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CONVERGENCE OF HIGH ORDER METHODS FOR MISCIBLE DISPLACEMENT

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Abstract. We derive error estimates for a fully discrete scheme using primal discontinuous Galerkin discretization in space and backward Euler discretization in time. The estimates in the energy norm are optimal with respect to the mesh size and suboptimal with respect to the polynomial degree. The proposed scheme is of high order as polynomial approximations of pressure and concentration can take any degree. In addition, the method can handle different types of boundary conditions and is well-suited for unstructured meshes.

Key Words. flow, transport, porous media, miscible displacement, NIPG, SIPG, IIPG, h and p-version, fully discrete scheme.

1. Introduction

A high order numerical method for solving miscible displacement is introduced and analyzed in this paper. Miscible displacement occurs in important applications such as remediation of contaminated groundwater and production of oil from petroleum reservoirs. The physical model that describes the miscible displacement phenomena arises from the natural law of conservation of mass. This law is applied to each component of the fluid mixture. The mathematical model consists of a coupled system of partial differential equations: a pressure equation and a concentration equation for each component. Since the components of the fluid mixture may react with each other, the numerical method must accurately solve the laws of conservation. In particular, it is important to solve the continuity equation that describes the flow phenomena with high accuracy.

In this work, we propose a fully discrete scheme that is locally mass conservative. The approximations of pressure and concentration at each time step are discontinuous piecewise polynomials of different degrees. We show convergence of the numerical method with respect to both the mesh size and the polynomial degree. The flexibility inherent to discontinuous approximation spaces allows the use of complicated geometries and unstructured meshes. The primal discontinuous Galerkin method, analyzed in this paper, encompasses the nonsymmetric interior penalty Galerkin (NIPG) method, the symmetric interior penalty Galerkin (SIPG) and the incomplete interior penalty Galerkin (IIPG) method introduced for elliptic problems in [18, 26, 4]. Discontinuous Galerkin methods have been recently popular in modeling complex flow and transport problems in porous media (see for instance [22, 6, 5, 10, 14]).

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Several methods for solving the miscible displacement are proposed and analyzed in the literature. When classical continuous finite element approximations are used for both the pressure and the concentration equations, optimal convergence rates are proved in the dispersion-free case and nearly optimal convergence rates in the dispersion case, under somewhat idealized circumstances [8]. However, this procedure does not handle the transport-dominated problem arising from the concentration equation. Strong improvement in the accuracy of the approximation of the concentration is obtained by considering interior penalty Galerkin methods that can be based on continuous piecewise polynomial spaces [27] or on discontinuous piecewise polynomial spaces [11]. In this case, the pressure equation is solved with a continuous finite element method and penalty terms involving the jumps in the normal derivative are introduced in the concentration equation.

In the miscible displacement problem, only the velocity enters the equation for the concentration and therefore a natural procedure for solving the pressure equation is the locally mass conservative mixed finite element method. The concentration equation can be handled either by a continuous finite element method [12, 13] or by a modified method of characteristics, which combines the time derivative and the advection terms as a directional derivative [16, 24, 3]. In [23], a combination of a continuous finite element method and the method of characteristics for the concentration equation and a standard continuous finite element method for the pressure equation is used. As in the above cases, time stepping is done along the characteristics.

More recently, primal discontinuous Galerkin methods have been applied and analyzed for solving the miscible displacement problem using a semi-discrete approach. The system of equations is discretized in space only. A combined mixed method for the pressure equation with NIPG for concentration equation is studied in [20]. Both pressure and concentration are approximated by the NIPG method in [21, 17]. However, the convergence result in [21] is valid only if the boundary condition for pressure is a Neumann type. The numerical scheme presented in this paper, is fully discrete and valid for both Dirichlet and Neumann boundary conditions for the pressure and Dirichlet, Neuman and mixed boundary conditions for the concentration.

The outline of the paper is as follows. Section 2 contains the model problem and assumptions on the data. The coupled discontinuous Galerkin scheme is formulated in Section 3. Existence and convergence of the numerical solution are obtained in Section 4. Extensions of the scheme to several types of boundary conditions are presented in Section 5.

2. Model Problem and Notation

Consider the miscible displacement of one incompressible fluid by another in a porous medium $\Omega \subset \mathbb{R}^2$ and over the time interval $(0,T)$. Let p denote the pressure in the fluid mixture and let c denote the concentration (fraction volume) of the displaced fluid in the fluid mixture. The partial differential equations describing this type of flow are:

$$-\nabla \cdot \left(\frac{K}{\mu(c)} \nabla p\right) = f_1, \quad \text{in} \quad \Omega \times (0, T), \tag{1}$$

$$u = -\frac{K}{\mu(c)} \nabla p, \quad \text{in} \quad \Omega \times (0, T), \tag{2}$$

$$\varphi \frac{\partial c}{\partial t} + \nabla \cdot (uc - D(u)\nabla c) = f_2, \quad \text{in} \quad \Omega \times (0, T), \tag{3}$$

subject to the following boundary conditions:

$$p = p_{\text{dir}} \quad \text{on} \quad \Gamma_{\text{D}} \times [0, T], \tag{4}$$

$$u \cdot n = u_{\text{dir}} \quad \text{on} \quad \Gamma_{\text{N}} \times [0, T], \tag{5}$$

$$c = c_{\text{dir}} \quad \text{on} \quad \partial\Omega \times [0, T], \tag{6}$$

where $\Gamma_{\text{D}} \cup \Gamma_{\text{N}}$ is a partition of the boundary $\partial\Omega$. Equation (1), referred to as the pressure equation, is coupled with equation (3) through the viscosity of the fluid mixture. Equation (3), referred to as the concentration equation, is coupled with equation (1) through the fluid velocity (2) and the dispersion-diffusion tensor $D(u)$:

$$D(u) = (\alpha_t \|u\|_2 + d_m)I + (\alpha_l - \alpha_t)\frac{uu^T}{\|u\|_2}.$$

The coefficient d_m is the molecular diffusivity, α_l and α_t are the longitudinal and transverse dispersivities, $\|u\|_2$ is the Euclidean norm of the velocity and I is the identity matrix. Let us also note, that the permeability K in the velocity equation (2) is obtained from a macroscopic averaging of the microscopic features of the medium. Hence, it can be discontinuous in space variable and can vary over several orders of magnitude. The coefficient ϕ in (3) is the porosity. Assumptions on the coefficients are made below.

Assumption H1. The function μ^{-1} is positive, bounded below and above by $\underline{\mu}$ and $\overline{\mu}$ respectively and it is also Lipschitz continuous.

$$\forall x_1, x_2 \in \mathbb{R}, \left|\frac{1}{\mu(x_1)} - \frac{1}{\mu(x_2)}\right| \le \mu_L |x_1 - x_2|. \tag{7}$$

Assumption H2. The matrix K is symmetric positive definite and uniformly bounded above and below. There are positive constants $\overline{k}, \underline{k}$ such that:

$$\forall x \in \mathbb{R}^2, \quad \underline{k} x^T x \le x^T K x \le \overline{k} x^T x. \tag{8}$$

Assumption H3. The diffusion coefficient is strictly positive and the dispersivities are bounded.

$$\forall x \in \mathbb{R}^2, \quad 0 \le \alpha_l(x) \le \overline{\alpha_l}, \quad 0 \le \alpha_t(x) \le \overline{\alpha_t}, \text{ and } 0 < \underline{d} \le d_m.$$

Under assumption H3 it was shown that $D(u)$ is uniformly positive definite in Ω and Lipschitz continuous [21]:

$$\forall u \in \mathbb{R}^2, \quad \forall x \in \mathbb{R}^2, \quad \underline{d} x^T x \le x^T D(u) x, \tag{9}$$

$$\forall u, v \in \mathbb{R}^2, \quad \|D(u) - D(v)\|_2 \le k_2 \|u - v\|_2, \tag{10}$$

Assumption H4. The matrix $D(u)$ is uniformly bounded above.

$$\forall u \in \mathbb{R}^2, \quad \forall x \in \mathbb{R}^2, x^T D(u) x \le \overline{d} x^T x. \tag{11}$$

We propose a discontinuous finite element discretization of (1)-(6). For this, we introduce a non-degenerate quasi-uniform subdivision of Ω, made of either triangles or quadrilaterals. The quasi-uniformity assumption is only needed for the p-version, i.e. for deriving error estimates in terms of the polynomial degree. As usual, the maximum diameter over all mesh elements is denoted by h. The set of interior edges is denoted by Γ_h. To each edge e in Γ_h, we associate a unit normal vector n_e. For a boundary edge, n_e is chosen so that it coincides with the outward normal. The space of discontinuous polynomials of degree $r \geq 1$ is denoted by $\mathcal{D}_r(\mathcal{E}_h)$:

$$\mathcal{D}_r(\mathcal{E}_h) = \{v \in L^2(\Omega) : \forall E \in \mathcal{E}_h : v|_E \in \mathbb{P}_r(E)\}.$$

For any function $v \in \mathcal{D}_r(\mathcal{E}_h)$, we denote the jump and average over a given edge e by $[v]$ and $\{v\}$ respectively. Assuming that n_e is outward to E_e^1, we can write:

$$\forall e = \partial E_e^1 \cap \partial E_e^2, \quad [v]|_e = v|_{E_e^1} - v|_{E_e^2}, \quad \{v\}|_e = 0.5 v|_{E_e^1} + 0.5 v|_{E_e^2},$$
$$\forall e = \partial E_e^1 \cap \partial\Omega, \quad [v]|_e = v|_{E_e^1}, \quad \{v\}|_e = v|_{E_e^1}.$$

Let N be a positive integer and let $\Delta t = T/N$ be the time step. Denote $t^i = i\Delta t$ for $0 \leq i \leq N$. Define the space

$$\mathcal{D}_{r,h}^N = \{\boldsymbol{v} = (v^i)_{0\leq i\leq N} : \quad \forall 0 \leq i \leq N \quad v^i \in \mathcal{D}_r(\mathcal{E}_h)\}.$$

We also denote by $\tilde{M}$ the constant that only depends on the maximum number of neighbors that one mesh element can have so that the following inequality holds. Let A be any quantity depending on E_e^1 or E_e^2:

$$\forall i = 1, 2, \quad (\sum_{e\in\Gamma_h} A(E_e^i))^{1/2} \leq \frac{\sqrt{\tilde{M}}}{2}(\sum_{E\in\mathcal{E}_h} A(E))^{1/2}. \tag{12}$$

$$(\sum_{e\in\Gamma_{\mathrm{D}}} A(E_e^1))^{1/2} \leq \sqrt{\tilde{M}}(\sum_{E\in\mathcal{E}_h} A(E))^{1/2}. \tag{13}$$

Let $H^k(\mathcal{O})$ be the usual Sobolev space on $\mathcal{O} \subset \mathbb{R}^d, d \geq 1$ with norm $\|\cdot\|_{k,\mathcal{O}}$. We also define the broken norm: $|||v|||_{k,\Omega} = \left(\sum_{E\in\mathcal{E}_h} \|v\|_{k,E}^2\right)^{1/2}$. We now recall well-known trace results and inverse inequality used in the error analysis [1, 25, 19].

