_{h,x})^{-1}dyd\tau\Big)^{1/2}\Big] \\
&\quad \le C\Delta t\|\theta_h(\cdot,t_n)\|_{L^2}^2 + C\Delta t h^4\|u\|^2_{L^\infty(0,T;H^2)} + C(\Delta t)^3\|u\|^2_{L^\infty(0,T;H^1)}.
\end{aligned} \tag{20}
$$

We decompose the second term on the right-hand side of (19) as

$$
\begin{aligned}
&\int_0^1\int_a^b \rho(x^*+\tau(x-x^*),t_{n-1})(x-x^*)\theta_{hx}(x,t_n)dxd\tau\\
&\quad=\Delta t\int_0^1\int_a^b \rho(x^*+\tau(x-x^*),t_{n-1})V(x,t_n)\theta_{hx}(x,t_n)dxd\tau\\
&\quad=\Delta t\int_a^b V(x,t_n)\theta_{hx}(x,t_n)\Big(\rho(x,t_{n-1})\\
&\qquad\quad+\int_0^1\int_0^1 \frac{d}{d\gamma}\rho(x+\gamma(1-\tau)(x^*-x),t_{n-1})d\gamma d\tau\Big)dx\\
&\quad=\Delta t\int_a^b V(x,t_n)\theta_{hx}(x,t_n)\rho(x,t_{n-1})dx\\
&\qquad+\Delta t\int_0^1\int_0^1\int_a^b V(x,t_n)\theta_{hx}(x,t_n)(1-\tau)(x^*-x)\\
&\qquad\quad\times\rho_x(x+\gamma(1-\tau)(x^*-x),t_{n-1})dxd\gamma d\tau.
\end{aligned}
\tag{21}
$$

We use the inverse inequality in (3) to bound the second term by

$$
\begin{aligned}
&\Big|\Delta t\int_0^1\int_0^1\int_a^b V(x,t_n)\theta_{hx}(x,t_n)(1-\tau)(x^*-x)\\
&\qquad\qquad\times\rho_x(x+\gamma(1-\tau)(x^*-x),t_{n-1})dxd\gamma d\tau\Big|\\
&\quad\le C(\Delta t)^2h\|\theta_{hx}(\cdot,t_n)\|_{L^2}\|u(\cdot,t_{n-1})\|_{H^2}\\
&\quad\le C\Delta t\ \|\theta_h(\cdot,t_n)\|_{L^2}^2+C(\Delta t)^3\ \|u\|_{L^\infty(0,T;H^2)}^2.
\end{aligned}
\tag{22}
$$

A standard estimate of the first term on the right-hand side of (21) yields

$$
\begin{aligned}
&\Big|\Delta t\int_a^b V(x,t_n)\rho(x,t_{n-1})\theta_{hx}(x,t_n)dx\Big|\\
&\quad\le C\Delta t\|\theta_{hx}(\cdot,t_n)\|_{L^2}\|\rho(\cdot,t_{n-1})\|_{L^2}\\
&\quad\le C\Delta t\ h^2\|\theta_{hx}(\cdot,t_n)\|_{L^2}\|u(\cdot,t_{n-1})\|_{H^2}\\
&\quad\le C\Delta t\|\theta_h(\cdot,t_n)\|_{L^2}^2+C\Delta t\ h^2\|u(\cdot,t_{n-1})\|_{H^2}.
\end{aligned}
$$

This will result in a suboptimal-order estimate of order $O(h+\Delta t)$. To derive an optimal-order estimate, we sum this term by parts to obtain

$$
\begin{aligned}
&\Delta t\int_a^b V(x,t_n)\rho(x,t_{n-1})\theta_{hx}(x,t_n)dx\\
&\quad=\Delta t\sum_{i=1}^I\int_{x_{i-1}}^{x_i} V(x,t_n)\rho(x,t_{n-1})\frac{\theta_h(x_i,t_n)-\theta(x_{i-1},t_n)}{h}dx\\
&\quad=-\frac{\Delta t}{h}\sum_{i=1}^I\int_{x_{i-1}}^{x_i}\Delta_hV(x,t_n)\rho(x,t_{n-1})\theta_h(x_i,t_n)dx\\
&\qquad-\frac{\Delta t}{h}\sum_{i=1}^I\int_{x_{i-1}}^{x_i}\Delta_h\rho(x,t_{n-1})V(x+h,t_n)\theta_h(x_i,t_n)dx.
\end{aligned}
\tag{23}
$$

Here $\Delta_hV(x,t_n):=V(x+h,t_n)-V(x,t_n)$ is the forward difference operator.

