INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 6, Number 1, Pages 161–176

NUMERICAL MODELING OF A DUAL VARIATIONAL INEQUALITY OF UNILATERAL CONTACT PROBLEMS USING THE MIXED FINITE ELEMENT METHOD

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Abstract. We study the dual mixed finite element approximation of unilateral contact problems. Based on the dual mixed variational formulation with three unknowns (stress, displacement and the displacement on the contact boundary), the a priori error estimates have been established for both conforming and nonconforming finite element approximations. A Uzawa type iterative algorithm is developed to solve the resulting linear system. Numerical example shows good performance of the method.

Key Words. mixed finite element method, dual variational inequality, Uzawa algorithm and error estimates.

1. Introduction

While contact problems are being solved and many finite element programs offer contact analysis capabilities for production and research, efforts to obtain more effective solutions are still made (cf. [2]). One reason is that many different kinds of contact problems can involve large relative motion, frictional forces, and static or dynamic condition. Another reason is that contact solution procedures only stay in research and easy-to-use finite element schemes for contact problems are still unavailable in applications.

Developing efficient computing tools for the numerical simulation of contact problem with unilateral Signorini boundary conditions is of a permanent growing interest in many physical areas (cf. [2, 3, 4, 15, 16, 18]). The particular feature of the unilateral problems is that the mathematical variational statement leads to variational inequalities set on closed convex functional cones. The modeling of the non-penetration condition in the discrete finite element level is of crucial importance. This condition may be imposed on the displacement and expressed in a weak sense. The way that enforced depends on the well-posedness of the discrete inequalities and the accuracy of the approximation algorithm. This point is addressed in many published papers, especially for Lagrangian finite element discretizations (cf. [18, 16, 17, 3, 19]). In these papers, either the displacement is the only unknown or the displacement and the stress on the contact zone are independent unknowns. The convergence rate of these methods have been established. Much attention has been paid to the numerical simulation of variational inequalities modeling for unilateral contact problems by finite element methods (cf. [3, 9, 11, 16]). Either from the accuracy point of view or from developing efficient algorithms to solve the resulting minimization problem(cf. [20]), the hardest task is the discretization of

Received by the editors May 10, 2007 and, in revised form, May 18, 2008.

2000 *Mathematics Subject Classification.* 35R35, 49J40 , 60G40.

The project was supported by the National Natural Sciences Foundation of China (No.40745033) and by the Natural Science Foundation of Hebei Province (No.A2007001027) .

the Signorini unilateral condition, which usually can not be satisfied *exactly* by the numerical solution. This often leads to nonconforming method (cf. [19]).

When high accuracy of the stress and the displacement on the contact boundary are desirable, a possible way is to successively refine the mesh. An alternative way is to resort to a new mixed variational formulation that includes stress, displacement and the displacement on the contact boundary as the main unknowns. However, the well-posedness of such mixed variational problem depends heavily on the so-called ellipticity condition and the B.-B. inequality, which is the source of trouble in constructing finite element approximation spaces. To overcome this difficulty, we modify the dual variational formulation by adding a new term that enhances the ellipticity and brings more freedom in choosing finite element spaces. Based on this modified variational formulation problem, we introduce two new types of finite element approximation spaces. A priori error estimates have been carried out for both methods and our numerical results confirm the efficiency of the methods.

The remaining part of this paper is as follows. In the next section we will state the functional setting and the unilateral contact problem. In Section 3, we shall derive the dual variational formulation of the unilateral contact problems by Lagrange multiplier method. In Section 4, we introduce the conforming and nonconforming finite element approximations of the dual mixed variational problem and derive the a priori error estimates. In Section 5, a Uzawa type iterative algorithm is presented to solve the approximation problem. Numerical study of two methods is described in Section 6. Finally, we give concluding remarks in Section 7.

2. Functional setting and contact Problem

Notation. Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain with generic point x. The Lebesgue space $L^p(\Omega)$ is endowed with the norm: $\forall \psi \in L^p(\Omega)$,

$$\|\psi\|_{L^p(\psi)} = \left(\int_\Omega |\psi(x)|^p dx \right)^{1/p}.$$

We make use of the standard Sobolev space $H^m(\Omega),\ m \geq 1$, equipped with the norm:

$$\|\psi\|_{H^m(\Omega)} = \left(\sum_{0 \leq |\alpha| \leq m} \|\partial^\alpha \psi(x)\|^2_{L^2(\Omega)} \right)^{1/2},$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index in N and the symbol ∂^α represents a partial derivative. In particular, $L^2(\Omega) = H^0(\Omega)$. The fractional order Sobolev space $H^\nu(\Omega),\ \nu \in \mathbb{R}_+ \setminus N$ is defined as in [1] and equipped with the norm

$$\|\psi\|_{H^\nu(\Omega)} = \left(\|\psi\|^2_{H^m(\Omega)} + \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{(\partial^\alpha \psi(x) - \partial^\alpha \psi(y))^2}{|x-y|^{2+2\theta}} dxdy \right)^{1/2},$$

where $\nu = m + \theta,\ m$ is the integer part of ν and $\theta \in [0,1]$ is the decimal part.

For any portion γ of the boundary $\partial\Omega$ and any $\nu \in R_+ \setminus N$, the Hilbert space $H^\nu(\gamma)$ is associated with the norm

$$\|\psi\|_{H^\nu(\gamma)} = \left(\|\psi\|^2_{H^m(\gamma)} + \int_\gamma \int_\gamma \frac{(\partial^m_\Gamma \psi(x) - \partial^m_\Gamma \psi(y))^2}{|x-y|^{2+2\theta}} d\Gamma d\Gamma \right)^{1/2},$$

where m is the integer part of ν, θ its decimal part. The symbol $\partial^m_\Gamma \psi$ stands for the m-th order derivative of ψ along γ and $d\Gamma$ denotes the linear measure on $\partial\Omega$. The space $H^{-\nu}(\gamma)$ stands for the topological dual space of $H^\nu(\gamma)$ and the duality pairing is denoted $< .,. >_{\nu,\gamma}$. The spacial space $H^{m+\frac{1}{2}}_{00}(\gamma)$ is defined as the set of the restrictions to γ of the function of $H^{m+\frac{1}{2}}(\partial\Omega)$ that vanish on $\partial\Omega \setminus \gamma$, it is also obtained by Hilbertian interpolation between $H_0^{m+1}(\gamma)$ and $H_0^m(\gamma)$. To be complete with the Sobolev functional tools used hereafter, recall that for $\nu > \frac{3}{2}$, the trace operator

$$T : \psi \longmapsto (\psi|_{\partial\Omega}, (\frac{\partial\psi}{\partial n})|_{\partial\Omega}),$$

is continuous from $H^\nu(\Omega)$ onto $H^{\nu-\frac{1}{2}}(\partial\Omega)$ (cf, [1, 10, 14]). Otherwise, if $1 \le \nu \le \frac{3}{2}$, define the space $X^\nu(\Omega)$ to be

$$X^\nu(\Omega) = \{\psi \in H^\nu(\Omega), \quad \Delta\psi \in L^2(\Omega)\},$$

equipped with the graph norm

$$\|\psi\|_{X^\nu(\Omega)} = (\|\psi\|^2_{H^\nu(\Omega)} + \|\Delta\psi\|^2_{L^2(\Omega)})^{\frac{1}{2}}.$$

Then the trace operator T is continuous from $X^\nu(\Omega)$ onto $H^{\nu-\frac{1}{2}}(\partial\Omega) \times H^{\nu-\frac{3}{2}}(\partial\Omega)$.

