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CONVERGENCE ANALYSIS OF A SPLITTING METHOD FOR STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we propose a fully drift-implicit splitting numerical scheme for the stochastic differential equations driven by the standard d-dimensional Brownian motion. We prove that its strong convergence rate is of the same order as the standard Euler-Maruyama method. Some numerical experiments are also carried out to demonstrate this property. This scheme allows us to use the latest information inside each iteration in the Euler-Maruyama method so that better approximate solutions could be obtained than the standard approach.

Key Words. Stochastic differential equation, drift-implicit splitting scheme, Brownian motion

1. Introduction

Let us consider the following stochastic differential equations (SDEs)

$$(1)\qquad \begin{cases} dy(t) = f(y(t))dt + g(y(t))dW(t), & 0 \le t \le T \\ y(0) = y_0 \end{cases}$$

where $T > 0$ is the terminal time, $y(t) : [0, T] \times \Omega \to R^m$, $f(y) : R^m \to R^m$, $g(y) : R^m \to R^{m\times d}$, and $W(t) = (W_1(t), \cdots, W_d(t))^*$ is a standard d-dimensional Brownian motion defined on a complete, filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{0\le t\le T})$. Stochastic differential equations are used in many fields, such as stock market, financial mathematics, stochastic controls, dynamic system, biological science, chemical reactive kinetics and hydrology, and so on. Thus, it is of importance to study the solution of SDEs. However, it is often very difficult or impossible to find the analytic solutions of SDEs, as a consequence, numerical methods for finding approximate solutions of SDEs have attracted much attentions.

There have been a lot of publications in which numerical methods for stochastic differential equations and their applications were studied and discussed. For instance, the Itô-Taylor type method proposed in [11] that makes use of the so-called Itô Taylor expansion to discretize the SDEs; the linearization type methods suggested in [3, 12, 17], that first linearize the drift and diffusion coefficients of the SDEs and then solve the pruned linear SDEs instead; the Runge-Kutta type methods [4, 5, 16, 20], in which the Runge-Kutta methods for solving ordinary differential equations are extended to solve the SDEs. Concerning the stability of the methods,

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some implicit discretization schemes were proposed in [6, 7] to stablize the numerical discretization. In order to improve the accuracy of the approximate solution, some high-order numerical methods for solving SDEs were studied in [1, 4, 5, 10, 11] and some splitting methods were also studied in [2].

The Euler-Maruyama (E-M) method is so far the most studied numerical method for solving SDEs and its strong convergence rate is $1/2$ for general cases. Due to its easy implementation, the E-M method and its modified versions have been very commonly used for applied stochastic problems, such as stochastic optimal control and stochastic partial differential equations. Since SDEs are often driven by a high-dimensional Brownian motion and coupled with other type stochastic problems [9], more efficient and accurate solvers for high-dimensional SDEs are urgently needed. In the past decades, the operator splitting scheme has been extensively studied and becomes one of the most popular and efficient ways to deal with multi-dimensional problems which are modeled by the deterministic ordinary or partial differential equations. In fact, the same idea also can be applied to the SDEs. In this paper, we will propose a new splitting scheme for numerical solutions of the SDEs (1), and show that the resulted approximate solution converges to the analytic solution of the SDEs with the same convergence rate as the one the E-M method has. Furthermore, this scheme allows us to use the latest information inside each iteration in the E-M method so that better approximate solutions could be obtained than the standard approach especially when d is large.

We organize this paper as follows. In Section 2, we first propose a fully drift-implicit splitting scheme for the discretization of the SDEs (1), then we prove the strong convergence of this scheme in Section 3. After presenting some computational experiments in Section 4, conclusions are given in Section 5.

2. A fully drift-implicit splitting scheme of SDEs

Let us rewrite the stochastic differential equations (1) in the following form:

$$\left\{ \begin{array}{ll} dy(t) & = f(y(t))dt + \sum_{i=1}^{d} g_i(y(t))dW_i(t) \qquad ,0 < t \le T \\ y(0) & = y_0, \end{array} \right. \tag{2}$$

where $W_i(t)$, $i = 1, 2, \cdots, d$ are independent one-dimensional Brownian motions and $g_i : R^m \to R^m$, $i = 1, 2, \cdots, d$.

