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NUMERICAL ANALYSIS FOR A NONLOCAL ALLEN-CAHN EQUATION

PETER W. BATES, SARAH BROWN, AND JIANLONG HAN

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Abstract. We propose a stable, convergent finite difference scheme to solve numerically a nonlocal Allen-Cahn equation which may model a variety of physical and biological phenomena involving long-range spatial interaction. We also prove that the scheme is uniquely solvable and the numerical solution will approach the true solution in the L^∞ norm.

Key Words. Finite difference scheme; Long range interaction.

1. Introduction

Consider the following problem

$$u_t = \int_\Omega J(x-y)u(y)dy - \int_\Omega J(x-y)dy\, u(x) - f(u) \tag{1}$$

in $(0,T)\times\Omega$, with initial condition

$$u(0,x) = u_0(x), \tag{2}$$

where $T>0$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain. The unknown u is a real-valued order parameter, the interaction kernel satisfies $J(-x)=J(x)$, and f is bistable.

The equation (1) can be derived as an L^2 gradient flow for the free energy

$$E = \frac{1}{4}\int\int J(x-y)\left(u(x)-u(y)\right)^2 dx\, dy + \int F\left(u(x)\right)dx, \tag{3}$$

where F is a double well function.

The L^2 gradient flow for the classical Ginzberg-Landau energy functional

$$E = \frac{1}{2}\int |\nabla u|^2 dx + \int F\left(u(x)\right)dx, \tag{4}$$

is the Allen-Cahn equation:

$$u_t = \triangle u(x) - f(u) \tag{5}$$

As mentioned in [3], the equations (1) and (5) are important for modelling a variety of physical and biological phenomena involving media with properties varying in space. There is by now a lot of work on equation (1) and (5) (see for example [1], [2], [5], [7], [8], [9], [11], [12], [13], [15], [16], [17], and the references therein).

To the best of our knowledge, there are very few results on the numerical solutions to (1). In this paper, we develop a finite difference scheme for equation (1) for $n=1$ and $n=2$. We also prove that the difference scheme is stable and that the numerical

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approximation converges to the solution of (1) as the spatial and temporal mesh size approaches zero. Our numerical results coincide with the theoretical results in [12].

2. Analysis of the proposed scheme

In this section, we consider finite difference approximations of equation (1) for $n = 1$ and $n = 2$. For the sake of exposition, we take $f(u) = u^3 - u$, but the analysis applies to the general smooth bistable function if care is taken in the choice of linearization.

We use the following notation:

For $n = 1$ with $\Omega = (-L, L)$,

$$\Omega_x = \{x_i |\, x_i = -L + i\triangle x,\, 0 \le i \le M\},$$
$$\Omega_t = \{t_k |\, t_k = k\triangle t,\, 0 \le k \le K\},$$

where $\triangle x = 2L/M$ and $\triangle t = T/K$. Our difference scheme for equation (1) for $n = 1$ is as follows:

$$u_i^0 = u_0(x_i), \text{ for } 0 \le i \le M, \tag{6}$$

$$\delta_t u_i^k = (J * u^k)_i - (J * 1)_i u_i^k + \psi(u_i^k, u_i^{k+1}) \text{ for } 0 \le i \le M,\, 0 \le k \le K-1, \tag{7}$$

where

$$\delta_t u_i^k = \frac{u_i^{k+1} - u_i^k}{\triangle t},$$

$$(J * u^k)_i = \triangle x \left[\frac{1}{2} J(x_0 - x_i) u_0^k + \sum_{m=1}^{M-1} J(x_m - x_i) u_m^k + \frac{1}{2} J(x_M - x_i) u_M^k \right],$$

and

$$\psi(u_i^k, u_i^{k+1}) = u_i^k - \left(u_i^k\right)^2 u_i^{k+1}.$$

For a rectangular domain $(-L, L) \times (-W, W) \subset \mathbb{R}^2$, we have

$$\Omega_{x,y} = \{(x_i, y_j) |\, x_i = -L + i\triangle x,\, y_j = -W + j\triangle y,\, 0 \le i \le M,\, 0 \le j \le N\},$$
$$\Omega_t = \{t_k |\, t_k = k\triangle t,\, 0 \le t \le K\},$$

where $\triangle x = 2L/M$ and $\triangle y = 2W/N$.

Our difference scheme in this case is

$$u_{i,j}^0 = u_0(x_i, y_j) \text{ for } 0 \le i \le M,\, 0 \le j \le N, \tag{8}$$

$$\delta_t u_{i,j}^k = (J * u^k)_{i,j} - (J * 1)_{i,j} u_{i,j}^k + \psi(u_{i,j}^k, u_{i,j}^{k+1}) \quad \text{for } 0 \le i \le M,\, 0 \le j \le N,\, 0 \le k \le K-1, \tag{9}$$

where

$$\delta_t u_{i,j}^k = \frac{u_{i,j}^{k+1} - u_{i,j}^k}{\triangle t},$$

$$
\begin{aligned}
(J*u^k)_{i,j} =& \triangle x \triangle y \left[\sum_{m=1}^{M-1} \sum_{n=1}^{N-1} J(x_m - x_i, y_n - y_j) u^k_{m,n} \right. \\
&+ \frac{1}{2} \sum_{m=1}^{M-1} \left(J(x_m - x_i, y_0 - y_j) u^k_{m,0} + J(x_m - x_i, y_N - y_j) u^k_{m,N} \right) \\
&+ \frac{1}{2} \sum_{n=1}^{N-1} \left(J(x_0 - x_i, y_n - y_j) u^k_{0,n} + J(x_M - x_i, y_n - y_j) u^k_{M,n} \right) \\
&+ \frac{1}{4} \left(J(x_0 - x_i, y_0 - y_j) u^k_{0,0} + J(x_M - x_i, y_0 - y_j) u^k_{M,0} \right. \\
&\left. \left. + J(x_0 - x_i, y_N - y_j) u^k_{0,N} + J(x_M - x_i, y_N - y_j) u^k_{M,N} \right) \right],
\end{aligned}
$$

and

$$\psi(u^k_{i,j}, u^{k+1}_{i,j}) = u^k_{i,j} - (u^k_{i,j})^2 u^{k+1}_{i,j}.$$

Remark 2.1. *The choice of $\psi(u^k_{i,j}, u^{k+1}_{i,j})$ implies that our method is semi-implicit, which gives us better stability properties.*

Theorem 2.2. *If $u(0,x,y) \in C(\bar{\Omega})$ and $J \in C([-2L,2L] \times [-2W,2W])$, then there exists a unique solution to the difference equations (8)-(9), and we have*

$$\max_{i,j} |u^k_{i,j}| \le C \tag{10}$$

for some constant C which is independent of k, i.e., the scheme is stable under the maximum norm.

Proof. From equations (8)-(9) and the form of ψ, it is clear that the scheme is uniquely solvable.

Set $\Lambda = \max |u_0(x,y)|$.