Lemma 1. *There is a constant M_t independent of h such that if E is a triangle or quadrilateral, for any $e \subset \partial E$:*

$$\forall v \in H^s(E), s \geq 1, \|v\|_{0,e} \leq M_t h^{-1/2}(\|v\|_{0,E} + h\,\|\nabla v\|_{0,E}), \tag{14}$$

$$\forall v \in H^s(E), s \geq 2, \|\nabla v \cdot n\|_{0,e} \leq M_t h^{-1/2}(\|\nabla v\|_{0,E} + h\,\left\|\nabla^2 v\right\|_{0,E}). \tag{15}$$

Lemma 2. *Let E be a mesh element. Let $g : \mathbb{N} \to \mathbb{N}$ be a function defined by $g(k) = (k+1)(k+2)$ if E is a triangle, and by $g(k) = k^2$ if E is a quadrilateral. There is a constant M_t independent of h and k such that:*

$$\forall v \in \mathbb{P}_k(E), \forall e \subset \partial E, \|v\|_{0,e} \leq M_t\sqrt{\frac{g(k)}{h}}\,\|v\|_{0,E}. \tag{16}$$

Lemma 3 (Inverse Inequalities)**.** *Let E be a mesh element and $v \in \mathrm{P}_r(E)$. Then there exists a constant C independent of h and r such that*

$$\|v\|_{L^\infty(E)} \leq Ch^{-1}r^2\|v\|_{0,E}, \tag{17}$$

$$\|v\|_{1,E} \leq Ch^{-1}r^2\|v\|_{0,E}. \tag{18}$$

3. Scheme

At each discrete time t^i, we will approximate the pressure $p(t^i,\cdot)$ and concentration $c(t^i,\cdot)$ by discontinuous piecewise polynomials P^i and C^i of degree r_p and r_c respectively. For the p-version, we assume that the degrees are related in the following fashion. There exist positive constants δ_0, δ_1 such that

$$\delta_0 \leq \frac{r_c}{r_p} \leq \delta_1. \tag{19}$$

Before formulating the scheme, we introduce additional notation. Let ε be a parameter that takes the value -1, 0 or 1. By changing the value of ε, we will obtain the NIPG, SIPG or IIPG method. Let $\sigma_p > 0$ and $\sigma_c > 0$ be the penalty parameters. Our numerical method is the following: find $\boldsymbol{P} = (P^i)_{0\leq i\leq N} \in \mathcal{D}^N_{r_p,h}$ and $\boldsymbol{C} = (C^i)_{0\leq i\leq N} \in \mathcal{D}^N_{r_c,h}$ such that

Initial Concentration

$$\forall v \in \mathcal{D}_{r_c}(\mathcal{E}_h), \quad \int_\Omega C^0 v = \int_\Omega c^0 v. \tag{20}$$

Pressure Equation: $\forall 0 \leq i \leq N-1$,

$$\begin{aligned}
&\forall z \in \mathcal{D}_{r_p}(\mathcal{E}_h), \quad \sum_{E\in\mathcal{E}_h} \int_E \frac{1}{\mu(C^{i+1})} K\nabla P^{i+1}\cdot\nabla z + \sigma_p \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}} \frac{g(r_p)}{|e|}\int_e [P^{i+1}][z] \\
&\quad - \sum_{e\in\Gamma_h}\int_e \{\frac{1}{\mu(C^{i+1})} K\nabla P^{i+1}\cdot n_e\}[z] - \sum_{e\in\Gamma_{\mathrm{D}}}\int_e \frac{1}{\mu(c_{\mathrm{dir}})} K\nabla P^{i+1}\cdot n_e z \\
&\quad +\varepsilon \sum_{e\in\Gamma_h}\int_e \{\frac{1}{\mu(C^{i+1})} K\nabla z\cdot n_e\}[P^{i+1}] + \varepsilon \sum_{e\in\Gamma_{\mathrm{D}}}\int_e \frac{1}{\mu(c_{\mathrm{dir}})} K\nabla z\cdot n_e P^{i+1} \\
&= \int_\Omega f_1 z + \sum_{e\in\Gamma_{\mathrm{N}}}\int_e u_{\mathrm{dir}} z + \sigma_p \sum_{e\in\Gamma_{\mathrm{D}}} \frac{g(r_p)}{|e|}\int_e p_{\mathrm{dir}} z + \varepsilon \sum_{e\in\Gamma_{\mathrm{D}}}\int_e \frac{1}{\mu(c_{\mathrm{dir}})} K\nabla z\cdot n_e p_{\mathrm{dir}}.
\end{aligned} \tag{21}$$

Concentration Equation: $\forall 0 \leq i \leq N-1$,

$$\begin{aligned}
&\forall v \in \mathcal{D}_{r_c}(\mathcal{E}_h), \quad \int_\Omega \frac{\varphi}{\Delta t}(C^{i+1}-C^i)v + \sum_{E\in\mathcal{E}_h}\int_E \frac{C^{i+1}}{\mu(C^{i+1})} K\nabla P^{i+1}\cdot\nabla v \\
&\quad + \sum_{E\in\mathcal{E}_h}\int_E D(U^{i+1})\nabla C^{i+1}\cdot\nabla v - \sum_{e\in\Gamma_h}\int_e \{\frac{C^{i+1}}{\mu(C^{i+1})} K\nabla P^{i+1}\cdot n_e\}[v] \\
&\quad - \sum_{e\in\Gamma_{\mathrm{D}}}\int_e \frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})} K\nabla P^{i+1}\cdot n_e v - \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}}\int_e \{D(U^{i+1})\nabla C^{i+1}\cdot n_e\}[v] \\
&\quad +\varepsilon \sum_{e\in\Gamma_h}\int_e \{\frac{C^{i+1}}{\mu(C^{i+1})} K\nabla v\cdot n_e\}[P^{i+1}] + \varepsilon \sum_{e\in\Gamma_{\mathrm{D}}}\int_e \frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})} K\nabla v\cdot n_e P^{i+1} \\
&\quad +\varepsilon \sum_{e\in\Gamma_h}\int_e \{D(U^{i+1})\nabla v\cdot n_e\}[C^{i+1}] + \sigma_c \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}} \frac{g(r_c)}{|e|}\int_e [C^{i+1}][v] \\
&= \varepsilon \sum_{e\in\Gamma_{\mathrm{D}}}\int_e \frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})} K\nabla v\cdot n_e p_{\mathrm{dir}} + \sigma_c \sum_{e\in\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}} \frac{g(r_c)}{|e|}\int_e c_{\mathrm{dir}} v + \int_\Omega f_2 v + \sum_{e\in\Gamma_{\mathrm{N}}}\int_e c_{\mathrm{dir}} u_{\mathrm{dir}} v,
\end{aligned} \tag{22}$$

with the definition of the discrete velocity U^{i+1} given by

$$U^{i+1} = -\frac{K}{\mu(C^{i+1})}\nabla P^{i+1}. \tag{23}$$

We obtain a nonlinear system of equations that can be written in short as

$$\forall z \in \mathcal{D}_{r_p}(\mathcal{E}_h), \quad \forall v \in \mathcal{D}_{r_c}(\mathcal{E}_h), \quad 0 \le i \le N, \quad \mathcal{L}(P^i, C^i; z, v) = 0.$$

It is easy to check, using standard techniques for Interior Penalty discontinuous Galerkin methods, that the scheme (20)-(22) is consistent, i.e. if the solution of (1)-(6) is smooth enough and if we denote $c^i = c(t^i, \cdot)$ and $p^i = p(t^i, \cdot)$, then

$$\forall 0 \le i \le N, \quad \forall z \in \mathcal{D}_{r_p}(\mathcal{E}_h), \quad \forall v \in \mathcal{D}_{r_c}(\mathcal{E}_h), \quad \mathcal{L}(p^i, c^i; z, v) = 0, \tag{24}$$

4. Existence and Convergence of the Discrete Solution

In this section, we prove the existence and show convergence of the numerical solution $(\boldsymbol{P}, \boldsymbol{C})$ by the use of the Schauder's fixed point theorem (see for example theorem 6.44 in [9]). Let $\tilde{p}$ and $\tilde{c}$ be approximations of p and c. We assume that

$$\tilde{p} \in L^\infty(0, T, W^{1,\infty}(\Omega)), \quad \tilde{c} \in L^\infty(0, T, L^\infty(\Omega)), \quad \tilde{c}_{tt} \in L^\infty(0, T, L^2(\Omega)). \tag{25}$$

We will denote $\tilde{p}^i(\cdot) = \tilde{p}(t^i, \cdot)$ and $\tilde{c}^i(\cdot) = \tilde{c}(t^i, \cdot)$. We assume that there are constants $\kappa_p, \kappa_c \ge 2$ such that

$$\forall 0 \le i \le N, \quad \forall t > 0, \quad p^i(t) \in H^{\kappa_p}(\Omega), \quad c^i(t) \in H^{\kappa_c}(\Omega).$$