Contact Problem. We consider a material body to be the closure of a set Ω in $\mathbb{R}^2$ of material particles x. Let $\partial\Omega = \Gamma$, the deformation of the body unilaterally supported by a frictionless rigid foundation and subjected to body force $\vec{f}$ and surface traction $\vec{t}$ applied to a portion Γ_F of the body's surface. The body is fixed along a portion Γ_D of its boundary and we denote by Γ_C a portion of the body which is a candidate contact surface. The actual surface on which the body comes in contact with the foundation is not know in advance but is contained in the portion Γ_C of Γ. We confine our attention to infinitesimal deformations of the body. In addition, we further assume that the measures of Γ_D and Γ_C are positive. To avoid technicalities arising from the special Sobolev space $H_{00}^{1/2}(\Gamma_C)$, we assume that Γ_D and Γ_C do not touch. Under the linear elasticity frame, let $\vec{u} = (u_1, u_2)$ denote displacement, and have the following basic relation:

$$\varepsilon(\vec{v}) = (\varepsilon_{ij})_{1\le i,j\le 2}, \quad \varepsilon_{ij} = \frac{1}{2}(\partial_j v_i + \partial_i v_j), \quad 1 \le i,j \le 2, \tag{2.1}$$

$$\sigma(\vec{v}) = (\sigma_{ij})_{1\le i,j\le 2}, \quad \sigma_{ij} = \sigma E_{ijkl}\varepsilon_{kl}(\vec{v}), \quad 1 \le i,j \le 2, \tag{2.2}$$

where $\varepsilon(\vec{v})$, $\sigma(\vec{v})$ denote strain tensor and stress tensor, respectively, and $E = \{E_{ijkl}\}, 1 \le i,j,k,l \le 2$, is the Hook tensor of elastic material. The frictionless unilateral contact problem in elasticity is described as follows (cf. [18])

$$\begin{cases} -\operatorname{div}\sigma(\vec{u}) = \vec{f} \quad \textit{in } \Omega, \\ \sigma(\vec{u}) = E\varepsilon(\vec{u}) \quad \textit{in } \Omega, \\ \vec{u} = 0 \quad \textit{on } \Gamma_D, \\ \sigma(\vec{u})\vec{n} = \vec{t} \qquad \textit{on } \Gamma_F, \\ \sigma_T(\vec{u}) = 0 \qquad \textit{on } \Gamma_C, \\ \sigma_n(\vec{u})(u_n - g) = 0, \quad u_n - g \le 0, \quad \sigma_n(\vec{u}) \le 0 \quad \textit{on } \Gamma_C, \end{cases} \tag{2.3}$$

where $u_n = \vec{u}\cdot\vec{n} = u_i n_i$, $\sigma_n(\vec{u}) = \sigma_{ij}(\vec{u})n_i n_j$, $\sigma_T(\vec{u}) = \sigma(\vec{u})\vec{n} - \sigma_n(\vec{u})\vec{n}$, $\vec{n}$ denotes the unit outward normal to Γ.

3. Dual Variational formulation of contact problem

Ref [22] gives the following dual variational formulation of (2.3)

Problem 3.1. *Find $\sigma \in K_*, \vec{u} \in (L^2(\Omega))^2$, such that*

$$\begin{cases} (C\sigma, \tau - \sigma) + (\operatorname{div}(\tau - \sigma), \vec{u}) \ge \langle \tau_n - \sigma_n, g\rangle_{\Gamma_C}, \qquad \forall \tau \in K_*, \\ (\operatorname{div}\sigma, \vec{v}) = -(\vec{f}, \vec{v}), \qquad \forall \vec{v} \in (L^2(\Omega))^2, \end{cases} \tag{3.1}$$

where

$$\begin{cases} K_* = \{\tau \in Q_t | \tau_n \leq 0, \ on \ \Gamma_C\}, \\ Q_t = \{\tau \in H(\mathrm{div}, \Omega) | \tau_{ij} = \tau_{ji}, \ in \ \Omega \ ; \tau \cdot \vec{n} = \vec{t}, \ on \ \Gamma_F; \ \vec{\tau}_T = 0, \ on, \Gamma_C\}. \end{cases}$$

$C = \{c_{ijkl}\}$ is the inverse of Hookean tensor $E = \{E_{ijkl}\}$, and satisfies the following conditions (cf. [18]):

$$(3.2) \qquad \begin{cases} C_{ijkl} \in L^\infty(\Omega), \quad Max\|C_{ijkl}\|_{0,\infty} \leq M', \\ C_{ijkl} = C_{jikl} = C_{klij}, \\ C_{ijkl}(x)\tau_{ij}\tau_{kl} \geq m'\tau_{ij}\tau_{ij}, \quad \forall x \in \Omega, \end{cases}$$

where M, M', m and m' are positive constants.

Using classical complementary energy principle, Problem 3.1 is equivalent to the following minimal problem (cf. [6]):

$$(3.3) \qquad \inf_{\tau \in K^*} \left\{ \frac{1}{2}(C\tau, \tau) - \langle \tau_n, g \rangle_{\Gamma_C} \right\},$$

where

$$\begin{cases} \langle \tau_n, g \rangle_{\Gamma_C} = \int_{\Gamma_C} \tau_n g ds, \\ K^* = \{\tau \in V^* | \tau_n \leq 0, \ on \ \Gamma_C\}, \\ V^* = \{\tau \in H(\mathrm{div}, \Omega) | \ \mathrm{div}\, \tau + \vec{f} = 0, \tau_{ij} = \tau_{ji}, \ in \ \Omega, \ \tau \cdot \vec{n} = 0, \\ \qquad on \ \Gamma_F, \ \vec{\tau}_T = 0, \ on \ \Gamma_C\}, \\ H(\mathrm{div}; \Omega) = \{\tau \in (L^2(\Omega))^2, \mathrm{div}\, \tau \in L^2(\Omega)\}. \end{cases}$$

In order to relax the constrains $\mathrm{div}\, \tau + \vec{f} = 0$ and $\tau_n|_{\Gamma_C} \leq 0$ from K^* simultaneously, we introduce Lagrange multiplier $\tilde{v}$ and μ,

$$(3.4) \qquad \begin{aligned} &\inf_{\tau \in K^*} \left\{ \frac{1}{2}(C\tau, \tau) - \langle \tau_n, g \rangle_{\Gamma_C} \right\} \\ &= \inf_{\tau \in Q_t} \sup_{\tilde{v} \in L^2(\Omega), \mu \in \Lambda} \{ \frac{1}{2}(C\tau, \tau) + (\mathrm{div}\, \tau + \vec{f}, \vec{v}) + \langle \tau_n, \mu - g \rangle_{\Gamma_C} \}, \end{aligned}$$

where

$$\begin{cases} \Lambda = \{\mu \in H_{00}^{1/2}(\Gamma_C) | \ \mu \geq 0, on \ \Gamma_C\}, \\ H_{00}^{1/2}(\Gamma_C) = \{\mu \in H^{1/2}(\Gamma_C) | \xi^{-1/2}\mu \in L^2(\Gamma_C)\}. \end{cases}$$

The norm of $H_{00}^{1/2}(\Gamma_C)$ is defined as follows:

$$_{00}\|\mu\|_{1/2,\Gamma_C} = \{\|\mu\|^2_{1/2,\Gamma_C} + \|\xi^{-1/2}\mu\|^2_{0,\Gamma_C}\}^{1/2},$$

here ξ is the distance from any point on Γ_C to two ends of Γ_C. In fact, $_{00}\|\mu\|_{1/2,\Gamma_C}$ is equivalent to $\|\mu\|_{1/2,\Gamma_C}$ (cf.[18]).

Assume that

$$(3.5) \qquad L(\tau; \tilde{v}, \mu) = \frac{1}{2}(C\tau, \tau) + (\mathrm{div}\, \tau, \tilde{v}) + \langle \tau_n, \mu - g \rangle_{\Gamma_C} + (\vec{f}, \vec{v}).$$

The saddle point $(\sigma; \tilde{u}, \lambda)$ of $L(\tau; \tilde{v}, \mu)$ on $Q_t \times ((L^2(\Omega))^2 \times \Lambda)$ satisfies the following variational formulation (cf [18]):

Find $\sigma \in Q_t, \tilde{u} \in (L^2(\Omega))^2, \lambda \in \Lambda$, such that, for $\forall \tau \in Q_0, \tilde{v} \in (L^2(\Omega))^2, \mu \in \Lambda$

$$(3.6) \qquad \begin{cases} (C\sigma, \tau) + (\mathrm{div}\, \tau, \tilde{u}) + \langle \tau_n, \lambda \rangle_{\Gamma_C} = \langle \tau_n, g \rangle_{\Gamma_C}, \\ (\mathrm{div}\, \sigma, \tilde{v} - \tilde{u}) + \langle \sigma_n, \mu - \lambda \rangle_{\Gamma_C} + (\vec{f}, \tilde{v} - \tilde{u}) \leq 0, \end{cases}$$

where $Q_0 = \{\tau \in H(\mathrm{div}, \Omega) | \tau_{ij} = \tau_{ji}, in \ \Omega, \tau \cdot \vec{n} = 0, on \ \Gamma_F, \vec{\tau}_T = 0, on \ \Gamma_C\}$,