For $k = 0$,

$$\max_{i,j} |u^0_{i,j}| = \max |u_0(x,y)| = \Lambda \tag{11}$$

For $0 \le k \le K$, denote $a_k = \max |u^k_{i,j}|$. Equation (9) implies

$$a_{k+1} \le a_k + (2C_1 + 1) \triangle t \, a_k \tag{12}$$

where $C_1 = \max |(J*1)_{i,j}|$. From equation (12), we have

$$
\begin{aligned}
a_{k+1} &\le (1 + (2C_1+1)\triangle t) a_k \le (1 + (2C_1+1)\triangle t)^2 a_{k-1} \\
&\vdots \\
&\le (1 + (2C_1+1)\triangle t)^k a_0 \le e^{(2C_1+1)\triangle t k} a_0 \\
&\le e^{(2C_1+1)2T} a_0 \le C(T)\Lambda.
\end{aligned} \tag{13}
$$

Therefore,

$$\max_{i,j,k} |u^k_{i,j}| \le C(T)\Lambda, \tag{14}$$

where $C(T)$ is independent of k and of the spatial mesh size. This completes the proof.

□

Remark 2.3. *A similar result holds for all values of n.*

Next we consider error estimates for the proposed scheme. In order to show that the numerical solution converges to the solution of (1), we have the following lemma.

Lemma 2.4. *Suppose that J and f satisfy the following assumptions:*

(A_1) $J \in W^{2,1}(\mathbb{R}^n)$ *and* $(L_1 \equiv \max\{\sup \int_\Omega |J(x-y)|dy,\ \sup \int_\Omega |J'(x-y)|dx,$ $\sup \int_\Omega |J''(x-y)|dy\} < \infty$.

(A_2) $f \in C^2(\mathbb{R}^n)$ *and there exist* $c_1,\ c_2,\ c_3,\ c_4 > 0$ *and* $r > 2$ *such that* $f(u)u \geq c_1|u|^r - c_2|u|$, *and* $|f(u)| \leq c_3|u|^{r-1} + c_4$.

If $u_0 \in C^2(\bar{\Omega})$, *then there exists a unique solution* $u \in C^{1,2}([0,T]\times\bar{\Omega})$ *to (1)-(2). Furthermore,* $u \in C^{1,2}([0,\infty)\times\bar{\Omega})$ *and*

$$\sup_{t\geq 0} ||u||_{C(\bar{\Omega})} \leq C(\Lambda),$$

where $C(\Lambda)$ *depends only on* $\Lambda = ||u_0||_\infty$.

Proof. The proof of the lemma is similar to that of Theorem 2.3 in [6], and so we omit it here. □

Theorem 2.5. *Take* $n = 2$ *and let* $\Omega = (-L, L) \times (-W, W)$. *If (1)-(2) has a solution such that* $u(t, x, y) \in C^{1,2}([0,T] \times \bar{\Omega})$, *then the solution of the difference scheme converges to the solution of (1) uniformly, as* $(\triangle t, \triangle x, \triangle y) \to 0$ *and the convergence rate is* $O(\triangle t + \triangle x^2 + \triangle y^2)$.

Proof. Let $u(t, x, y)$ be the solution of equation (1). We use the following notation: For $0 \leq k \leq K, 0 \leq i \leq M$, and $0 \leq j \leq N$, let

$$U^k_{i\,j} = u(t_k, x_i, y_j). \tag{15}$$

From the Taylor expansion of u, we also have the following approximations:

$$u_t(t_k, x_i, y_j) = \frac{U^{k+1}_{i\,j} - U^k_{i,\,j}}{\triangle t} + O(\triangle t) \tag{16}$$

for $k \geq 0$. Also,

$$\int_\Omega J(x_i - x, y_j - y)u(t_k, x, y)dx\,dy = (J * U^k)_{i\,j} + O(\triangle x^2 + \triangle y^2) \tag{17}$$

and

$$\int_\Omega J(x_i - x, y_j - y)dx\,dy = (J * 1)_{i\,j} + O(\triangle x^2 + \triangle y^2). \tag{18}$$

Equations (1), (16)-(18) imply the following estimate holds:

$$\begin{aligned}\frac{U^{k+1}_{i\,j} - U^k_{i\,j}}{\triangle t} &= (J * U^k)_{i\,j} - (J * 1)_{i\,j}U^k_{i\,j} \\ &\quad + U^k_{i\,j} - (U^k_{i\,j})^3 + O(\triangle t + \triangle x^2 + \triangle y^2).\end{aligned} \tag{19}$$

We define the error as

(20)

$$V^k_{i,\,j} = U^k_{i\,j} - u^k_{i,\,j},\ \text{for } k = 0, 1, \cdots, K,\ i = 0, 1 \cdots, M,\ \texttt{and}\ j = 0, 1, \cdots, N.$$

From (8) and (15), we have

$$V_{i,j}^0 = 0 \text{ for } i = 0, 1 \cdots, M, \quad \texttt{and } j = 0, 1, \cdots, N. \tag{21}$$

For $k > 0$, (9) and (19) imply that

$$\begin{aligned} V_{i,j}^{k+1} &= \triangle t \left[(J * V^k)_{i,j} - (J*1)_{ij} V_{i,j}^k + V_{i,j}^k - \left((U_{ij}^k)^3 - (u_{i,j}^k)^2 u_{i,j}^{k+1} \right) \right] \\ &\quad + V_{i,j}^k + 2\triangle t O(\triangle t + \triangle x^2 + \triangle y^2). \end{aligned} \tag{22}$$

Since

$$\begin{aligned} (U_{ij}^k)^3 - (u_{i,j}^k)^2 u_{i,j}^{k+1} &= (U_{ij}^k)^2 (U_{ij}^k - U_{ij}^{k+1}) \\ &\quad + V_{i,j}^k (U_{ij}^k + u_{i,j}^k) U_{ij}^{k+1} + (u_{i,j}^k)^2 V_{i,j}^{k+1}, \end{aligned} \tag{23}$$

equation (22) implies

$$\begin{aligned} &\left(1 + 2\triangle t (u_{i,j}^k)^2\right) V_{i,j}^{k+1} \\ &= \triangle t \left[(J * V^k)_{i,j} - (J*1)_{ij} V_{i,j}^k + V_{i,j}^k - (U_{ij}^k)^2 (U_{ij}^k - U_{ij}^{k+1}) \right. \\ &\quad \left. - V_{i,j}^k (U_{ij}^k + u_{i,j}^k) U_{ij}^{k+1} \right] \quad + V_{i,j}^k + 2\triangle t O(\triangle t^2 + \triangle x^2 + \triangle y^2). \end{aligned} \tag{24}$$

Equation (24) and the boundedness of $|U_{ij}^k|$ and $|u_{i,j}^k|$ imply that

$$\max_{i,j} |V_{i,j}^{k+1}| \le C(\Lambda_1, \Lambda_2, \Lambda_3) \triangle t \max_{i,j} |V_{i,j}^k| + \max_{i,j} |V_{i,j}^k| + |\triangle t R_2|, \tag{25}$$

where $C(\Lambda_1, \Lambda_2, \Lambda_3)$ depends only on $\Lambda_1 = \max_{i,j,k} |u_{i,j}^k|$, $\Lambda_2 = \max_{i,j,k} |U_{i,j}^k|$ and $\Lambda_3 = \sup \int_\Omega |J(x-y)| dy$, and $R_2 = O(\triangle t + \triangle x^2 + \triangle y^2)$.

