A DOMAIN DECOMPOSITION METHOD WITH LAGRANGE MULTIPLIER BASED ON THE POINTWISE MATCHING CONDITION

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(Communicated by Jun Zou)

Abstract. In this paper, we are concerned with a non-overlapping domain decomposition method with nonmatching grids. In this method, a new pointwise matching condition is used to define weak continuity of approximate solutions on the interface. The main merit of the new method is that numerical integrations can be avoided when calculating interface matrices. We derive an almost optimal error estimate of the resulting approximate solutions for two kinds of applicable situations. Some numerical experiments confirm the theoretical result.

Key Words. domain decomposition, nonmatching grids, pointwise matching, error estimate.

1. Introduction

The domain decomposition method (DDM) with nonmatching grids is now popular in engineering and scientific computing (see [1], [2], [5], [9], [10], [12], [13], [14], [15] and [17]). A key ingredient in this method is the choice of a suitable interface matching condition, which defines the discrete variational problem associated with this DDM. The convergence of the resulting approximate solution, which is only weak continuous across the interface, strongly depends on such interface matching condition.

There are two kinds of interface matching conditions in literature: the integral matching condition (see [4], [3], [6] and [13]), and the pointwise matching condition (see [4] and [6]). When using the integral matching condition, calculation of numerical integrations on the interface will be in general expensive, especially for three-dimensional problems. Use of the pointwise matching condition can remove this difficulty, but it may generate unsatisfactory approximate solutions (see [4] and [6]).

In the present paper we investigate when the pointwise matching condition works well in DDM with nonmatching grids for the second-order elliptic problem in three dimensions. The main difficulty is the design of a weak conformity on the *wire-basket* set to the approximation. For two-dimensional problems, one can require that the approximation is continuous at the cross-points (see [4]). But, one can not impose the same continuity on the *wire-basket* set, since the grids on the *wire-basket*

set are still nonmatching. If no constraint is added on the *wire-basket* set, the resulting approximation has low convergence. To remove this difficulty, we propose a combination between the pointwise matching condition and the integral matching condition. The idea is to define a suitable *discrete L^2* projection into the Lagrange multiplier space defined on the common face of two neighboring subdomains. The new matching condition implies that the restrictions of the underlying approximation on two neighboring subdomains has the same *discrete L^2* projection. In essence, the new matching condition involves a set of internal nodes on each local face. We require that the approximation has point to point continuity at the nodes not closing the boundary of the face, and possesses weak continuity in the sense of average at the nodes closing the boundary of the face. The whole matching condition can be expressed in a unified manner by defining an interpolation type operator. It will be shown that the resulting approximate solution possesses almost the optimal error estimate for two practical situations.

The outline of this paper is as follows. In Section 2, we introduce DDM with the new pointwise matching condition. In Section 3, we show that this pointwise matching condition can result in almost the optimal error estimate for two applied cases. In Section 4, we give some numerical results, which confirm the effectiveness of this new interface matching condition.

2. DDMs with Pointwise Matching Condition

In this paper, we consider the following model problem

$$(1) \qquad \begin{cases} -\nabla(\omega\nabla u) = & f, \quad in \ \Omega, \\ u = & 0, \quad on \ \partial\Omega, \end{cases}$$

where $\Omega \subset R^3$ is a bounded, connected Lipschitz domain, and $\omega \in L^\infty(\Omega)$ is a positive real function.

Let $H_0^1(\Omega)$ be the standard Sobolev space and define the following bilinear form:

$$a(u,v) = \int_\Omega \omega\nabla u \cdot \nabla v dx, \ u,v \in H_0^1(\Omega).$$

Then the corresponding weak form of (1) is: Find $u \in H_0^1(\Omega)$, such that:

$$(2) \qquad a(u,v) = (f,v), \ \forall \ v \in H_0^1(\Omega),$$

where $(\cdot,\cdot)$ denotes the $L^2(\Omega)$-inner product.

In the following, we define the discrete problem of (2) based on DDMs with a new pointwise matching condition.

As usual, we decompose the domain Ω into the union of some subdomains $\overline{\Omega} = \sum_{k=1}^N \overline{\Omega}_k$, which satisfies that $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$. For convenience, we assume that the decomposition is geometrically conforming:

(1) if Ω_i and Ω_j are two neighboring subdomains, then $\partial\Omega_i \cap \partial\Omega_j$ is just a common vertex, or a common edge or a common face of Ω_i and Ω_j.

(2) each subdomain has the same "size" d in the usual way (refer to [20]).

In particular, when $\partial\Omega_i \cap \partial\Omega_j$ is a common face of Ω_i and Ω_j, we set $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$ and call Γ_{ij} to be a local interface.

For each Ω_k, we introduce a partition $\mathcal{T}_k$ which is made of elements that are either hexahedra or tetrahedra. Let h_k be the mesh size of $\mathcal{T}_k$, i.e., h_k denotes the maximum diameter of all elements in $\mathcal{T}_k$. Define $h = \min_{1 \leq k \leq N} h_k$. The triangulation

of Ω generally does not match on the interface Γ. So each Γ_{ij} relates to two different 2D meshes $\mathcal{T}_{ij}$ and $\mathcal{T}_{ji}$, which are the restriction of $\mathcal{T}_i$ and $\mathcal{T}_j$ on Γ_{ij} respectively.

Remark 2.1 For each Ω_k, we assume the partition on $\partial\Omega_k$ is regular and quasi-uniform because our theoretical analysis involves some mathematical tools such as inverse estimates and discrete norms on interfaces. But it is not an essential restriction for some applications such as solving the elliptic problems with singularity. In general, we can divide Ω into more subdomains to meet this assumption.

For $1 \leq k \leq N$, define

$$V(\Omega_k) = \{v : v \in C(\Omega_k), \ v|_{\partial\Omega \cap \partial\Omega_k} = 0; \ \forall \, e \in \mathcal{T}_k, \ v|_e \in P_1(e)\}$$

and

$$V(\partial\Omega_k) = \{v|_{\partial\Omega_k} : \ v \in V(\Omega_k)\}.$$

where e is any element in $\mathcal{T}_k$ and $P_1(e)$ denotes the space consisting of continuous linear (or trilinear) functions on e.

The definition of the finite element space on Ω involves a suitable matching condition on each Γ_{ij}. To describe the idea more clearly, we want to use a local multiplier space $W(\Gamma_{ij})$ on Γ_{ij}. There are many ways to define such local multiplier space (see [3], [4], [6], [13], [16] and [19]). As an example, we consider only the local multiplier space defined in [3].

Without loss of generality, we assume that $h_i \leq h_j$ for each face Γ_{ij}. Then, the local multiplier space on Γ_{ij} is defined by the triangulation $\mathcal{T}_{ij}$ (instead of $\mathcal{T}_{ji}$). For convenience, we only consider the case that $\mathcal{T}_{ij}$ is made up of triangles here.

