INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 5, Supp , Pages 36–46

BROWNIAN MOTION AND ENTROPY GROWTH ON IRREGULAR SURFACES

C. CHEVALIER AND F. DEBBASCH

Abstract. Many situations of physical and biological interest involve diffusions on manifolds. It is usually assumed that irregularities in the geometry of these manifolds do not influence diffusions. The validity of this assumption is put to the test by studying Brownian motions on nearly flat 2D surfaces. It is found by perturbative calculations that irregularities in the geometry have a cumulative and drastic influence on diffusions, and that this influence typically grows exponentially with time. The corresponding characteristic times are computed and discussed. Conditional entropies and their growth rates are considered too.

Key Words. Brownian motion, stochastic processes on manifolds, lateral diffusions.

1. Introduction

Stochastic process theory is one of the most popular tools used in modelling time-asymmetric phenomena, with applications as diverse as economics ([21, 22]), traffic management ([20, 15]), biology ([16, 2, 10, 8]), physics ([23]) and cosmology ([5]). Many diffusions of biological interest, for example the lateral diffusions ([4, 17]), can be modelled by stochastic processes defined on differential manifolds ([12, 13, 9, 18]). In practice, the geometry of the manifold is never known with infinite precision, and it is common to ascribe to the manifold an approximate, mean geometry and to assume irregularities in the geometry have, in the mean, a negligible influence on diffusion phenomena ([4, 1, 3, 6, 19]). The aim of this article is to investigate if this last assumption is indeed warranted.

To this end, we fix a base manifold $\mathcal{M}$ and focus on Brownian motion. We introduce two metrics on $\mathcal{M}$. The first one, g, represents the real, irregular geometry of the manifold; what an observer would consider as the approximate, mean geometry is represented by another metric, which we call $\bar{g}$; to keep the discussion as general as possible, both metrics are allowed to depend on time.

We compare the Brownian motions in the approximate metric $\bar{g}$ to those in the real, irregular metric g by comparing their respective densities with respect to a reference volume measure, conveniently chosen as the volume measure associated to $\bar{g}$. Explicit computations are presented for diffusions on nearly flat 2D surfaces whose geometry fluctuates on spatial scales much smaller than the scales on which these diffusions are observed. We investigate in particular if the densities generated by Brownian motions in the real, irregular metric g coincide on large scales with the densities generated by Brownian motions in the approximate metric $\bar{g}$. We perform a perturbative calculation and find that, generically, these densities differ, even on large scales, and that the relative differences of their spatial

Received by the editors and, in revised form, .
2000 *Mathematics Subject Classification.* 60J65,8J65,60-xx:60Gxx,82Cxx:82C70.

Fourier components grow exponentially in time; on a given surface, the characteristic time τ at which the perturbative terms become comparable (in magnitude) to the zeroth order terms depends on the amplitude ε of the irregularities and on the large scale wave vector k at which diffusions are observed; we find that τ generally scales as $-\left(\nu^{-2}\ln(\varepsilon/\nu^{1/2})\right)\times\left(1/\mid K^*\mid^2\chi\right)$, where χ is the diffusion coefficient and $\nu = \mid k\mid / \mid K^*\mid$, K^* being a typical wave-vector characterizing the metric irregularities. Our general conclusion is that geometry fluctuations have a cumulative effect on Brownian motion and that their influence on diffusions cannot be neglected.

2. Brownian motions on a manifold

2.1. Brownian motion in a time-independent metric. Let $\mathcal{M}$ be a fixed real base manifold of dimension d. Let g be a time-independent metric on $\mathcal{M}$. This metric endows $\mathcal{M}$ with a natural volume measure which will be denoted hereafter by dVol_g. If $\mathcal{C}$ is a chart on $\mathcal{M}$ with coordinates $x = (x^i), i = 1, ..., d$, integrating against dVol_g comes down to integrating against $\sqrt{\det g_{ij}}\, d^d x$, where the g_{ij}'s are the components of g in the coordinate basis associated to $\mathcal{C}$.

There is a canonical definition of a Brownian motions on $\mathcal{M}$ equipped with metric g ([14, 9, 11, 18]). Quite intuitively, these Brownian motions are defined through the diffusion equation obeyed by their densities n with respect to dVol_g. Given an arbitrary positive diffusion constant χ, this equation reads:

$$\partial_t n = \chi \Delta_g n, \tag{1}$$

where Δ_g is the Laplace-Beltrami operator associated to g ([7]); given a chart $\mathcal{C}$ with coordinates x, one can write:

$$\Delta_g n = \frac{1}{\sqrt{\det g_{kl}}}\partial_i\left(\sqrt{\det g_{kl}}\, g^{ij}\partial_j n\right), \tag{2}$$

where ∂_i represents partial derivation with respect to x^i and the g^{ij}'s are the components of the inverse of g in the coordinate basis associated to $\mathcal{C}$. Observe that one of the reasons why this definition makes sense is that the diffusion equation (1) conserves the normalization of n with respect to dVol_g.