The existence and uniqueness of the solution of (3.6) depends on the ellipticity condition and the B-B condition [18]. However, it is not easy to find finite element spaces satisfying these two conditions simultaneously. We instead enhance the ellipticity of $(C\sigma, \tau)$ by adding an extra term $(\operatorname{div}\sigma, \operatorname{div}\tau)$. Therefore, we define

$$a(\sigma, \tau) = (C\sigma, \tau) + (\operatorname{div}\sigma, \operatorname{div}\tau).$$

It is clear that for any $\tau \in H(\operatorname{div})$,

$$a(\tau, \tau) \geq c\|\tau\|^2_{H(\operatorname{div},\Omega)}.$$

This shows that the modified bilinear form $a(\cdot, \cdot)$ satisfies the coercive condition over the whole space $H(\operatorname{div})$ instead of the kernel space, which shall bring more freedom for constructing the finite element spaces as we will show later on. Similar idea has been exploited by Brezzi et al in [7] to obtain continuous stress approximation in mixed finite element set-up. We reshape (3.6) into another dual variational formulation of unilateral contact problem:

Problem 3.2. *Find $\sigma \in Q_t, \tilde{u} \in (L^2(\Omega))^2, \lambda \in \Lambda$ such that*

$$\begin{cases} a(\sigma, \tau) + b(\tau; \tilde{u}, \lambda) = \langle \tau_n, g\rangle_{\Gamma_C} - (\vec{f}, \operatorname{div}\tau), & \forall \tau \in Q_0, \\ b(\sigma; \tilde{v} - \tilde{u}, \mu - \lambda) \leq -(\vec{f}, \tilde{v} - \tilde{u}), & \forall \tilde{v} \in (L^2(\Omega))^2,\ \mu \in \Lambda, \end{cases} \tag{3.7}$$

where

$$b(\tau; \tilde{v}, \mu) = (\operatorname{div}\tau, \tilde{v}) + \langle \tau_n, \mu\rangle_{\Gamma_C}.$$

It is easy to obtain

$$\begin{cases} a(\tau, \tau) \geq C_0\|\tau\|^2_{H(\operatorname{div},\Omega)}, \\ a(\sigma, \tau) \leq M_0\|\sigma\|_{H(\operatorname{div},\Omega)}\|\tau\|_{H(\operatorname{div},\Omega)}. \end{cases} \tag{3.8}$$

It follows from [21] the following B.-B. condition and the existence result.

Lemma 3.1. *If $meas\Gamma_D > 0$, then there exists a constant $\beta > 0$, independent of $\tilde{v}$, μ, such that*

$$\sup_{\tau \in Q_0} \frac{b(\tau; \tilde{v}, \mu)}{\|\tau\|_{H(\operatorname{div},\Omega)}} \geq \beta(\|\tilde{v}\|_{0,\Omega} + \|\mu\|_{1/2,\Gamma_C}), \quad \forall \tilde{v} \in (L^2(\Omega))^2,\ \mu \in \Lambda. \tag{3.9}$$

Theorem 3.1. *If the conditions in Lemma 3.1 and condition (3.8) are satisfied, then Problem 3.2 has one and only one solution.*

Remark 3.1: The relation of solutions between Problem 3.1 and Problem 3.2 can be derived as follows. Substituting

$$(C\sigma, \pi) = \langle u_n, \pi_n\rangle_{\Gamma_C} + \langle \tau, \vec{u}\rangle_{\Gamma_C} - (\operatorname{div}\pi, \vec{u})$$

into the first equation of Problem 3.2, we obtain

$$\begin{cases} \langle \pi_n, u_n + \lambda - g\rangle_{\Gamma_C} = 0,\ \forall \pi \in Q_0, \\ \langle \tau, \vec{u}\rangle_{\Gamma_D} + (\operatorname{div}\tau, \tilde{u} - \vec{u}) = 0,\ \forall \tau \in Q_0. \end{cases}$$

This immediately implies $\lambda = g - u_n$ on Γ_C, namely, the meaning of Lagrange multiplier μ. When $\vec{u}|_{\Gamma_D} = 0$ and $meas\Gamma_D > 0$, $\tilde{u} = \vec{u}$, i.e. $\tilde{u}$ denotes elastic displacement $\vec{u}$. The advantage of Problem 3.2 is that the three unknowns can be solved simultaneously.

4. The Mixed Finite Element of Dual Variational Formulation

In this section we present a mixed finite element approximation of Problem 3.2 by conforming and nonconforming finite element methods, respectively. For conforming FEM, we use standard RT_1 element to approximate the stress, the piecewise linear element to approximate the displacement and the piecewise continuous linear element to approximate the boundary displacement. As to the nonconforming FEM, we use the piecewise constant element to approximate both the boundary displacement and the displacement, the standard RT_0 to approximate the stress. We derive the error estimate in what follows.

4.1 The Abstract Framework

Let Ω be a convex polygon. The triangulation $\mathcal{T}_h$ of Ω consists of triangular elements denoted e so that

$$\bar{\Omega} = \bigcup_{e\in\mathcal{T}_h} \bar{e}.$$

The discretization parameter h on Ω is given by

$$h = \max_{e\in\mathcal{T}_h} h_e,$$

where h_e denotes the diameter of the triangle e.

Let $Q_h, (L_h^2)^2, M_h$ be the finite element approximating space of $H(\mathrm{div},\Omega), (L^2(\Omega))^2$ and $H^{1/2}(\Gamma_C)$, respectively. Let $Q_0^h = Q_0 \cap Q_h,\ Q_t^h = Q_t \cap Q_h$ and $\Lambda_h = \{\mu \in M_h | \mu \geq 0\}$. The finite element approximation of Problem 3.2 is as follows:

Problem 4.1. *Find $\sigma_h \in Q_t^h, \tilde{u}_h \in (L_h^2)^2, \lambda_h \in \Lambda_h$, such that*

$$\left\{ \begin{array}{l} a(\sigma_h, \tau_h) + b(\tau_h; \tilde{u}_h, \lambda_h) = \langle \tau_{hn}, g\rangle_{\Gamma_C} - (\vec{f}, \mathrm{div}\, \tau_h), \quad \forall \tau_h \in Q_0^h, \\ b(\sigma_h; \tilde{v}_h - \tilde{u}_h, \mu_h - \lambda_h) \leq -(\vec{f}, \tilde{v}_h - \tilde{u}_h), \quad \forall \tilde{v}_h \in (L_h^2)^2,\ \mu_h \in \Lambda_h. \end{array} \right. \tag{4.1}$$

For convenience, we assume that

$$\left\{ \begin{array}{l} q = (\tilde{v}, \mu),\ p = (\tilde{u}, \lambda),\ \mathcal{N} = H(\mathrm{div},\Omega) \times \mathcal{R},\ \mathcal{R} = (L^2(\Omega))^2 \times H^{1/2}(\Gamma_C), \\ U = (\sigma, p),\ V = (\tau, q),\ (\tilde{f}, q - p) = -(\vec{f}, \tilde{v} - \tilde{u}), \end{array} \right.$$

and definite that the following norms

$$\left\{ \begin{array}{l} \|q\|_{\mathcal{R}} = \{\|\tilde{v}\|_{0,\Omega}^2 + \|\mu\|_{1/2,\Gamma_C}^2\}^{1/2}, \\ \|V\|_{\mathcal{N}} = \{\|\tau\|_{H(\mathrm{div},\Omega)}^2 + \|q\|_{\mathcal{R}}^2\}^{1/2}. \end{array} \right.$$

We write Problem 3.2 as follows:

Find $\sigma \in Q_t,\ p \in (L^2(\Omega))^2 \times \Lambda$ such that

$$\left\{ \begin{array}{l} a(\sigma, \tau) + b(\tau; p) = \langle \tau_n, g\rangle_{\Gamma_C} - (\vec{f}, \mathrm{div}\, \tau), \quad \forall \tau \in Q_0, \\ b(\sigma; q - p) \leq (\tilde{f}, q - p), \quad \forall q \in (L^2(\Omega))^2 \times \Lambda. \end{array} \right. \tag{4.2}$$

Similarly, the finite element approximating Problem 4.1 is written as: Find $\sigma_h \in Q_t^h, p_h \in (L_h^2)^2 \times \Lambda_h$, such that

$$\left\{ \begin{array}{l} a(\sigma_h, \tau_h) + b(\tau_h; p_h) = \langle \tau_{hn}, g\rangle_{\Gamma_C} - (\vec{f}, \mathrm{div}\, \tau_h), \quad \forall \tau_h \in Q_0^h, \\ b(\sigma_h; q_h - p_h) \leq (\tilde{f}, q_h - p_h), \quad \forall q_h \in (L_h^2)^2 \times \Lambda_h. \end{array} \right. \tag{4.3}$$