Let $\{x_k\}_{k=1}^{N_{ij}^0} \subset \Gamma_{ij}$ be all interior nodes associated with $\mathcal{T}_{ij}$, and let $\{x_k\}_{k=N_{ij}^0+1}^{N_{ij}} \subset \partial\Gamma_{ij}$ be all the boundary nodes associated with $\mathcal{T}_{ij}$. Then, basis functions $\{\phi_k\}_{k=1}^{N_{ij}^0}$ of $W(\Gamma_{ij})$ can be defined as in [3], with ϕ_k corresponding to a interior node x_k. For the quadrangle case, the definition is similar. It is known that we have, for both cases, $\sum_{k=1}^{N_{ij}^0} \phi_k = 1$, and

$$W(\Gamma_{ij}) \subset V_i(\Gamma_{ij}), \ \ dim(W(\Gamma_{ij})) = dim(V_i^0(\Gamma_{ij})) = N_{ij}^0.$$

In the following we will use the discrete $L^2(\Gamma_{ij})$-inner product (refer to [20])

$$\langle v, w\rangle_{0,\Gamma_{ij},\, h} = h_{ij}^2 \sum_{k=1}^{N_{ij}} u(x_k) \cdot v(x_k), \ \forall \, u, v \in C(\Gamma_{ij}),$$

where h_{ij} is the mesh size of $\mathcal{T}_{ij}$.

Let $\|\cdot\|_{0,\,\Gamma_{ij},\, h}$ denote the discrete norm induced by the inner-product $\langle\cdot,\cdot\rangle_{0,\Gamma_{ij},\, h}$. It is well known that

$$\|v\|_{0,\,\Gamma_{ij},\, h}^2 \eqsim \|v\|_{0,\Gamma_{ij}}^2, \quad \forall \, v \in V_i(\Gamma_{ij}).$$

For two neighboring subdomains Ω_i and Ω_j, let $v_i \in V(\Omega_i)$ and $v_j \in V(\Omega_j)$ be the restriction of the solution on Ω_i and Ω_j respectively. We require v_i and v_j satisfy the following weak continuous condition on the interface Γ_{ij}

$$\langle v_i|_{\Gamma_{ij}} - v_j|_{\Gamma_{ij}}, \ \phi\rangle_{\Gamma_{ij},\, h} = 0, \ \forall \, \phi \in \ W(\Gamma_{ij}).$$

Define $\Pi_{ij} : \ C(\Gamma_{ij}) \to W(\Gamma_{ij})$ as

$$\Pi_{ij} \, v = \sum_{k=1}^{N_{ij}^0} v(x_k) \cdot \phi_k, \ \forall \, v \in \ C(\Gamma_{ij}).$$

By the definition of the discrete $L^2(\Gamma_{ij})$-inner product, one can verify directly that

$$\langle v_i|_{\Gamma_{ij}} - v_j|_{\Gamma_{ij}}, \; \phi \rangle_{\Gamma_{ij}, \; h} = 0, \; \forall \, \phi \in \; W(\Gamma_{ij}) \Leftrightarrow \Pi_{ij} \; (v_i|_{\Gamma_{ij}}) = \Pi_{ij} \; (v_j|_{\Gamma_{ij}}).$$

Remark 2.2 The above interface continuous condition is a variant of the original pointwise matching condition. Here, we require that $v_i|_{\Gamma_{ij}}$ and $v_j|_{\Gamma_{ij}}$ have the same interpolation-type projection in the local multiplier space $W(\Gamma_{ij})$. The difference from that of [4] is that we require the continuity of $v_i|_{\Gamma_{ij}}$ and $v_j|_{\Gamma_{ij}}$ only at most of the interior nodes on Γ_{ij}.

Remark 2.3 The pointwise matching condition can avoid calculation of complicated integrations on the local faces in the process of generating coupling matrixes. Compared with the integral matching condition, it will reduce the arithmetic complexity greatly.

Set $V(\Omega) = \prod\limits_{k=1}^{N} V(\Omega_k)$, and define

$$\widetilde{V}(\Omega) = \{v = (v_1, \cdots, v_N) \in V(\Omega) : \; \Pi_{ij} \; (v_i|_{\Gamma_{ij}}) = \Pi_{ij} \; (v_j|_{\Gamma_{ij}}), \; \forall \, \Gamma_{ij} \subset \Gamma\}.$$

Note that we do not require the conformity $\widetilde{V}(\Omega) \subset H^1(\Omega)$.

Define the local bilinear form

$$a_k(u,v) = \int_{\Omega_k} \omega \nabla u \cdot \nabla v dx \; u,v \in H^1(\Omega_k).$$

The discrete problem of (2) is: Find $u_h = (u_{h_1}, \cdots, u_{h_N}) \in \widetilde{V}(\Omega)$, such that

$$(3) \qquad \sum_{k=1}^{N} a_k(u_{h_k}, v_{h_k}) = (f, v_h), \; \forall \, v_h \in \widetilde{V}(\Omega).$$

It is easy to see that the bilinear form defining (3) is coercive in $\widetilde{V}(\Omega)$. Then the existence of the solution u_h can be guaranteed.

3. The Main Result

For simplicity, we will frequently use the notations $\lesssim$ and $\overline{\overline{\sim}}$. For any two non-negative quantities x and y, $x \lesssim y$ means that $x \leq Cy$ for some constant C independent of mesh size h, subdomain size d and the related parameters. Similarly, $x \overline{\overline{\sim}} y$ means $x \lesssim y$ and $y \lesssim x$.

Define

$$\|v\|_A = (\sum_{k=1}^{N} a_k(v_k, \; v_k))^{\frac{1}{2}}, \; v = (v_1, \cdots, v_N) \in \prod_{k=1}^{N} H^1(\Omega_k).$$

Theorem 3.1 Assume that $u|_{\Omega_k} \in H^{1+\alpha_k}(\Omega_k)$ with $\alpha_k \in (\frac{1}{2}, 1]$ $(k = 1, \cdots, N)$. Let u_h be the solution of (3). When one of the following two conditions holds:
 (i) $V_j(\Gamma_{ij}) \subset V_i(\Gamma_{ij})$;
 (ii) $\alpha_i \leq \alpha_j$, and $h_i \leq h_j^{2\alpha_j}$,
we have

$$(4) \qquad \|u - u_h\|_A \lesssim (\sum_{k=1}^{N} (1 + \log \frac{d}{h_k}) h_k^{2\alpha_k} \|u\|_{1+\alpha_k, \; \Omega_k}^2)^{\frac{1}{2}}.$$

Remark 3.1 The error estimate described in Theorem 3.1 is different slightly from the most existing results: there is a logarithm factor in (4). It seems that this

logarithm factor can not be eliminated.

Remark 3.2 It is known that fine mesh would be used in the subdomain associated with a low regularity. Thus, the assumption $\alpha_i \leq \alpha_j$ in the condition (ii) is natural, since the multiplier space $W(\Gamma_{ij})$ is defined by $\mathcal{T}_{ij}$ with $h_i \leq h_j$.

As usual, the result can be proved by Strang lemma. The main difficulty of the proof lies in how to derive various interpolation errors of the operator Π_{ij}. The proof of Theorem 3.1 will be given in Subsection 3.2. In Subsection 3.1, we first give some lemmas.

3.1. Some lemmas. For an element of $\mathcal{T}_{ij}$, we call the element to be an **interior element**, if all of its vertexes are in the interior of Γ_{ij}. In addition, for each Γ_{ij}, let $\mathcal{N}_{ij}$ and $\mathcal{N}_{ji}$ denote the sets of all nodes of the triangulation $\mathcal{T}_{ij}$ and of $\mathcal{T}_{ji}$ respectively.