2.2. Brownian motion in a time-dependent metric. The preceding definition of Brownian motion cannnot be used in this case because the diffusion equation (1) does not conserve the normalization of $n(t)$ with respect to the volume measure $\mathrm{dVol}_{g(t)}$ associated to a time-dependent metric. To proceed, we introduce an arbitrary, time-independent metric γ on $\mathcal{M}$, denote by $\mu_{g(t)|\gamma}$ the density of $\mathrm{dVol}_{g(t)}$ with respect to dVol_γ, and define the Brownian motion in the time-dependent metric $g(t)$ as the stochastic process whose density n with respect to $\mathrm{dVol}_{g(t)}$ obeys the following generalized diffusion equation:

$$\frac{1}{\mu_{g(t)|\gamma}}\partial_t\left(\mu_{g(t)|\gamma} n\right) = \chi \Delta_{g(t)} n. \tag{3}$$

Given an arbitrary coordinate system (x), equation (3) transcribes into:

$$\partial_t\left(\sqrt{\det g_{kl}}\, n\right) = \chi \partial_i\left(\sqrt{\det g_{kl}}\, g^{ij}\partial_j n\right), \tag{4}$$

which shows that the Brownian motion in $g(t)$ does not actually depend on γ. Moreover,

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{M}} \mathrm{dVol}_{g(t)} n &= \frac{d}{dt}\int_{\mathcal{M}} \mathrm{dVol}_\gamma \mu_{g(t)|\gamma} n \\ &= \int_{\mathcal{M}} \mathrm{dVol}_\gamma \partial_t \left(\mu_{g(t)|\gamma} n\right) \\ &= \chi \int_{\mathcal{M}} \mathrm{dVol}_\gamma \mu_{g(t)|\gamma} \Delta_{g(t)} n \\ &= \chi \int_{\mathcal{M}} \mathrm{dVol}_{g(t)} \Delta_{g(t)} n \\ &= 0. \end{aligned} \tag{5}$$

Thus, contrary to (1), equation (3) conserves the normalization of $n(t)$.

2.3. Entropies of Brownian motion in a time-dependent metric. Let n and $\tilde{n}$ be two solutions of (3). We define the time-dependent conditional entropy $S_{n|\tilde{n}}$ of n with respect to $\tilde{n}$ by:

$$S_{n/\tilde{n}}(t) = -\int_{\mathcal{M}} \mathrm{dVol}_{g(t)} n \ln(\frac{n}{\tilde{n}}). \tag{6}$$

This entropy is a non decreasing function of t. This can be seen by the following calculation. One can write:

$$S_{n|\tilde{n}}(t) = -\int_{\mathcal{M}} \mathrm{dVol}_\gamma\ \mu_{g(t)|\gamma} n\ \ln(\frac{\mu_{g(t)|\gamma} n}{\mu_{g(t)|\gamma} \tilde{n}}), \tag{7}$$

which leads to

$$\begin{aligned}\frac{dS_{n|\tilde{n}}}{dt} &= -\int_{\mathcal{M}} \mathrm{dVol}_\gamma \left(\partial_t(\mu_{g(t)|\gamma} n) \ln(\frac{\mu_{g(t)|\gamma} n}{\mu_{g(t)|\gamma}\tilde{n}}) \right. \\ &+ \left. \mu_{g(t)|\gamma} n \frac{\mu_{g(t)|\gamma}\tilde{n}\partial_t(\mu_{g(t)|\gamma} n) - \mu_{g(t)|\gamma} n \partial_t(\mu_{g(t)|\gamma}\tilde{n})}{\mu^2_{g(t)|\gamma} n\tilde{n}} \right). \end{aligned} \tag{8}$$

Equation (3) can then be used to transform all temporal derivatives into spatial ones and one obtains:

$$\frac{dS_{n|\tilde{n}}}{dt} = -\chi \int_{\mathcal{M}} \mathrm{dVol}_{g(t)} \left(\Delta_{g(t)} n \left(\ln(\frac{n}{\tilde{n}}) + 1\right) - \frac{n}{\tilde{n}} \Delta_{g(t)} \tilde{n}\right). \tag{9}$$

Integrating by parts delivers:

$$\frac{dS_{n|\tilde{n}}}{dt} = +\chi \int_{\mathcal{M}} \mathrm{dVol}_{g(t)} \tilde{n} \left(\nabla(\ln\frac{n}{\tilde{n}})\right)^2, \tag{10}$$

which proves the expected result. Conditional entropies of Brownian motions thus obey a very simple H-theorem, even in time-dependent geometries.

The Gibbs entropy $S_G[n]$ of a density n is defined by

$$S_G[n](t) = -\int_{\mathcal{M}} \mathrm{dVol}_{g(t)} n \ln n. \tag{11}$$

The H-theorem above applies to $S_G[n]$ only if the $\tilde{n} = 1$ is a solution of the transport equation (3). This is automatically the case if the metric g is time-independent, but may not be true in time-dependent metrics. Note also that a uniform density is not normalizable on non compact manifolds.

3. How to compare Brownian motions in different metrics

Let $\mathcal{M}$ be a real differential manifold of dimension d. We first introduce on $\mathcal{M}$ a metric $\bar{g}(t)$ which describes what an observer would consider as the approximate, mean geometry of the manifold. The real, irregular geometry of $\mathcal{M}$ is described by a different metric $g(t)$.