Let

$$V_k = \max_{i,j} |V_{i,j}^k|.$$

From (25), we have

$$\begin{aligned} V_{k+1} &\le (1 + C(\Lambda_1, \Lambda_2, \Lambda_3)\triangle t) V_k + \triangle t R_2 \\ &\le (1 + C(\Lambda_1, \Lambda_2, \Lambda_3)\triangle t) \left[(1 + C(\Lambda_1, \Lambda_2, \Lambda_3)\triangle t) V_{k-1} + \triangle t R_2 \right] + \triangle t R_2 \\ &\le (1 + C(\Lambda_1, \Lambda_2, T)\triangle t)^2 V_{k-1} + \left[(1 + C(\Lambda_1, \Lambda_2, \Lambda_3)\triangle t) + 1 \right] \triangle t R_2 \\ &\vdots \\ &\le (1 + C(\Lambda_1, \Lambda_2, \Lambda_3)\triangle t)^k V_1 + \Big[(1 + C(\Lambda_1, \Lambda_2, \Lambda_3)\triangle t)^{k-1} \\ &\quad + (1 + C(\Lambda_1, \Lambda_2, \Lambda_3)\triangle t)^{k-2} + \cdots + 1 \Big] \triangle t R_2 \\ &\le e^{C(\Lambda_1, \Lambda_2, \Lambda_3)\triangle t k} V_1 + \frac{[1 + C(\Lambda_1, \Lambda_2, \Lambda_3)\triangle t]^k - 1}{C(\Lambda_1, \Lambda_2, \Lambda_3)} R_2 \\ &\le e^{C(\Lambda_1, \Lambda_2, \Lambda_3)T} V_1 + \frac{e^{C(\Lambda_1, \Lambda_2, \Lambda_3)T} - 1}{C(\Lambda_1, \Lambda_2, \Lambda_3)} R_2 \\ &= O(\triangle t + \triangle x^2 + \triangle y^2). \end{aligned} \tag{26}$$

Therefore, the error rate is $O(\triangle t + \triangle x^2 + \triangle y^2)$. □

3. Applications to Allen-Cahn equation

The method for the nonlocal Allen-Cahn equation can also be applied to the Allen-Cahn equation.

Consider the following problem:

$$
\begin{aligned}
&u_t = \triangle u - f(u) \text{ in } (0,T)\times\Omega \\
&\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0 \\
&u(0,x) = u_0(x)
\end{aligned}
\tag{27}
$$

where $f(u) = u^3 - u$.

Our difference scheme for equation (27) with $n = 1$ is as follows:

$$
\delta_t u_i^k = \delta_x^2 u_i^k + \psi(u_i^k, u_i^{k+1}) \text{ for } 0 \le i \le M,\ 1 \le k \le K-1 \tag{28}
$$

with time and spatial discretization as before. The Neumann boundary condition is expressed by

$$
\frac{u_1^k - u_{-1}^k}{2\triangle x} = 0,\ \frac{u_{M+1}^k - u_{M-1}^k}{2\triangle x} = 0 \text{ for } 0 \le k \le K-1, \tag{29}
$$

where

$$
\delta_t u_i^k = \frac{u_i^{k+1} - u_i^k}{\triangle t},
$$

$$
\delta_x^2 u_i^k = \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{\triangle x^2},
$$

and

$$
\psi(u_i^k, u_i^{k+1}) = u_i^k - \left(u_i^k\right)^2 u_i^{k+1}.
$$

Equations (28)-(29) imply

$$
\left(1 + \triangle t (u_0^k)^2\right) u_0^{k+1} = r_x(2u_1^k - 2u_0^k) + (\triangle t + 1)u_0^k \text{ for } 0 \le k \le K-1, \tag{30}
$$

$$
\begin{aligned}
\left(1 + \triangle t (u_i^k)^2\right) u_i^{k+1} &= r_x(u_{i+1}^k - 2u_i^k + u_{i-1}^k) + (\triangle t + 1)u_i^k \\
&\qquad \text{for } 1 \le i \le M-1,\ 0 \le k \le K-1,
\end{aligned}
\tag{31}
$$

$$
\left(1 + \triangle t (u_M^k)^2\right) u_i^{k+1} = r_x(2u_{M-1}^k - 2u_M^k) + \triangle t u_M^k \text{ for } 0 \le k \le K-1, \tag{32}
$$

where $r_x = \frac{\triangle t}{\triangle x^2}$.

Similarly, the difference scheme for $n = 2$ is

$$
u_{i,j}^0 = u_0(x_i, y_j),\ \text{ for } 0 \le i \le M,\ 0 \le j \le N, \tag{33}
$$

$$
\begin{aligned}
\delta_t u_{i,j}^k &= \delta_x^2 u_{i,j}^k + \delta_y^2 u_{i,j}^k + \psi(u_{i,j}^k,\ u_{i,j}^{k+1}) \\
&\qquad \text{for } 0 \le i \le M,\ 0 \le j \le N,\ 1 \le k \le K-1
\end{aligned}
\tag{34}
$$

with Neumann boundary condition

$$\begin{aligned} &\frac{u^k_{1,j}-u^k_{-1,j}}{2\triangle x}=0,\ \frac{u^k_{M+1,j}-u^k_{M-1,j}}{2\triangle x}=0, \quad \text{for } 0\le j\le N,\\ &\frac{u^k_{i,1}-u^k_{i,-1}}{2\triangle y}=0,\ \frac{u^k_{i,N+1}-u^k_{i,N-1}}{2\triangle y}=0, \quad \text{for } 0\le i\le M, \end{aligned} \tag{35}$$

where

$$\begin{aligned} \delta_t u^k_{i,\,j} &= \frac{u^{k+1}_{i,\,j}-u^k_{i,\,j}}{\triangle t},\\ \delta_x^2 u^k_{i,\,j} &= \frac{u^k_{i+1,\,j}-2u^k_{i,\,j}+u^k_{i-1,\,j}}{\triangle x^2},\\ \delta_y^2 u^k_{i,\,j} &= \frac{u^k_{i,\,j+1}-2u^k_{i,\,j}+u^k_{i,\,j-1}}{\triangle y^2}, \end{aligned}$$

and

$$\psi(u^k_{i,\,j},u^{k+1}_{i,\,j}) = u^k_{i,\,j} - \left(u^k_{i,\,j}\right)^2 u^{k+1}_{i,\,j}.$$

Let $r_x = \frac{\triangle t}{\triangle x^2}$ and $r_y = \frac{\triangle t}{\triangle y^2}$. From (33)-(35), we have

$$\begin{aligned} \left(1+\triangle t(u^k_{i,j})^2\right) u^{k+1}_{i,j} &= r_x(u^k_{i+1,j}+u^k_{i-1,j}) + (1-2r_x-2r_y)u^k_{i,j}\\ &\quad + r_y(u^k_{i,j+1}+u^k_{i,j-1}) + \triangle t u^k_{i,j}\\ &\quad \text{for } 1\le i\le M-1,\ 1\le j\le N-1, \end{aligned} \tag{36}$$

$$\left(1+\triangle t(u^k_{0,0})^2\right) u^{k+1}_{0,0} = 2r_x u^k_{1,0} + (1-2r_x-2r_y)u^k_{0,0} + 2r_y u^k_{0,1} + \triangle t u^k_{0,0}, \tag{37}$$

$$\begin{aligned} \left(1+\triangle t(u^k_{0,N})^2\right) u^{k+1}_{0,N} &= 2r_x u^k_{1,N} + (1-2r_x-2r_y)u^k_{0,N}\\ &\quad + 2r_y u^k_{0,N-1} + \triangle t u^k_{0,N}, \end{aligned} \tag{38}$$