It is well-known that the problem (2) is equivalent to the following Itô integral equation

$$y(t) \quad = \quad y_0 + \int_0^t f(y(s))ds + \sum_{i=1}^{d} \int_0^t g_i(y(s))dW_i(s). \tag{3}$$

To discretize the equation (2), we first partition the time interval $[0, T]$ by

$$0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T. \tag{4}$$

Let $\Delta t_n = t_{n+1} - t_n$ denote the discrete time step at the time t_n, and set $\Delta t = \max_{n=0}^{N-1} \Delta t_n$. For the simplicity of description, we only discuss the case of uniform time partition, but all results obtained in this paper still remain valid for general partition (4).

From the equation (3), we have exactly

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(y(s))ds + \sum_{i=1}^{d} \int_{t_n}^{t_{n+1}} g_i(y(s))dW_i(s). \tag{5}$$

Suppose that Δt is sufficiently small. If we use $f(y(t_n))\Delta t$ to approximate the integral $\int_{t_n}^{t_{n+1}} f(y(s))ds$ and use $g_i(y(t_n))\Delta W_i(t_n)$ with $\Delta W_i(t_n) = W_i(t_{n+1}) - W_i(t_n)$ to approximate the Itô integral $\int_{t_n}^{t_{n+1}} g_i(y(s))dW_i(s)$, then we obtain the following standard explicit E-M scheme:

$$y(t_{n+1}) = y(t_n) + f(y(t_n))\Delta t + \sum_{i=1}^{d} g_i(y(t_n))\Delta W_i(t_n). \tag{6}$$

for $n = 0, 1, \cdots, N-1$. Again, if we use $f(y(t_{n+1}))\Delta t$ to approximate the integral $\int_{t_n}^{t_{n+1}} f(y(s))ds$, we obtain the following drift-implicit E-M scheme:

$$y(t_{n+1}) = y(t_n) + f(y(t_{n+1}))\Delta t + \sum_{i=1}^{d} g_i(y(t_n))\Delta W_i(t_n). \tag{7}$$

for $n = 0, 1, \cdots, N-1$. In [6,7], the following implicit backward Euler method was proposed and discussed for discretizing the SDEs (2):

$$\left\{ \begin{array}{rcl} Y_n^* &=& Y_n + f(Y_n^*)\Delta t, \\ Y_{n+1} &=& Y_n^* + \sum_{i=1}^{d} g_i(Y_n^*)\Delta W_i(t_n) \end{array} \right. \tag{8}$$

with $Y_0 = y_0$. In this approach, a partial operator splitting has been used to separate the drift term of dt and the diffusion term of $dW(t)$ so that an intermediate variable Y_n^* can be first found implicitly and only involves f and then used to compute Y_{n+1} by an explicit Euler scheme only involving $g = (g_1, g_2, \cdots, g_d)$.

In this paper, we propose a fully drift-implicit splitting approximation scheme for the SDEs (2) that can be described in the following:

$$\left\{ \begin{array}{rcl} Y_n^0 &=& Y_n + f(Y^0)\Delta t \\ Y_n^1 &=& Y^0 + g_1(Y_n^0)\Delta W_1(t_n) \\ Y_n^2 &=& Y^1 + g_2(Y_n^1)\Delta W_2(t_n) \\ &\vdots& \\ Y_{n+1} &=& Y_n^{d-1} + g_d(Y_n^{d-1})\Delta W_d(t_n). \end{array} \right. \tag{9}$$

for $n = 0, 1, \cdots, N-1$ with $Y_0 = y_0$. Notice that in this scheme $g(y(t))dW(t)$ is furtherly splitted into d sub-terms.

We now introduce some notations used in the following sections. We use $< \cdot, \cdot >$ to denote the inner product in the Euclidean space R^m, use $|X|$ to denote both the Euclidean vector norm for the vector X or the Frobenius (or trace) matrix norm for the matrix X, use $a \vee b$ to denote the bigger one of the two real number a and b, and use $\mathbb{E}[\cdot]$ to denote the mathematical expectation.

3. Convergence analysis

For simple presentations, in the following sections, we only consider the SDEs (2) with $d = 2$. All results obtained later hold true for the general d. Now we rewrite the SDEs (2) with $d = 2$ by

$$\left\{ \begin{array}{rcl} dy(t) &=& f(y(t))dt + g_1(y(t))dW_1(t) + g_2(y(t))dW_2(t), \quad 0 < t \le T, \\ y(0) &=& y_0, \end{array} \right. \tag{10}$$

where W_1 and W_2 are two independent standard one-dimensional Brownian motions, $f : R^m \to R^m$ and $g_i : R^m \to R^m$ with $i = 1, 2$ are vector functions. Then the splitting scheme (9) for the equations (10) becomes

$$\left\{\begin{array}{rcl} Y_n^* & = & Y_n + \Delta t f(Y_n^*) \\ Y_n^{**} & = & Y_n^* + \Delta W_1(t_n) g_1(Y_n^*) \\ Y_{n+1} & = & Y_n^{**} + \Delta W_2(t_n) g_2(Y_n^{**}), \end{array}\right. \tag{11}$$

with $Y_0 = y_0$.