Throughout the paper, we set $v_i = v_h|_{\Omega_i}$ and $v_j = v_h|_{\Omega_j}$ for $v_h \in \widetilde{V}(\Omega)$. Let $\pi_{ij} : C(\Gamma_{ij}) \to V_i(\Gamma_{ij})$ be the standard nodal interpolation operator.

The following Lemma 3.1~3.5 will be used to analyze the approximate error.

Lemma 3.1 If $h_i \leq h_j$, then the following estimate holds

$$\|(\pi_{ij} - \Pi_{ij})v_j\|_{0, \ \Gamma_{ij}} \lesssim h_i^{\frac{1}{2}} |v_j|_{1, \ \Omega_j},$$

where h_i, h_j denotes the mesh sizes of $\mathcal{T}_{ij}$ and $\mathcal{T}_{ji}$ respectively.

Proof: It is clear that $(\pi_{ij} - \Pi_{ij})v_j \in V_i(\Gamma_{ij})$. Thus,

$$\|(\pi_{ij} - \Pi_{ij})v_j\|_{0, \ \Gamma_{ij}}^2 \ \overline{\overline{\sim}} \ \|(\pi_{ij} - \Pi_{ij})v_j\|_{0, \ \Gamma_{ij},h}^2.$$

We further get by the definition of the discrete $L^2(\Gamma_{ij})$ norm

$$\|(\pi_{ij} - \Pi_{ij})v_j\|_{0, \ \Gamma_{ij}}^2 \ \overline{\overline{\sim}} \ h_i^2 \sum_{x_k \in \mathcal{N}_{ij}} (\pi_{ij}v_j(x_k) - \Pi_{ij}v_j(x_k))^2.$$

For each internal node $x_k \in \mathcal{N}_{ij}$, we have by the definition of π_{ij} and Π_{ij}

$$\pi_{ij}v_j(x_k) - \Pi_{ij}v_j(x_k) = 0.$$

Thus,

$$(5) \qquad \|(\pi_{ij} - \Pi_{ij})v_j\|_{0, \ \Gamma_{ij}}^2 \ \overline{\overline{\sim}} \ h_i^2 \sum_{x_k \in \ \partial\Gamma_{ij} \cap \mathcal{N}_{ij}} (\pi_{ij}v_j(x_k) - \Pi_{ij}v_j(x_k))^2.$$

In the following we estimate the right side of (5) by three steps.

Step 1: Transform the sum in (5) into a sum over the edges close to Γ_{ij}.

Let x_k be a boundary node on $\partial\Gamma_{ij}$, and let $x_{k_1}, x_{k_2}, x_{k_3} \in \mathcal{N}_{ij}$ denote the interior nodes which are connected to x_k by an edge. Then, there are three different location relations between x_k and the interior nodes neighboring to x_k (see Figure 1 (a)-(c)).

The definition of Π_{ij} shows that for each interior node x in $\mathcal{N}_{ij}$

$$\Pi_{ij}v_j(x) = v_j(x),$$

and for all cases showed in Figure 1,

$$\Pi_{ij}v_j(x_k) = a_1\Pi_{ij}v_j(x_{k_1}) + a_2\Pi_{ij}v_j(x_{k_2})$$

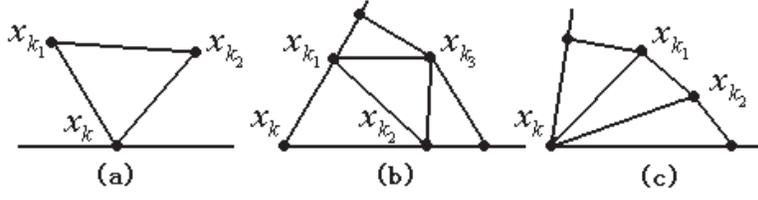where $a_1 > 0, a_2 > 0$ and $a_1 + a_2 = 1$.

FIGURE 1. (a) x_k is not a vertex. (b) x_k is a vertex, and all edges connected to x_k lie on $\partial\Gamma_{ij}$. (c) x_k is a vertex, and there is at least one internal node which is connected to x_k by an edge.

Specially, for the case showed in Figure 1 (b), we have by the definition of Π_{ij}

$$\Pi_{ij}v_j(x_{k_1}) = \Pi_{ij}v_j(x_{k_3}),\ \Pi_{ij}v_j(x_{k_2}) = \Pi_{ij}v_j(x_{k_3}).$$

Thus, we derive for the case (a) and (c)

$$\begin{aligned}
(\pi_{ij}v_j(x_k) - \Pi_{ij}v_j(x_k))^2 &= (v_j(x_k) - \Pi_{ij}v_j(x_k))^2 \\
&\leq a_1^2(v_j(x_k) - \Pi_{ij}v_j(x_{k_1}))^2 + a_2^2(v_j(x_k) - \Pi_{ij}v_j(x_{k2}))^2 \\
&= a_1^2(v_j(x_k) - v_j(x_{k_1}))^2 + a_2^2(v_j(x_k) - v_j(x_{k2}))^2 \\
&\leq (v_j(x_k) - v_j(x_{k_1}))^2 + (v_j(x_k) - v_j(x_{k2}))^2,
\end{aligned}$$

and for the case (b)

$$\begin{aligned}
(\pi_{ij}v_j(x_k) - \Pi_{ij}v_j(x_k))^2 &= (v_j(x_k) - \Pi_{ij}v_j(x_k))^2 \\
&\leq a_1^2(v_j(x_k) - \Pi_{ij}v_j(x_{k_1}))^2 + a_2^2(v_j(x_k) - \Pi_{ij}v_j(x_{k_2}))^2 \\
&= a_1^2(v_j(x_k) - \Pi_{ij}v_j(x_{k_3}))^2 + a_2^2(v_j(x_k) - \Pi_{ij}v_j(x_{k_3}))^2 \\
&= a_1^2(v_j(x_k) - v_j(x_{k_3}))^2 + a_2^2(v_j(x_k) - v_j(x_{k_3}))^2 \\
&\leq (v_j(x_k) - v_j(x_{k_3}))^2 \\
&\leq (v_j(x_k) - v_j(x_{k_1}))^2 + (v_j(x_{k_1}) - v_j(x_{k_3}))^2.
\end{aligned}$$

Note that we can also get for the case (b)

$$(\pi_{ij}v_j(x_k) - \Pi_{ij}v_j(x_k))^2 \leq (v_j(x_k) - v_j(x_{k_2}))^2 + (v_j(x_{k_2}) - v_j(x_{k_3}))^2.$$

Let $\mathcal{T}_{ij}^b$ denote the set of elements in $\mathcal{T}_{ij}$, which have either one single vertex or one edge lying on $\partial\Gamma_{ij}$, and let $\mathcal{E}_{ij}^b$ be the set of edges of all elements in $\mathcal{T}_{ij}^b$. Then, we obtain by (5) and above discussions

$$\begin{aligned}
\|(\pi_{ij} - \Pi_{ij})v_j\|_{0,\ \Gamma_{ij}}^2 &\lesssim h_i^2 \sum_{\tau \in \mathcal{T}_{ij}^b} \sum_{y,\ z \in \tau \cap \mathcal{N}_{ij}} (v_j(y) - v_j(z))^2 \\
&\lesssim h_i^2 \sum_{e \in \mathcal{E}_{ij}^b} (v_j(x_1^e) - v_j(x_2^e))^2.
\end{aligned}$$

(6)

Hereafter, we use x_1^e and x_2^e to denote the two endpoints of an edge e associated with $\mathcal{T}_{ij}$.