Furthermore, we define the following bilinear function $F: \quad \mathcal{N} \times \mathcal{N} \to R^1$ as

$$F(U, V) = a(\sigma, \tau) + b(\tau; p) - b(\sigma; q), \quad \forall U, V \in \mathcal{N}, \tag{4.4}$$

and linear function $\mathcal{L}: \quad \mathcal{N} \to R^1$ as

$$\langle \mathcal{L}, V\rangle = \langle \tau_n, g\rangle_{\Gamma_C} - (\vec{f}, \mathrm{div}\, \tau) - (\tilde{f}, q), \quad \forall V \in \mathcal{N}. \tag{4.5}$$

From the definition of F, it is easy to get

$$F(V, V) = a(\tau, \tau), \quad \forall V \in \mathcal{N}, \tag{4.6}$$

$$|F(U, V)| \leq c\|U\|_{\mathcal{N}}\|V\|_{\mathcal{N}}. \tag{4.7}$$

Lemma 4.1. *Mixed variational problem 3.2 is equivalent to the following problem: Find $U = (\sigma, p) \in \mathcal{Q} = Q_t \times \mathcal{M}$, such that*

$$F(U, V - U) \geq \langle \mathcal{L}, V - U\rangle, \quad \forall V \in Q_0 \times \mathcal{M}, \tag{4.8}$$

where $\mathcal{M} = (L^2(\Omega))^2 \times \Lambda$.

PROOF.

(i) Let (σ, p) be the solution of (4.2), then for $\forall V \in Q_o \times \mathcal{M}$, we get

$$\begin{aligned} F(U, V-U) &= a(\sigma, \tau-\sigma) + b(\tau-\sigma; p) - b(\sigma; q-p) \\ &\geq \langle \tau_n - \sigma_n, g\rangle_{\Gamma_C} - (\vec{f}, \operatorname{div}(\tau-\sigma)) - (\tilde{f}, q-p) \\ &= \langle \mathcal{L}, V-U\rangle. \end{aligned}$$

(ii) Contrarily, if $U = (\sigma, p)$ is the solution of (4.8), then

$$\begin{aligned} F(U, V-U) &= a(\sigma, \tau-\sigma) + b(\tau-\sigma; p) - b(\sigma; q-p) \\ &\geq \langle \mathcal{L}, V-U\rangle = \langle \tau_n - \sigma_n, g\rangle_{\Gamma_C} - (\vec{f}, \operatorname{div}(\tau-\sigma)) \\ &\quad -(\tilde{f}, q-p), \qquad \forall \tau \in Q_0, q \in (L^2(\Omega))^2 \times \Lambda. \end{aligned}$$

Taking $\tau = \sigma$ in (4.9), we obtain

$$b(\sigma; q-p) \leq (\tilde{f}, q-p), \qquad \forall q \in (L^2(\Omega))^2 \times \Lambda. \tag{4.9}$$

If taking $q = p$ in (4.9), we have

$$\begin{aligned} &a(\sigma, \tau-\sigma) + b(\tau-\sigma; p) \\ &\geq \langle \tau_n - \sigma_n, g\rangle_{\Gamma_C} - (\vec{f}, \operatorname{div}(\tau-\sigma)), \qquad \forall \tau \in Q_0. \end{aligned}$$

Furthermore, taking $\tau = 0$ and $\tau = 2\sigma$, respectively in (4.9), we can deduce

$$a(\sigma, \sigma) + b(\sigma; p) = \langle \sigma_n, g\rangle_{\Gamma_C} - (\vec{f}, \operatorname{div}\sigma), \tag{4.10}$$

which implies

$$a(\sigma, \tau) + b(\tau; p) \geq \langle \tau_n, g\rangle_{\Gamma_C} - (\vec{f}, \operatorname{div}\tau), \qquad \forall \tau \in Q_0. \tag{4.11}$$

By $\pm\tau \in Q_0$, we obtain

$$a(\sigma, \tau) + b(\tau; p) = \langle \tau_n, g\rangle_{\Gamma_C} - (\vec{f}, \operatorname{div}\tau), \qquad \forall \tau \in Q_0. \tag{4.12}$$

Combining (i) and (ii), we finish the proof. □

Similarly, we can prove that Problem 4.1 is equivalent to the following problem

Find $U_h = (\sigma_h, p_h) \in \mathcal{Q}_h = Q_t^h \times \mathcal{M}_h$, such that

$$F(U_h, V_h - U_h) \geq \langle \mathcal{L}, V_h - U_h\rangle, \qquad \forall V_h \in Q_0^h \times \mathcal{M}_h, \tag{4.13}$$

where $\mathcal{M}_h = (L_h^2)^2 \times \Lambda_h$, in general, $\mathcal{Q}_h \not\subset \mathcal{Q}$.

Lemma 4.2. *Let (σ, p) and (σ_h, p_h) be the solution of (4.2) and (4.3) respectively, then (cf.* [21]*), for all $q_h \in \mathcal{M}_h$, $q \in \mathcal{M}, \tau_h \in Q_0^h, \tau \in Q_0$,*

$$\begin{aligned} \|\sigma - \sigma_h\|^2_{H(\operatorname{div},\Omega)} \leq C_1 \Big\{ &\|\sigma - \tau_h\|^2_{H(\operatorname{div},\Omega)} + \|p - q_h\|^2_{\mathcal{R}} \\ &+ A_1(\tau_h) + b(\sigma, p - q_h) - (\tilde{f}, p - q_h) \Big\} \\ &+ C_2\{A_2(\tau) + b(\sigma, p_h - q) - (\tilde{f}, p_h - q)\} + \|p - p_h\|^2_{\mathcal{R}}, \end{aligned} \tag{4.14}$$

where

$$\begin{cases} A_1(\tau_h) = a(\sigma, \tau_h - \sigma) + b(\tau_h - \sigma, p) + \langle \sigma_n - \tau_{hn}, g\rangle_{\Gamma_C} - (\vec{f}, \operatorname{div}(\sigma - \tau_h)), \\ A_2(\tau) = a(\sigma, \tau - \sigma_h) + b(\tau - \sigma_h, p) + \langle \sigma_{hn} - \tau_h, g\rangle_{\Gamma_C} - (\vec{f}, \operatorname{div}(\sigma_h - \tau)). \end{cases}$$

Remark 4.1: The first term of the bound given in Lemma 4.2 is the approximation error. The second term is the consistency error, it is the " variational crime" and is due to the nonconformity of the approximation.

Theorem 4.1. *If $Q_0^h \subset Q_0$, and $b(\tau_h, q_h)$ satisfies B.–B. condition, i.e. exist $\beta = const. > 0$ such that*

$$\sup_{\tau_h \in Q_0^h} \frac{b(\tau_h; q_h)}{\|\tau_h\|_{H(\mathrm{div},\Omega)}} \geq \beta \|q_h\|_{\mathcal{R}}, \qquad \forall q_h \in \mathcal{M}_h, \tag{4.15}$$

then

$$\begin{aligned} (1) \quad & \|\sigma - \sigma_h\|^2_{H(\mathrm{div},\Omega)} \\ & \leq c_1\{\|\sigma - \tau_h\|^2_{H(\mathrm{div},\Omega)} + \|p - q_h\|^2_{\mathcal{R}}\} + \{b(\sigma; p_h - q) - (\tilde{f}, p_h - q)\} \\ & \quad + \{b(\sigma; p - q_h) - (\tilde{f}, p - q_h)\}, \\ (2) \quad & \|p - p_h\|_{\mathcal{R}} \leq c_2\{\|\sigma - \sigma_h\|_{H(\mathrm{div},\Omega)} + \|p - q_h\|_{\mathcal{R}}\}, \end{aligned}$$

for any $\tau_h \in Q_0^h$, $q \in \mathcal{M}$, $q_h \in \mathcal{M}_h$.