We assume that the following standard hp-type approximation results hold [2]

$$\forall 0 \le i \le N, \quad \|\tilde{p}^i - p^i\|_{H^s(\Omega)} \le M \frac{h^{\min(r_p+1, \kappa_p) - s}}{r_p^{\kappa_p - s}} \|p^i\|_{H^{\kappa_p}(\Omega)}, \tag{26}$$

$$\forall 0 \le i \le N, \quad \|\tilde{c}^i - c^i\|_{H^s(\Omega)} \le M \frac{h^{\min(r_c+1, \kappa_c) - s}}{r_c^{\kappa_c - s}} \|c^i\|_{H^{\kappa_c}(\Omega)}, \tag{27}$$

Here and throughout the paper, M is a generic constant independent of h, r_c, r_p and Δt, that takes different values at different places. In addition, in the case of the p-version, we assume that $\kappa_p, \kappa_c \ge 3$.
Next we prove existence and convergence of the solution using an idea similar to idea in [15]. Let us define the following subset of the broken Sobolev space:

$$\mathcal{W} = \Big\{ (\boldsymbol{\psi}, \boldsymbol{\phi}) \in \mathcal{D}^N_{r_p,h} \times \mathcal{D}^N_{r_c,h} : \phi^0 = \tilde{c}^0, \text{ and there exist positive constants}$$

$$K_1, K_2, \ldots, K_6, \Delta t_0 \text{ independent of } h \text{ such that for } \Delta t \le \Delta t_0 \text{ and } 0 \le i \le N-1:$$

$$(\frac{1}{\Delta t} - K_1) \left\|\phi^{i+1} - \widetilde{c}^{i+1}\right\|^2_{0,\Omega} - \frac{1}{\Delta t} \left\|\phi^i - \widetilde{c}^i\right\|^2_{0,\Omega}$$

$$+|||\phi^{i+1} - \widetilde{c}^{i+1}|||_1^2 \le K_2 \frac{h^{2r_p}}{r_p^{2\kappa_p - 4}} + K_3 \frac{h^{2r_c}}{r_c^{2\kappa_c - 4}} + K_4 \Delta t^2,$$

$$|||\psi^{i+1} - \widetilde{p}^{i+1}|||_1^2 \le K_5 \frac{h^{2r_p}}{r_p^{2\kappa_p - 4}} + K_6 \frac{h^{2r_c}}{r_c^{2\kappa_c - 4}} \Big\}.$$

The set $\mathcal{W}$ is closed, convex subset of the broken Sobolev space and it is not empty since it contains the element $(\tilde{p}^i, \tilde{c}^i)_{0 \le i \le N}$.

Lemma 4. *For any $(\boldsymbol{\psi}, \boldsymbol{\phi})$ in $\mathcal{W}$, if Δt is small enough (namely $\Delta t = \mathcal{O}(h/r_c) < 1/K_1$), there exist positive constant M_1, M_2, M_3 for any $1 \le i \le N$*

$$\left\|\phi^i - \widetilde{c}^i\right\|_{0,\Omega} \le M_1 \big(\frac{h^{r_p}}{r_p^{\kappa_p - 2}} + \frac{h^{r_c}}{r_c^{\kappa_c - 2}} + \Delta t \big), \tag{28}$$

$$||\phi^i||_{\infty,\Omega} \le M_2, \quad |||\psi^i|||_1 \le M_3. \tag{29}$$

The constants M_1, M_2 are independent of h, r_p, r_c and Δt but depend on $K_1, \ldots, K_4$. The constant M_3 is independent of h, r_p, r_c and Δt but depends on K_5, K_6.

Proof. We will show that (29) is a consequence of the definition of $\mathcal{W}$. We first prove (28), which will yield (29). From the definition of the space $\mathcal{W}$, we have for $0 \le i \le N-1$:

$$\left\|\phi^{i+1} - \widetilde{c}^{i+1}\right\|_{0,\Omega}^2 - \left\|\phi^{i} - \widetilde{c}^{i}\right\|_{0,\Omega}^2 + \Delta t |||\phi^{i+1} - \widetilde{c}^{i+1}|||_1^2$$

$$\le \Delta t K_2 \frac{h^{2r_p}}{r_p^{2\kappa_p - 4}} + \Delta t K_3 \frac{h^{2r_c}}{r_c^{2\kappa_c - 4}} + K_4 \Delta t^3 + \Delta t K_1 \left\|\phi^{i+1} - \widetilde{c}^{i+1}\right\|_{0,\Omega}^2.$$

Fix $n \ge 1$, sum from $i = 0$ to $i = n-1$ and note that $\sum_{i=0}^{n-1} \Delta t \le T$ and $\phi^0 = \tilde{c}^0$:

$$\|\phi^n - \widetilde{c}^n\|_{0,\Omega}^2 + \Delta t \sum_{i=0}^{n-1} |||\phi^{i+1} - \widetilde{c}^{i+1}|||_1^2$$

$$\le K_2 T \frac{h^{2r_p}}{r_p^{2\kappa_p - 4}} + K_3 T \frac{h^{2r_c}}{r_c^{2\kappa_c - 4}} + K_4 T \Delta t^2 + \Delta t K_1 \sum_{i=0}^{n-1} \left\|\phi^{i+1} - \widetilde{c}^{i+1}\right\|_{0,\Omega}^2.$$

From Gronwall's lemma, if $\Delta t < 1/K_1$, there is a constant M independent of h and Δt such that

$$\|\phi^n - \widetilde{c}^n\|_{0,\Omega}^2 + \Delta t \sum_{i=0}^{N-1} |||\phi^{i+1} - \widetilde{c}^{i+1}|||_1^2 \le M\left(\frac{h^{2r_p}}{r_p^{2\kappa_p - 4}} + \frac{h^{2r_c}}{r_c^{2\kappa_c - 4}} + \Delta t^2\right).$$

This yields (28). Besides, from (19) and choosing $\Delta t = \mathcal{O}(\frac{h}{r_c^2})$, we conclude that

$$\|\phi^n - \widetilde{c}^n\|_{0,\Omega} \le M \frac{h}{r_c^2}.$$

Using an inverse inequality (17), we have

$$\|\phi^n - \widetilde{c}^n\|_{\infty,\Omega} \le M r_c^2 h^{-1} \|\phi^n - \widetilde{c}^n\|_{0,\Omega} \le M.$$

This implies that

$$\|\phi^n\|_{\infty,\Omega} \le M + \|\tilde{c}^n\|_{\infty,\Omega} \le M_2,$$

which with (25) yields gives the result (29). $\square$

We now define an operator $\mathcal{F}$ that acts on elements in $\mathcal{W}$.

$$\forall (\boldsymbol{\psi}, \boldsymbol{\phi}) \in \mathcal{W}, \quad \mathcal{F}(\boldsymbol{\psi}, \boldsymbol{\phi}) = (\boldsymbol{\psi}_L, \boldsymbol{\phi}_L),$$

where (ψ_L, ϕ_L) satisfies initial conditions:

$$(\psi_L^0, \phi_L^0) = (\psi^0, \phi^0), \tag{30}$$

and for $0 \le i \le N-1$, $\psi_L^{i+1} \in \mathcal{D}_{r_p}(\mathcal{E}_h)$ and $\phi_L^{i+1} \in \mathcal{D}_{r_c}(\mathcal{E}_h)$ such that

$$\forall z \in \mathcal{D}_{r_p}(\mathcal{E}_h), \quad \sum_{E \in \mathcal{E}_h} \int_E \frac{1}{\mu(\phi^{i+1})} K \nabla \psi_L^{i+1} \cdot \nabla z + \sigma_p \sum_{e \in \Gamma_h \cup \Gamma_{\mathrm{D}}} \frac{g(r_p)}{|e|} \int_e [\psi_L^{i+1}][z]$$

$$- \sum_{e \in \Gamma_h} \int_e \{\frac{1}{\mu(\phi^{i+1})} K \nabla \psi_L^{i+1} \cdot n_e\}[z] - \sum_{e \in \Gamma_{\mathrm{D}}} \int_e \frac{1}{\mu(c_{\mathrm{dir}})} K \nabla \psi_L^{i+1} \cdot n_e z$$

$$+ \varepsilon \sum_{e \in \Gamma_h} \int_e \{\frac{1}{\mu(\phi^{i+1})} K \nabla z \cdot n_e\}[\psi_L^{i+1}] + \varepsilon \sum_{e \in \Gamma_{\mathrm{D}}} \int_e \frac{1}{\mu(c_{\mathrm{dir}})} K \nabla z \cdot n_e \psi_L^{i+1}$$

$$= \int_\Omega f_1 z + \sum_{e \in \Gamma_{\mathrm{N}}} \int_e u_{\mathrm{dir}} z + \sigma_p \sum_{e \in \Gamma_{\mathrm{D}}} \frac{g(r_p)}{|e|} \int_e p_{\mathrm{dir}} z + \varepsilon \sum_{e \in \Gamma_{\mathrm{D}}} \int_e \frac{1}{\mu(c_{\mathrm{dir}})} K \nabla z \cdot n_e p_{\mathrm{dir}}. \tag{31}$$

$$\forall v \in \mathcal{D}_{r_c}(\mathcal{E}_h), \quad \int_\Omega \frac{\varphi}{\Delta t}(\phi_L^{i+1} - \phi_L^i) v + \sum_{E \in \mathcal{E}_h} \int_E \frac{\phi^{i+1}}{\mu(\phi^{i+1})} K \nabla \psi_L^{i+1} \cdot \nabla v$$

$$+\sum_{E\in\mathcal{E}_h}\int_E D(\zeta^{i+1})\nabla\phi_L^{i+1}\cdot\nabla v-\sum_{e\in\Gamma_h}\int_e\{\frac{\phi^{i+1}}{\mu(\phi^{i+1})}K\nabla\psi_L^{i+1}\cdot n_e\}[v]$$

$$-\sum_{e\in\Gamma_{\rm D}}\int_e\frac{c_{\rm dir}}{\mu(c_{\rm dir})}K\nabla\psi_L^{i+1}\cdot n_e v-\sum_{e\in\Gamma_h\cup\Gamma_{\rm D}\cup\Gamma_{\rm N}}\int_e\{D(\zeta^{i+1})\nabla\phi_L^{i+1}\cdot n_e\}[v]$$

$$+\varepsilon\sum_{e\in\Gamma_h}\int_e\{\frac{\phi^{i+1}}{\mu(\phi^{i+1})}K\nabla v\cdot n_e\}[\psi_L^{i+1}]+\varepsilon\sum_{e\in\Gamma_{\rm D}}\int_e\frac{c_{\rm dir}}{\mu(c_{\rm dir})}K\nabla v\cdot n_e\psi_L^{i+1}$$

$$+\varepsilon\sum_{e\in\Gamma_h}\int_e\{D(\zeta^{i+1})\nabla v\cdot n_e\}[\phi_L^{i+1}]+\sigma_c\sum_{e\in\Gamma_h\cup\Gamma_{\rm D}\cup\Gamma_{\rm N}}\frac{g(r_c)}{|e|}\int_e[\phi_L^{i+1}][v]$$