Step 2: Extend the sum in (6) into a multiple sums involving elements in $\mathcal{T}_{ji}$.

Consider an edge $e \in \mathcal{E}_{ij}^b$. There are different location relations between the two endpoints x_1^e and x_2^e and the elements in $\mathcal{T}_{ji}$: x_1^e and x_2^e lie in the same element τ^e of $\mathcal{T}_{ji}$ (see Figure 2 (a), (b) and (c)), or x_1^e and x_2^e lie in two different elements τ_1^e and τ_2^e of $\mathcal{T}_{ji}$ respectively (see Figure 2 (d), (e) and (f)). Let $\mathcal{T}_{ji}^e \subset \mathcal{T}_{ji}$ be the set of elements containing the endpoints of e. Namely, $\mathcal{T}_{ji}^e = \{\tau^e\}$ or $\mathcal{T}_{ji}^e = \{\tau_1^e,\ \tau_2^e\}$.
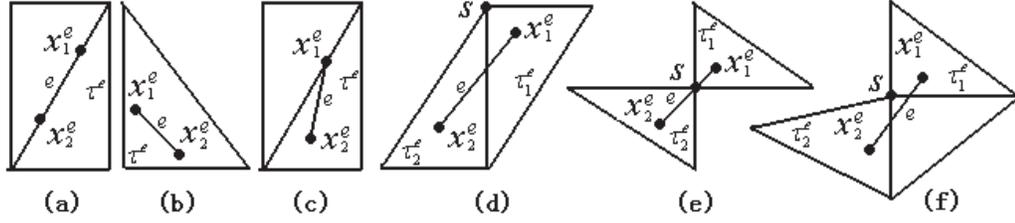
FIGURE 2. various location relations between the two endpoints x_1^e and x_2^e and the elements in $\mathcal{T}_{ji}$.

By the triangle inequality, we have

$$(v_j(x_1^e) - v_j(x_2^e))^2 \lesssim (v_j(x_1^e) - v_j(x_s))^2 + (v_j(x_s) - v_j(x_2^e))^2$$

for any $x_s \in \mathcal{N}_{ji}$. Noting that v_j is linear on each element of $\mathcal{T}_{ji}$, we can derive for all cases showed in Figure 2

(7) $$(v_j(x_1^e) - v_j(x_2^e))^2 \lesssim \sum_{\tau \in \mathcal{T}_{ji}^e} \sum_{y, z \in \tau \cap \mathcal{N}_{ji}} (v_j(y) - v_j(z))^2.$$

Thus, we obtain by (6) $\sim$ (7)

(8) $$\|(\pi_{ij} - \Pi_{ij})v_j\|_{0,\ \Gamma_{ij}}^2 \lesssim h_i^2 \sum_{e \in \mathcal{E}_{ij}^b} \sum_{\tau \in \mathcal{T}_{ji}^e} \sum_{y, z \in \tau \cap \mathcal{N}_{ji}} (v_j(y) - v_j(z))^2.$$

Step 3: Eliminate the sum over $\mathcal{E}_{ij}^b$.

For an element τ of $\mathcal{T}_{ji}$, there are at most $O(\frac{h_j}{h_i})$ edges $e \in \mathcal{E}_{ij}^b$ such that an endpoint of e contained in τ. Namely, each element $\tau \in \mathcal{T}_{ji}$ repeats at most $O(\frac{h_j}{h_i})$ times in the sums $\sum_{e \in \mathcal{E}_{ij}^b} \sum_{\tau \in \mathcal{T}_{ji}^e}$. Thus, we derive by (8)

(9) $$\|(\pi_{ij} - \Pi_{ij})v_j\|_{0,\ \Gamma_{ij}}^2 \lesssim \frac{h_j}{h_i} \cdot h_i^2 \sum_{\tau \in \mathcal{T}_{ji}} \sum_{y, z \in \tau \cap \mathcal{N}_{ji}} (v_j(y) - v_j(z))^2.$$

The definition of the discrete $H^1(\Gamma_{ij})$ semi-norm shows

(10) $$\sum_{\tau \in \mathcal{T}_{ji}} \sum_{y, z \in \tau \cap \mathcal{N}_{ji}} (v_j(y) - v_j(z))^2 \eqsim |v_j|_{1,\ \Gamma_{ij}}^2.$$

Thus, we get by (9), (10), the inverse estimate and the trace theorem

$$\|(\pi_{ij} - \Pi_{ij})v_j\|_{0,\ \Gamma_{ij}}^2 \lesssim h_i^2 \cdot \frac{h_j}{h_i} |v_j|_{1,\ \Gamma_{ij}}^2 \lesssim h_i |v_j|_{\frac{1}{2},\ \Gamma_{ij}}^2 \lesssim h_i |v_j|_{1,\ \Omega_j}^2.$$

The proof is then finished. ∎

Remark 3.3 For $v_i = v_h|_{\Omega_i}$, the following estimate also holds

$$\|(\pi_{ij} - \Pi_{ij})v_i\|_{0,\ \Gamma_{ij}} \lesssim h_i^{\frac{1}{2}} |v_i|_{1,\ \Omega_i}.$$

In fact, the above estimate is a direct result of Lemma 3.1 when $V_j(\Gamma_{ij}) = V_i(\Gamma_{ij})$.

Lemma 3.2 Assume that $\alpha_i \in (\frac{1}{2}, 1]$. Then,

$$\text{(11)} \qquad \|\Pi_{ij}v_i - v_i\|_{\frac{1}{2}-\alpha_i,\ \Gamma_{ij}} \lesssim (1 + \log \frac{d}{h_i})^{\frac{1}{2}} h_i^{\alpha_i} |v_i|_{1,\ \Omega_i}.$$

Proof: By the inverse estimate, we get

$$\text{(12)} \qquad \|v_i - \Pi_{ij}v_i\|_{\frac{1}{2}-\alpha_i,\ \Gamma_{ij}} \lesssim h_i^{\alpha_i-1} \|v_i - \Pi_{ij}v_i\|_{-\frac{1}{2},\ \Gamma_{ij}}.$$

The definition of the operator Π_{ij} indicates that the following equality

$$v_i - \Pi_{ij}v_i = 0$$

holds in all internal elements on Γ_{ij}.