PROOF. Using Lemma 4.1 and $Q_0^h \subset Q_0$, and taking $\tau = \tau_h$ in (4.2), we have

$$a(\sigma, \tau_h) + b(\tau_h; p) = \langle \tau_{hn}, g \rangle_{\Gamma_C} - (\vec{f}, \mathrm{div}\, \tau_h). \tag{4.16}$$

From (4.2) we have

$$a(\sigma, \sigma) + b(\sigma; p) = \langle \sigma_n, g \rangle_{\Gamma_C} - (\vec{f}, \mathrm{div}\, \sigma). \tag{4.17}$$

It follows from the above two inequalities, we obtain

$$a(\sigma, \tau_h - \sigma) + b(\tau_h - \sigma, p) = \langle \tau_{hn} - \sigma_n, g \rangle_{\Gamma_C} - (\vec{f}, \mathrm{div}(\tau_n - \sigma)). \tag{4.18}$$

Therefore, $A_1(\tau_h) = 0$.
Similarly, we may get $A_2(\tau) = 0$.
Using discrete B.–B. condition, we obtain

$$\beta \|p_h - q_h\|_{\mathcal{R}} \leq \sup_{\tau_h \in Q_0^h} \frac{b(\tau_h; q_h - p_h)}{\|\tau_h\|_{H(\mathrm{div},\Omega)}}, \tag{4.19}$$

and

$$\begin{aligned} b(\tau_h; q_h - p_h) &= b(\tau_h; q_h) - b(\tau_h, p_h) \\ &= b(\tau_h; q_h) + a(\sigma_h, \tau_h) - \langle \tau_{hn}, g \rangle_{\Gamma_C} + (\vec{f}, \mathrm{div}\, \tau_h) \\ &= b(\tau_h; q_h) + a(\sigma_h, \tau_h) - a(\sigma, \tau_h) - b(\tau_h; p) \\ &= b(\tau_h; q_h - p) + a(\sigma_h - \sigma, \tau_h) \\ &\leq c\{\|q_h - p\|_{\mathcal{R}} + \|\sigma_h - \sigma\|_{H(\mathrm{div},\Omega)}\}\|\tau_h\|_{H(\mathrm{div},\Omega)}. \end{aligned}$$

Combining the above two inequalities we get

$$\|p_h - q_h\|_{\mathcal{R}} \leq c\{\|q_h - p\|_{\mathcal{R}} + \|\sigma_h - \sigma\|_{H(\mathrm{div},\Omega)}\}.$$

By triangle inequality, it is easy to get

$$\begin{aligned} \|p - p_h\|_{\mathcal{R}} &\leq \|p - q_h\|_{\mathcal{R}} + \|q_h - p_h\|_{\mathcal{R}} \\ &\leq c\{\|p - q_h\|_{\mathcal{R}} + \|\sigma - \sigma_h\|_{H(\mathrm{div},\Omega)}\}. \end{aligned}$$

Substituting the estimate of $\|p - p_h\|_{\mathcal{R}}$ into (4.14), and taking ε small enough, we get the estimate of $\|\sigma - \sigma_h\|_{H(\mathrm{div},\Omega)}$. □

4.2 Conforming discretization of dual variational formulation

Now we are ready to give the error estimates of finite element solution for some special finite element subspace. Let $\mathcal{T}_h$ be the triangulation of Ω. Denote by $(L_h^2)^2$ the piecewise linear function subspace of $(L^2(\Omega))^2$ to approximate the displacement, M_h the continuous piecewise linear function subspace of $H^{1/2}(\Gamma_C)$ to approximate the displacement on Γ_C and Q_h the RT_1 subspace of $H(\mathrm{div}, \Omega)$ to approximate the stress. For any element $e \in \mathcal{T}_h$, RT_1 element have the following properties (cf. [8, 6]):

(i) $\tau_h \cdot \vec{n}|_{\partial e}$ is piecewise linear on ∂e, $\vec{n}$ is the unit outward normal to ∂e.

(ii) $\operatorname{div} \tau_h|_e$ is a linear function.

Theorem 4.2. *If the dual variational Problem 3.2 has unique solution, and Q_h, $(L_h^2)^2$ are the finite element subspace of $H(\operatorname{div}, \Omega)$, $(L^2(\Omega))^2$, respectively. $\Lambda_h = M_h \cap \Lambda$, then discrete Problem4.1 exists a unique solution.*

PROOF. Obviously, we only need to prove the discrete B.-B. condition.

Define the interpolation of $\pi_h : \quad H(\operatorname{div}, \Omega) \longrightarrow Q_h$ as follows(cf.[6]):
For any given $\tau \in H(\operatorname{div}, \Omega)$, $\pi_h \tau$ is solved by the following equation

$$b(\tau - \pi_h \tau; \tilde{v}_h, \mu_h) = 0, \qquad \forall \tilde{v}_h \in (L_h^2)^2, \mu_h \in \Lambda_h. \tag{4.20}$$

Let e_i(i=1,2,3) be the three edges of element e, ℓ stands for the element on Γ_C corresponding to the triangulation. In order to make $b(\tau - \pi_h \tau; \tilde{v}_h, \mu_h) = 0$, i.e.

$$\int_\Omega \operatorname{div}(\tau - \pi_h \tau) \tilde{v}_h dx + \int_{\Gamma_C} (\tau_n - \pi_h \tau_n) \mu_h ds = 0.$$

We assume that

$$\int_e \operatorname{div}(\tau - \pi_h \tau) \tilde{v}_h dx = 0, \quad \int_\ell (\tau_n - \pi_h \tau_n) \mu_h ds = 0.$$

Because $\tilde{v}_h$ is a piecewise linear function and μ_h is a continuous piecewise linear function on Γ_C, then

$$\int_{e_i} (\tau - \pi_h \tau) \vec{n} \cdot \tilde{v}_h ds = 0, \quad \int_e (\tau - \pi_h \tau) \nabla \tilde{v}_h dx = 0.$$

For any given τ, we can get the expression of $\pi_h \tau$ from above equations, and have the following inequality (cf. [6]):

$$\|\pi_h \tau\|_{H(\operatorname{div},\Omega)} \le c \|\tau\|_{H(\operatorname{div},\Omega)}. \tag{4.21}$$

In order to make $\pi_h \tau \in Q_0^h$, when e_i is located on Γ_F, we may assume that

$$(\pi_h \tau) \vec{n}|_{\Gamma_F} = \tau \vec{n}|_{\Gamma_F}.$$

Combining (4.20) and (4.21), we obtain

$$\sup_{\tau_h \in Q_0^h} \frac{b(\tau_h; \tilde{v}_h, \mu_h)}{\|\tau_h\|_{H(\operatorname{div},\Omega)}} \ge \sup_{\tau \in Q_0} \frac{b(\pi_h \tau; \tilde{v}_h, \mu_h)}{\|\pi_h \tau\|_{H(\operatorname{div},\Omega)}}$$
$$\ge \sup_{\tau \in Q_0} \frac{b(\tau; \tilde{v}_h, \mu_h)}{c\|\tau\|_{H(\operatorname{div},\Omega)}} \ge \beta(\|\tilde{v}_h\|_{0,\Omega} + \|\mu_h\|_{1/2,\Gamma_C}).$$

This gives the discrete B.-B. condition. □

Theorem 4.3. *If Ω is a convex polygon, $\vec{f} \in (L^2(\Omega))^2, g \in H^{3/2}(\Gamma_C)$; $(\sigma; \tilde{u}, \lambda)$ and $(\sigma_h; \tilde{u}_h, \lambda_h)$ are the solutions of Problem 3.2 and Problem 4.1, respectively, then when Q_h, $(L_h^2)^2$ and Λ_h are chosen as the above, we have the following error estimate*

$$\|\sigma - \sigma_h\|_{H(\operatorname{div},\Omega)} + \|\tilde{u} - \tilde{u}_h\|_{0,\Omega} + \|\lambda - \lambda_h\|_{1/2,\Gamma_C} \le ch^{3/4}.$$

PROOF. By the construction of $\pi_h \tau$, we have the following basic estimates:

$$\|\sigma - \pi_h \sigma\|_{0,\Omega} \le ch\|\sigma\|_{1,\Omega},$$
$$\|\operatorname{div}(\sigma - \pi_h \sigma)\|_{0,\Omega} \le ch\|\operatorname{div} \sigma\|_{1,\Omega}.$$

Let $\tilde{u}_I \in (L_h^2)^2$ and $\lambda_I \in \Lambda_h$ be the Lagrangian interpolants of $\tilde{u}$ and λ, respectively. By the standard approximation theory (cf. [10]), we have

$$\|\tilde{u} - \tilde{u}_I\|_{0,\Omega} \leq ch\|\tilde{u}\|_{1,\Omega},$$
$$\|\lambda - \lambda_I\|_{1/2,\Gamma_C} \leq ch\|\lambda\|_{3/2,\Gamma_C}.$$