(32)

$$=\varepsilon\sum_{e\in\Gamma_{\rm D}}\int_e\frac{c_{\rm dir}}{\mu(c_{\rm dir})}K\nabla v\cdot n_e p_{\rm dir}+\sigma_c\sum_{e\in\Gamma_{\rm D}\cup\Gamma_{\rm N}}\frac{g(r_c)}{|e|}\int_e c_{\rm dir}v+\int_\Omega f_2 v+\sum_{e\in\Gamma_{\rm N}}\int_e c_{\rm dir}u_{\rm dir}v,$$

where

$$\zeta^{i+1}=-\frac{K}{\mu(\phi^{i+1})}\nabla\psi^{i+1}. \tag{33}$$

We show that $\mathcal{F}$ is well-defined by proving existence and uniqueness of $(\boldsymbol{\psi}_L,\boldsymbol{\phi}_L)$.

Lemma 5. *There exists a unique solution $(\boldsymbol{\psi}_L,\boldsymbol{\phi}_L)\in\mathcal{D}^N_{r_p,h}\times\mathcal{D}^N_{r_c}$ that satisfies (30)-(32).*

Proof. Since the problem (30)-(32) is linear and finite-dimensional, it suffices to show uniqueness of the solution. Let $(\boldsymbol{\psi}_{L1},\boldsymbol{\phi}_{L1})$ and $(\boldsymbol{\psi}_{L2},\boldsymbol{\phi}_{L2})$ be two solutions and let $(\bar{\boldsymbol{\psi}},\bar{\boldsymbol{\phi}})$ denote their differences. Then, the pair $(\bar{\boldsymbol{\psi}},\bar{\boldsymbol{\phi}})$ satisfies (30)-(32) with zero data $f_1=p_{\rm dir}=u_{\rm dir}=c_{\rm dir}=f_2=0$ and $\phi_L^i=0$. Clearly, we have $(\bar\psi^0,\bar\phi^0)=(0,0)$. Fix $i\in\{0,\dots,N-1\}$ and choose the test function $z=\bar\psi^{i+1}$ in (31).

$$|||\frac{1}{\mu(\phi^{i+1})^{1/2}}K^{1/2}\nabla\bar\psi^{i+1}|||^2_{0,\Omega}+\sigma_p\sum_{e\in\Gamma_h\cup\Gamma_{\rm D}}\frac{g(r_p)}{|e|}\|[\bar\psi^{i+1}]\|^2_{0,e}$$

$$-(1-\epsilon)\sum_{e\in\Gamma_h}\int_e\{\frac{1}{\mu(\phi^{i+1})}K\nabla\bar\psi^{i+1}\cdot n_e\}[\bar\psi^{i+1}]-(1-\epsilon)\sum_{e\in\Gamma_{\rm D}}\int_e\frac{1}{\mu(c_{\rm dir})}K\nabla\bar\psi^{i+1}\cdot n_e\bar\psi^{i+1}=0.$$

If $\epsilon=1$, we directly have that $\bar\psi^{i+1}=0$. Otherwise, using assumption H1 and trace and inverse inequalities, we can bound the last two terms of the left-hand side of the equation above by

$$\frac{1}{2}|||\frac{1}{\mu(\phi^{i+1})^{1/2}}K^{1/2}\nabla\bar\psi^{i+1}|||^2_{0,\Omega}+M\sum_{e\in\Gamma_h\cup\Gamma_{\rm D}}\frac{1}{|e|}\|[\bar\psi^{i+1}]\|^2_{0,e},$$

which implies that $\bar\psi^{i+1}=0$ if the penalty value σ_p is large enough. Next, we choose the test function $v=\bar\phi^{i+1}$ in (32). A similar argument gives that $\bar\phi^{i+1}=0$ if the penalty σ_c is large enough. □

We now show that the range of $\mathcal{F}$ is included in the space $\mathcal{W}$. The same technique can be used to show that $\mathcal{F}$ is continuous.

Theorem 1.

$$\forall(\boldsymbol{\psi},\boldsymbol{\phi})\in\mathcal{W},\quad\mathcal{F}(\boldsymbol{\psi},\boldsymbol{\phi})\in\mathcal{W}.$$

Proof. Let $(\boldsymbol{\psi},\boldsymbol{\phi})\in\mathcal{W}$, $(\boldsymbol{\psi}_L,\boldsymbol{\phi}_L)=\mathcal{F}(\boldsymbol{\psi},\boldsymbol{\phi})$ and denote

$$\forall 0\le i\le N,\quad \tau^i=\psi_L^i-\widetilde{p}^i,\theta^i=p^i-\widetilde{p}^i,\xi^i=\phi_L^i-\tilde{c}^i,\chi^i=c^i-\tilde{c}^i.$$

From the consistency equations (24), we have

$$\begin{aligned}
&\sum_{E\in\mathcal{E}_h}\int_E\frac{1}{\mu(\phi^{i+1})}K\nabla\widetilde{p}^{i+1}\cdot\nabla z+\sigma_p\sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}}\frac{g(r_p)}{|e|}\int_e[\widetilde{p}^{i+1}][z]\\
&-\sum_{e\in\Gamma_h}\int_e\{\frac{1}{\mu(\phi^{i+1})}K\nabla\widetilde{p}^{i+1}\cdot n_e\}[z]-\sum_{e\in\Gamma_{\mathrm{D}}}\int_e\frac{1}{\mu(c_{\mathrm{dir}})}K\nabla\widetilde{p}^{i+1}\cdot n_e z-\sum_{e\in\Gamma_{\mathrm{N}}}\int_e u_{\mathrm{dir}}z\\
&+\varepsilon\sum_{e\in\Gamma_h}\int_e\{\frac{1}{\mu(\phi^{i+1})}K\nabla z\cdot n_e\}[\widetilde{p}^{i+1}]+\varepsilon\sum_{e\in\Gamma_{\mathrm{D}}}\int_e\frac{1}{\mu(c_{\mathrm{dir}})}K\nabla z\cdot n_e\widetilde{p}^{i+1}\\
&-\int_\Omega f_1 z-\sigma_p\sum_{e\in\Gamma_{\mathrm{D}}}\frac{g(r_p)}{|e|}\int_e p_{\mathrm{dir}}z-\varepsilon\sum_{e\in\Gamma_{\mathrm{D}}}\int_e\frac{1}{\mu(c_{\mathrm{dir}})}K\nabla z\cdot n_e p_{\mathrm{dir}}\\
&=-\sum_{E\in\mathcal{E}_h}\int_E\frac{1}{\mu(c^{i+1})}K\nabla\theta^{i+1}\cdot\nabla z-\sigma_p\sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}}\frac{g(r_p)}{|e|}\int_e[\theta^{i+1}][z]\\
&+\sum_{e\in\Gamma_h}\int_e\{\frac{1}{\mu(c^{i+1})}K\nabla\theta^{i+1}\cdot n_e\}[z]+\sum_{e\in\Gamma_{\mathrm{D}}}\int_e\frac{1}{\mu(c_{\mathrm{dir}})}K\nabla\theta^{i+1}\cdot n_e z\\
&-\varepsilon\sum_{e\in\Gamma_h}\int_e\{\frac{1}{\mu(c^{i+1})}K\nabla z\cdot n_e\}[\theta^{i+1}]-\varepsilon\sum_{e\in\Gamma_{\mathrm{D}}}\int_e\frac{1}{\mu(c_{\mathrm{dir}})}K\nabla z\cdot n_e\theta^{i+1}\\
&+\sum_{E\in\mathcal{E}_h}\int_E(\frac{1}{\mu(\phi^{i+1})}-\frac{1}{\mu(c^{i+1})})K\nabla\widetilde{p}^{i+1}\cdot\nabla z-\sum_{e\in\Gamma_h}\int_e\{(\frac{1}{\mu(\phi^{i+1})}-\frac{1}{\mu(c^{i+1})})K\nabla\widetilde{p}^{i+1}\cdot n_e\}[z]\\
&+\varepsilon\sum_{e\in\Gamma_h}\int_e\{(\frac{1}{\mu(\phi^{i+1})}-\frac{1}{\mu(c^{i+1})})K\nabla z\cdot n_e\}[\widetilde{p}^{i+1}].
\end{aligned}\tag{34}$$