Then, we have by Lemma 6.8 in [11]

$$\text{(13)} \qquad \|v_i - \Pi_{ij}v_i\|_{-\frac{1}{2},\ \Gamma_{ij}} \lesssim (1 + \log \frac{d}{h_i})^{\frac{1}{2}} h_i^{\frac{1}{2}} \|v_i - \Pi_{ij}v_i\|_{0,\ \Gamma_{ij}}.$$

On the other hand, we derive by Remark 3.3, the inverse estimate and the trace theorem

$$\text{(14)} \qquad \begin{aligned} \|v_i - \Pi_{ij}v_i\|_{0,\ \Gamma_{ij}} &= \|v_i - \pi_{ij}v_i\|_{0,\ \Gamma_{ij}} + \|\pi_{ij}v_i - \Pi_{ij}v_i\|_{0,\ \Gamma_{ij}} \\ &\lesssim h_i^{\alpha_i+\frac{1}{2}} |v_i|_{\alpha_i+\frac{1}{2},\ \Gamma_{ij}} + h_i^{\frac{1}{2}} |v_i|_{1,\ \Omega_i} \\ &\lesssim h_i^{\alpha_i+\frac{1}{2}} h_i^{-\alpha_i} |v_i|_{\frac{1}{2},\ \Gamma_{ij}} + h_i^{\frac{1}{2}} |v_i|_{1,\ \Omega_i} \\ &\lesssim h_i^{\frac{1}{2}} |v_i|_{1,\ \Omega_i}. \end{aligned}$$

Combining (12), (13) with (14), yields the desired result. ∎

Lemma 3.3 Assume that $V_j(\Gamma_{ij}) \subset V_i(\Gamma_{ij})$ and $\alpha_j \in (\frac{1}{2}, 1]$. Then,

$$\text{(15)} \qquad \|v_j - \Pi_{ij}v_j\|_{\frac{1}{2}-\alpha_j,\ \Gamma_{ij}} \lesssim (1 + \log \frac{d}{h_j})^{\frac{1}{2}} h_j^{\alpha_j} |v_j|_{1,\ \Omega_j}.$$

Proof: By the definition of the operator Π_{ij}, the equality

$$v_j - \Pi_{ij}v_j = 0$$

holds on all interior elements in Γ_{ij}. Here, we have used the assumption $V_j(\Gamma_{ij}) \subset V_i(\Gamma_{ij})$.

For $v_j \in V_j(\Gamma_{ij}) \subset V_i(\Gamma_{ij})$, we get by the inverse estimate

$$\text{(16)} \qquad \|v_j - \Pi_{ij}v_j\|_{\frac{1}{2}-\alpha_j,\ \Gamma_{ij}} \lesssim h_i^{\alpha_j-1} \|v_j - \Pi_{ij}v_j\|_{-\frac{1}{2},\ \Gamma_{ij}}.$$

By Lemma 6.8 in [11], we have

$$\text{(17)} \qquad \|v_j - \Pi_{ij}v_j\|_{-\frac{1}{2},\ \Gamma_{ij}} \lesssim (1 + \log \frac{d}{h_i})^{\frac{1}{2}} h_i^{\frac{1}{2}} \|v_j - \Pi_{ij}v_j\|_{0,\ \Gamma_{ij}}.$$

Note that $h_j \geq h_i$, we obtain by Lemma 3.1, the inverse estimate and the trace theorem

$$\text{(18)} \qquad \begin{aligned} \|v_j - \Pi_{ij}v_j\|_{0,\ \Gamma_{ij}} &= \|v_j - \pi_{ij}v_j\|_{0,\ \Gamma_{ij}} + \|\pi_{ij}v_j - \Pi_{ij}v_j\|_{0,\ \Gamma_{ij}} \\ &\lesssim h_i^{\alpha_j+\frac{1}{2}} |v_j|_{\alpha_j+\frac{1}{2},\ \Gamma_{ij}} + h_i^{\frac{1}{2}} |v_j|_{1,\ \Omega_j} \\ &\lesssim h_i^{\alpha_j+\frac{1}{2}} h_j^{-\alpha_j} |v_j|_{\frac{1}{2},\ \Gamma_{ij}} + h_i^{\frac{1}{2}} |v_j|_{1,\ \Omega_j} \\ &\lesssim h_i^{\frac{1}{2}} |v_j|_{1,\ \Omega_j}. \end{aligned}$$

Since $0 < h_i \le h_j < 1$ and $\alpha_j > \frac{1}{2}$, we have

$$(19) \qquad (1 + \log \frac{d}{h_i})^{\frac{1}{2}} \lesssim (1 + \log \frac{d}{h_j})^{\frac{1}{2}} (\frac{h_j}{h_i})^{\frac{1}{2}} \lesssim (1 + \log \frac{d}{h_j})^{\frac{1}{2}} (\frac{h_j}{h_i})^{\alpha_j}.$$

Combining (16), (17), (18) with (19), gives the desired result. ∎

Lemma 3.4 Assume that $h_i \le h_j^{2\alpha_j}$ $(\alpha_j \in (\frac{1}{2}, 1])$. Then,

$$(20) \qquad \|v_j - \Pi_{ij}v_j\|_{\frac{1}{2}-\alpha_j, \ \Gamma_{ij}} \lesssim h_j^{\alpha_j} |v_j|_{1, \ \Omega_j}.$$

Proof: It is obvious that

$$(21) \qquad \|v_j - \Pi_{ij}v_j\|_{\frac{1}{2}-\alpha_j, \ \Gamma_{ij}} \lesssim \|v_j - \Pi_{ij}v_j\|_{0, \ \Gamma_{ij}}.$$

By the inverse estimate, the trace theorem and the assumption that $h_i \le h_j^{2\alpha_j}$, we obtain

$$(22) \qquad \begin{aligned} \|v_j - \pi_{ij}v_j\|_{0, \ \Gamma_{ij}} &\lesssim h_i^{\alpha_j+\frac{1}{2}} |v_j|_{\alpha_j+\frac{1}{2}, \ \Gamma_{ij}} \\ &\lesssim h_i^{\alpha_j+\frac{1}{2}} \cdot h_j^{-\alpha_j} |v_j|_{\frac{1}{2}, \ \Gamma_{ij}} \\ &\lesssim h_j^{\alpha_j} |v_j|_{1, \ \Omega_j}. \end{aligned}$$

By the above inequality, Lemma 3.1 and $h_i \le h_j^{2\alpha_j}$, we can derive

$$(23) \qquad \begin{aligned} \|v_j - \Pi_{ij}v_j\|_{0, \ \Gamma_{ij}} &= \|v_j - \pi_{ij}v_j\|_{0, \ \Gamma_{ij}} + \|\pi_{ij}v_j - \Pi_{ij}v_j\|_{0, \ \Gamma_{ij}} \\ &\lesssim h_j^{\alpha_j} |v_j|_{1, \ \Omega_j} + h_i^{\frac{1}{2}} |v_j|_{1, \ \Omega_j} \\ &\lesssim h_j^{\alpha_j} |v_j|_{1, \ \Omega_j}. \end{aligned}$$

By (21), (22) and (23), yields the inequality (20). ∎

For convenience, set $\omega = \omega_k$ on Ω_k. Let $\frac{\partial u}{\partial n_i}$ denote the unit outward normal derivative of $u|_{\Omega_i}$ on $\partial \Omega_i$.