Taking $(\tau_h; \tilde{v}_h, \mu_h) = (\pi_h \sigma; \tilde{u}_I, \lambda_I),\ q = p_h$, and $\mu_h = \lambda_I$, we obtain

$$\begin{aligned} b(\sigma; p - q_h) - (\tilde{f}, p - q_h) &= (\operatorname{div} \sigma, \tilde{u} - \tilde{v}_h) + \langle \sigma_n, \lambda - \mu_h \rangle_{\Gamma_C} + (\vec{f}, \tilde{u} - \tilde{v}_h) \\ &= \langle \sigma_n, \lambda - \mu_h \rangle_{\Gamma_C} \\ &\leq \|\sigma_n\|_{0,\Gamma_C} \|\lambda - \lambda_I\|_{0,\Gamma_C} \\ &\leq ch^{3/2} \|\sigma_n\|_{0,\Gamma_C} \|\lambda\|_{3/2,\Gamma_C}. \end{aligned}$$

Using Theorem 4.1, we obtain

$$\begin{aligned} &\|\sigma - \sigma_h\|^2_{H(\operatorname{div},\Omega)} \\ &\leq ch^{3/2}(\|\sigma\|_{1,\Omega} + \|\tilde{u}\|_{1,\Omega} + \|\operatorname{div} \sigma\|_{1,\Omega} + \|\lambda\|_{3/2,\Gamma_C}), \end{aligned}$$

and

$$\begin{aligned} &\|\tilde{u} - \tilde{u}_h\|_{0,\Omega} + \|\lambda - \lambda_h\|_{0,\Gamma_C} \\ &\leq ch^{3/4}(\|\sigma\|_{1,\Omega} + \|\tilde{u}\|_{1,\Omega} + \|\operatorname{div} \sigma\|_{1,\Omega} + \|\lambda\|_{3/2,\Gamma_C}). \end{aligned}$$

This gives the proof. □

4.3 Nonconforming discretization of dual variational formulation

We define Q_h as the standard RT_0 space to approximate the stress, and $(L_h^2)^2$ as the piecewise constant space to approximate the displacement, and Λ_h as the piecewise constant subset of $L^2(\Gamma_C)$ to approximate the displacement on Γ_C. Obviously, $\Lambda_h \not\subset \Lambda$. Therefore, it is a nonconforming approximation of Λ.

Similar to the conforming element case, we firstly define the interpolation operator $\pi_h : H(\operatorname{div}, \Omega) \to Q_h$. For any given $\tau \in H(\operatorname{div}, \Omega)$

$$b(\tau - \pi_h \tau; \tilde{v}_h, \mu_h) = 0, \quad \forall \tilde{v}_h \in (L_h^2)^2,\ \mu_h \in \Lambda_h.$$

In order to make $b(\tau - \pi_h \tau; \tilde{v}_h, \mu_h) = 0$, let

$$\begin{cases} (\operatorname{div}(\tau - \pi_h \tau), \tilde{v}_h) = 0, \quad \forall \tilde{v}_h \in (L_h^2)^2, \\ \int_{\Gamma_C} (\tau_n - \pi_h \tau_n) \mu_h d\Gamma = 0, \quad \forall \mu_h \in \Lambda_h. \end{cases}$$

i.e.

$$\sum_{e \in \mathcal{T}_h} \int_e \operatorname{div}(\tau - \pi_h \tau) \tilde{v}_h dx = 0, \qquad \forall \tilde{v}_h \in (L_h^2)^2,$$

$$\sum_{\ell \in \Gamma_{Ch}} \int_\ell (\tau_n - \pi_h \tau_n) \mu_h ds = 0, \qquad \forall \mu_h \in \Lambda_h,$$

where Γ_{Ch} denotes the partition of Γ_C, and ℓ is corresponding partition on Γ_C. Let $\tilde{v}_h$ and μ_h be piecewise constants in Ω and on Γ_C, respectively.

For any given τ, we have the explicit expression of $\pi_h \tau$. Therefore, it is easy to prove the following inequality (see[6])

$$\|\pi_h \tau\|_{H(\operatorname{div},\Omega)} \leq c\|\tau\|_{H(\operatorname{div},\Omega)}.$$

Theorem 4.4. *If the dual variational Problem 3.2 has a unique solution, and Q_h, $(L_h^2)^2$ are the finite element subspace of $H(\operatorname{div}, \Omega)$, $(L^2(\Omega))^2$, respectively. $\Lambda_h \subset L^2(\Gamma_C)$, is the external approximation of Λ, then Problem 4.1 has a unique solution.*

PROOF. Similar to the proof of Theorem 4.2, we only need to verify the discrete B.-B. condition.

According to the definition of interpolation operator π_h above, we have

$$\sup_{\tau_h \in Q_0^h} \frac{b(\tau_h; \tilde{v}_h, \mu_h)}{\|\tau_h\|_{H(\mathrm{div},\Omega)}} \geq \sup_{\tau \in Q_0} \frac{b(\pi_h \tau; \tilde{v}_h, \mu_h)}{\|\pi_h \tau\|_{H(\mathrm{div},\Omega)}} \geq \sup_{\tau \in Q_0} \frac{b(\tau; \tilde{v}_h, \mu_h)}{c\|\tau\|_{H(\mathrm{div},\Omega)}}.$$

By Lemma 3.1 we have

$$\sup_{\tau \in Q_0} \frac{b(\tau; \tilde{v}_h, \mu)}{\|\tau\|_{H(\mathrm{div},\Omega)}} \geq \beta\{\|\tilde{v}_h\|_{0,\Omega}^2 + \|\mu\|_{1/2,\Gamma_C}^2\}^{1/2}, \quad \forall \tilde{v}_h \in (L_h^2)^2, \mu \in H^{1/2}(\Gamma_C).$$

Because $H^{1/2}(\Gamma_C)$ is dense in $L^2(\Gamma_C)$, further, we deduce

$$\sup_{\tau \in Q_0} \frac{b(\tau; \tilde{v}_h, \mu_h)}{\|\tau\|_{H(\mathrm{div},\Omega)}} \geq \beta\{\|\tilde{v}_h\|_{0,\Omega}^2 + \|\mu_h\|_{0,\Gamma_C}^2\}^{1/2}, \quad \forall \vec{v}_h \in (L_h^2)^2, \mu_h \in L^2(\Gamma_C).$$

Finally, we have

$$\sup_{\tau_h \in Q_0^h} \frac{b(\tau_h; \tilde{v}_h, \mu_h)}{\|\tau_h\|_{H(\mathrm{div},\Omega)}} \geq \beta\{\|\tilde{v}_h\|_{0,\Omega}^2 + \|\mu_h\|_{0,\Gamma_C}^2\}^{1/2}, \quad \forall \vec{v}_h \in (L_h^2)^2, \mu_h \in \Lambda_h.$$

This has proven the discrete B.-B. condition. □

Theorem 4.5. *If Ω is a convex polygon, $f \in (L^2(\Omega))^2, g \in H^{3/2}(\Gamma_C)$; $(\sigma; \tilde{u}, \lambda)$ and $(\sigma_h; \tilde{u}_h, \lambda_h)$ are the solution Problem 3.2 and Problem 4.1, respectively, then when Q_h, $(L_h^2)^2$ are respectively chosen as the subspace of $H(\mathrm{div},\Omega)$ and $(L^2(\Omega))^2$; $\Lambda_h = \{\mu \in M_h | \mu \geq 0\}$ is the nonconforming approximation of Λ, we have the following estimates*

$$\|\sigma - \sigma_h\|_{H(\mathrm{div},\Omega)} + \|\vec{u} - \vec{u}_h\|_{0,\Omega} + \|\lambda - \lambda_h\|_{0,\Gamma_C} \leq ch^{1/2}.$$

PROOF. Here we mainly estimate the consistence error

$$b(\sigma; p_h - q) - (\vec{f}, p_h - q).$$

Other terms can be estimated as that in Theorem 4.3. A direct calculation gives

$$\begin{aligned}
&b(\sigma; p_h - q) - (\vec{f}, p_h - q) \\
&= (\mathrm{div}\,\sigma, \vec{u}_h - \vec{v}) + \langle \sigma_n, \lambda_h - \mu \rangle_{\Gamma_C} + (f, \vec{u}_h - \vec{v}) \\
&= \langle \sigma_n, \lambda_h - \mu \rangle_{\Gamma_C}.
\end{aligned}$$

To estimate $\langle \sigma_n, \lambda_h - \mu \rangle_{\Gamma_C}$, we define the following interpolation operator(see [5]) $r_h : L^2(\Gamma_C) \to M_h \subset H^{1/2}(\Gamma_C)$, which preserves the non-positivity, i.e. if $\tau_n \leq 0$ then $r_h \tau_n \leq 0$. Using the above derivation and choosing $\mu = \lambda$, we have: $\forall r_h \sigma_n \in M_h$,

$$\int_{\Gamma_C} \sigma_n(\lambda_h - \lambda) d\Gamma = \int_{\Gamma_C} (\sigma_n - r_h \sigma_n)(\lambda_h - \lambda) d\Gamma + \int_{\Gamma_C} r_h \sigma_n (\lambda_h - \lambda) d\Gamma.$$