Consider an arbitrary point O in $\mathcal{M}$ and let B_t be the Brownian motion in $g(t)$ that starts at O. The density n of B_t with respect to $\mathrm{dVol}_{g(t)}$ obeys the diffusion equation:

$$\frac{1}{\mu_{g(t)|\gamma}}\partial_t\left(\mu_{g(t)|\gamma}n\right) = \chi\Delta_{g(t)}n. \tag{12}$$

We denote by $\bar{B}_t$ the Brownian motion in $\bar{g}(t)$ that starts at point O and by $\bar{n}$ its density with respect to $\mathrm{dVol}_{\bar{g}(t)}$; this density obeys:

$$\frac{1}{\mu_{\bar{g}(t)|\gamma}}\partial_t\left(\mu_{\bar{g}(t)|\gamma}\bar{n}\right) = \chi\Delta_{\bar{g}(t)}\bar{n}. \tag{13}$$

We will compare the two Brownian motions by comparing on large scales their respective densities with respect to a reference volume measure on $\mathcal{M}$. From an observational point of view, the best choice is clearly $\mathrm{dVol}_{\bar{g}(t)}$, the volume measure associated to the approximate, mean geometry of the manifold. The density N of B_t with respect to $\mathrm{dVol}_{\bar{g}(t)}$ is given in terms of n by:

$$N = \mu_{g(t)|\bar{g}(t)}n, \tag{14}$$

where $\mu_{g(t)|\bar{g}(t)}$ is the density of $\mathrm{dVol}_{g(t)}$ with respect to $\mathrm{dVol}_{\bar{g}(t)}$. The transport equation obeyed by N can be deduced from (12) and reads:

$$\frac{1}{\mu_{g(t)|\gamma}}\partial_t\left(\mu_{\bar{g}(t)|\gamma}N\right) = \chi\Delta_{g(t)}\left(\frac{1}{\mu_{g(t)|\bar{g}(t)}}N\right). \tag{15}$$

In a chart $\mathcal{C}$ with coordinates (x), (14) transcribes into:

$$N(t,x) = \frac{\sqrt{\mathrm{det}g_{ij}(t,x)}}{\sqrt{\mathrm{det}\bar{g}_{ij}(t,x)}}n(t,x) \tag{16}$$

and (15) becomes:

$$\partial_t\left(\sqrt{\mathrm{det}\bar{g}_{kl}(t,x)}N(t,x)\right) = \chi\partial_i\left(\sqrt{\mathrm{det}g_{kl}(t,x)}g^{ij}(t,x)\partial_j\frac{\sqrt{\mathrm{det}\bar{g}_{kl}(t,x)}}{\sqrt{\mathrm{det}g_{kl}(t,x)}}N(t,x)\right). \tag{17}$$

Let N and $\tilde{N}$ be the densities with respect to $\mathrm{dVol}_{\bar{g}(t)}$ corresponding to two solutions n and $\tilde{n}$ of equation (3). The conditional entropy of n with respect to $\tilde{n}$ can also be written:

$$S_{n|\tilde{n}}(t) = -\int_{\mathcal{M}}\mathrm{dVol}_{\bar{g}(t)}N\ln(\frac{N}{\tilde{N}}). \tag{18}$$

This entropy can thus be also interpreted as the conditional entropy of N with respect to $\tilde{N}$ on the manifold equiped with metric $\bar{g}(t)$. Note however that the Gibbs entropy of n in $g(t)$ does not coincide with the Gibbs entropy of N in $\bar{g}(t)$.

The main question investigated in this article is: how does the density N obeying (15) differ on large scales from the density $\bar{n}$ obeying (13)? Since this question is extremely difficult to solve in its full generality, we now concentrate on nearly flat 2D surfaces.

4. Brownian motions on nearly flat 2D surfaces

4.1. The problem. We choose $\mathbb{R}^2$ as base manifold $\mathcal{M}$ and retain $\bar{g} = \eta$, the flat Euclidean metric on $\mathbb{R}^2$. The real, irregular metric of the manifold is still denoted by $g(t)$ and we define $h(t)$ by $g^{-1}(t) = \eta^{-1} + \varepsilon h(t)$, where ε is a small parameter (infinitesimal) tracing the nearly flat character of the surface. From now on, we will use the metric η (*resp.* the inverse of η) to lower (*resp.* raise) all indices.

Let us choose a chart $\mathcal{C}$ where $\eta_{ij} = \text{diag}(1,1)$. The tensor field $h(t)$ is then represented by its components $h^{ij}(t,x)$. A particularly simple but very illustrative form for these components is:

$$h^{ij}(t,x) = \sum_{nn'} h^{ij}_{nn'} \cos\left(\omega_{n'} t - k_n.x + \phi_{nn'}\right), \tag{19}$$

where $k_n.x = k_{n\,1}x^1 + k_{n\,2}x^2$ and both integer indices run through arbitrary finite sets. This choice has the double advantage of leading to conclusions which are sufficiently robust to remain qualitatively valid for all sorts of physically interesting perturbations h while making all technical aspects of the forthcoming computations and discussions as simple as possible. The *Ansatz* (19) will therefore be retained in the remainder of this article. Let us remark that perturbations $h(t)$ proportional to η amount to a simple modification of the conformal factor linking the 2D metric $g(t)$ to the flat metric η.