$$\begin{aligned} \left(1+\triangle t(u^k_{M,0})^2\right) u^{k+1}_{M,0} &= 2r_x u^k_{M-1,0} + (1-2r_x-2r_y)u^k_{M,0}\\ &\quad + 2r_y u^k_{M,1} + \triangle t u^k_{M,0}, \end{aligned} \tag{39}$$

$$\begin{aligned} \left(1+\triangle t(u^k_{M,N})^2\right) u^{k+1}_{M,N} &= 2r_x u^k_{M-1,N} + (1-2r_x-2r_y)u^k_{M,N}\\ &\quad + 2r_y u^k_{M,N-1} + \triangle t u^k_{M,N}, \end{aligned} \tag{40}$$

$$\begin{aligned} \left(1+\triangle t(u^k_{0,j})^2\right) u^{k+1}_{0,j} &= 2r_x u^k_{1,j} + (1-2r_x-2r_y)u^k_{0,j}\\ &\quad + r_y(u^k_{0,j+1}+u^k_{0,j-1}) + \triangle t u^k_{0,j}\\ &\quad \text{for } 1\le j\le N-1, \end{aligned} \tag{41}$$

$$\begin{aligned} \left(1+\triangle t(u^k_{M,j})^2\right) u^{k+1}_{M,j} &= 2r_x u^k_{M-1,j} + (1-2r_x-2r_y)u^k_{M,j}\\ &\quad + r_y(u^k_{M,j+1}+u^k_{M,j-1}) + \triangle t u^k_{M,j}\\ &\quad \text{for } 1\le j\le N-1, \end{aligned} \tag{42}$$

$$\begin{aligned} \left(1+\triangle t(u^k_{i,0})^2\right) u^{k+1}_{i,0} &= r_x(u^k_{i+1,0}+u^k_{i-1,0}) + (1-2r_x-2r_y)u^k_{i,0}\\ &\quad + 2r_y u^k_{i,1} + \triangle t u^k_{i,0} \quad \text{for } 1\le i\le M-1, \end{aligned} \tag{43}$$

$$\begin{aligned}(1+\triangle t(u_{i,N}^{k})^{2})\,u_{i,N}^{k+1} &= r_x(u_{i+1,N}^{k}+u_{i-1,N}^{k})+(1-2r_x-2r_y)u_{i,N}^{k}\\ &\quad +2r_y u_{i,N-1}^{k}+\triangle t u_{i,N}^{k}\quad \text{for } 1\le i\le M-1.\end{aligned} \tag{44}$$

Theorem 3.1. *If $u(0,x,y)\in C(\bar{\Omega})$, then there exists a unique solution to the difference equations (28)-(29), and if $r_x+r_y\le \frac{1}{2}$ we have*

$$\max_{i,\,j}|u_{i,\,j}^{k}|\le C \tag{45}$$

for some constant C which is independent of k, i.e., the scheme is stable under the maximum norm.

Proof. From equations (33), (36)-(44), it is clear that the scheme is uniquely solvable.

Set $\Lambda=\max|u_0(x,y)|$.

For $k=0$,

$$\max_{i,j}|u_{i,\,j}^{0}|=\max|u_0(x,y)|=\Lambda \tag{46}$$

For $0\le k\le K$, denote $a_k=\max|u_{i,\,j}^{k}|$. If $r_x+r_y\le\frac{1}{2}$, equations (36)-(44) imply

$$\begin{aligned}a_{k+1} &\le (1+\triangle t)a_k\le(1+\triangle t)^2a_{k-1}\\ &\cdots\\ &\le(1+\triangle t)^{k+1}a_0\le C(T)\Lambda.\end{aligned} \tag{47}$$

Therefore,

$$\max_{i,\,j,\,k}|u_{i,\,j}^{k}|\le C(T)\Lambda, \tag{48}$$

where $C(T)$ is independent of k and of the spatial mesh size.

□

Next we consider error estimates for the proposed scheme.

Theorem 3.2. *If (27) has a solution such that $u(t,x,y)\in C^{1,2}([0,T]\times\bar{\Omega})$, then the solution of the difference scheme converges to the solution of (27) uniformly, as $(\triangle t,\triangle x,\triangle y)\to 0$ and the convergence rate is $O(\triangle t+\triangle x^2+\triangle y^2)$.*

Proof. Let $u(t,x,y)$ be the solution of equation (27). As before, for $0\le k\le K,\ 0\le i\le M$, and $0\le j\le N$, let

$$U_{i\,j}^{k}=u(t_k,x_i,y_j). \tag{49}$$

We have for $k\ge 0$,

$$u_t(t_k,x_i,y_j)=\frac{U_{i,j}^{k+1}-U_{i,j}^{k}}{\triangle t}+O(\triangle t), \tag{50}$$

$$u_{xx}(t_k,x_i,y_j)=\frac{U_{i+1,\,j}^{k}-2U_{i,\,j}^{k}+U_{i-1,j}^{k}}{\triangle x^2}+O(\triangle x^2), \tag{51}$$

$$u_{yy}(t_k,x_i,y_j)=\frac{U_{i,\,j+1}^{k}-2U_{i,\,j}^{k}+U_{i,j-1}^{k}}{\triangle y^2}+O(\triangle y^2). \tag{52}$$

Equations (27), (50)-(52) imply the following difference equations:

$$\begin{aligned}\frac{U_{i,j}^{k+1}-U_{i,j}^{k}}{\triangle t} &= \frac{U_{i+1,j}^{k}-2U_{i,j}^{k}+U_{i-1,j}^{k}}{\triangle x^2} \\ &+ \frac{U_{i,j+1}^{k}-2U_{i,j}^{k}+U_{i,j-1}^{k}}{\triangle y^2}+U_{i,j}^{k}-(U_{i,j}^{k})^3+R,\end{aligned} \tag{53}$$

$$\begin{aligned}&U_{1,j}^{k}=U_{-1,j}^{k}+O(\triangle x^2),\ U_{M+1,j}^{k}=U_{M-1,j}^{k}+O(\triangle x^2), \quad \text{for } 0\le j\le N,\\ &U_{i,1}^{k}=U_{i,-1}^{k}+O(\triangle y^2),\ U_{i,N+1}^{k}=U_{i,N-1}^{k}+O(\triangle y^2) \ \ \text{for } 0\le i\le M\end{aligned} \tag{54}$$

where $R=O(\triangle t+\triangle x^2+\triangle y^2)$.