The first integral is easily bounded as

$$\int_{\Gamma_C} (\sigma_n - r_h \sigma_n)(\lambda_h - \lambda) d\Gamma \leq ch^{1/2} \|\sigma\|_{1,\Omega} \|\lambda_h - \lambda\|_{0,\Gamma_C}. \tag{4.22}$$

By $\sigma_n|_{\Gamma_C} \leq 0$ and $\lambda = g - u_n$, we get

$$\begin{aligned}
\int_{\Gamma_C} r_h \sigma_n (\lambda_h - \lambda) d\Gamma \quad &\leq \quad -\int_{\Gamma_C} r_h \sigma_n \lambda d\Gamma = \int_{\Gamma_C} (\sigma_n - r_h \sigma_n) \lambda d\Gamma \\
&\leq \quad \|\sigma_n - r_h \sigma_n\|_{0,\Gamma_C} \|\lambda\|_{0,\Gamma_C} \\
&\leq \quad ch^{1/2} \|\sigma\|_{1,\Omega} \|\lambda\|_{0,\Gamma_C}.
\end{aligned} \tag{4.23}$$

Combining the estimates (4.22) and (4.23), we obtain

$$b(\sigma; p_h - p) - (\vec{f}, p_h - p) \le ch^{1/2}\|\sigma\|_{1,\Omega}(\|\lambda_h - \lambda\|_{0,\Gamma_C} + \|\lambda\|_{0,\Gamma_C}). \tag{4.24}$$

Substituting (4.24) into (1) of Theorem4.1, we get

$$\|\sigma - \sigma_h\|^2_{H(\mathrm{div},\Omega)} \le ch^{1/2}(\|\sigma\|_{1,\Omega} + \|\vec{u}\|_{1,\Omega} + \|\operatorname{div}\sigma\|_{1,\Omega} + \|\lambda\|_{0,\Gamma_C}).$$

Substituting (4.25) into (2) of Theorem 4.1 and using Young's inequality, we obtain

$$\|\tilde{u} - \tilde{u}_h\|_{0,\Omega} + \|\lambda - \lambda_h\|_{0,\Gamma_C} \le ch^{1/2}. \tag{4.25}$$

This finishes the proof. □

Remark 4.2: The convergence rate of nonconforming method is lower than the conforming method, because we have not found a refined estimate for the consistence terms, e.g. (4.24). We will discuss this problem in another paper.

5. Optimization Algorithm for Mixed Finite Element Discrete Problem

In this section, the Uzawa type algorithm is presented for solving Problem 4.1. If we assume that

$$\begin{cases} L(\tau; v, \mu) = J_0(\tau) + (\operatorname{div}\tau, v) + \langle \tau_n, v\rangle_{\Gamma_C}, \\ J_0 = \frac{1}{2}(\tau,\tau) - \langle \tau_n, g\rangle_{\Gamma_C} + (f, \operatorname{div}\tau + v), \end{cases}$$

then Problem 3.2 and Problem 4.1 can be written as the following equivalent saddle-point problems, respectively.

Problem 5.1. *Find $\sigma \in Q_t,\ \tilde{u} \in (L^2(\Omega))^2,\ \lambda \in \Lambda$ such that*

$$L(\sigma; \tilde{v}, \mu) \le L(\sigma; \tilde{u}, \lambda) \le L(\tau; \tilde{u}, \lambda), \quad \forall \tau \in Q_0,\ \tilde{v} \in (L^2(\Omega))^2,\ \mu \in \Lambda.$$

Problem 5.2. *Find $\sigma_h \in Q_t^h,\ \tilde{u}_h \in (L_h^2)^2,\ \lambda_h \in \Lambda_h$ such that*

$$L(\sigma_h; \tilde{v}_h, \mu) \le L(\sigma_h; \tilde{u}_h, \lambda_h) \le L(\tau_h; \tilde{u}_h, \lambda_h), \quad \forall \tau_h \in Q_t^h,\ \tilde{v}_h \in (L_h^2)^2,\ \mu_h \in \Lambda_h.$$

The above two inequalities immediately implies

Lemma 5.1. *If $\{\sigma_h; u_h, \lambda_h\}$ is the saddle-point of the problem 5.2, then*

$$\begin{aligned} (i)\quad & J_0(\sigma_h) + (\operatorname{div}\sigma_h, \tilde{u}_h) + \langle \sigma_{hn}, \lambda_h\rangle_{\Gamma_C} \\ & \le J_0(\tau_h) + (\operatorname{div}\tau_h, \tilde{u}_h) + \langle \tau_{hn}, \lambda_h\rangle_{\Gamma_C}, \quad \forall \tau_h \in Q_0^h. \\ (ii)\quad & \langle \sigma_{hn}, \mu_h - \lambda_h\rangle_{\Gamma_C} + (\operatorname{div}\sigma_h + f, \tilde{v}_h - \tilde{u}_h) \le 0,\ \forall \mu_h \in \Lambda_h,\ \tilde{v}_h \in (L_h^2)^2. \end{aligned}$$

Lemma 5.2. *The following variational formulation*

$$\langle \sigma_{hn}, \mu_h - \lambda_h\rangle_{\Gamma_C} + (\operatorname{div}\sigma_h + f, \tilde{v}_h - \tilde{u}_h) \le 0, \qquad \forall \mu_h \in \Lambda_h,\ \tilde{v}_h \in (L_h^2)^2,$$

is equivalent to

$$\operatorname{div}\sigma_h + f = 0, \qquad \lambda_h = P_\Lambda(\rho\sigma_{hn} + \lambda_h),$$

where P_Λ is the project operator: $L^2(\Gamma_C) \to \Lambda_h$, $\rho > 0$.

The proof of Lemma 5.2 can be found in [20].

Basing on Lemma 5.1 and Lemma 5.2 , we define the following Uzawa type iterative algorithm

(i) Given $\tilde{u}_h^m \in (L_h^2)^2,\ \lambda_h^m \in \Lambda_h$, we define $\sigma_h^m \in Q_t^h$ such that

$$\begin{aligned} & J_0(\sigma_h^m) + (\operatorname{div}\sigma_h^m, \tilde{u}_h^m) + \langle \sigma_{hn}^m, \lambda_h^m\rangle_{\Gamma_C} \\ & \le J_0(\tau_h) + (\operatorname{div}\tau_h, \tilde{u}_h^m) + \langle \tau_{hn}, \lambda_h^m\rangle_{\Gamma_C}, \forall \tau_h \in Q_0^h. \end{aligned}$$

(ii) Find $\tilde{u}_h^{m+1}$ and λ_h^{m+1} by using the following iterative method

$$\begin{cases} \tilde{u}_h^{m+1} = \tilde{u}_h^m + \rho_m(\operatorname{div}\sigma_h^m + \vec{f}), \\ \lambda_h^{m+1} = P_\Lambda(\rho_m\sigma_{hn}^m + \lambda_h^m), \end{cases}$$

where $\rho_m > 0$ is a constant chosen properly and P_Λ is defined in Lemma 5.2.

Theorem 5.1. *There exist $a_0,\ a_1 = constant,\ 0 < a_0 \le \rho_n \le a_1$ such that above Uzawa type iterative algorithm is convergent according to the following sense*

(i) $\sigma_h^m \to \sigma_h$ *strongly in* $H(\operatorname{div}, \Omega)$.

(ii) $\lim_{m\to\infty} \|\tilde{u}_h^{m+1} - \tilde{u}_h^m\|_{0,\Omega} = 0, \quad \lim_{m\to\infty} \|\lambda_h^{m+1} - \lambda_h^m\|_{0,\Gamma_C} = 0.$

(iii) $\{\tilde{u}_h^m, \lambda_h^m\}_m \to \{\tilde{u}_h, \lambda_h\}$ *weakly in* $(L_h^2)^2 \times \Lambda_h$.

The proof of the above theorem is similar to [13, Theorem 3.1].