3.1. Existence and uniqueness of the discrete solution. In order to derive the error estimate, we first need to make some assumption about the regularity of the functions f and g.

Assumption 3.1. *We assume that f and g_i $(i = 1, 2)$ are global Lipschitz vector functions, that is, for any a and b in R^m, there exists a constant $K > 0$ such that*

$$|f(a) - f(b)|^2 \vee |g_1(a) - g_1(b)|^2 \vee |g_2(a) - g_2(b)|^2 \le K|a - b|^2. \tag{12}$$

From the Lipschitz condition (12), it is easy to deduce that there are two positive constants α_0 and β_0 such that

$$|f(a)|^2 \vee |g_1(a)|^2 \vee |g_2(a)|^2 \le \alpha_0 + \beta_0|a|^2, \quad \forall\, a \in R^m. \tag{13}$$

Now we give the first lemma.

Lemma 3.1. *Suppose Assumption 3.1 hold and let $\Delta t \in (0, \Delta t_c)$ with $\Delta t_c \le 1/(2\sqrt{K})$ where the constant K is defined in (12). Then for a given constant $d \in R^m$, the implicit equation*

$$c = d + \Delta t f(c) \tag{14}$$

has a unique solution c. Define $F^{\Delta t}(d)$, $f^{\Delta t}(d)$, and $g_1^{\Delta t}(d)$ by

$$F^{\Delta t}(d) = c, \quad f^{\Delta t}(d) = f(F^{\Delta t}(d)), \quad \text{and} \quad g_1^{\Delta t}(d) = g_1(F^{\Delta t}(d)), \tag{15}$$

respectively where c is the unique solution of the equations (14). Then $F^{\Delta t}$, $f^{\Delta t}$ and $g_1^{\Delta t}$ are C^1 vector functions, and

$$f^{\Delta t}(\cdot) \to f(\cdot), \qquad g_1^{\Delta t}(\cdot) \to g_1(\cdot)$$

uniformly in C^1 as $\Delta t \to 0$. Furthermore, there exist three positive constants L, α and β such that the functions $f^{\Delta t}$ and $g_1^{\Delta t}$ satisfy

$$|f^{\Delta t}(a) - f^{\Delta t}(b)|^2 \vee |g_1^{\Delta t}(a) - g_1^{\Delta t}(b)|^2 \le L|a - b|^2, \tag{16}$$

and

$$|f^{\Delta t}(a)|^2 \vee |g_1^{\Delta t}(a)|^2 \le \alpha + \beta|a|^2 \tag{17}$$

for any a and b in R^m.

Proof: The original proof can be found in [6]. For the completeness of the paper, we give the proof here with some modifications. The existence and the uniqueness of the solution for the equation (14) can be proved via a contraction mapping theorem, which also establishes the C^1 smoothness of the functions $f^{\Delta t}(\cdot)$ and $F^{\Delta t}(\cdot)$, and the convergence of the function $f^{\Delta t}(\cdot)$ to the function $f(\cdot)$. Then the smoothness and the convergence of the function $g_1^{\Delta t}(\cdot)$ follow from the fact $g_1^{\Delta t}(\cdot) = g_1(F^{\Delta t}(\cdot))$.

Let $c^{(1)} = d^{(1)} + \Delta t f(c^{(1)})$ and $c^{(2)} = d^{(2)} + \Delta t f(c^{(2)})$. Then we have

$$|c^{(1)} - c^{(2)}|^2 - \Delta t < f(c^{(1)}) - f(c^{(2)}), c^{(1)} - c^{(2)} > = < d^{(1)} - d^{(2)}, c^{(1)} - c^{(2)} > .$$

From the Lipschitz condition (12), we obtain

$$(1 - \sqrt{K}\Delta t)|c^{(1)} - c^{(2)}|^2 \le \frac{1}{2}|d^{(1)} - d^{(2)}|^2 + \frac{1}{2}|c^{(1)} - c^{(2)}|^2,$$

which implies

$$|c^{(1)} - c^{(2)}|^2 \le \frac{1}{1 - 2\sqrt{K}\Delta t}|d^{(1)} - d^{(2)}|^2. \tag{18}$$