Equation (17) reads, in the chart $\mathcal{C}$:

$$\partial_t N = \chi \partial_i \left(\sqrt{\text{det} g_{kl}(t,x)}\, g^{ij}(t,x) \partial_j \frac{N}{\sqrt{\text{det} g_{kl}(t,x)}} \right) \tag{20}$$

or, alternately,

$$\partial_t N = \chi \partial_i \left(g^{ij}(t,x) \left(\partial_j N - N \partial_j l \right) \right), \tag{21}$$

where

$$l(t,x) = \ln \sqrt{\text{det} g_{kl}(t,x)}. \tag{22}$$

4.2. General perturbative solution. The solution of (21) will be searched for as a perturbation series in the amplitude ε of the fluctuations:

$$N(t,x) = \sum_{m \in N} \varepsilon^m N_m(t,x). \tag{23}$$

Setting to 0 both coordinates of the point O where the diffusion starts from, we further impose, for all x, that $N_0(0,x) = \delta(x)$ and $N_m(0,x) = 0$ for all $m > 0$.

The function $l(t,x)$ can be expanded in ε, so that $l(t,x) = \sum_{m \in N} \varepsilon^m l_m(t,x)$ and one finds, for the first three contributions:

$$\begin{aligned} l_0(t,x) &= 0 \\ l_1(t,x) &= -\frac{1}{2}\, \eta_{ij} h^{ij}(t,x) \\ l_2(t,x) &= \frac{1}{4} \eta_{ik} \eta_{jl} h^{ij}(t,x) h^{kl}(t,x). \end{aligned} \tag{24}$$

Equation (21) can then be rewritten as the system

$$\partial_t N_m = \chi \Delta_\eta N_m + \chi S_m[h, N_r], m \in \mathbb{N}, r \in \mathbb{N}_{m-1} \tag{25}$$

where the source term S_m is a functional of the fluctuation h and of the contributions to N of order strictly lower than m. In particular,

$$\begin{aligned} S_0 &= 0 \\ S_1 &= \partial_i \left(h^{ij}\partial_j N_0 + \frac{1}{2} N_0 \eta^{ij}\eta_{kl}\partial_j h^{kl} \right) \\ S_2 &= \partial_i \left(h^{ij}\partial_j N_1 + \frac{1}{2}\left(N_0 h^{ij} + N_1 \eta^{ij}\right)\eta_{kl}\partial_j h^{kl} - \frac{1}{4} N_0 \eta^{ij}\eta_{mk}\eta_{nl}\partial_j \left(h^{mn}h^{kl}\right) \right). \end{aligned} \tag{26}$$

Two remarks are now in order. Taken together, $S_0(t,x) = 0$ and $N_0(t,x) = \delta(x)$ imply that N_0 coincides with the Green function of the standard diffusion equation on the flat plane:

$$N_0(t,x) = \frac{1}{4\pi\chi t} \exp\left(-\frac{x^2}{4\chi t}\right). \tag{27}$$

Moreover, the fact that S_m is a divergence for all m implies that the normalizations of all N_m's are conserved in time. The initial condition $N_m(0,x) = 0$ for all x and $m > 0$ then implies that all N_m's with $m > 0$ remain normalized to zero and only contribute to the local density of particles, and not to the total density. This is perfectly coherent with the fact that N_0 is normalized to unity.

Define now spatial Fourier transforms by

$$\hat{f}(t,k) = \int_{\mathbb{R}^2} f(t,x)\exp(-ik.x)d^2x, \tag{28}$$

where $k.x = k_1x^1 + k_2x^2$. A direct calculation then delivers:

$$\hat{S}_1(t,k) = -k_i \int_{\mathbb{R}^2} A^i(t,k,k')\hat{N}_0(t,k-k')d^2k' \tag{29}$$

where

$$\hat{N}_0(t,k) = \exp\left(-\chi k^2 t\right) \tag{30}$$

and

$$A^i(t,k,k') = (k_j - k'_j)\hat{h}^{ij}(t,k') + \frac{1}{2}\,\eta^{ij}k'_j\eta_{kl}\hat{h}^{kl}(t,k'). \tag{31}$$

The first order density fluctuation N_1 is then obtained by solving equation (25) with (29) as source term, taking into account the initial condition $N_1(0,x) = 0$ for all x. One thus obtains:

$$\hat{N}_1(t,k) = I_1(t,k)\exp(-\chi k^2 t) \tag{32}$$

with

$$I_1(t,k) = \int_0^t \hat{S}_1(t',k)\exp(\chi k^2 t')dt'. \tag{33}$$

Equation (26) then gives:

$$\begin{aligned} \hat{S}_2(t,k) &= -k_i \int_{\mathbb{R}^2} A^i(t,k,k')\hat{N}_1(t,k-k')d^2k' \\ &+ \int_{\mathbb{R}^2\times\mathbb{R}^2} B^i(t,k',k'')\hat{N}_0(t,k-k')d^2k'd^2k'' \end{aligned} \tag{34}$$

with

$$B^i(t,k,k') = \frac{1}{2}\, k'_j \eta_{kl} \hat{h}^{kl}(t,k') \hat{h}^{ij}(t,k-k')$$

$$- \frac{1}{4}\, \eta^{ij} k'_j \eta_{mk} \eta_{nl} \left(\hat{h}^{mn}(t,k') \hat{h}^{kl}(t,k-k') + \hat{h}^{kl}(t,k') \hat{h}^{mn}(t,k-k') \right). \quad (35)$$