We bound the first term on the right-hand side of (23) by

$$
\begin{aligned}
&\Big|\frac{\Delta t}{h}\sum_{i=1}^{I}\int_{x_{i-1}}^{x_i}\Delta_h V(x,t_n)\rho(x,t_{n-1})\theta_h(x_i,t_n)dx\Big| \\
&\quad=\Big|\frac{\Delta t}{h}\sum_{i=1}^{I}\int_{x_{i-1}}^{x_i}\int_{x}^{x+h}V_y(y,t_n)dy\ \rho(x,t_{n-1})\theta_h(x_i,t_n)dx\Big| \\
&\quad\le C\Delta t\ \|\theta_h(\cdot,t_n)\|_{L^2}\ \|\rho(\cdot,t_{n-1})\|_{L^2} \\
&\quad\le C\Delta t\ \|\theta_h(\cdot,t_n)\|_{L^2}^2+C\Delta t\ h^4\ \|u\|^2_{L^\infty(0,T;H^2)},
\end{aligned}
\tag{24}
$$

where we have used the equivalence between the discrete and continuous L^2 norms.

However, if we similarly bound the second term on the right side of Eq. (23), we can only obtain a suboptimal-order estimate. To derive an optimal-order estimate, we introduce an auxiliary function $\psi(x,t)$ by

$$
\psi(x,t)=\Delta_h u(x,t)=\int_0^h u_\alpha(\alpha+x,t)d\alpha.
$$

We note that the spatial partition is uniform and $\rho(x+h,t_{n-1})$ is a shift of $\rho(x,t_{n-1})$ by one grid point, so the forward difference operator and the shift operator are commutative. This leads to the following expansion

$$
\begin{aligned}
\Delta_h\rho(x,t_{n-1})\ &=(\Pi_h-\mathrm{I})u(x+h,t_{n-1})-(\Pi_h-\mathrm{I})u(x,t_{n-1}) \\
&=(\Pi_h-\mathrm{I})\Delta_h u(x,t_{n-1})=(\Pi_h-\mathrm{I})\psi(x,t_{n-1}).
\end{aligned}
$$

Inserting this identity into the second term on the right-hand side of (23) gives

$$
\begin{aligned}
&\Big|\frac{\Delta t}{h}\sum_{i=1}^{I}\int_{x_{i-1}}^{x_i}\Delta_h\rho(x,t_{n-1})V(x+h,t_n)\theta_h(x_i,t_n)dx\Big| \\
&\le\frac{C\Delta t}{h}\|\theta_h(\cdot,t_n)\|_{L^2}\Big(\sum_{i=1}^{I}\int_{x_{i-1}}^{x_i}((\Pi-\mathrm{I})\psi(x,t_{n-1}))^2dx\Big)^{1/2} \\
&\le C\Delta t\ h\|\theta_h(\cdot,t_n)\|_{L^2}\ \Big\|\int_0^h u_\alpha(\alpha+x,t)d\alpha\Big\|_{H^2} \\
&\le C\Delta t\ h^2\|\theta_h(\cdot,t_n)\|_{L^2}\|u(\cdot,t_{n-1})\|_{H^3} \\
&\le C\Delta t\|\theta(\cdot,t_n)\|_{L^2}^2+C\Delta t\ h^4\|u\|^2_{L^\infty(0,T;H^3)}.
\end{aligned}
\tag{25}
$$

Combining all these estimates we have proved (13). □

5.2. The Proof of Estimate (14).

Lemma 5.2. *Then following superconvergence estimate holds*

$$
\begin{aligned}
&\Big|\Delta t\int_a^b D(x,t_n)\rho_x(x,t_n)\theta_{hx}(x,t_n)dx\Big| \\
&\quad\le C\Delta t\|\theta_h(\cdot,t_n)\|_{L^2}^2+\frac{\Delta t}{2}\int_a^b D(x,t_n)\theta_{hx}^2(x,t_n)dx \\
&\qquad+C\Delta t\min\{h^2,(\Delta t)^2\}\|u\|^2_{L^\infty(0,T;H^2)}+C\lambda\Delta t h^4\|u\|^2_{L^\infty(0,T;H^3)}.
\end{aligned}
$$