The inequality (18) and the definition of the vector function $F^{\Delta t}(\cdot)$ lead to

$$|F^{\Delta t}(d_1) - F^{\Delta t}(d_2)|^2 \le \frac{1}{1 - 2\sqrt{K}\Delta t}|d_1 - d_2|^2. \tag{19}$$

From the inequality (19), the definitions of the vector functions $f^{\Delta t}$ and $g_1^{\Delta t}$ in (15), and the Lipschitz condition (12), we deduce that the functions $f^{\Delta t}$ and $g_1^{\Delta t}$ are two global Lipschitz functions with the Lipschitz $L = \frac{K}{1-2\sqrt{K}\Delta t}$. The inequality (17) holds with the constants $\alpha = 2|f^{\Delta t}(0)|^2 \vee 2|g_1^{\Delta t}(0)|^2$ and $\beta = 2L$ from the facts

$$|f^{\Delta t}(a)|^2 \le 2|f^{\Delta t}(a) - f^{\Delta t}(0)|^2 + 2|f^{\Delta t}(0)|^2 \le 2L|a|^2 + 2|f^{\Delta t}(0)|^2$$

and $|g_1^{\Delta t}(a)|^2 \le 2L|a|^2 + 2|g_1^{\Delta t}(0)|^2$. □

Lemma 3.2. *Suppose $\Delta t \in (0, \Delta t_c)$ with $\Delta t_c \le 1/(2\sqrt{K})$ where the K is the Lipschitz constant defined in (12). Then the splitting scheme (11) is equivalent to the E-M scheme for solving the modified SDEs*

$$\left\{ \begin{array}{rcl} dy^{\Delta t} & = & f^{\Delta t}(y^{\Delta t})dt + g_1^{\Delta t}(y^{\Delta t})dW_1(t) + g_2^{\Delta t}(y^{\Delta t})dW_2(t) \\ y^{\Delta t}(0) & = & y_0, \end{array} \right. \tag{20}$$

for $0 < t \le T$, where the functions $f^{\Delta t}(\cdot)$ and $g_1^{\Delta t}(\cdot)$ are defined in Lemma 3.1, and the function $g_2^{\Delta t}$ is defined by

$$g_2^{\Delta t}(y^{\Delta t}(t)) = g_2(y^{\Delta t}(t)) + \Delta t f^{\Delta t}(y^{\Delta t}(t)) + \Delta W_1(t_n) g_1^{\Delta t}(y^{\Delta t}(t))$$

for $t \in [\,t_n, t_{n+1})$ for $n = 0, 1, \cdots, N-1$.

Proof: Using the Lemma 3.1 and the definition of the scheme (11), we obtain

$$\begin{array}{rcl} F^{\Delta t}(Y_n) & = & Y_n^*, \\ F^{\Delta t}(Y_n) & = & Y_n + \Delta t f^{\Delta t}(Y_k), \\ g_1(Y_n^*) & = & g_1^{\Delta t}(Y_n), \\ g_2(Y_n^*) & = & g_2(Y_n + \Delta t f^{\Delta t}(Y_n) + \Delta W_1(t_n) g_1^{\Delta t}(Y_n)) = g_2^{\Delta t}(Y_n). \end{array}$$

Then the splitting scheme (11) can be rewritten as

$$Y_{n+1} = Y_n + \Delta t f^{\Delta t}(Y_n) + \Delta W_1(t_n) g_1^{\Delta t}(Y_n) + \Delta W_2(t_n) g_2^{\Delta t}(Y_n),$$

which is exactly the E-M scheme for the SDEs (20). □

From the Lemma 3.2, we directly obtain the existence and uniqueness of the discrete solution of the splitting scheme (11). So does the the splitting scheme (9) for general d-dimensional case. We also would like to remark on Assumption 3.1 and Euler discretizations that complete implicit schemes are specially useful when one needs to run long time simulations, e.g., to approximate invariant measures or Lyapunov exponents, and when the drift coefficients are not globally Lipschitz, see [13, 19].

3.2. Error Estimates. Let us introduce two time-continuous random vector processes $Y(t)$ and $\bar{Y}(t)$, which are, respectively, defined by

$$Y(t) := Y_n, \quad \text{for} \quad t \in [\,t_n, t_{n+1}), \tag{21}$$

(22)
$$\bar{Y}(t) := \quad Y_0 + \int_0^t f^{\Delta t}(Y(s))ds + \int_0^t g_1^{\Delta t}(Y(s))dW_1(s) + \int_0^t g_2^{\Delta t}(Y(s))dW_2(s),$$

where the vector functions $f^{\Delta t}(\cdot)$ and $g_1^{\Delta t}(\cdot)$ are defined in Lemma 3.1, and the vector function $g_2^{\Delta t}(Y(s))$ is defined by

$$g_2^{\Delta t}(Y(s)) = g_2(Y(s)) + \Delta t f^{\Delta t}(Y(s)) + \Delta\hat{W}_1(s) g_1^{\Delta t}(Y(s))$$

where $\Delta\hat{W}_1(s) = \Delta W_1(t_n)$ for $s \in [t_n, t_{n+1})$.