6. Numerical Discussion

We consider the following unilateral problem (cf.[13]):

$$\begin{cases} -\Delta u = f \quad in\Omega, \\ u = 0 \quad on\Gamma_D, \\ u \ge 0,\ \frac{\partial u}{n} \ge 0, \quad on\Gamma_C \\ \frac{\partial u}{n} \cdot u = 0,\ on\Gamma_C, \end{cases} \tag{6.1}$$

where

$$\begin{cases} \Omega = [0,1]\times[0,1], \quad \partial\Omega = \Gamma_C \cup \Gamma_D, \\ \Gamma_C = \{(x,y)|0 \le x \le 1,\ y = 0\} \cup \{(x,y)|0 \le x \le 1,\ y = 1\}, \end{cases} \tag{6.2}$$

$$f(x,y) = \begin{cases} 10, & \text{if } (x,y) \in [0,1/2]\times[0,1], \\ -10, & \text{if } (x,y) \in [1/2,1]\times[0,1]. \end{cases} \tag{6.3}$$

6.1 Numerical Modeling for Conforming Finite Element

(i) Given any initial value $u_{hi}^m \in L_h^2,\ \lambda_{hj}^m \in \Lambda_h$, find σ_{hi}^m as the solution of the following linear equations:

$$(\sigma_h, \tau_h) - (\operatorname{div}\tau_h, u_h) + \langle \tau_{nh}, \lambda_h - g\rangle_{\Gamma_C} = 0, \qquad \forall \tau_h \in Q_0^h.$$

(ii) Using σ_{hi}^m, find u_{hi}^{m+1} and λ_{hj}^{m+1}

$$\begin{cases} u_{hi}^{m+1} = u_{hi}^m + \frac{1}{S_{\Delta_i}}\int_{\Delta_i} \rho_m(\operatorname{div}\sigma_{hi}^m + f)dxdy, \\ \lambda_{hj}^{m+1} = min\{0, \lambda_{hj}^m + s_i\}, \end{cases}$$

where S_{Δ_i} is the area of i-th element, $s_i = \frac{1}{l_i}\int_{l_i} \rho(\operatorname{div}\sigma_{hi}^n + f_i)ds$.

(iii) The criterion of stopping iteration is

$$error = \frac{\sum_i |u_i^{m+1} - u_i^m|}{\sum_i |u_i^{m+1}|} \le 10^{-5}.$$

In this case, we use 8×8 triangulation, $\rho_m = 0.05$ and the iterative number is 200. u_h and λ_h are depicted in Figure 6.2. From these figures we see that u_h is very close to the value of u on fixed boundary $\Gamma_D = \{(x,y)|0 \le y \le 1, x = 0\} \cup \{(x,y)|0 \le y \le 1, x = 1\}$. The dentation in the graph of u_h is caused by the triangular meshes, and decreases with the refinement of the mesh. For λ, we give the graph of λ_h when $y = 1$. λ_h on Γ_C is very close to that of u_h on Γ_C.

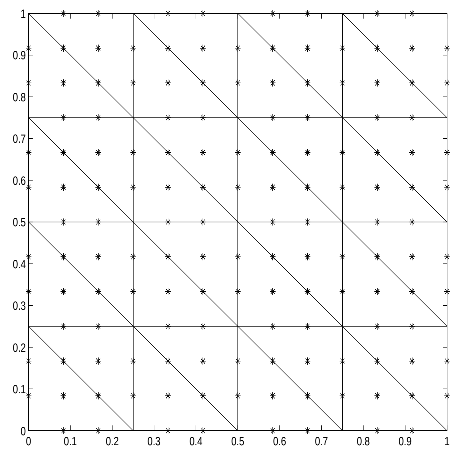
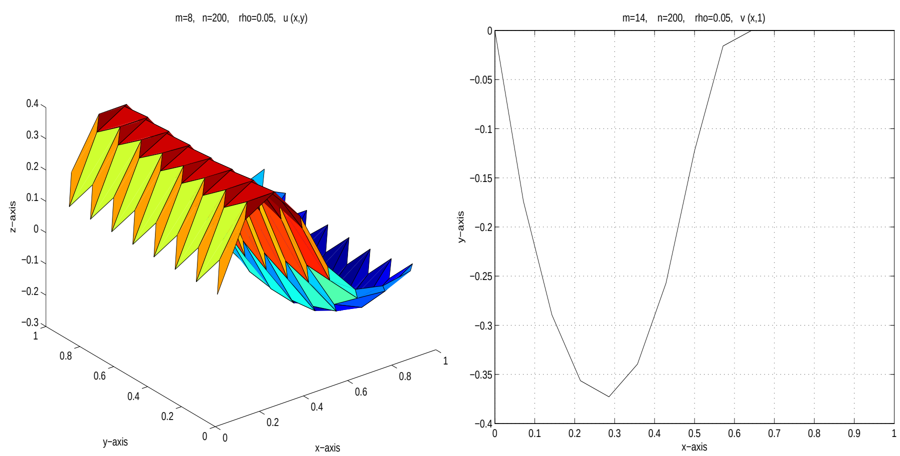


FIGURE 6.1. Triangulations and nodes employed in the conforming finite element method



u λ

FIGURE 6.2. Computed u and λ using conforming finite element method.

6.2 Numerical Modeling for Nonconforming Finite Element

The iterative formulations are similar to (i)-(iii) of Section 6.1. In this case, we use 14×14 triangulation, $\rho_m = 0.05$ and the computing time is the same as that of Section 6.1. u_h and λ_h are depicted in Figure 6.4. It follows from Table 6.1 that both conforming element and nonconforming element are numerically stable. Therefore, we see that u_h and λ_h in Figure 6.4 have better approximation than those of Figure 6.2. In addition, Table 6.1 shows that the nonconforming method converges faster than the conforming method.

7. Concluding Remarks

This paper includes the finite element error estimates and the numerical simulation of the dual mixed variational formulation of unilateral contact problem. We obtain the convergence rate of both the conforming and nonconforming methods. The numerical example shows that the nonconforming method converges faster than the conforming method.

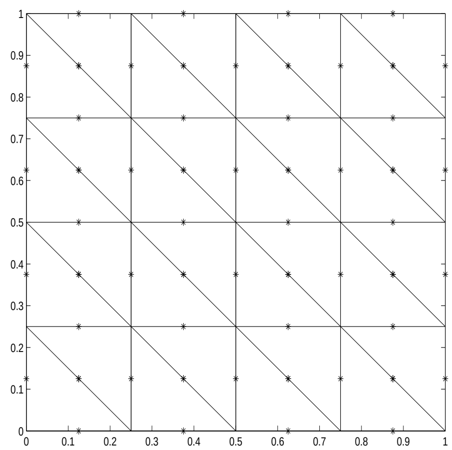
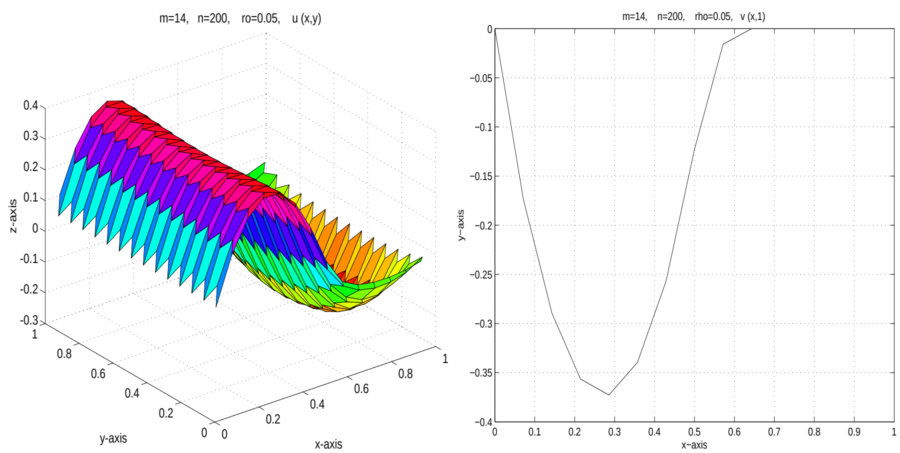


FIGURE 6.3. Triangulations and nodes employed in the nonconforming method



u λ

FIGURE 6.4. Computed u and λ using nonconforming finite element.

itearative number	*conforming FEM*	*nonconforming FEM*
10	0.0391011	0.0667215
20	0.0395721	0.0354249
30	0.0365382	0.0131121
40	0.0315310	0.0047859
50	0.0227896	0.0017652
60	0.0149436	0.0006545
70	0.0094068	0.0002429

TABLE 6.1. The comparisons of **errors** for two finite element approximations when the partition of Ω is 12×12.

Acknowledgments

The authors thank the anonymous referees for their valuable comments and suggestions.

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