We define the error as

(55)

$$V_{i,j}^{k}=U_{ij}^{k}-u_{i,j}^{k},\ \text{for } k=0,\,1,\cdots,K,\ i=0,\,1\,\cdots,M,\ \texttt{and}\ j=0,\,1,\cdots,N.$$

From (33), (36)-(44) and (53)-(54), we have

$$\begin{aligned}V_{i,j}^{k+1} &= r_x(V_{i+1,j}^{k}+V_{i-1,j}^{k})+r_y(V_{i,j+1}^{k}+V_{i,j-1}^{k})\\ &+(1-2r_x-2r_y)V_{i,j}^{k}+\triangle t V_{i,j}^{k}\\ &-\triangle t\left((U_{i,j}^{k})^3-(u_{i,j}^{k})^2u_{i,j}^{k+1}\right)+\triangle t R_1(\triangle t,\triangle x,\triangle y).\end{aligned} \tag{56}$$

Since

$$\begin{aligned}(U_{ij}^{k})^3-(u_{i,j}^{k})^2u_{i,j}^{k+1} &= (U_{ij}^{k})^2(U_{ij}^{k}-U_{ij}^{k+1})\\ &+V_{i,j}^{k}(U_{ij}^{k}+u_{i,j}^{k})U_{ij}^{k+1}\\ &+(u_{i,j}^{k})^2V_{i,j}^{k+1},\end{aligned} \tag{57}$$

equations (54), (55) and (56) imply

$$\begin{aligned}&(1+2\triangle t(u_{i,j}^{k})^2)V_{i,j}^{k+1}\\ &=r_x(V_{i+1,j}^{k}+V_{i-1,j}^{k})+r_y(V_{i,j+1}^{k}+V_{i,j-1}^{k})+(1-2r_x-2r_y)V_{i,j}^{k}\\ &\quad+\triangle t V_{i,j}^{k}-\triangle t(U_{ij}^{k})^2(U_{ij}^{k}-U_{ij}^{k+1})\\ &\quad-\triangle t V_{i,j}^{k}(U_{ij}^{k}+u_{i,j}^{k})U_{ij}^{k+1}+\triangle t R_1(\triangle t,\triangle x,\triangle y).\end{aligned} \tag{58}$$

If $r_x+r_y\le\frac{1}{2}$, equation (57) and the boundedness of $|U_{i,j}^{k}|$ and $|u_{i,j}^{k}|$ imply that

$$\max_{i,j}|V_{i,j}^{k+1}|\le(1+C(\Lambda_1,\Lambda_2)\triangle t)\max_{i,j}|V_{i,j}^{k}|+|\triangle t R_2|, \tag{59}$$

where $C(\Lambda_1,\Lambda_2)$ depends only on $\Lambda_1=\max_{i,j,k}|u_{i,j}^{k}|$ and $\Lambda_2=\max_{i,j,k}|U_{i,j}^{k}|$, and $R_2=O(\triangle t+\triangle x^2+\triangle y^2)$.

Equation (59) implies

$$\max_{i,j}|V_{i,j}^{k+1}|\le C(\Lambda_1,\Lambda_2)(\triangle t+\triangle x^2+\triangle y^2). \tag{60}$$

Therefore, the error is $O(\triangle t+\triangle x^2+\triangle y^2)$. □

4. Numerical results

In this section, we present some results of computational experiments to show that the proposed difference scheme is stable and gives reasonable solutions.

Case 1: The diffusion part is small, i.e., ϵ is small.

In this case, we consider the following two equations:

$$\begin{cases} u_t = \int_\Omega J_\epsilon(x-y)u(y)dy - \int_\Omega J_\epsilon(x-y)dy\, u(x) - f(u) \\ u(0,x) = u_0(x), \end{cases} \tag{61}$$

$$\begin{cases} u_t = \epsilon^2 \triangle u - f(u) \\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{62}$$

where $f(u) = u^3 - u$ and $J_\epsilon = \frac{1}{\epsilon^n} J(\frac{x}{\epsilon})$ for a fixed kernel J. Note that for a large class of kernels J, the linear operator on the RHS of (61) is asymptotically close to $\epsilon^2\Delta$ in a certain sense (see [4]).

Consider the case $n = 1$ and $\Omega = (-1, 1)$.

For the nonlocal Allen-Cahn equation (61), $J_\epsilon(x) = \frac{1}{\epsilon} e^{-(\frac{x}{\epsilon})^2}$, $\epsilon = 0.1$, $\triangle t = 0.001$, and $\triangle x = 0.01$. Figure 1 shows the initial data $u_0(x) = 0.1\cos(2\pi x)$ and the numerical results at $t = 0.5$, $t = 1$, $t = 5$.

For the Allen-Cahn equation (62), $\epsilon = 0.1$, $\triangle t = 0.0001$, and $\triangle x = 0.02$. Figure 2 shows the numerical results for the Allen-Cahn equation at $t = 0$, $t = 0.5$, $t = 1$, and $t = 5$.

For $n = 2$, let $\Omega = (0, 1) \times (0, 1)$.

For the nonlocal Allen-Cahn equation (61), $J_\epsilon(x, y) = \frac{1}{\epsilon} e^{-\frac{x^2+y^2}{\epsilon^2}}$, $\epsilon = 0.1$, $\triangle t = 0.001$, $\triangle x = 0.08$, and $\triangle y = 0.08$. Figure 3 shows the initial data $u_0(x) = 0.1\cos(2\pi x) * \cos(2\pi y)$ and the numerical results at $t = 0.5$, $t = 2.5$, $t = 5$.

For the Allen-Cahn equation (62), $\epsilon = 0.1$, $\triangle t = 0.00001$, $\triangle x = 0.02$, and $\triangle y = 0.02$. Figure 4 shows the initial data $u_0(x) = 0.1\cos(2\pi x) * \cos(2\pi y)$ and the numerical results at $t = 0.5$, $t = 4$, $t = 10$.

From the computational experiments for the nonlocal Allen-Cahn equation for $n = 1$ and $n = 2$, we observed that solutions u corresponding to equations (61) tend to patterns that are nearly piecewise constant (Figure 1 and Figure 3) for initial data $-1 \le u_0(x) \le 1$. This is consistent with the result in [12]. It is also consistent with the result in [10] that when $t \to \infty$, u will approach a steady state solution of (1). For the steady state solution of (1), see [3] and [14].

From the computational experiments for the Allen-Cahn equation, we observe that although the solution u corresponding to (62) initially increases in time in the region $\{x|u_0(x) > 0\}$ and decreases in time in the region $\{x|u_0(x) < 0\}$, it will not tend to a piecewise constant function. This is because, although the linear operators are somewhat similar, diffusion is still an unbounded operator and the nonlocal operator is bounded.

Case 2: The diffusion part is large relative to the reaction term.

In this case, we consider the following two equations:

$$\begin{cases} u_t = D\left[\int_\Omega J_\epsilon(x-y)u(y)dy - \int_\Omega J_\epsilon(x-y)dy\, u(x)\right] - D_1 f(u) \\ u(0,x) = u_0(x), \end{cases} \tag{63}$$

$$\begin{cases} u_t = D_2 \triangle u - f(u) \\ \frac{\partial u}{\partial n}\big|_{\partial\Omega} = 0 \\ u(0,x) = u_0(x) \end{cases} \tag{64}$$

where $f(u)$ and $J_\epsilon(x)$ are defined as before.

For $n = 1$, Figure 5 shows the initial data $u_0(x) = 2\cos(2\pi x)$ and the numerical results for the nonlocal Allen-Cahn equation at $t = 0.5$, $t = 2$, and $t = 80$ with $D = 10$, $D_1 = 0.01$, $\epsilon = 0.1$, $\triangle t = 0.001$, and $\triangle x = 0.02$.

Figure 6 shows the initial data $u_0(x) = 2\cos(2\pi x)$ and the numerical results for the Allen-Cahn equation at $t = 0.05$, $t = 0.1$ and $t = 0.2$. with $D_2 = 2$, $\triangle t = 0.00001$, and $\triangle x = 0.02$.