From the definitions (21) and (22) and the definition of $\Delta\hat{W}_1(s)$, we know that $\Delta\hat{W}_1(s)$ and $Y(s)$ ($\forall s \in [0,T]$) are independent, and that $\bar{Y}(t_n) = Y(t_n) = Y_n$. So we call the two random processes $\bar{Y}(t)$ and $Y(t)$ the two time continuous extensions of the discontinuous random variables $\{Y_n\}_{n=0}^N$.

Now we present our main result of this paper about the strong convergence and error estimates for the splitting scheme (11).

Theorem 3.1. *Let $y(t)$ and $\bar{Y}(t)$ be the solutions of (10) and (22), respectively. Then under the Assumption 3.1, if $\Delta t < \Delta t_c < 1/(2\sqrt{K})$ with the Lipschitz constant K in (12), we have the estimate*

$$\mathbb{E}\left[|\bar{Y}(t) - y(t)|^2\right] \leq C\Delta t \tag{23}$$

where the constant C does not depend on $y(t)$, $\bar{Y}(t)$, and the time partition.

The proof of the Theorem 3.1 will be postponed to the end of this section, let us first prove some preliminary lemmas and propositions.

Lemma 3.3. *Let Y_n $(0 \leq n \leq N)$ be the solution of the splitting scheme (11). Then under the conditions of the Theorem 3.1, there exists a constant $A = A(T) > 0$ independent of Δt such that*

$$\mathbb{E}\left[|Y_n|^2\right] \leq A$$

for $n = 0, 1, 2, \cdots, N$.

Proof: Using Itô's formula to the process $|\bar{Y}(t)|^2$ with $\bar{Y}(t)$ defined by (22), we obtain

$$\begin{aligned} |\bar{Y}(t)|^2 \;&=\; |Y_0|^2 + 2\int_0^t < \bar{Y}(s), f^{\Delta t}(Y(s)) > ds \\ &\quad +2\int_0^t < \bar{Y}(s), g_1^{\Delta t}(Y(s)) > dW_1(s) + \int_0^t |g_1^{\Delta t}(Y(s))|^2 ds \\ &\quad +2\int_0^t < \bar{Y}(s), g_2^{\Delta t}(Y(s)) > dW_2(s) + \int_0^t |g_2^{\Delta t}(Y(s))|^2 ds. \end{aligned} \tag{24}$$

Take the mathematical expectation on the two sides of the equation (24), then

$$\begin{aligned}\mathbb{E}[|\bar{Y}(t)|^2] &= |Y_0|^2 + 2\int_0^t \mathbb{E}[< \bar{Y}(s), f^{\Delta t}(Y(s)) >]ds \\ &\quad + \int_0^t \mathbb{E}[|g_1^{\Delta t}(Y(s))|^2]ds + \int_0^t \mathbb{E}[|g_2^{\Delta t}(Y(s))|^2]ds \\ &\leq |Y_0|^2 + \int_0^t \mathbb{E}[|\bar{Y}(s)|^2]ds + \int_0^t \mathbb{E}[|f^{\Delta t}(Y(s))|]^2 ds \\ &\quad + \int_0^t \mathbb{E}[|g_1^{\Delta t}(Y(s))|^2]ds + \int_0^t \mathbb{E}[|g_2^{\Delta t}(Y(s))|^2]ds. \end{aligned} \tag{25}$$

Let $t = t_n$ in the inequality (25) with n satisfying $n\Delta t \in [0, T]$, then we get

$$\begin{aligned}\mathbb{E}[|Y_n|^2] &\leq |Y_0|^2 + \int_0^{t_n} \mathbb{E}[|\bar{Y}(s)|^2]ds + \int_0^{t_n} \mathbb{E}[|f^{\Delta t}(Y(s))|^2]ds \\ &\quad + \int_0^{t_n} \mathbb{E}[|g_1^{\Delta t}(Y(s))|^2]ds + \int_0^{t_n} \mathbb{E}[|g_2^{\Delta t}(Y(s))|^2]ds \\ &\leq |Y_0|^2 + \int_0^{t_n} \mathbb{E}[|\bar{Y}(s)|^2]ds + \sum_{i=1}^{n} \mathbb{E}[|f^{\Delta t}(Y_i)|^2]\Delta t \\ &\quad + \sum_{i=1}^{n} \mathbb{E}[|g_1^{\Delta t}(Y_i)|^2]\Delta t + \sum_{i=1}^{n} \mathbb{E}[|g_2^{\Delta t}(Y_i)|^2]\Delta t. \end{aligned} \tag{26}$$