Proof. When $Cr \geq 1$ that implies $h \leq C\Delta t$, we bound the left side of Eq. (14) by

$$
\begin{aligned}
&\Big|\Delta t \int_a^b D(x,t_n)\rho_x(x,t_n)\theta_{hx}(x,t_n)dx\Big| \\
&\leq \frac{1}{2}\Delta t \int_a^b D(x,t_n)\theta_{hx}^2(x,t_n)dx + C\Delta t h^2 \|u\|^2_{L^\infty(0,T;H^2)} \\
&\leq \frac{1}{2}\Delta t \int_a^b D(x,t_n)\theta_{hx}^2(x,t_n)dx + C\Delta t \min\{h^2,(\Delta t)^2\} \|u\|^2_{L^\infty(0,T;H^2)}.
\end{aligned} \tag{26}
$$

For $Cr < 1$, note that $\theta_{hx}(x,t_n)$ is constant on each interval $[x_{i-1}, x_i]$ and that ρ satisfies $\rho(x_{i-1},t_n) = \rho(x_i,t_n) = 0$ for $i = 1,\ldots,I$, we rewrite the left side of Eq. (14) as follows:

$$
\begin{aligned}
&\Delta t \int_a^b D(x,t_n)\rho_x(x,t_n)\theta_{hx}(x,t_n)dx \\
&= \Delta t \sum_{i=1}^I \theta_{hx}(x_{i-\frac12},t_n) \int_{x_{i-1}}^{x_i} (D(x,t_n) - D(x_{i-\frac12},t_n))\rho_x(x,t_n)dx \\
&= \Delta t \sum_{i=1}^I \theta_{hx}(x_{i-\frac12},t_n) \Big[\int_{x_{i-1}}^{x_i} D_x(x_{i-\frac12},t_n)(x - x_{i-\frac12}) \\
&\quad \Big(\rho_x(x_{i-\frac12},t_n) - \int_{x_{i-\frac12}}^x u_{yy}(y,t_n)dy\Big)dx \\
&\quad + \int_{x_{i-1}}^{x_i}\int_{x_{i-\frac12}}^x \int_{x_{i-\frac12}}^y D_{zz}(z,t_n)dzdy\rho_x(x,t_n)dx\Big] \\
&= -\Delta t \sum_{i=1}^I \theta_{hx}(x_{i-\frac12},t_n) \Big[\int_{x_{i-1}}^{x_i} D_x(x_{i-\frac12},t_n)(x - x_{i-\frac12}) \\
&\quad \Big(u_{xx}(x_{i-\frac12},t_n)(x - x_{i-\frac12}) + \int_{x_{i-\frac12}}^x \int_{x_{i-\frac12}}^y u_{zzz}(z,t_n)dzdy\Big)dx \\
&\quad - \int_{x_{i-1}}^{x_i}\int_{x_{i-\frac12}}^x \int_{x_{i-\frac12}}^y D_{zz}(z,t_n)dzdy\rho_x(x,t_n)dx\Big].
\end{aligned} \tag{27}
$$

Here we have used the fact that the integral involving $\rho_x(x_{i-\frac12},t_n)$ vanishes. This expression can further be simplified as follows:

$$
\begin{aligned}
&\Delta t \int_a^b D(x,t_n)\rho_x(x,t_n)\theta_{hx}(x,t_n)dx \\
&= -\frac{\Delta t h^3}{24} \sum_{i=1}^I \theta_{hx}(x_{i-\frac12},t_n) D_x(x_{i-\frac12},t_n) u_{xx}(x_{i-\frac12},t_n) \\
&\quad + \Delta t \sum_{i=1}^I \theta_{hx}(x_{i-\frac12},t_n) \Big[\int_{x_{i-1}}^{x_i}\int_{x_{i-\frac12}}^x \int_{x_{i-\frac12}}^y D_{zz}(z,t_n)dzdy\rho_x(x,t_n)dx \\
&\quad - \int_{x_{i-1}}^{x_i} D_x(x_{i-\frac12},t_n))(x - x_{i-\frac12}) \int_{x_{i-\frac12}}^x \int_{x_{i-\frac12}}^y u_{zzz}(z,t_n)dzdydx\Big].
\end{aligned} \tag{28}
$$

We sum the first term on the right side of the preceding equation by parts and utilize the periodicity of the problem to cancel the corresponding boundary terms.