Lemma 3.5 Assume that $u|_{\Omega_k} \in H^{1+\alpha_k}(\Omega_k)$ with $\alpha_k \in (\frac{1}{2}, 1]$ ($k = 1, \cdots, N$). Then,

$$(24) \qquad \begin{aligned} |\int_{\Gamma_{ij}} \omega_i \frac{\partial u}{\partial n_i} \cdot (v_i - v_j) ds| &\lesssim \sum_{\Gamma_{ij}} \|\omega_j u\|_{1+\alpha_j, \ \Omega_j} \cdot \|v_j - \Pi_{ij}v_j\|_{\frac{1}{2}-\alpha_j, \ \Gamma_{ij}} \\ &+ \sum_{\Gamma_{ij}} \|\omega_i u\|_{1+\alpha_i, \ \Omega_i} \cdot \|v_i - \Pi_{ij}v_i\|_{\frac{1}{2}-\alpha_i, \ \Gamma_{ij}}. \end{aligned}$$

Proof : Using the pointwise matching condition, yields

$$(25) \qquad \begin{aligned} |\int_{\Gamma_{ij}} \omega_i \frac{\partial u}{\partial n_i} \cdot (v_i - v_j) ds| &\le |\int_{\Gamma_{ij}} \omega_i \frac{\partial u}{\partial n_i} \cdot (v_i - \Pi_{ij}v_i + \Pi_{ij}v_j - v_j) ds| \\ &\le |\int_{\Gamma_{ij}} \omega_i \frac{\partial u}{\partial n_i} \cdot (\Pi_{ij}v_i - v_i) ds| \\ &+ |\int_{\Gamma_{ij}} \omega_j \frac{\partial u}{\partial n_j} \cdot (\Pi_{ij}v_j - v_j) ds|. \end{aligned}$$

Here, we have used the fact that $\omega_i \frac{\partial u}{\partial n_i} = -\omega_j \frac{\partial u}{\partial n_j}$. By the definition of the negative norm and the trace theorem, we derive

$$(26) \qquad \begin{aligned} |\int_{\Gamma_{ij}} \omega_i \frac{\partial u}{\partial n_i} \cdot (v_i - \Pi_{ij}v_i) ds| &\lesssim \|\omega_i \frac{\partial u}{\partial n_i}\|_{\alpha_i-\frac{1}{2}, \ \Gamma_{ij}} \cdot \|v_i - \Pi_{ij}v_i\|_{\frac{1}{2}-\alpha_i, \ \Gamma_{ij}} \\ &\lesssim \|\omega_i u\|_{1+\alpha_i, \ \Omega_i} \cdot \|v_i - \Pi_{ij}v_i\|_{\frac{1}{2}-\alpha_i, \ \Gamma_{ij}} \end{aligned}$$

and

$$(27) \qquad |\int_{\Gamma_{ij}} \omega_j \frac{\partial u}{\partial n_j} \cdot (v_j - \Pi_{ij}v_j) ds| \lesssim \|\omega_j u\|_{1+\alpha_j, \ \Omega_j} \cdot \|v_j - \Pi_{ij}v_j\|_{\frac{1}{2}-\alpha_j, \ \Gamma_{ij}}.$$

Substituting (26), (27) into (25), yields the inequality (24). ∎

The following results will be used to analyze the consistence error.

Lemma 3.6 The following inequality holds

$$(28) \qquad \|\Pi_{ij}v\|_{0,\,\Gamma_{ij}} \lesssim \|\pi_{ij}v\|_{0,\,\Gamma_{ij}}, \quad \forall v \in C(\Gamma_{ij}).$$

Proof : By the definition of the discrete $L^2(\Gamma_{ij})$-norm, we have

$$\|\Pi_{ij}\ v\|_{0,\,\Gamma_{ij}} \overset{\equiv}{\sim} \|\Pi_{ij}\ v\|_{0,\,\Gamma_{ij},\,h} \lesssim \|\pi_{ij}\ v\|_{0,\,\Gamma_{ij},\,h} \overset{\equiv}{\sim} \|\pi_{ij}\ v\|_{0,\,\Gamma_{ij}}.$$

This gives the desired result. ∎

Let us number all internal nodes in $\mathcal{N}_{ij}$ from 1 to N_{ij}^0. For the convenience, we define the operator $\widehat{\Pi}_{ij} : W(\Gamma_{ij}) \to V_i^0(\Gamma_{ij})$ by

$$\widehat{\Pi}_{ij}\ v = \sum_{i=1}^{N_{ij}^0} v(x_i) \cdot \psi_i,$$

where $\psi_i \in V_i^0(\Gamma_{ij})$ is the nodal basis function related to x_i.

The following result gives the stability of the operator $\widehat{\Pi}_{ij}$ with respect to L^2 norm.

Lemma 3.7 The operator $\widehat{\Pi}_{ij}$ satisfies

$$(29) \qquad \|\widehat{\Pi}_{ij}v\|_{0,\,\Gamma_{ij}} \lesssim \|v\|_{0,\,\Gamma_{ij}}, \quad \forall\, v \in\, W(\Gamma_{ij}).$$

Proof : Note that

$$\|\widehat{\Pi}_{ij}v\|_{0,\,\Gamma_{ij}} \overset{\equiv}{\sim} \|\widehat{\Pi}_{ij}v\|_{0,\,\Gamma_{ij},\,h} \quad \text{and} \quad \|v\|_{0,\,\Gamma_{ij}} \overset{\equiv}{\sim} \|v\|_{0,\,\Gamma_{ij},\,h}.$$

By the definition of the discrete $L^2(\Gamma_{ij})$-norm and $\widehat{\Pi}_{ij}$, we can derive the inequality (29). ∎

Lemma 3.8 For $v_i \in V(\Omega_i)$ and $v_j \in V(\Omega_j)$, we have

$$\Pi_{ij}(v_j|_{\Gamma_{ij}}) - \Pi_{ij}(v_i|_{\Gamma_{ij}}) = \Pi_{ij}(\widehat{\Pi}_{ij}(\Pi_{ij}(v_j|_{\Gamma_{ij}}) - (v_i|_{\Gamma_{ij}}))).$$

Proof : For convenience, set

$$v = \Pi_{ij}(v_j|_{\Gamma_{ij}}) - \Pi_{ij}(v_i|_{\Gamma_{ij}}) \in W(\Gamma_{ij}).$$

Then, by the definitions of Π_{ij} and $\widehat{\Pi}_{ij}$, we know that

$$\Pi_{ij}\widehat{\Pi}_{ij}v(x_k) = \widehat{\Pi}_{ij}v(x_k) = v(x_k)$$

at each interior nodes x_k. Let ϕ_k denote the basis function of $W(\Gamma_{ij})$ associated with the interior node x_k $(k = 1, \cdots, N_{ij}^0)$. Thus, the definition of $W(\Gamma_{ij})$ implies that

$$\Pi_{ij}\widehat{\Pi}_{ij}v(x) = \sum_{k=1}^{N_{ij}^0} \Pi_{ij}\widehat{\Pi}_{ij}v(x_k)\phi_k(x) = \sum_{k=1}^{N_{ij}^0} v(x_k)\phi_k(x) = v(x)$$

for any $x \in \Gamma_{ij}$. Here, we have used the fact that both v and $\Pi_{ij}\widehat{\Pi}_{ij}v$ belong to $W(\Gamma_{ij})$. This gives the desired result. ∎

Lemma 3.9 Assume that $u|_{\Omega_k} \in H^{1+\alpha_k}(\Omega_k)$ with $\alpha_k \in (\frac{1}{2}, 1]$ $(\,k = 1, \cdots, N)$. Let $\widetilde{v}_k \in V(\Omega_k)$ be the nodal interpolation of $u|_{\Omega_k}$. Then, we have for each Γ_{ij}

$$\|\Pi_{ij}(\widetilde{v}_i|_{\Gamma_{ij}} - \widetilde{v}_j|_{\Gamma_{ij}})\|_{0,\,\Gamma_{ij}} \lesssim h_i^{\alpha_i+\frac{1}{2}}\|u\|_{1+\alpha_i,\,\Omega_i} + h_j^{\alpha_j+\frac{1}{2}}\|u\|_{1+\alpha_j,\,\Omega_j}.$$