The second order density fluctuation N_2 then reads:

$$\hat{N}_2(t,k) = I_2(t,k) \exp(-\chi k^2 t) \quad (36)$$

with

$$I_2(t,k) = \int_0^t \hat{S}_2(t',k) \exp(\chi k^2 t') dt'. \quad (37)$$

4.3. How the irregularities influence diffusions.

4.3.1. First order terms. Let us now insert *Ansatz* (19) in the above expressions (29) and (32) for $\hat{S}_1$ and $\hat{N}_1$. One finds:

$$\hat{S}_1(t,k) = \sum_{nn'\sigma} A^\sigma_{nn'}(k) \exp\left(i\sigma(\omega_{n'} t + \phi_{nn'}) - (k + \sigma k_n)^2 \chi t\right) \quad (38)$$

with

$$A^\sigma_{nn'}(k) = -\, \frac{1}{2} \left[k_i (k_j + \sigma k_{nj}) h^{ij}_{nn'} - \frac{1}{2}\, \sigma \eta^{ij} k_i k_{nj} \eta_{kl} h^{kl}_{nn'} \right] \quad (39)$$

and $\sigma \in \{-1, +1\}$. This leads to:

$$\frac{\hat{N}_1(t,k)}{\hat{N}_0(t,k)} = \sum_\sigma \Bigg\{ \sum_{(n,n') \notin \Sigma^\sigma(k)} I^\sigma_{nn'}(k) \left[\exp\left(\sigma i \omega_{n'} t - (k_n^2 + 2\sigma k.k_n)\chi t\right) - 1 \right]$$

$$+ \; t \sum_{(n,n') \in \Sigma^\sigma(k)} A^\sigma_{nn'}(k) \exp\left(i\phi_{nn'}\right) \Bigg\}. \quad (40)$$

with

$$I^\sigma_{nn'}(k) = \frac{A^\sigma_{nn'}(k) \exp(i\phi_{nn'})}{i\sigma\omega_{n'} + (k^2 - (k + \sigma k_n)^2)\,\chi}, \quad (41)$$

and $\Sigma^\sigma(k) = \left\{ (n,n'), \sigma i \omega_{n'} + \left(k^2 - (k\sigma k_n)^2\right)\chi = 0 \right\}$. Note that both sets are disjoint, unless there is an (n,n') for which $k_n = 0$ and $\omega_{n'} = 0$.

This expression characterizes how Brownian motions in the irregular metric differ, at first order, from Brownian motions on the flat Euclidean plane. The dependence on the wave vector k indicates that the influence of the irregularities varies with the spatial scale at which the diffusion is observed. Two opposite situations are particularly worth commenting upon. Take a certain k_n and consider $\hat{N}_1$ at scales characterized by wave vectors much smaller than k_n, say $\mid k \mid = \mid k_n \mid O(\nu)$, where ν is an infinitesimal (small parameter). Neglecting the contributions of the frequencies $\omega_{n'}$, the amplitudes $A^\pm_{nn'}(k)$ typically scale as $\mid k \mid \mid k_n \mid$, so that the $I^\pm_{nn'}(k)$'s scale as $O(\nu)$. Note however that perturbations h proportional to η do not obey this typical scaling, but rather $A^\pm_{nn'}(k) \sim k^2$, and $I^\pm_{nn'}(k) \sim O(\nu^2)$. The sets $\sigma^\pm$ are empty and the time-dependence of $\mid \hat{N}_1/\hat{N}_0 \mid$ is controled by the real exponentials in (40), which essentially decrease as $\exp(-k_n^2 \chi t)$. The first order relative contribution $\varepsilon \hat{N}_1/\hat{N}_0$ thus tends towards a quantity $L_1(k)$ which is linear in the $I^\pm_{nn'}(k)$; the typical relaxation time is $\tau_1 \sim 1/(\chi k_n^2)$, which is much smaller than the diffusion time $1/(\chi k^2)$ associated to scale k. Moreover, the limit $L_1(k)$ scales as

$O(\varepsilon\nu)$, except for perturbations h proportional to η, for which it scales as $O(\varepsilon\nu^2)$; $L_1(k)$ is therefore always much smaller than ε and, in particular, tends to zero with ν *i.e.* as the scale separation tends to infinity. The effect of the k_n Fourier mode on scales characterized by a wave vector k verifying $\mid k \mid \ll \mid k_n \mid$ is thus in practice negligible.