This leads to the following equation

$$
\begin{aligned}
&\Big|\frac{\Delta t h^3}{24}\sum_{i=1}^{I}\theta_{hx}(x_{i-\frac{1}{2}},t_n)D_x(x_{i-\frac{1}{2}},t_n)u_{xx}(x_{i-\frac{1}{2}},t_n)\Big| \\
&\quad = \Big|\Delta t\sum_{i=1}^{I}\theta_h(x_i,t_n)\frac{1}{24}h^2D_x(x_{i-\frac{1}{2}},t_n)u_{xx}(x_{i-\frac{1}{2}},t_n) \\
&\qquad -\Delta t\sum_{i=1}^{I}\theta_h(x_i,t_n)\frac{1}{24}h^2D_x(x_{i+1/2},t_n)u_{xx}(x_{i+1/2},t_n)\Big| \\
&\quad = \Big|\frac{\Delta t h^2}{24}\sum_{i=1}^{I}\theta_h(x_i,t_n)\int_{x_{i-\frac{1}{2}}}^{x_{i+1/2}}(D_xu_{xx})_x(x,t_n)dx\Big| \\
&\quad \le \Delta t h^{5/2}\sum_{i=1}^{I}|\theta_h(x_i,t_n)|\ \|u\|_{L^\infty(0,T;H^3(x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}))} \\
&\quad \le C\Delta t\|\theta(\cdot,t_n)\|_{L^2}^2 + C\Delta t h^4\|u\|_{L^\infty(0,T;H^3)}.
\end{aligned}
\tag{29}
$$

We bound the remaining terms on the right side of (28) by

$$
\begin{aligned}
&\Big|\Delta t\sum_{i=1}^{I}\theta_{hx}(x_{i-\frac{1}{2}},t_n)\Big[\int_{x_{i-1}}^{x_i}\int_{x_{i-\frac{1}{2}}}^{x}\int_{x_{i-\frac{1}{2}}}^{y}D_{zz}(z,t_n)dzdy\rho_x(x,t_n)dx \\
&\quad -\int_{x_{i-1}}^{x_i}D_x(x_{i-\frac{1}{2}},t_n)(x-x_{i-\frac{1}{2}})\int_{x_{i-\frac{1}{2}}}^{x}\int_{x_{i-\frac{1}{2}}}^{y}u_{zzz}(z,t_n)dzdydx\Big]\Big| \\
&\le C\Delta t h^{\frac{5}{2}}\sum_{i=1}^{I}|\theta_{hx}(x_{i-\frac{1}{2}},t_n)|\ \|\rho_x(\cdot,t_n)\|_{L^\infty(0,T;L^2(x_{i-1},x_i))} \\
&\quad +C\Delta t h^{\frac{7}{2}}\sum_{i=1}^{I}|\theta_{hx}(x_{i-\frac{1}{2}},t_n)|\ \|u\|_{L^\infty(0,T;H^3(x_{i-1},x_i))} \\
&\le C\Delta t h^{5/2}\|\theta_h(\cdot,t_n)\|_{L^\infty}\|u\|_{L^\infty(0,T;H^3)} \\
&\le C\Delta t\|\theta_h(\cdot,t_n)\|_{L^2}^2 + C\Delta t h^4\|u\|_{L^\infty(0,T;H^3)}^2.
\end{aligned}
\tag{30}
$$

We combine the estimates (26) - (30) to finish the proof. □

6. Concluding Remarks

In this paper we prove *a priori* error estimates in a weighted energy norm to the MMOC scheme for time-dependent convection-diffusion equations with degenerate diffusion. The convergence rates are independent of the lower bound of the diffusion or any norms of the true solution. In other words, these estimates hold uniformly with respect to the degenerate diffusion.

Finally, we point out that the estimates in this paper were derived on a uniform space-time partition with no upstream weighting or local grid refinement or any other special arrangements of the grid, so these estimates justify the strength of the MMOC scheme. The analysis fully utilizes the simplicity of the one space dimension and the periodic boundary conditions. However, a multidimensional analogue of problem (1) with general boundary conditions presents much more severe challenges, due to the complication of multiple space dimensions, the solution structures, and the appearance of boundary and interior layers. These issues will be investigated in the near future.

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