Subtracting equation (34) from (31) and choosing $z=\tau^{i+1}$, we obtain:

$$\begin{aligned}
&|||\frac{1}{\mu(\phi^{i+1})^{1/2}}K^{1/2}\nabla\tau^{i+1}|||_{0,\Omega}^2+\sigma_p\sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}}\frac{g(r_p)}{|e|}\|[\tau^{i+1}]\|_{0,e}^2\\
&=(1-\varepsilon)\sum_{e\in\Gamma_h}\int_e\{\frac{1}{\mu(\phi^{i+1})}K\nabla\tau^{i+1}\cdot n_e\}[\tau^{i+1}]+(1-\varepsilon)\sum_{e\in\Gamma_{\mathrm{D}}}\int_e(\frac{1}{\mu(c_{\mathrm{dir}})}K\nabla\tau^{i+1}\cdot n_e)\tau^{i+1}\\
&-\sum_{E\in\mathcal{E}_h}\int_E\frac{1}{\mu(c^{i+1})}K\nabla\theta^{i+1}\cdot\nabla\tau^{i+1}-\sigma_p\sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}}\frac{g(r_p)}{|e|}\int_e[\theta^{i+1}][\tau^{i+1}]\\
&+\sum_{e\in\Gamma_h}\int_e\{\frac{1}{\mu(c^{i+1})}K\nabla\theta^{i+1}\cdot n_e\}[\tau^{i+1}]+\sum_{e\in\Gamma_{\mathrm{D}}}\int_e\frac{1}{\mu(c_{\mathrm{dir}})}K\nabla\theta^{i+1}\cdot n_e\tau^{i+1}\\
&-\varepsilon\sum_{e\in\Gamma_h}\int_e\{\frac{1}{\mu(c^{i+1})}K\nabla\tau^{i+1}\cdot n_e\}[\theta^{i+1}]-\varepsilon\sum_{e\in\Gamma_{\mathrm{D}}}\int_e\frac{1}{\mu(c_{\mathrm{dir}})}K\nabla\tau^{i+1}\cdot n_e\theta^{i+1}\\
&+\sum_{E\in\mathcal{E}_h}\int_E(\frac{1}{\mu(\phi^{i+1})}-\frac{1}{\mu(c^{i+1})})K\nabla\widetilde{p}^{i+1}\cdot\nabla\tau^{i+1}\\
&-\sum_{e\in\Gamma_h}\int_e\{(\frac{1}{\mu(\phi^{i+1})}-\frac{1}{\mu(c^{i+1})})K\nabla\widetilde{p}^{i+1}\cdot n_e\}[\tau^{i+1}]\\
&+\varepsilon\sum_{e\in\Gamma_h}\int_e\{(\frac{1}{\mu(\phi^{i+1})}-\frac{1}{\mu(c^{i+1})})K\nabla z\cdot n_e\}[\widetilde{p}^{i+1}]=T_1+\cdots+T_{11}.
\end{aligned}\tag{35}$$

Next, we bound each term in the right-hand side of (35) using techniques standard for discontinuous Galerkin methods. In what follows, the quantities ε_i are positive real numbers to be defined later. Using Assumptions $H1$ and $H2$ and Cauchy-Schwarz inequality, we have

$$|T_1| \leq (1-\varepsilon)\overline{\mu} \sum_{e\in\Gamma_h} \left\|\{K^{\frac{1}{2}}\nabla\tau^{i+1}\}\right\|_{0,e} \left\|[\tau^{i+1}]\right\|_{0,e}$$

We now fix an interior edge e and denote E_e^1 and E_e^2 two elements sharing the edge e. Using (12) and the trace inequality (16), we have:

$$\sum_{e\in\Gamma_h} \left\|\{K^{\frac{1}{2}}\nabla\tau^{i+1}\}\right\|_{0,e} \left\|[\tau^{i+1}]\right\|_{0,e} \leq \sum_{e\in\Gamma_h} \frac{1}{2}(\left\|K^{\frac{1}{2}}\nabla\tau^{i+1}|_{E_e^1}\right\|_{0,e} + \left\|K^{\frac{1}{2}}\nabla\tau^{i+1}|_{E_e^2}\right\|_{0,e}) \left\|[\tau^{i+1}]\right\|_{0,e}$$

$$\leq \frac{1}{2}M_t\sqrt{\frac{g(r_p)}{h}} \sum_{e\in\Gamma_h}(\left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E_e^1} + \left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E_e^2}) \left\|[\tau^{i+1}]\right\|_{0,e}$$

$$\leq (\sum_{e\in\Gamma_h} \frac{M_t^2 g(r_p)}{4h} \left\|[\tau^{i+1}]\right\|_{0,e}^2)^{\frac{1}{2}} \Big((\sum_{e\in\Gamma_h} \left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E_e^1}^2)^{\frac{1}{2}} + (\sum_{e\in\Gamma_h} \left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E_e^2}^2)^{\frac{1}{2}}\Big)$$

$$\leq (\sum_{e\in\Gamma_h} \frac{\widetilde{M}M_t^2 g(r_p)}{4h} \left\|[\tau^{i+1}]\right\|_{0,e}^2)^{\frac{1}{2}} (\sum_{E\in\mathcal{E}_h} \left\|K^{\frac{1}{2}}\nabla\tau^{i+1}\right\|_{0,E}^2)^{\frac{1}{2}}.$$

Therefore, we have the following bound on T_1:

$$|T_1| \leq \frac{\mu}{24}|||K^{\frac{1}{2}}\nabla\tau^{i+1}|||_{0,\Omega}^2 + (1-\varepsilon)^2 \frac{3(\overline{\mu})^2\overline{k}\widetilde{M}M_t^2}{2\underline{\mu}} \sum_{e\in\Gamma_h} \frac{g(r_p)}{h} \left\|[\tau^{i+1}]\right\|_{0,e}^2.$$

Similarly, using (13) and (16), we have for T_2:

$$|T_2| \leq \frac{\mu}{24}|||K^{\frac{1}{2}}\nabla\tau^{i+1}|||_{0,\Omega}^2 + (1-\varepsilon)^2 \frac{6(\overline{\mu})^2\overline{k}\widetilde{M}M_t^2}{\underline{\mu}} \sum_{e\in\Gamma_{\mathrm{D}}} \frac{g(r_p)}{|e|} \left\|[\tau^{i+1}]\right\|_{0,e}^2.$$

The term T_3 is bounded using assumption H1 and (8), Cauchy-Schwarz and Young's inequalities:

$$|T_3| \leq \frac{\mu}{12}|||K^{\frac{1}{2}}\nabla\tau^{i+1}|||_{0,\Omega}^2 + M|||\nabla\theta^{i+1}|||_{0,\Omega}^2.$$

Using the trace inequality (14), we have for the term T_4:

$$|T_4| \leq \frac{\sigma_p}{8} \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}} \frac{g(r_p)}{|e|} \left\|[\tau^{i+1}]\right\|_{0,e}^2 + Mg(r_p) \sum_{E\in\mathcal{E}_h} (h^{-2}\left\|\theta^{i+1}\right\|_{0,E}^2 + \left\|\nabla\theta^{i+1}\right\|_{0,E}^2).$$

The terms T_5 and T_6 are bounded in a similar way as the terms T_1 and T_2, except that the trace inequality (15) is used instead of (16).

$$|T_5|+|T_6| \leq \frac{\sigma_p}{8} \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}} \frac{g(r_p)}{|e|} \left\|[\tau^{i+1}]\right\|_{0,e}^2 + \frac{M}{g(r_p)} \sum_{E\in\mathcal{E}_h} (\left\|\nabla\theta^{i+1}\right\|_{0,E}^2 + h^2\left\|\nabla^2\theta^{i+1}\right\|_{0,E}^2).$$

The terms T_7 and T_8 are handled in the same way as terms T_1 and T_2, with the exception that the trace inequality (14) is used to handle the approximation error term.

$$|T_7| + |T_8| \leq \frac{\mu}{6}|||K^{1/2}\nabla\tau^{i+1}|||_{0,\Omega}^2 + Mg(r_p) \sum_{E\in\mathcal{E}_h} (h^{-2}\left\|\theta^{i+1}\right\|_{0,E}^2 + \left\|\nabla\theta^{i+1}\right\|_{0,E}^2).$$

Using (7), (8), Cauchy-Schwarz inequality and assumption on $\widetilde{p}^{i+1}$ (25), we have:

$$|T_9| \leq \frac{\mu}{12}|||K^{\frac{1}{2}}\nabla\tau^{i+1}|||_{0,\Omega}^2 + M\|\nabla\widetilde{p}^{i+1}\|_\infty^2 \|\phi^{i+1} - \widetilde{c}^{i+1}\|_{0,\Omega}^2 + M\|\nabla\widetilde{p}^{i+1}\|_\infty^2 \left\|\chi^{i+1}\right\|_{0,\Omega}^2.$$

The term T_{10} is a summation term over interior edges. We assume that the edge e is shared by the elements E_e^1 and E_e^2. Thus, we have using (7), (8), (25) and Cauchy-Schwarz inequality:

$$|T_{10}| \leq \left\|\nabla\widetilde{p}^{i+1}\right\|_\infty \overline{k}\frac{\mu_L}{2}\sum_{e\in\Gamma_h}\Big((\left\|(\phi^{i+1}-\widetilde{c}^{i+1})|_{E_e^1}\right\|_{0,e}+\left\|(\phi^{i+1}-\widetilde{c}^{i+1})|_{E_e^2}\right\|_{0,e})\left\|[\tau^{i+1}]\right\|_{0,e}$$

$$+(\left\|\chi^{i+1}|_{E_e^1}\right\|_{0,e}+\left\|\chi^{i+1}|_{E_e^2}\right\|_{0,e})\left\|[\tau^{i+1}]\right\|_{0,e}\Big).$$

Using the trace inequality (14), (16), we have:

$$|T_{10}| \leq \frac{\sigma_p}{8}\sum_{e\in\Gamma_h}\frac{g(r_c)}{|e|}\left\|[\tau^{i+1}]\right\|^2_{0,e} + M\|\nabla\widetilde{p}^{i+1}\|^2_\infty\|\phi^{i+1}-\widetilde{c}^{i+1}\|^2_{0,\Omega}$$

$$+\frac{M\left\|\nabla\widetilde{p}^{i+1}\right\|^2_\infty}{g(r_c)}\sum_{E\in\mathcal{E}_h}(\left\|\chi^{i+1}\right\|^2_{0,E}+h^2\left\|\nabla\chi^{i+1}\right\|^2_{0,E}).$$

The term T_{11} vanishes if the approximation $\widetilde{p}$ is continuous. Otherwise, we can bound exactly like the term T_5.

$$|T_{11}| \leq \frac{\mu}{12}|||K^{1/2}\nabla\tau^{i+1}|||^2_{0,\Omega} + Mg(r_p)\sum_{E\in\mathcal{E}_h}(h^{-2}\left\|\theta^{i+1}\right\|^2_{0,E}+\left\|\nabla\theta^{i+1}\right\|^2_{0,E}).$$