Consider now the opposite case, *i.e.* $\mid k \mid$ comparable to, or larger than $\mid k_n \mid$. Neglecting again the contribution of the frequencies $\omega_{n'}$, the amplitudes $A^{\pm}_{nn'}(k)$ then scale as k^2, and $I^{\pm}_{nn'}(k) \sim \mid k \mid / \mid k_n \mid$. Let now θ be the angle between k and k_n and suppose, to simplify the discussion, that $\cos\theta$ does not vanish. At least one of the exponentials in (40) will then be an increasing function of time provided $\mid k \mid > \mid k_n \mid /(2\cos\theta)$. Take for example $k = k_n$; the second exponential in (40) then increases with a characteristic time-scale $1/(k^2\chi)$. This means that the first order contributions of the irregularities to the density actually become comparable to unity at this scale at characteristic times $\tau_1 \sim -(\ln\varepsilon)/(k^2\chi)$; this time probably also signals the break down of the perturbative expansion in ε for the scale $k = k_n$. As for the linear terms in t appearing on the right-hand side of (40), they actually contribute to $\hat{N}_1/\hat{N}_0$ if at least one of the sets $\sigma^{\pm}(k)$ is not empty. This condition is realized if $\omega_{n'} = 0$ and $\mid k \mid = \mid k_n \mid /(2\cos\theta)$.

The conclusion of this discussion is that, at first order in the amplitude of the perturbation h, a given Fourier mode k_n of h dramatically influences diffusions on scales characterized by wave vectors with modulus comparable to or larger than $\mid k_n \mid$, but has a negligible influence on scales characterized by wave vectors with modulus much smaller than $\mid k_n \mid$. We will now show that this conclusion cannot be extended to all perturbation orders and that taking into account terms of orders higher than 1 proves that h generally influences diffusions on all scales.

4.3.2. Second order terms. It is straightforward to obtain from equations (34), (36) , (40) and (30) explicit expressions for $\hat{S}_2(t,k)$ and $\hat{N}_2(t,k)/\hat{N}_0(t,k)$. These are extremely complicated and do not warrant full reproduction here.

Of interest is that $\hat{N}_2/\hat{N}_0$ contains contributions whose amplitudes potentially grows exponentially in time. One of these reads

$$D_1(t,k) = \sum_{n,n',\sigma_1,\sigma_2,\sigma_3} I^{\sigma_1}_{nn'}(k+\sigma_2 k_p) A^{\sigma_2}_{pp'}(k) J^{\sigma_1\sigma_3}_{nn'pp'}(k) \times$$

$$\left[\exp\left(i(\sigma_1\omega_{n'} + \sigma_2\omega_{p'})t - \left((k_n+\sigma_1\sigma_2 k_p)^2 + 2k.(k_n+\sigma_1\sigma_2 k_p)\right)\chi t\right) - 1\right] \quad (42)$$

with $\sigma_i = \pm 1$ $(i = 1,2,3)$ and

$$J^{\sigma_1\sigma_3}_{nn'pp'}(k) = \frac{\exp\left(i\sigma_1\sigma_3\phi_{pp'}\right)}{i\sigma_1(\omega_{n'}+\sigma_3\omega_{p'}) + (k^2 - (k+\sigma_1(k_n+\sigma_3 k_p))^2)\chi}. \quad (43)$$

The right-hand sides of (42) contains four exponentials of given (n,n',p,p'); these involve the wave vectors $K^{\pm}_{np} = k_n \pm k_p$. Let us for the moment ignore the factors in front of these exponentials. Let k be an arbitrary wave vector and let $\theta^{\pm}$ be the angle between k and $K^{\pm}_{np}$. Each of the conditions $2k \mid \cos\theta^{\pm} \mid > \mid K^{-}_{np} \mid$ makes one of the four exponentials an increasing function of t. At second order, the spatial scales at which diffusions are influenced by the perturbation h are thus determined, not by the k_n's, but by the combinations $K^{\pm}_{np} = k_n \pm k_p$. Indeed, quite generally, the temporal behaviour of terms of order q, $q \geq 1$, will be determined by combinations of q wave vectors k_n. For perturbations h with a rich enough spectrum, these combinations correspond to all sorts of spatial scales and, in particular, to scales much larger than those over which h itself varies. Thus, h will generally influence diffusions on all spatial scales.

Let us elaborate quantitatively on this conclusion by further exploring the behaviour of $D_1(t,k)$. Suppose for example that the moduli of all k_n's are of the same order of magnitude, say k^*, but that there are some n and p for which $\mid K^-_{np} \mid \sim K^* O(\nu)$, where $\nu << 1$. The condition introduced above, which ensures that one of the exponentials involving K^-_{np} grows with t, then translates into $\mid k \mid > (2/\cos\theta^-) K^* O(\nu)$, and is realized for $\mid k \mid = K^* O(\nu)$ provided $\cos\theta^- \lesssim 1$ Let us check now that the factors in front of the exponentials do not tend towards zero with ν. Ignoring as before the influence of the frequencies ω_q, the quantity $I^-_{nn'}(k+k_p)$ (see (41)) scales as $A^-_{nn'}(k+k_p)/k_p^2$ *i.e.* as $k_p^2/k_p^2 = 1$. The quantity $\tilde{J}^-_{nn'pp'}(k)$ scales as $(Q_{np}(k))^{-1} = \left[2k.K^-_{np} - (K^-_{np})^2\right]^{-1}$. The factor in front of the exponential thus scales as $\mid k \mid\mid k_p \mid (Q_{np}(k))^{-1}$ for perturbations h not proportional to η, and as $k^2 (Q_{np}(k))^{-1}$ otherwise. Taking into account that $\mid k \mid \sim \mid K_{np} \mid$ and putting $\cos\theta^- = 1$ to simplify the discussion, one finds that the factor in front of the exponentials scales as $\mid k_p \mid / \mid k \mid = O(1/\nu)$ if h is not proportional to η and as $O(1)$ otherwise. This factor therefore does not tend to zero with the separation scale parameter ν. Actually, for perturbations which are not proportional to η, this factor tends to infinity as ν tends to zero, a fact which only increases the influence of h on diffusions.