Now we estimate the second term on the right hand side of the inequality (26). From the definitions of the random processes $\bar{Y}(t)$ and $Y(t)$, we have

$$\begin{aligned}&\int_0^{t_n} \mathbb{E}\left[|\bar{Y}(s)|^2\right] ds \leq 2\int_0^{t_n} \mathbb{E}\left[|\bar{Y}(s) - Y(s)|^2\right] ds + 2\int_0^{t_n} \mathbb{E}\left[|Y(s)|^2\right] ds \\ &= 2\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[|\bar{Y}(s) - Y_i|^2\right] ds + 2\sum_{i=0}^{n-1} \mathbb{E}\left[|Y_i|^2\right] \Delta t \\ &= 2\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[| \int_{t_i}^{s} f^{\Delta t}(Y(r))dr + \int_{t_i}^{s} g_1^{\Delta t}(Y(r))dW_1(r) \right. \\ &\quad \left. + \int_{t_i}^{s} g_2^{\Delta t}(Y(r))dW_2(r)|^2 \right] ds + 2\sum_{i=0}^{n-1} \mathbb{E}\left[|Y_i|^2\right] \Delta t \\ &\leq 6\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[| \int_{t_i}^{s} f^{\Delta t}(Y(r))dr|^2 \right] ds \\ &\quad +6\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[| \int_{t_i}^{s} g_1^{\Delta t}(Y(r))dW_1(r)|^2 \right] ds \\ &\quad +6\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\left[| \int_{t_i}^{s} g_2^{\Delta t}(Y(r))dW_2(r)|^2 \right] ds + 2\sum_{i=0}^{n-1} \mathbb{E}\left[|Y_i|^2\right] \Delta t \\ &\leq 6\sum_{i=0}^{n-1} \mathbb{E}\left[|f^{\Delta t}(Y_i)|^2\right] (\Delta t)^3 + 6\sum_{i=0}^{n-1} \mathbb{E}\left[|g_1^{\Delta t}(Y_i)|^2\right] (\Delta t)^2 \\ &\quad +6\sum_{i=0}^{n-1} \mathbb{E}\left[|g_2^{\Delta t}(Y_i)|^2\right] (\Delta t)^2 + 2\sum_{i=0}^{n-1} \mathbb{E}\left[|Y_i|^2\right] \Delta t. \end{aligned} \tag{27}$$

From Lemma 3.1, we have the estimates

$$\mathbb{E}\left[|f^{\Delta t}(Y_i)|^2\right] \leq \alpha + \beta\mathbb{E}\left[|Y_i|^2\right], \tag{28}$$

$$\mathbb{E}\left[|g_1^{\Delta t}(Y_i)|^2\right] \leq \alpha + \beta\mathbb{E}\left[|Y_i|^2\right], \tag{29}$$

and

$$\begin{aligned}
\mathbb{E}\left[|g_2^{\Delta t}(Y_i)|^2\right] &= \mathbb{E}\left[|g_2(Y_i + \Delta t f^{\Delta t}(Y_i) + \Delta\hat{W}_1(t_i) g_1^{\Delta t}(Y_i))|^2\right] \\
&\leq \alpha + \beta\mathbb{E}\left[|Y_i + \Delta t f^{\Delta t}(Y_i) + \Delta\hat{W}_1(t_i) g_1^{\Delta t}(Y_i)|^2\right] \\
&\leq \alpha + 3\beta\left(\mathbb{E}\left[|Y_i|^2\right] + (\Delta t)^2\mathbb{E}\left[|f^{\Delta t}(Y_i)|^2\right] + \Delta t\mathbb{E}\left[|g_1^{\Delta t}(Y_i)|^2\right]\right) \\
&\leq C_1\left(1 + \mathbb{E}\left[|Y_i|^2\right]\right),
\end{aligned} \tag{30}$$

where $C_1 > 0$ is a constant independent on Y_n and Δt. Inserting the estimates (28), (29) and (30) into the estimate (27), we get