Combining all the bounds above, using the fact that $1 \leq r^2 \leq g(r) \leq 6r^2$, we have the pressure:

$$\frac{\mu}{2}|||K^{\frac{1}{2}}\nabla\tau^{i+1}|||^2_{0,\Omega} + (\frac{\sigma_p}{2}-(1-\varepsilon)^2\frac{3(\overline{\mu})^2\overline{k}\widetilde{M}M_t^2}{2\underline{\mu}})\sum_{e\in\Gamma_h}\frac{g(r_p)}{|e|}\left\|[\tau^{i+1}]\right\|^2_{0,e}$$

$$+(\frac{7}{8}\sigma_p-(1-\varepsilon)^2\frac{6(\overline{\mu})^2\overline{k}\widetilde{M}M_t^2}{\underline{\mu}})\sum_{e\in\Gamma_{\mathrm{D}}}\frac{g(r_p)}{|e|}\left\|[\tau^{i+1}]\right\|^2_{0,e} \leq M\frac{g(r_p)}{h^2}\left\|\theta^{i+1}\right\|^2_{0,\Omega}$$

$$+M(1+g(r_p))|||\nabla\theta^{i+1}|||^2_{0,\Omega}+M\left\|\chi^{i+1}\right\|^2_{0,\Omega}+\frac{Mh^2}{g(r_c)}|||\nabla\chi^{i+1}|||^2_{0,\Omega}+M\|\phi^{i+1}-\widetilde{c}^{i+1}\|^2_{0,\Omega}.$$

Define the limiting value of the penalty parameter:

$$\sigma_p^* = (1-\varepsilon)^2\frac{48(\overline{\mu})^2\overline{k}\widetilde{M}M_t^2}{7\underline{\mu}}.$$

Assuming that $\sigma_p > \sigma_p^*$, using the approximation results and the fact that $\boldsymbol{\phi}$ belongs to $\mathcal{W}$, we obtain:

$$|||\nabla\tau^{i+1}|||^2_{0,\Omega} + \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}}\frac{g(r_p)}{|e|}\left\|[\tau^{i+1}]\right\|^2_{0,e}$$

$$\leq M\max(\frac{1}{\underline{\mu}\underline{k}},\frac{1}{\sigma_p-\sigma_p^*})\Big(\frac{h^{2\min(r_p+1,\kappa_p)-2}}{r_p^{2\kappa_p-4}}\|p^{i+1}\|^2_{H^{\kappa_p}(\Omega)}$$

$$+\frac{h^{2\min(r_c+1,\kappa_c)-2}}{r_c^{2\kappa_c-2}}\|c^{i+1}\|^2_{H^{\kappa_c}(\Omega)}+M_1(\frac{h^{2r_p}}{r_p^{2\kappa_p-4}}+\frac{h^{2r_c}}{r_c^{2\kappa_c-4}}+\Delta t^2)\Big). \tag{36}$$

Next, we consider the concentration equation in the system (24). The same way as for the pressure equation, the concentration equation can be rewritten as:

$$\int_\Omega\frac{\phi}{\Delta t}(\widetilde{c}^{i+1}-\widetilde{c}^i)v + \sum_{E\in\mathcal{E}_h}\int_E\frac{\phi^{i+1}}{\mu(\phi^{i+1})}K\nabla\widetilde{p}^{i+1}\cdot\nabla v + \sum_{E\in\mathcal{E}_h}\int_E D(\zeta^{i+1})\nabla\widetilde{c}^{i+1}\cdot\nabla v$$

$$
\begin{aligned}
&- \sum_{e\in\Gamma_h} \int_e \{ \frac{\phi^{i+1}}{\mu(\phi^{i+1})} K\nabla \widetilde{p}^{i+1}\cdot n_e\}[v] - \sum_{e\in\Gamma_{\mathrm{D}}} \int_e \frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})} K\nabla \widetilde{p}^{i+1}\cdot n_e v - \sum_{e\in\Gamma_{\mathrm{N}}} \int_e c_{\mathrm{dir}} u_{\mathrm{dir}} v \\
&\quad - \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}} \int_e \{D(\zeta^{i+1})\nabla\widetilde{c}^{i+1}\cdot n_e\}[v] + \varepsilon \sum_{e\in\Gamma_h} \int_e \{ \frac{\phi^{i+1}}{\mu(\phi^{i+1})} K\nabla v\cdot n_e\}[\widetilde{p}^{i+1}] \\
&\quad\quad +\varepsilon \sum_{e\in\Gamma_{\mathrm{D}}} \int_e \frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})} K\nabla v\cdot n_e \widetilde{p}^{i+1} + \varepsilon \sum_{e\in\Gamma_h} \int_e \{D(\zeta^{i+1})\nabla v\cdot n_e\}[\widetilde{c}^{i+1}] \\
&\quad\quad +\sigma_c \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}} \frac{g(r_c)}{|e|}\int_e [\widetilde{c}^{i+1}][v] - \varepsilon \sum_{e\in\Gamma_{\mathrm{D}}} \int_e \frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})} K\nabla v\cdot n_e p_{\mathrm{dir}} \\
&\quad\quad -\sigma_c \sum_{e\in\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}} \frac{g(r_c)}{|e|}\int_e c_{\mathrm{dir}} v - \int_\Omega f_2 v = \int_\Omega \phi\Delta t \rho^{i+1} v - \int_\Omega \phi \chi_t^{i+1} v \\
&\quad\quad - \sum_{E\in\mathcal{E}_h} \int_E \frac{c^{i+1}}{\mu(c^{i+1})} K\nabla\theta^{i+1}\cdot\nabla v - \sum_{E\in\mathcal{E}_h}\int_E D(u^{i+1})\nabla\chi^{i+1}\cdot\nabla v \\
&\quad\quad + \sum_{e\in\Gamma_h}\int_e \{\frac{c^{i+1}}{\mu(c^{i+1})} K\nabla\theta^{i+1}\cdot n_e\}[v] + \sum_{e\in\Gamma_{\mathrm{D}}}\int_e \frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})} K\nabla\theta^{i+1}\cdot n_e v \\
&\quad + \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}} \int_e \{D(u^{i+1})\nabla\chi^{i+1}\cdot n_e\}[v] - \varepsilon \sum_{e\in\Gamma_h}\int_e \{\frac{c^{i+1}}{\mu(c^{i+1})} K\nabla v\cdot n_e\}[\theta^{i+1}] \\
&\quad\quad -\varepsilon \sum_{e\in\Gamma_{\mathrm{D}}}\int_e \frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})} K\nabla v\cdot n_e\theta^{i+1} - \varepsilon\sum_{e\in\Gamma_h}\int_e \{D(u^{i+1})\nabla v\cdot n_e\}[\chi^{i+1}] \\
&\quad -\sigma_c \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}} \frac{g(r_c)}{|e|}\int_e [\chi^{i+1}][v] + \sum_{E\in\mathcal{E}_h}\int_E (D(\zeta^{i+1}) - D(u^{i+1}))\nabla\widetilde{c}^{i+1}\cdot\nabla v \\
&\quad\quad\quad + \sum_{E\in\mathcal{E}_h}\int_E \Big(\frac{\phi^{i+1}}{\mu(\phi^{i+1})} - \frac{c^{i+1}}{\mu(c^{i+1})}\Big) K\nabla\widetilde{p}^{i+1}\cdot\nabla v \\
&\quad\quad\quad - \sum_{e\in\Gamma_h}\int_e \{\Big(\frac{\phi^{i+1}}{\mu(\phi^{i+1})} - \frac{c^{i+1}}{\mu(c^{i+1})}\Big) K\nabla\widetilde{p}^{i+1}\cdot n_e\}[v] \\
&\quad\quad\quad - \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}} \int_e \{\Big(D(\zeta^{i+1}) - D(u^{i+1})\Big)\nabla\widetilde{c}^{i+1}\cdot n_e\}[v] \\
&\quad\quad\quad +\varepsilon \sum_{e\in\Gamma_h}\int_e \{\Big(\frac{\phi^{i+1}}{\mu(\phi^{i+1})} - \frac{c^{i+1}}{\mu(c^{i+1})}\Big) K\nabla v\cdot n_e\}[\widetilde{p}^{i+1}] \\
&\quad\quad\quad +\varepsilon \sum_{e\in\Gamma_h}\int_e \{\Big(D(\zeta^{i+1}) - D(u^{i+1})\Big)\nabla v\cdot n_e\}[\widetilde{c}^{i+1}].
\end{aligned}
\tag{37}
$$