These estimates can be used to evaluate some characteristic times. For perturbations proportional to η, the second order term $\varepsilon^2 D_1$ reaches unity after a characteristic time $\tau_2^\eta \sim -(2/\nu^2 K^{*\,2}\chi)\ln\varepsilon$; for perturbations not proportional to η, the corresponding characteristic time is $\tau_2 \sim -(2/\nu^2 K^{*\,2}\chi)\ln(\varepsilon/\nu^{1/2}) << \tau_2^\eta$. These characteristic times are probably upper bound for the time at which the perturbation expansion ceases to be valid for scale k.

5. Influence of the irregularities on the entropies

The influence of generic metric irregularities on conditional entropies is technically extremely difficult to investigate in detail. We therefore restrict our discussion by considering only time-independent perturbations h proportional to η and write:

$$h^{ij}(x) = \sum_n a_n \eta^{ij} \cos\left(k_n.x - \phi_n\right). \tag{44}$$

Let us focus on the Gibbs entropy $S_G[n]$ of the density n evaluated in Section 4. This entropy reads

$$S_G[n](t) = -\int_{\mathbb{R}^2} d^2x\ N(t,x) \ln\left(\frac{N(t,x)}{\sqrt{\det g(t,x)}}\right). \tag{45}$$

The perturbative expansion of both g and N, together with the normalization conditions $\int_{\mathbb{R}^2} N_m d^2x = 0$ for $m = 1, 2$, leads to:

$$S_G[n](t) = \sum_{m\in N} \varepsilon^m S_{Gm}[n](t) \tag{46}$$

with

$$\begin{aligned} &S_{G0}[n](t) = -\int_{\mathbb{R}^2} d^2x\ N_0(t,x)\ln N_0(t,x),\\ &S_{G1}[n](t) = -\int_{\mathbb{R}^2} d^2x \left(N_1(t,x)\ln N_0(t,x) - N_0(t,x) l_1(x)\right),\\ &S_{G2}[n](t) = \\ &-\int_{\mathbb{R}^2} d^2x \left(N_2(t,x)\ln N_0(t,x) - N_0(t,x) l_2(x) - N_1(t,x) l_1(x) + \frac{N_1^2(t,x)}{2N_0(t,x)}\right). \end{aligned} \tag{47}$$

Expression (27) for N_0 leads to $S_{G0}[n](t) = 1 + \ln(4\pi\chi t)$, which is, as expected, an increasing function of t. This function is also strictly positive. A direct computation shows that N_1 is an uneven function of x. SInce N_0 is even in x, so is $\ln N_0$ and the contribution of $N_1 \ln N_0$ to $S_{G1}[n]$ vanishes identically. One finds, using (24), that:

$$S_{G1}[n](t) = -\sum_n a_n \cos\phi_n \exp\left(-k_n^2 \chi t\right). \tag{48}$$

The first order contribution to the Gibbs entropy may thus be a decreasing or an increasing function of time, and its sign is not fixed either. Each term in the sum tends to zero on a characteristic time $T_n = 1/(\chi k_n^2)$. Suppose the diffusion is observed at scale k with $\mid k \mid = \mid k_n \mid O(\nu)$. The relaxation time T_n is then much smaller than the typical diffusive time $T = 1/(\chi k^2)$ at scale k and the first order contribution to $S_G[n]$ can then be neglected. This echoes the conclusion obtained above in Section 4 that, at first order in ε, the effects of metric perturbations are confined to scales comparable to the variation scales of the perturbations.

The second order term $S_{G2}[n]$ cannot be computed exactly. Considering the conclusions of Section 4, one nevertheless expects increasing, possibly exponential functions of the time t to contribute to $S_{G2}[n]$, the characteristic time scale $T_{nn'}$ of these functions being related to the differences $k_n - k_{n'}$ in wave numbers of the metric perturbation by $T_{nn'} = 1/(\chi(k_n - k_{n'})^2)$. This expectation can be confirmed by computing exactly the contribution of $N_1 l_1$ to $S_{G2}[n]$. Naturally, given a certain wave number k, $T_{nn'}$ may be comparable to $T = 1/(\chi k^2)$, even if both $\mid k_n \mid$ and $\mid k_{n'} \mid$ are much larger than $\mid k \mid$. The behaviour of the Gibbs entropy thus confirms that, at second order, metric irregularities an influence diffusions at all scales, including scales much larger than the typical variation scales of the metric perturbation.