$$\begin{aligned}
\int_0^{t_n} \mathbb{E}\left[|\bar{Y}(s)|^2\right] ds &\leq C_2(1 + \sum_{i=0}^{n-1} \mathbb{E}\left[|Y_i|^2\right])\Delta t \\
&\leq C_2\left[\left(1 + |Y_0|^2 + \sum_{i=1}^{n} \mathbb{E}\left[|Y_i|^2\right]\right)\right]\Delta t
\end{aligned} \tag{31}$$

where $C_2 > 0$ is a constant independent of Y_n and Δt. From the inequalities (26), (28), (29), (30) and(31), we deduce

$$\mathbb{E}\left[|Y_n|^2\right] \leq C_3\left(1 + \sum_{i=1}^{n} \mathbb{E}\left[|Y_i|^2\right]\Delta t\right). \tag{32}$$

where the constant $C_3 > 0$ depends on α, β, C_1, and C_2. Finally applying Gronwall inequality to (32) leads to the conclusion of the lemma. □

Lemma 3.4. *Let $\bar{Y}(t)$ be defined by the equation (22). Then under the conditions of the Theorem 3.1, there exists a constant $B = B(T)$ independent of Δt and $\bar{Y}(t)$, such that*

$$\mathbb{E}\left[|\bar{Y}(t)|^2\right] \leq B, \quad \forall\, t \in [0, T].$$

Proof: The proof of this lemma is almost the same as the proof of the Lemma 3.3. So we omit it here. □

In the next lemma, we will show that $y^{\Delta t}(t)$, the solution of the modified SDEs (20), has a bounded second moment.

Lemma 3.5. *Let $y^{\Delta t}$ be the solution of the SDEs (20). Then under the conditions of the Theorem 3.1, there exists a constant $M = M(T) > 0$ independent of Δt such that*

$$\mathbb{E}\left[|y^{\Delta t}(t)|^2\right] \leq M, \quad \forall\, t \in [0, T]. \tag{33}$$

Proof: From the SDEs (20) and using Itô's formula to $|y^{\Delta t}(t)|^2$, we deduce

$$\begin{aligned}|y^{\Delta t}(t)|^2 &= |Y_0|^2 + 2\int_0^t < y^{\Delta t}(s), f^{\Delta t}(y^{\Delta t}(s)) > ds \\ &+2\int_0^t < y^{\Delta t}(s), g_1^{\Delta t}(y^{\Delta t}(s)) > dW_1(s) \\ &+2\int_0^t < y^{\Delta t}(s), g_2^{\Delta t}(y^{\Delta t}(s)) > dW_2(s) \\ &+\int_0^t |g_1^{\Delta t}(y^{\Delta t}(s))|^2 ds + \int_0^t |g_2^{\Delta t}(y^{\Delta t}(s))|^2 ds.\end{aligned}$$

Take mathematical expectation on the two sides of the above equation, we have

$$\begin{aligned}\mathbb{E}\left[|y^{\Delta t}(t)|^2\right] &\leq |Y_0|^2 + \int_0^t \mathbb{E}\left[|y^{\Delta t}(s)|^2\right] ds + \int_0^t \mathbb{E}\left[|f^{\Delta t}(y^{\Delta t}(s))|^2\right] ds \\ &+\int_0^t \mathbb{E}\left[|g_1^{\Delta t}(y^{\Delta t}(s))|^2\right] ds + \int_0^t \mathbb{E}\left[|g_2^{\Delta t}(y^{\Delta t}(s))|^2\right] ds.\end{aligned} \tag{34}$$

Using Assumption 3.1 and Lemma 3.1, we obtain

$$\int_0^t \mathbb{E}\left[|f^{\Delta t}(y^{\Delta t}(s))|^2\right] ds \leq \alpha T + \beta\int_0^t \mathbb{E}\left[|y^{\Delta t}(s)|^2\right] ds, \tag{35}$$

$$\int_0^t \mathbb{E}\left[|g_1^{\Delta t}(y^{\Delta t}(s))|^2\right] ds \leq \alpha T + \beta\int_0^t \mathbb{E}\left[|y^{\Delta t}(s)|^2\right] ds. \tag{36}$$