For $n = 2$, Figure 7 shows the initial data $u_0(x, y) = 2\cos(2\pi x)\cos(2\pi y)$ and the numerical results for the nonlocal Allen-Cahn equation at $t = 1$, $t = 5$, and $t = 10$ with $D = 6$, $D_1 = 0.01$, $\epsilon = 0.1$, $\triangle t = 0.001$, $\triangle x = 0.05$, and $\triangle y = 0.05$.

Figure 8 shows the initial data $u_0(x, y) = 2\cos(2\pi x)\cos(2\pi y)$ and the numerical results for the Allen-Cahn equation at $t = 1$, $t = 2$ and $t = 5$. with $D_2 = 2$, $\triangle t = 0.00001$, $\triangle x = 0.02$, and $\triangle y = 0.02$.

From these numerical experiments for the Allen-Cahn equation and the nonlocal Allen-Cahn equation, we observe that the solution u corresponding to the Allen-Cahn equation converges to zero rapidly since the diffusion term is very large. We also observe that although $u(t, x)$ corresponding to the nonlocal Allen-Cahn equation decreases when t increases, the speed is slower than that of the Allen-Cahn equation.

Remark 4.1. *Since we consider both large and small diffusion cases for equations (1) and (5) in this section, instead of equations (1) and (5), we consider equations (61)and (62). By checking the proof of Theorem 2.5 and Theorem 3.2, we see that the convergence rate for the nonlocal Allen-Cahn equation (61) and the Allen-Cahn equation (62) also depends on ϵ. Actually, for the nonlocal Allen-Cahn equation (61), if $J_\epsilon(x, y) = \frac{1}{\epsilon^2} j(\frac{x}{\epsilon}, \frac{y}{\epsilon})$ in Theorem 2.5, we have*

$$\begin{aligned} &\int_\Omega J_\epsilon(x_i - x, y_j - y) dx\, dy \\ &= \int_{-L}^{L} \int_{-W}^{W} \frac{1}{\epsilon^2} j\left(\frac{x_i - x}{\epsilon}, \frac{y_j - y}{\epsilon}\right) dxdy \\ &= \int_{\frac{x_i - L}{\epsilon}}^{\frac{x_i + L}{\epsilon}} \int_{\frac{x_i - W}{\epsilon}}^{\frac{x_i + W}{\epsilon}} j(u, v) dudv. \end{aligned} \tag{65}$$

When we use the trapezoidal method to approximate $\int_{\frac{x_i - L}{\epsilon}}^{\frac{x_i + L}{\epsilon}} \int_{\frac{x_i - W}{\epsilon}}^{\frac{x_i + W}{\epsilon}} j(u, v) dudv$, the truncation error is $O(\triangle u^2 + \triangle v^2)\frac{1}{\epsilon}$. Since $u = \frac{x_i - x}{\epsilon}$, $v = \frac{y_j - y}{\epsilon}$, we have $O(\triangle u^2 + \triangle v^2)\frac{1}{\epsilon} = O(\triangle x^2 + \triangle y^2)\frac{1}{\epsilon^3}$. Therefore, we have

$$\int_{\frac{x_i - L}{\epsilon}}^{\frac{x_i + L}{\epsilon}} \int_{\frac{x_i - W}{\epsilon}}^{\frac{x_i + W}{\epsilon}} j(u, v) dudv = (j * 1)_{i\,j} + O(\triangle x^2 + \triangle y^2)\frac{1}{\epsilon^3} \tag{66}$$

Similarly,

$$\int_{\Omega} J_\epsilon(x_i - x, y_j - y)u(t_k, x, y)dx\,dy = (J * U^k)_{i\,j} + O(\triangle x^2 + \triangle y^2)\frac{1}{\epsilon^3}. \tag{67}$$

Following the proof of Theorem 2.5((19)-(25)), we have

$$V_{k+1} \le O\left(\triangle t + (\triangle x^2 + \triangle y^2)\frac{1}{\epsilon^3}\right). \tag{68}$$

Therefore, the convergence rate for equation (61) is $O(\triangle t + (\triangle x^2 + \triangle y^2)\frac{1}{\epsilon^3})$*. If* ϵ *is small, the mesh size* $\triangle x$ *and* $\triangle y$ *should be chosen correspondingly small.*

For the Allen-Cahn equation (62), if we check the proof of Theorem 3.2, we have

$$\begin{aligned} \frac{U^{k+1}_{i,\,j} - U^k_{i,\,j}}{\triangle t} &= \epsilon^2 \frac{U^k_{i+1,\,j} - 2U^k_{i,\,j} + U^k_{i-1,j}}{\triangle x^2} \\ &+ \epsilon^2 \frac{U^k_{i,\,j+1} - 2U^k_{i,\,j} + U^k_{i,j-1}}{\triangle y^2} \\ &+ U^k_{i,\,j} - (U^k_{i,\,j})^3 + O(\triangle t + \epsilon^2 \triangle x^2 + \epsilon^2 \triangle y^2). \end{aligned} \tag{69}$$

Instead of $r_x + r_y \le \frac{1}{2}$*, we require* $\epsilon(r_x + r_y) \le \frac{1}{2}$*, where* r_x *and* r_y *are defined as before.*

We have

$$\max_{i,j} |V^{k+1}_{i,\,j}| \le C(\triangle t + \epsilon^2 \triangle x^2 + \epsilon^2 \triangle y^2). \tag{70}$$

Therefore, the convergence rate for equation (62) is $O(\triangle t + \epsilon^2 \triangle x^2 + \epsilon^2 \triangle y^2)$*.*

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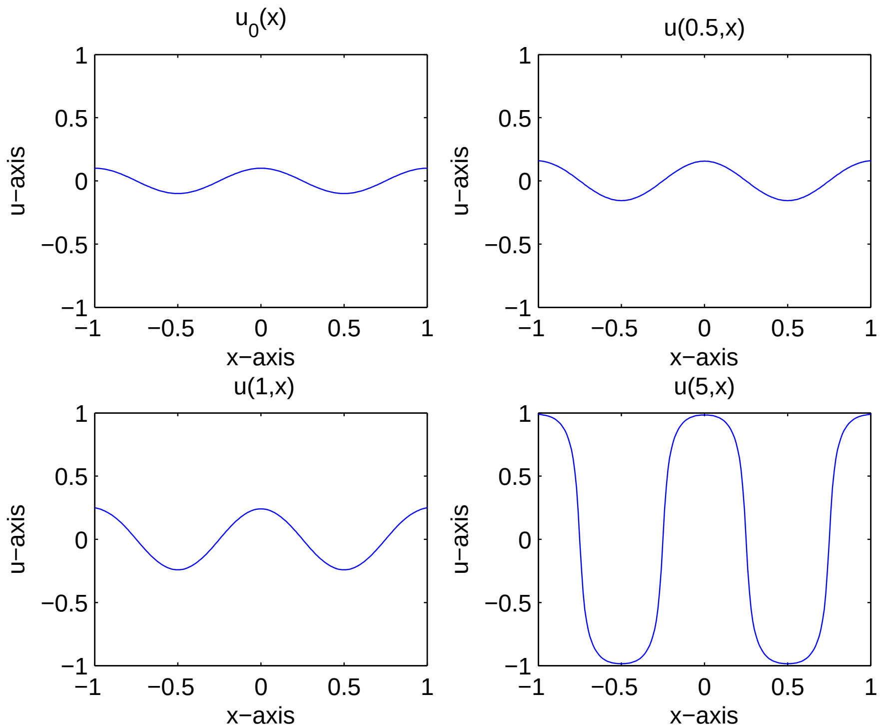

FIGURE 1. Nonlocal Allen Equation for $n = 1$ with a small diffusion coefficient.