Proof : We derive by Lemma 3.6

$$\|\Pi_{ij}(\widetilde{v}_i|_{\Gamma_{ij}} - \widetilde{v}_j|_{\Gamma_{ij}})\|_{0,\ \Gamma_{ij}} \quad \lesssim \|\pi_{ij}(\widetilde{v}_i|_{\Gamma_{ij}} - \widetilde{v}_j|_{\Gamma_{ij}})\|_{0,\ \Gamma_{ij}}$$

$$\overline{\overline{\approx}} \ \|\pi_{ij}(\widetilde{v}_i|_{\Gamma_{ij}} - \widetilde{v}_j|_{\Gamma_{ij}}) - u|_{\Gamma_{ij}} + u|_{\Gamma_{ij}}\|_{0,\ \Gamma_{ij}}$$

$$\overline{\overline{\approx}} \ \|\widetilde{v}_i|_{\Gamma_{ij}} - u|_{\Gamma_{ij}} + u|_{\Gamma_{ij}} - \pi_{ij}(\widetilde{v}_j|_{\Gamma_{ij}})\|_{0,\ \Gamma_{ij}}$$

$$\lesssim \|u|_{\Gamma_{ij}} - \widetilde{v}_i|_{\Gamma_{ij}}\|_{0,\ \Gamma_{ij}} + \|u|_{\Gamma_{ij}} - \pi_{ij}(\widetilde{v}_j|_{\Gamma_{ij}})\|_{0,\ \Gamma_{ij}}$$

$$\lesssim h_i^{\frac{1}{2}+\alpha_i}\|u|_{\Gamma_{ij}}\|_{\frac{1}{2}+\alpha_i,\ \Gamma_{ij}} + \|u|_{\Gamma_{ij}} - \pi_{ij}(\widetilde{v}_j|_{\Gamma_{ij}})\|_{0,\ \Gamma_{ij}}.$$

In addition, we have

$$\|u|_{\Gamma_{ij}} - \pi_{ij}(\widetilde{v}_j|_{\Gamma_{ij}})\|_{0,\ \Gamma_{ij}} \quad = \|u|_{\Gamma_{ij}} - \widetilde{v}_j|_{\Gamma_{ij}} + \widetilde{v}_j|_{\Gamma_{ij}} - \pi_{ij}(\widetilde{v}_j|_{\Gamma_{ij}})\|_{0,\ \Gamma_{ij}}$$

$$\lesssim \|u|_{\Gamma_{ij}} - \widetilde{v}_j|_{\Gamma_{ij}}\|_{0,\ \Gamma_{ij}} + \|\widetilde{v}_j|_{\Gamma_{ij}} - \pi_{ij}(\widetilde{v}_j|_{\Gamma_{ij}})\|_{0,\ \Gamma_{ij}}$$

$$\lesssim h_j^{\frac{1}{2}+\alpha_j}\|u|_{\Gamma_{ij}}\|_{\frac{1}{2}+\alpha_j,\ \Gamma_{ij}} + \|\widetilde{v}_j|_{\Gamma_{ij}} - \pi_{ij}(\widetilde{v}_j|_{\Gamma_{ij}})\|_{0,\ \Gamma_{ij}}.$$

Then, we get for any constant $s_{ij} \in (1, \frac{1}{2} + \alpha_j)$ (note that $h_i \leq h_j$)

$$\|\widetilde{v}_j|_{\Gamma_{ij}} - \pi_{ij}(\widetilde{v}_j|_{\Gamma_{ij}})\|_{0,\ \Gamma_{ij}} \quad \lesssim \|(\pi_{ij} - I)(u|_{\Gamma_{ij}} - \widetilde{v}_j|_{\Gamma_{ij}})\|_{0,\ \Gamma_{ij}} + \|u|_{\Gamma_{ij}} - \widetilde{v}_i|_{\Gamma_{ij}}\|_{0,\ \Gamma_{ij}}$$

$$\lesssim h_i^{s_{ij}}\|u|_{\Gamma_{ij}} - \widetilde{v}_j|_{\Gamma_{ij}}\|_{s_{ij},\ \Gamma_{ij}} + h_i^{\frac{1}{2}+\alpha_i}\|u|_{\Gamma_{ij}}\|_{\frac{1}{2}+\alpha_i,\ \Gamma_{ij}}$$

$$\lesssim h_i^{s_{ij}} \cdot h_j^{\frac{1}{2}+\alpha_j-s_{ij}}\|u|_{\Gamma_{ij}}\|_{\frac{1}{2}+\alpha_j,\ \Gamma_{ij}} + h_i^{\frac{1}{2}+\alpha_i}\|u|_{\Gamma_{ij}}\|_{\frac{1}{2}+\alpha_i,\ \Gamma_{ij}}$$

$$\lesssim h_j^{\frac{1}{2}+\alpha_j}\|u|_{\Gamma_{ij}}\|_{\frac{1}{2}+\alpha_j,\ \Gamma_{ij}} + h_i^{\frac{1}{2}+\alpha_i}\|u|_{\Gamma_{ij}}\|_{\frac{1}{2}+\alpha_i,\ \Gamma_{ij}},$$

where I denotes an identical operator.

Combining above three inequalities, we can derive the desired result. ∎

3.2. The proof of Theorem 3.1. For each Ω_i, let $J(i)$ denote the set of all faces Γ_{ij} satisfying

(a) $\Gamma_{ij} \subset \partial\Omega_i$

and

(b) $W(\Gamma_{ij}) \subset V_i(\Gamma_{ij})$.

The condition (a) implies that Γ_{ij} is a face of Ω_i. The condition (b) indicates that we take $\mathcal{T}_{ij}$ (instead of $\mathcal{T}_{ji}$) as the triangulations defining the nodal basis functions of $W(\Gamma_{ij})$.

The proof of Theorem 3.1 is based on the following Strang Lemma

$$(30) \qquad \|u - u_h\|_A \lesssim \inf_{v_h \in \widetilde{V}(\Omega)} \|u - v_h\|_A + \sup_{v_h \in \widetilde{V}(\Omega)} \frac{|\sum_{k=1}^{N} \int_{\partial\Omega_k} \omega_k \frac{\partial u}{\partial n_k}(v_h|_{\Omega_k})ds|}{\|v_h\|_A}.$$

Step 1: Estimate the first term (the consistence error) in the right of (30).