Subtracting equation (37) from (32), using (33) and choosing $z = \xi^{i+1}$, we obtain:

$$
\begin{aligned}
&\int_\Omega \frac{\phi}{\Delta t}(\xi^{i+1} - \xi^i)\xi^{i+1} + |||D(\zeta^{i+1})^{1/2}\nabla\xi^{i+1}|||_{0,\Omega}^2 + \sigma_c \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}} \frac{g(r_c)}{|e|}\|[\xi^{i+1}]\|_{0,e}^2 \\
&\quad = \sum_{E\in\mathcal{E}_h}\int_E \frac{\phi^{i+1}}{\mu(\phi^{i+1})} K\nabla\tau^{i+1}\cdot\nabla\xi^{i+1} - \sum_{e\in\Gamma_h}\int_e \{\frac{\phi^{i+1}}{\mu(\phi^{i+1})} K\nabla\tau^{i+1}\cdot n_e\}[\xi^{i+1}] \\
&\quad - \sum_{e\in\Gamma_{\mathrm{D}}}\int_e \frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})} K\nabla\tau^{i+1}\cdot n_e\xi^{i+1} + (1-\varepsilon)\sum_{e\in\Gamma_h}\int_e \{D(\zeta^{i+1})\nabla\xi^{i+1}\cdot n_e\}[\xi^{i+1}]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{e\in\Gamma_{\mathrm D}\cup\Gamma_{\mathrm N}}\int_e D(\zeta^{i+1})\nabla\xi^{i+1}\cdot n_e\xi^{i+1}-\varepsilon\sum_{e\in\Gamma_h}\int_e\{\frac{\phi^{i+1}}{\mu(\phi^{i+1})}K\nabla\xi^{i+1}\cdot n_e\}[\tau^{i+1}]\\
&-\varepsilon\sum_{e\in\Gamma_{\mathrm D}}\int_e\frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})}K\nabla\xi^{i+1}\cdot n_e\tau^{i+1}-\int_\Omega\phi\Delta t\rho^{i+1}\xi^{i+1}+\int_\Omega\phi\chi_t^{i+1}\xi^{i+1}\\
&+\sum_{E\in\mathcal E_h}\int_E\frac{c^{i+1}}{\mu(c^{i+1})}K\nabla\theta^{i+1}\cdot\nabla\xi^{i+1}+\sum_{E\in\mathcal E_h}\int_E D(u^{i+1})\nabla\chi^{i+1}\cdot\nabla\xi^{i+1}\\
&-\sum_{e\in\Gamma_h}\int_e\{\frac{c^{i+1}}{\mu(c^{i+1})}K\nabla\theta^{i+1}\cdot n_e\}[\xi^{i+1}]-\sum_{e\in\Gamma_{\mathrm D}}\int_e\frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})}K\nabla\theta^{i+1}\cdot n_e\xi^{i+1}\\
&-\sum_{e\in\Gamma_h\cup\Gamma_{\mathrm D}\cup\Gamma_{\mathrm N}}\int_e\{D(u^{i+1})\nabla\chi^{i+1}\cdot n_e\}[\xi^{i+1}]+\varepsilon\sum_{e\in\Gamma_h}\int_e\{\frac{c^{i+1}}{\mu(c^{i+1})}K\nabla\xi^{i+1}\cdot n_e\}[\theta^{i+1}]\\
&+\varepsilon\sum_{e\in\Gamma_{\mathrm D}}\int_e\frac{c_{\mathrm{dir}}}{\mu(c_{\mathrm{dir}})}K\nabla\xi^{i+1}\cdot n_e\theta^{i+1}+\varepsilon\sum_{e\in\Gamma_h}\int_e\{D(u^{i+1})\nabla\xi^{i+1}\cdot n_e\}[\chi^{i+1}]\\
&+\sigma_c\sum_{e\in\Gamma_h\cup\Gamma_{\mathrm D}\cup\Gamma_{\mathrm N}}\frac{g(r_c)}{|e|}\int_e[\chi^{i+1}][\xi^{i+1}]-\sum_{E\in\mathcal E_h}\int_E(D(\zeta^{i+1})-D(u^{i+1}))\nabla\widetilde{c}^{i+1}\cdot\nabla\xi^{i+1}\\
&-\sum_{E\in\mathcal E_{\langle}}\int_E\Big(\frac{\phi^{i+1}}{\mu(\phi^{i+1})}-\frac{c^{i+1}}{\mu(c^{i+1})}\Big)K\nabla\widetilde{p}^{i+1}\cdot\nabla\xi^{i+1}\\
&+\sum_{e\in\Gamma_h}\int_e\{\Big(\frac{\phi^{i+1}}{\mu(\phi^{i+1})}-\frac{c^{i+1}}{\mu(c^{i+1})}\Big)K\nabla\widetilde{p}^{i+1}\cdot n_e\}[\xi^{i+1}]\\
&+\sum_{e\in\Gamma_h\cup\Gamma_{\mathrm D}\cup\Gamma_{\mathrm N}}\int_e\{\Big(D(\zeta^{i+1})-D(u^{i+1})\Big)\nabla\widetilde{c}^{i+1}\cdot n_e\}[\xi^{i+1}]\\
&-\varepsilon\sum_{e\in\Gamma_h}\int_e\{\Big(\frac{\phi^{i+1}}{\mu(\phi^{i+1})}-\frac{c^{i+1}}{\mu(c^{i+1})}\Big)K\nabla\xi^{i+1}\cdot n_e\}[\widetilde{p}^{i+1}]
\end{aligned}
$$

$$
-\varepsilon\sum_{e\in\Gamma_h}\int_e\{\Big(D(\zeta^{i+1})-D(u^{i+1})\Big)\nabla\xi^{i+1}\cdot n_e\}[\widetilde{c}^{i+1}]=S_1+\dots S_{24}. \tag{38}
$$

The term S_8 contains the numerical error in the time discretization:

$$
\rho^{i+1}=\frac{1}{\Delta t}\Big(\frac{\tilde c^{i+1}-\tilde c^{i}}{\Delta t}-\frac{\partial\tilde c^{i+1}}{\partial t}\Big).
$$

We now bound each term in the right-hand side of (38). The terms $S_1,\dots,S_{18}$ are bounded like the terms T_i's. We skip the details (see [7]). Consider the term S_{19} using the assumptions H1, H3 and that $(\boldsymbol\psi,\boldsymbol\phi)\in\mathcal W$ we have:

$$
\begin{aligned}
S_{19}\le{}&\frac{d}{28}|||\nabla\xi^{i+1}|||_0^2+M\|\nabla\tilde c^{i+1}\|_\infty^2(|||\nabla(\psi^{i+1}-\tilde p^{i+1})|||_0^2+|||\nabla\theta^{i+1}|||_0^2)\\
&+M\|\nabla\tilde c^{i+1}\|_\infty^2\|\nabla p^{i+1}\|_\infty^2(\|\phi^{i+1}-\tilde c^{i+1}\|_0^2+\|\chi^{i+1}\|_0^2).
\end{aligned}
$$

Before bounding the term S_{20} we remark that

$$
\Big|\frac{\phi^{i+1}}{\mu(\phi^{i+1})}-\frac{c^{i+1}}{\mu(c^{i+1})}\Big|\le\bar\mu^2\Big(\frac{1}{\bar\mu}+\mu_L\|c^{i+1}\|_\infty\Big)|\phi^{i+1}-c^{i+1}|.
$$

Therefore we have

$$
S_{20}\le\frac{d}{28}|||\nabla\xi^{i+1}|||_0^2+M\|\nabla\tilde p^{i+1}\|_\infty^2(1+\|c^{i+1}\|_\infty)^2(\|\phi^{i+1}-\tilde c^{i+1}\|_0^2+\|\chi^{i+1}\|_0^2).
$$

The term S_{21} is bounded similarly to the term T_{10}.Consider the term S_{22}, using the assumptions H1 and H3 we have:

$$S_{22} \leq \frac{\sigma_c}{18} \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}} \frac{g(r_c)}{|e|} \|[\xi^{i+1}]\|_{0,e}^2 + \frac{M g(r_p)\|\nabla\tilde{c}^{i+1}\|_\infty^2}{g(r_c)} |||\nabla(\psi^{i+1} - \tilde{p}^{i+1})|||_1^2$$

$$+ \frac{M\|\nabla\tilde{c}^{i+1}\|_\infty^2}{g(r_c)} \sum_{E\in\mathcal{E}_h} (\|\nabla\theta^{i+1}\|_E^2 + h^2\|\nabla^2\theta^{i+1}\|_E^2) + M\|\nabla\tilde{c}^{i+1}\|_\infty^2\|\nabla p^{i+1}\|_\infty^2\|\phi^{i+1} - \tilde{c}^{i+1}\|_0^2$$

$$+ \frac{M\|\nabla\tilde{c}^{i+1}\|_\infty^2\|\nabla p^{i+1}\|_\infty^2}{g(r_c)} \sum_{E\in\mathcal{E}_h} (\|\chi^{i+1}\|_E^2 + h^2\|\nabla\chi^{i+1}\|_E^2).$$

The terms S_{23} and S_{24} are bounded like the term T_{11}. Combining the bounds above, using (36) and the fact that $1 \leq r^1 \leq g(r) \leq 6r^2$, we obtain the following estimate:

$$\frac{1}{2\Delta t}(\left\|\phi^{1/2}\xi^{i+1}\right\|_{0,\Omega}^2 - \left\|\phi^{1/2}\xi^{i}\right\|_{0,\Omega}^2) + \frac{d}{2}|||\nabla\xi^{i+1}|||_{0,\Omega}^2$$

$$+(\frac{\sigma_c}{3} - (1-\varepsilon)^2\frac{7(\overline{d})^2\widetilde{M}M_t^2}{4\underline{d}}) \sum_{e\in\Gamma_h} \frac{g(r_c)}{|e|} \left\|[\xi^{i+1}]\right\|_{0,e}^2$$

$$+(\frac{\sigma_c}{3} - \frac{7(\overline{d})^2\widetilde{M}M_t^2}{\underline{d}}) \sum_{e\in\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}} \frac{g(r_c)}{h} \left\|[\xi^{i+1}]\right\|_{0,e}^2$$

$$\leq M|||\nabla\tau^{i+1}|||_0^2 + M \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}} \frac{g(r_p)}{|e|} \left\|[\tau^{i+1}]\right\|_{0,e}^2 + \frac{1}{2}\left\|\xi^{i+1}\right\|_{0,\Omega}^2 + M\Delta t^2\left\|\rho^{i+1}\right\|_{0,\Omega}^2$$

$$+M\left\|\chi_t^{i+1}\right\|_{0,\Omega}^2 + M(\frac{g(r_c)}{h^2}+1)\|\chi^{i+1}\|_{0,\Omega}^2 + M\frac{g(r_c)}{h^2}\|\theta^{i+1}\|_{0,\Omega}^2$$

$$+M(1+g(r_c))|||\nabla\chi^{i+1}|||_{0,\Omega}^2 + M(1+g(r_c))|||\nabla\theta^{i+1}|||_{0,\Omega}^2 + M\|\phi^{i+1} - \tilde{c}^{i+1}\|_{0,\Omega}^2$$

$$+M\frac{h^2}{g(r_c)}|||\nabla^2\chi^{i+1}|||_{0,\Omega}^2 + M\frac{h^2}{g(r_c)}|||\nabla^2\theta^{i+1}|||_{0,\Omega}^2 + M|||\nabla(\tilde{\psi}^{i+1} - \tilde{p}^{i+1})|||_{0,\Omega}^2.$$