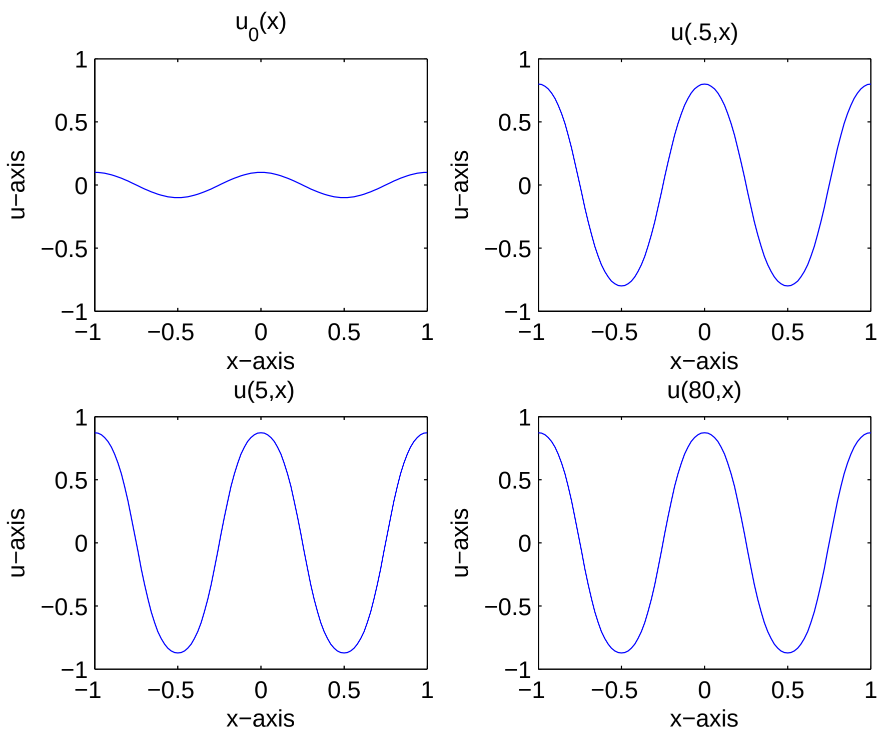

FIGURE 2. Allen-Cahn equation for $n = 1$ with a small diffusion coefficient.

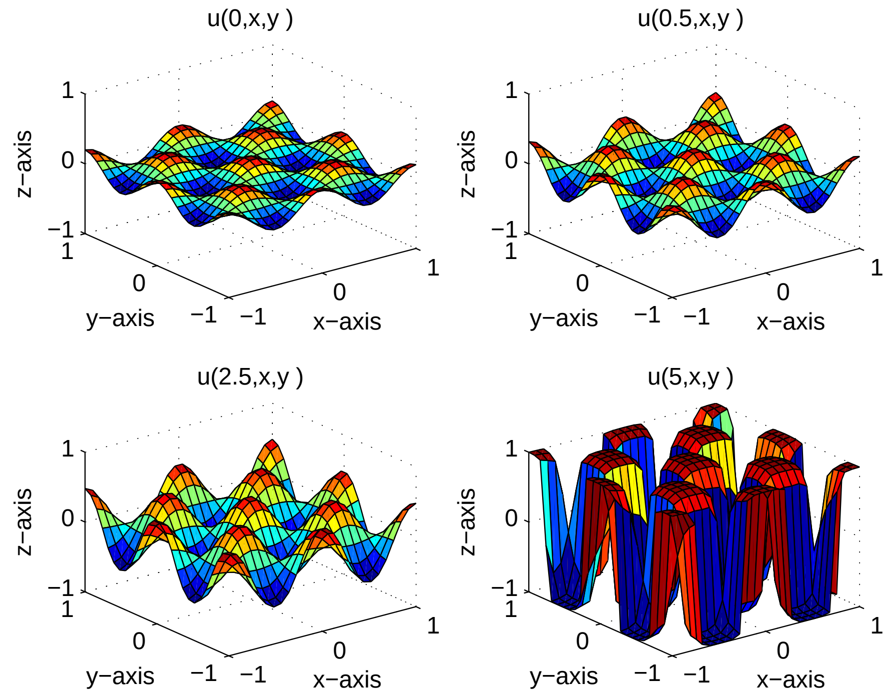

FIGURE 3. Nonlocal Allen-Cahn equation for $n = 2$ with a small diffusion coefficient.

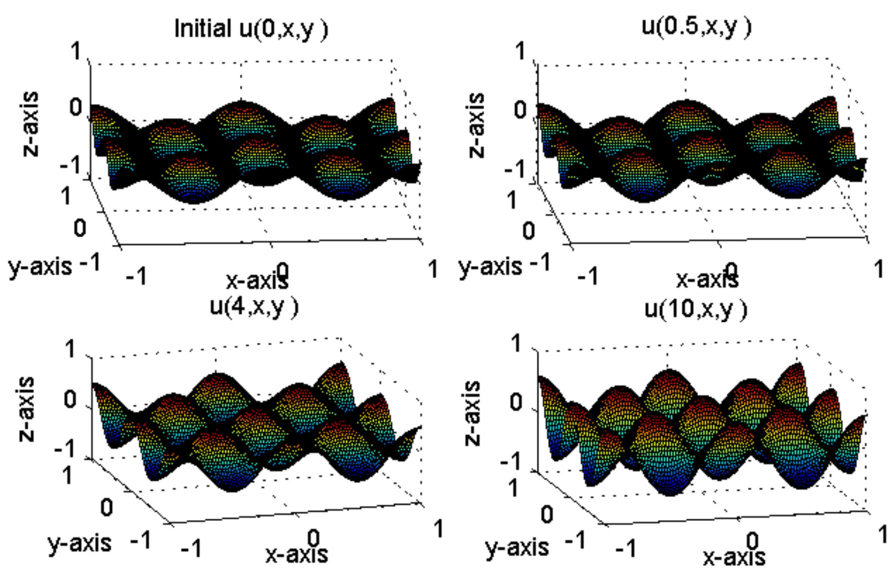

FIGURE 4. Allen-Cahn equation for $n = 2$ with a small diffusion coefficient.