Let $u_{h_k} \in V(\Omega_k)$ be the standard nodal interpolation of $u|_{\Omega_k}$. For each Γ_{ij}, set

$$t_{ij} = \widehat{\Pi}_{ij}(\Pi_{ij}(u_{h_j}|_{\Gamma_{ij}}) - \Pi_{ij}(u_{h_i}|_{\Gamma_{ij}})).$$

Then, $t_{ij} \in V_i^0(\Gamma_{ij})$. Let $\widetilde{t}_{ij}^i \in V(\partial\Omega_i)$ be the zero extension of t_{ij}. Let $R_{ij}^i : V(\partial\Omega_i) \to V(\Omega_i)$ denote the discrete harmonic extension operator, and set

$$r_{ij}^i = R_{ij}^i(\widetilde{t}_{ij}^i).$$

Define $v_h \in V(\Omega)$ by

$$v_h|_{\Omega_i} = u_{h_i} + \sum_{\Gamma_{ij} \in\ J(i)} r_{ij}^i,\ i = 1, \cdots, N.$$

It can be verified by Lemma 3.8 that such v_h belongs to $\tilde{V}(\Omega)$.

Using the stability of the discrete harmonic extension operator and the inverse estimate, we deduce

$$(31) \qquad \|r_{ij}^i\|_{1,\ \Omega_i} \lesssim \|\tilde{t}_{ij}^i\|_{\frac{1}{2},\ \partial\Omega_i} \lesssim h_i^{-\frac{1}{2}} \|\tilde{t}_{ij}^i\|_{0,\ \partial\Omega_i} = h_i^{-\frac{1}{2}} \|t_{ij}\|_{0,\ \Gamma_{ij}}.$$

This, together with Lemma 3.6, yields

$$
\begin{aligned}
\|u - v_h\|_{1,\ \Omega_i} \quad &\lesssim \|u - u_{h_i}\|_{1,\ \Omega_i} + \sum_{\Gamma_{ij}\in J(i)} \|r_{ij}^i\|_{1,\ \Omega_i} \\
&\lesssim \|u - u_{h_i}\|_{1,\ \Omega_i} + \sum_{\Gamma_{ij}\in J(i)} h_i^{-\frac{1}{2}} \|t_{ij}\|_{0,\ \Gamma_{ij}} \\
&= \|u - u_{h_i}\|_{1,\ \Omega_i} + \sum_{\Gamma_{ij}\in J(i)} h_i^{-\frac{1}{2}} \|\widehat{\Pi}_{ij}(\Pi_{ij}(u_{h_i}|_{\Gamma_{ij}} - u_{h_j}|_{\Gamma_{ij}}))\|_{0,\ \Gamma_{ij}} \\
&\lesssim h_i^{\alpha_i}\|u\|_{1+\alpha_i,\ \Omega_i} + \sum_{\Gamma_{ij}\in J(i)} h_i^{-\frac{1}{2}} \|\Pi_{ij}(u_{h_i}|_{\Gamma_{ij}} - u_{h_j}|_{\Gamma_{ij}})\|_{0,\ \Gamma_{ij}}.
\end{aligned}
$$

By the above inequality and Lemma 3.9, we obtain

$$(32) \qquad \inf_{v_h\in\tilde{V}(\Omega)} \|u - v_h\|_A \lesssim \Big(\sum_{k=1}^{N} h_k^{2\alpha_k}\|u\|_{1+\alpha_k,\ \Omega_k}^2\Big)^{\frac{1}{2}}.$$

Here, we have used the fact that $h_i \leq h_j$.

Step 2: Estimate the second term (the approximate error) in the right of (30).

Since $\omega_i\frac{\partial u}{\partial n_i} = -\omega_j\frac{\partial u}{\partial n_j}$ for each face Γ_{ij}, we derive

$$(33) \qquad \sum_{k=1}^{N} \int_{\partial\Omega_k} \omega_k\frac{\partial u}{\partial n_k}(v_h|_{\Omega_k})ds = \sum_{\Gamma_{ij}} \int_{\Gamma_{ij}} \omega_i\frac{\partial u}{\partial n_i}(v_i - v_j)ds.$$

It follows by Lemma 3.1~3.5 that

$$
(34) \quad
\begin{aligned}
|\int_{\Gamma_{ij}} \omega_i\frac{\partial u}{\partial n_i}(v_i - v_j)ds|^2 \quad &\lesssim \sum_{k=i,\ j} (1+\log\tfrac{d}{h_k})h_k^{2\alpha_k}\|\omega_k u\|_{1+\alpha_k,\ \Omega_k}^2 \cdot |v_k|_{1,\ \Omega_k}^2 \\
&\lesssim \|v_k\|_A^2 \cdot \sum_{k=i,\ j} (1+\log\tfrac{d}{h_k})h_k^{2\alpha_k}\|u\|_{1+\alpha_k,\ \Omega_k}^2.
\end{aligned}
$$

Then, we obtain by (34) and (33)

$$(35) \qquad \sup_{v_h\in\tilde{V}(\Omega)} \frac{|\sum_{k=1}^{N}\int_{\partial\Omega_k} \omega_k\frac{\partial u}{\partial n_k}(v_h|_{\Omega_k})ds|}{\|v_h\|_A} \lesssim \Big(\sum_{k=1}^{N}(1+\log\tfrac{d}{h_k})h_k^{2\alpha_k}\|u\|_{1+\alpha_k,\ \Omega_k}^2\Big)^{\frac{1}{2}}.$$

Now combining (35) with (32), we get Theorem 3.1. ∎

4. Numerical Experiment

Consider the model problem

$$(36) \qquad \begin{cases} -\nabla(\omega\nabla u) = f, \text{ in } \Omega, \\ u = g, \text{ on } \partial\Omega, \end{cases}$$

where $\Omega = [0,1]\times[0,1]\times[0,1]$.

We decompose Ω into $N_p \times N_p \times N_p$ cubes with the same size, and number all subdomains in the usual way, where each cube corresponds to a subdomain. Let each subdomain closing the boundary $\partial\Omega$ be divided into $n_b \times n_b \times n_b$ smaller cubes,

and let each interior subdomain be divided into $n_f \times n_f \times n_f$ smaller cubes. Then, we divide further each small cube into six tetrahedral elements.

As usual, we transform the resulting discrete problem (3) into a saddle-point system, in which a singular subproblem is involved for each interior subdomain (floating subdomain). We use the regularization method (see [8]) to solve such saddle-point system, where we choose the regularization parameter $\eta = 1.0 \times 10^{-5}$ to handle the singularity on the floating subdomains.

For convenience, we take $N_p = 3$ here. The l^2 errors of the approximate solutions are shown in the following tables for different n_f and n_b.

Example 4.1 Let f and g be defined such that the exact solution $u = \sin \pi x \cdot \sin \pi y \cdot \sin \pi z$ and $\omega = 1 + xyz$.

Table 4.1

N_p	n_b	n_f	l^2 error	n_f	l^2 error
3	4	8	1.28×10^{-2}	7	1.32×10^{-2}
3	8	16	3.25×10^{-3}	14	3.34×10^{-3}
3	16	32	7.53×10^{-4}	28	8.27×10^{-4}

Example 4.2 Let f and g be defined such that the exact solution $u = (x^2 + y^2 + z^2)^{\frac{1}{5}}$ and $\omega = 1$.

Table 4.2

N_p	n_b	n_f	l^2 error	n_f	l^2 error
3	4	8	2.32×10^{-3}	7	2.42×10^{-3}
3	8	16	6.46×10^{-4}	14	6.74×10^{-4}
3	16	32	1.70×10^{-4}	28	1.78×10^{-4}

Remark 4.1 The above tables indicate that the errors of the approximate solutions are almost optimal, which confirm our theoretical results.

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