The error $\|\rho^{i+1}\|_{0,\Omega}$ is bounded using a Taylor expansion with integral remainder:

$$\tilde{c}^i = \tilde{c}^{i+1} - \Delta t\frac{\partial\tilde{c}^{i+1}}{\partial t} + \frac{1}{2}\int_{t^i}^{t^{i+1}} (t-t^i)\frac{\partial^2\tilde{c}^{i+1}}{\partial t^2}dt,$$

which yields

$$\|\rho^{i+1}\|_{0,\Omega} \leq M\|\tilde{c}_{tt}\|_{L^\infty(t^i,t^{i+1},L^2(\Omega))}.$$

Define

$$\sigma_c^* = \max\Big((1-\varepsilon)^2\frac{21(\overline{d})^2\widetilde{M}M_t^2}{4\underline{d}}, \frac{21(\overline{d})^2\widetilde{M}M_t^2}{\underline{d}}\Big).$$

Under the condition $\sigma_c > \sigma_c^*$ and using the approximation result, we obtain the following estimate:

$$\frac{\left\|\varphi^{1/2}\xi^{i+1}\right\|_{0,\Omega}^2}{\Delta t} - \frac{\left\|\varphi^{1/2}\xi^{i}\right\|_{0,\Omega}^2}{\Delta t} + |||\nabla\xi^{i+1}|||_\Omega^2 + \sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}} \frac{g(r_c)}{|e|}\left\|[\xi^{i+1}]\right\|_{0,e}$$

$$\leq \max(1, \frac{1}{\underline{d}}, \frac{3}{2(\sigma_c - \sigma_*)})\Big(\left\|\xi^{i+1}\right\|_{0,\Omega}^2 + K_2\frac{h^{2r_p}}{r_p^{2\kappa_p-4}} + K_3\frac{h^{2r_c}}{r_c^{2\kappa_c-4}} + K_4\Delta t\Big). \tag{39}$$

Equations (36) and (39) imply that $(\boldsymbol{\psi}, \boldsymbol{\phi})$ belongs to $\mathcal{W}$. $\square$

From Lemma 4 and Theorem 1, we have that the set $\mathcal{F}(\mathcal{W})$ is bounded. Using similar techniques as in Lemma 5 and Theorem 1 it can be shown that $\mathcal{F}$ is continuous. Since we are in finite dimension, this means that the operator $\mathcal{F}$ is compact. Therefore, by Schauder's fixed point theorem there is a solution $(\boldsymbol{\psi},\boldsymbol{\phi}) \in \mathcal{W}$ such that

$$(\boldsymbol{\psi},\boldsymbol{\phi}) = \mathcal{F}(\boldsymbol{\psi},\boldsymbol{\phi}).$$

This fixed point solution is the DG solution to (20)-(22). Using the definition of the space $\mathcal{W}$, the approximation results (26), (27) and Lemma 4, we obtain the following *a priori* error estimates.

Theorem 2. *Let $(\boldsymbol{P},\boldsymbol{C})$ be a solution to (20)-(22). Assume that the solution (p,c) to (1)-(6) belongs to $L^\infty(0,T;H^{\kappa_p}(\Omega)) \times L^\infty(0,T;H^{\kappa_c}(\Omega))$. Assume that the penalty parameters satisfy:*

$$\sigma_p \geq \sigma_p^*, \quad \sigma_p^* = (1-\varepsilon)^2 \frac{48(\overline{\mu})^2 \overline{k} \widetilde{M} M_t^2}{7\underline{\mu}}$$

$$\sigma_c \geq \sigma_c^*, \quad \sigma_c^* = \max\Big((1-\varepsilon)^2 \frac{21(\overline{d})^2 \widetilde{M} M_t^2}{4\underline{d}}, \frac{21(\overline{d})^2 \widetilde{M} M_t^2}{\underline{d}}\Big).$$

Then, there exists a constant M independent of $h, r_p, r_c, \Delta t$ such that for all $i \geq 1$

$$\left\|C^i - c^i\right\|_{0,\Omega} + (\Delta t \sum_{j=1}^{i} |||C^j - c^j|||_1^2)^{1/2} + |||P^i - p^i|||_1 \leq M_1\big(\frac{h^{r_p}}{r_p^{\kappa_p-2}} + \frac{h^{r_c}}{r_c^{\kappa_c-2}} + \Delta t\big).$$

More technical details of the convergence analysis presented above, can be found in [7].

5. Extensions and Concluding Remarks

We studied the application of primal discontinuous Galerkin methods, namely NIPG, IIPG, SIPG, and backward Euler discretization to solve the miscible displacement problem. We gave explicit expressions of the limiting values of the penalty parameters above which the method is stable and convergent. The methods, presented above can be modified slightly to consider several other boundary conditions. The convergence analysis developed in Section 4 is independent of the choice of the boundary conditions and it can be applied in the same way as above to show the stability and convergence of the scheme introduced below. For instance, we may have

$$u \cdot n = u_{\mathrm{dir}} \quad \forall (x,t) \in \partial\Omega \times \bar{J}, \tag{40}$$

$$c = c_{\mathrm{dir}} \quad \text{on} \quad \Gamma_{\mathrm{D}} \times \bar{J}, \tag{41}$$

$$D(u)\nabla c \cdot n = 0, \quad \Gamma_{\mathrm{N}} \times \bar{J}. \tag{42}$$

If (6) and (40) hold, the scheme becomes:
Pressure Equation: $\forall 0 \leq i \leq N-1$,

$$\forall z \in \mathcal{D}_{r_p}(\mathcal{E}_h), \quad \sum_{E\in\mathcal{E}_h} \int_E \frac{1}{\mu(C^{i+1})} K\nabla P^{i+1} \cdot \nabla z + \sigma_p \sum_{e\in\Gamma_h} \frac{g(r_p)}{|e|} \int_e [P^{i+1}][z]$$

$$- \sum_{e\in\Gamma_h} \int_e \{\frac{1}{\mu(C^{i+1})} K\nabla P^{i+1} \cdot n_e\}[z] + \varepsilon \sum_{e\in\Gamma_h} \int_e \{\frac{1}{\mu(C^{i+1})} K\nabla z \cdot n_e\}[P^{i+1}] = \int_\Omega f_1 z + \int_{\partial\Omega} u_{\mathrm{dir}} z.$$

Concentration Equation: $\forall 0 \le i \le N-1$,

$$\forall v \in \mathcal{D}_{r_c}(\mathcal{E}_h),\ \int_\Omega \frac{\varphi}{\Delta t}(C^{i+1} - C^i)v + \sum_{E\in\mathcal{E}_h}\int_E \frac{C^{i+1}}{\mu(C^{i+1})}K\nabla P^{i+1}\cdot\nabla v$$

$$+\sum_{E\in\mathcal{E}_h}\int_E D(U^{i+1})\nabla C^{i+1}\cdot\nabla v - \sum_{e\in\Gamma_h}\int_e \{\frac{C^{i+1}}{\mu(C^{i+1})}K\nabla P^{i+1}\cdot n_e\}[v]$$

$$-\sum_{e\in\Gamma_h\cup\partial\Omega}\int_e \{D(U^{i+1})\nabla C^{i+1}\cdot n_e\}[v] + \varepsilon\sum_{e\in\Gamma_h}\int_e \{\frac{C^{i+1}}{\mu(C^{i+1})}K\nabla v\cdot n_e\}[P^{i+1}]$$

$$+\varepsilon\sum_{e\in\Gamma_h}\int_e \{D(U^{i+1})\nabla v\cdot n_e\}[C^{i+1}] + \sigma_c\sum_{e\in\Gamma_h\cup\partial\Omega}\frac{g(r_c)}{|e|}\int_e [C^{i+1}][v]$$

$$= \sigma_c\sum_{e\in\partial\Omega}\frac{g(r_c)}{|e|}\int_e c_{\mathrm{dir}}v + \int_\Omega f_2 v + \sum_{e\in\partial\Omega}\int_e c_{\mathrm{dir}}u_{\mathrm{dir}}v,$$

If (40), (41), (42) hold, the concentration equation becomes:
Concentration Equation: $\forall 0 \le i \le N-1$,

$$\forall v \in \mathcal{D}_{r_c}(\mathcal{E}_h),\quad \int_\Omega \frac{\varphi}{\Delta t}(C^{i+1} - C^i)v + \sum_{E\in\mathcal{E}_h}\int_E \frac{C^{i+1}}{\mu(C^{i+1})}K\nabla P^{i+1}\cdot\nabla v$$

$$+\sum_{E\in\mathcal{E}_h}\int_E D(U^{i+1})\nabla C^{i+1}\cdot\nabla v - \sum_{e\in\Gamma_h}\int_e \{\frac{C^{i+1}}{\mu(C^{i+1})}K\nabla P^{i+1}\cdot n_e\}[v] - \sum_{e\in\Gamma_{\mathrm{N}}}\int_e C^{i+1}u_{\mathrm{dir}}v$$

$$-\sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}}\int_e \{D(U^{i+1})\nabla C^{i+1}\cdot n_e\}[v] + \varepsilon\sum_{e\in\Gamma_h}\int_e \{\frac{C^{i+1}}{\mu(C^{i+1})}K\nabla v\cdot n_e\}[P^{i+1}]$$

$$+\varepsilon\sum_{e\in\Gamma_h}\int_e \{D(U^{i+1})\nabla v\cdot n_e\}[C^{i+1}] + \sigma_c\sum_{e\in\Gamma_h\cup\Gamma_{\mathrm{D}}\cup\Gamma_{\mathrm{N}}}\frac{g(r_c)}{|e|}\int_e [C^{i+1}][v] =$$

$$+\sigma_c\sum_{e\in\Gamma_{\mathrm{D}}}\frac{g(r_c)}{|e|}\int_e c_{\mathrm{dir}}v + \int_\Omega f_2 v + \sum_{e\in\Gamma_{\mathrm{N}}}\int_e c_{\mathrm{dir}}u_{\mathrm{dir}}v,$$

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