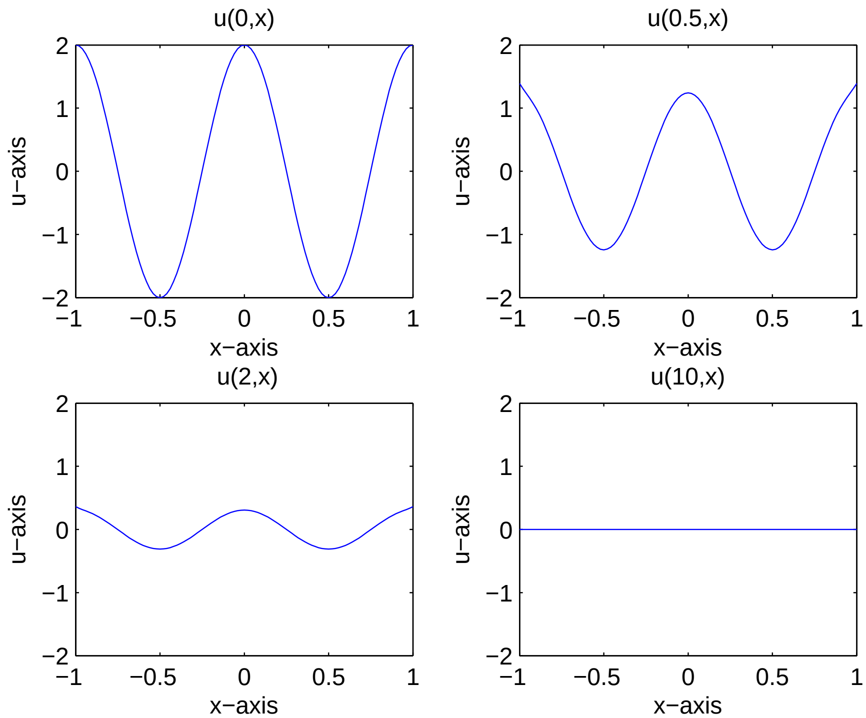

FIGURE 5. Nonlocal Allen Equation for $n = 1$ with a large diffusion coefficient.

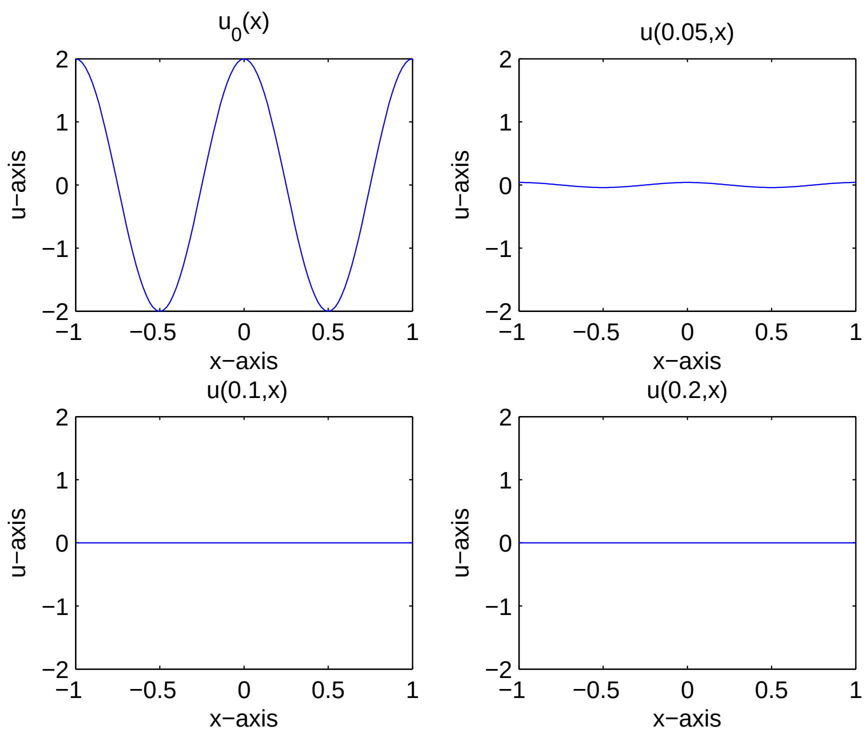

FIGURE 6. Allen-Cahn equation for $n = 1$ with a large diffusion coefficient.

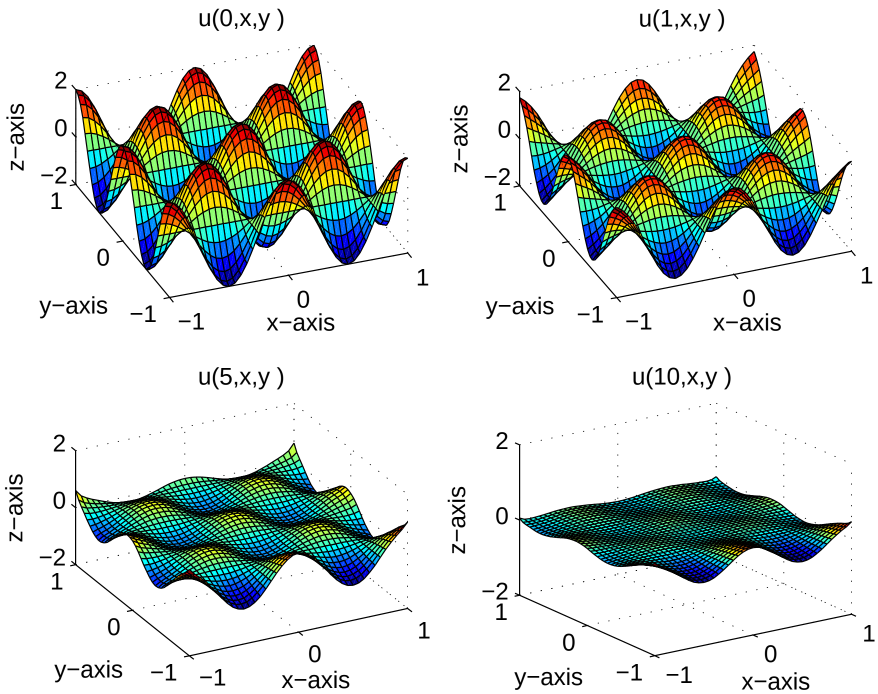

FIGURE 7. Nonlocal Allen-Cahn equation for $n = 2$ with a large diffusion coefficient.

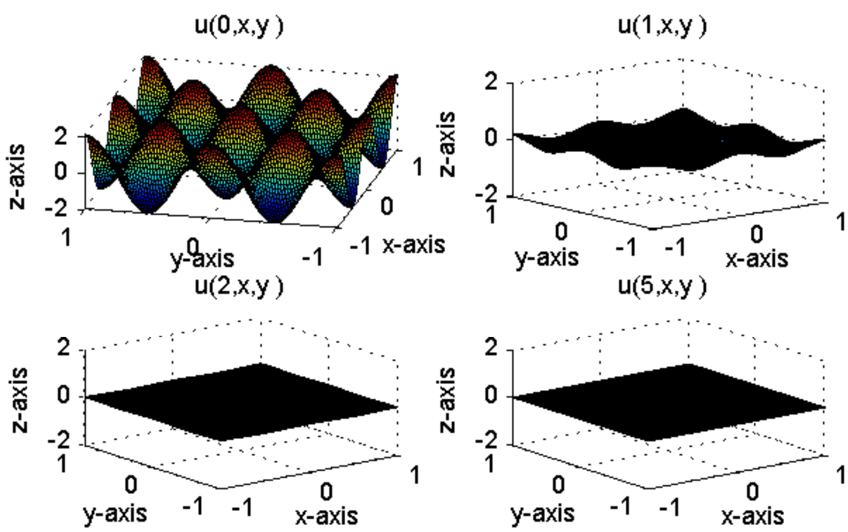

FIGURE 8. Allen-Cahn equation for $n = 2$ with a large diffusion coefficient.

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Department of Mathematics, Michigan State University East Lansing, MI 48824, USA
E-mail: bates@math.msu.edu

Department of Mathematics, Southern Utah University, Cedar City, UT 84720, USA
E-mail: brown_s@suu.edu and han@suu.edu