6. Conclusion

We have investigated how metric irregularities influence Brownian motion on a differential manifold. We have performed explicit perturbative calculations for nearly flat 2D manifolds and reached the conclusion that the metric irregularities have a cumulative effect on Brownian motion; more precisely, we have found that the relative difference of the spatial Fourier components of the densities generated by a Brownian motion on the flat surface and a Brownian motion on the irregular surface grows exponentially with time on all spatial scales, including scales much larger than those characteristic of the metric perturbation; entropy behavior has also been considered and characteristic times have been derived.

Let us conclude this article by mentioning some problems left open for further study. As stated in the introduction, many biological phenomena involve lateral diffusions on 2D interfaces. The results of this article suggest that the fluctuations of the interfaces profoundly affect these lateral diffusions; the discrepancies between real diffusions on irregular interfaces and idealized diffusions on highly regular surfaces are therefore probably observable and the biological consequences of these discrpancies should be carefully studied. On the theoretical side, one should envisage a non perturbative treatment of at least some of the problems studied in this article; this is probably best achieved through numerical simulations; a first step would be to confirm numerically, at least for 2D diffusions, the characteristic time estimates we have derived here. Finally, the case of relativistic diffusions in fluctuating space-times is certainly worth investigating, notably in a cosmological context.

References

[1] S. Abarbanel and A. Ditkowski, Asymptotically stable fourth-order accurate schemes for the diffusion equation on complex shapes, J. Comput. Phys., 133 (1996) 279.
[2] L. J. S. Allen, An Introduction to Stochastic Processes with Applications to Biology, Prentice Hall, 2003.
[3] J. Braga, J. M. P. Desterro and M. Carmo-Fonseca, Mol. Bio. Cell, 15 (2004) 4749-4760.
[4] A. Brünger, R. Peters and K. Schulten, Continuous fluorescence microphotolysis to observe lateral diffusion in membranes: theoretical methods and applications, J. Chem. Phys., 82 (1984) 2147.
[5] C. Chevalier and F. Debbasch, Fluctuation-Dissipation Theorems in an expanding Universe, J. Math. Phys., 48 (2007) 023304.
[6] M. Christensen, How to simulate anisotropic diffusion processes on curved surfaces, J. Comput. Phys., 201 (2004) 421-435.
[7] B. A. Dubrovin, S. P. Novikov and A. T. Fomenko, Modern geometry - Methods and applications, Springer-Verlag, New-York, 1984.
[8] L. Edelstein-Keshet, Mathematical Models in Biology, Classics in Applied Mathematics 46, SIAM, 2005.
[9] M. Emery, Stochastic calculus in manifolds, Springer-Verlag, 1989.
[10] N. S. Goel and N. Richter-Dyn, Stochastic Models in Biology, The Blackburn Press, 2004.
[11] N. Ikeda and S. Watanabe, Stochastic differential equations and diffusion processes, North-Holland Mathematical Library, 2nd edition, 1989.
[12] K. Itô, On stochastic differential equations on a differentiable manifold i., Nagoya Math. J., 1 (1950) 35-47.
[13] K. Itô, On stochastic differential equations on a differentiable manifold ii., MK, 28 (1953) 82-85.
[14] H. P. McKean, Stochastic integrals, Academic Press, New York and London, 1969.
[15] D. Mitra and Q. Wang, Stochastic traffic engineering for demand uncertainty and risk-aware network revenue management, IEEE/ACM Transactions on Networking, 13(2) (2005) 221-233.
[16] J. D. Murray, Mathematical Biology I: An Introduction, 3rd Edition, Interdisciplinary Applied Mathematics, Mathematical Biology, Springer, 2002.
[17] S. Nehls et al, Dynamics and retention of misfolded proteines in native er membranes, Nat. Cell. Bio., 2 (2000) 288-295.
[18] B. Øksendal, Stochastic Differential Equations, Universitext. Springer-Verlag, Berlin, 5th edition, 1998.
[19] I. F. Sbalzarini, A. Hayer, A. Helenius and P. Koumoutsakos, Simulations of (an)isotropic diffusion on curved biological interfaces. Biophysical J, 90(3) (2006) 878-885.
[20] M. Schreckenberg, A. Schadschneider, K. Nagel and N. Ito, Discrete stochastic models for traffic flow, Phys. Rev. E, 51(4) (1995) 2939-2949.
[21] S. E. Shreve, Stochastic Calculus for Finance I: The Binomial Asset Pricing Model, Springer Finance, Springer-Verlag, New-York, 2004.
[22] S. E. Shreve, Stochastic Calculus for Finance II: Continuous-Time Models, Springer Finance, Springer-Verlag, New-York, 2004.
[23] N. G. van Kampen, Stochastic Processes in Physics and Chemistry, North-Holland, Amsterdam, 1992.

Université Pierre et Marie Curie-Paris6, UMR 8112, ERGA-LERMA, 3 rue Galilée, 94200 Ivry, France.

E-mail: `chevalier_claire@yahoo.fr and fabrice.debbasch@gmail.com`