Combine (34), (35) and (36), and use the Cauchy-Schwartz inequality, then we get

$$\begin{aligned}&\int_0^t \mathbb{E}\left[|g_2^{\Delta t}(y^{\Delta t}(s))|^2\right] ds \\ &= \int_0^t \mathbb{E}\left[|g_2(y^{\Delta t}(s) + \Delta t f^{\Delta t}(y^{\Delta t}(s)) + \Delta\hat{W}_1(s) g_1^{\Delta t}(y^{\Delta t}(s)))|^2\right] ds \\ &\leq \alpha_0 T + \beta_0 \int_0^t \mathbb{E}\left[|y^{\Delta t}(s) + \Delta t f^{\Delta t}(y^{\Delta t}(s)) + \Delta\hat{W}_1(s) g_1^{\Delta t}(y^{\Delta t}(s))|^2\right] ds \\ &\leq \alpha_0 T + 3\beta_0 \int_0^t \mathbb{E}\left[|y^{\Delta t}(s)|^2\right] ds + 3\beta_0 \Delta t^2 \int_0^t \mathbb{E}\left[|f^{\Delta t}(y^{\Delta t}(s))|^2\right] ds \\ &\quad +3\beta_0 \int_0^t \mathbb{E}\left[|\Delta\hat{W}_1(s) g_1^{\Delta t}(y^{\Delta t}(s))|^2\right] ds \\ &\leq C_1\left(1 + \int_0^t \mathbb{E}\left[|y^{\Delta t}(s)|^2\right] ds\right) + C_2 \int_0^t \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2 |y^{\Delta t}(s)|^2\right] ds\end{aligned} \tag{37}$$

where the constants $C_1 > 0$ and $C_2 > 0$ are independent of $y^{\Delta t}$ and Δt. From the independence of $\Delta\hat{W}_1(s)$ and $y^{\Delta t}(s-\Delta t)$, we deduce

$$\begin{aligned}
&\int_0^t \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2|y^{\Delta t}(s)|^2\right] ds \\
&= \int_0^{\Delta t} \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2|y^{\Delta t}(s)|^2\right] ds + \int_{\Delta t}^t \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2|y^{\Delta t}(s)|^2\right] ds \\
&\le 2\int_0^{\Delta t} \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2|y^{\Delta t}(0)|^2\right] ds \\
&\quad +2\int_0^{\Delta t} \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2|y^{\Delta t}(s)-y^{\Delta t}(0)|^2\right] ds \\
&\quad +2\int_{\Delta t}^t \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2|y^{\Delta t}(s-\Delta t)|^2\right] ds \\
&\quad +2\int_{\Delta t}^t \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2|y^{\Delta t}(s)-y^{\Delta t}(s-\Delta t)|^2\right] ds
\end{aligned}$$

and then

$$\begin{aligned}
&\int_0^t \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2|y^{\Delta t}(s)|^2\right] ds \\
&\le 2(\Delta t)^2|Y_0|^2 + \int_0^{\Delta t} \mathbb{E}\left[|\Delta\hat{W}_1(s)|^4\right] ds + \int_0^{\Delta t} \mathbb{E}\left[|y^{\Delta t}(s)-y^{\Delta t}(0)|^4\right] ds \\
&\quad +4\Delta t\int_{\Delta t}^t \mathbb{E}\left[|y^{\Delta t}(s)-y^{\Delta t}(s-\Delta t)|^2\right] ds + 4\Delta t\int_{\Delta t}^t \mathbb{E}\left[|y^{\Delta t}(s)|^2\right] ds \\
&\quad +\int_{\Delta t}^t \mathbb{E}\left[|\Delta\hat{W}_1(s)|^4\right] ds + \int_{\Delta t}^t \mathbb{E}\left[|y^{\Delta t}(s)-y^{\Delta t}(s-\Delta t)|^4\right] ds \\
&\le 2(\Delta t)^2|Y_0|^2 + (\Delta t)^3 + \int_0^{\Delta t} \mathbb{E}\left[|y^{\Delta t}(s)-y^{\Delta t}(0)|^4\right] ds \\
&\quad +4\Delta t\int_0^t \mathbb{E}\left[|y^{\Delta t}(s)-y^{\Delta t}(s-\Delta t)|^2\right] ds + 4\Delta t\int_0^t \mathbb{E}\left[|y^{\Delta t}(s)|^2\right] ds \\
&\quad +(\Delta t)^2T + \int_0^t \mathbb{E}\left[|y^{\Delta t}(s)-y^{\Delta t}(s-\Delta t)|^4\right] ds. \qquad (38)
\end{aligned}$$

Using Assumption 3.1 and the standard arguments in [1,15] about the estimates of the solution of SDEs, we obtain the estimate

$$\mathbb{E}\left[|y^{\Delta t}(s)-y^{\Delta t}(t)|^p\right] \le C_{p,T}|s-t|^{p/2}, \qquad p\ge 2,\ \ 0\le s,t\le T \tag{39}$$

where the constant $C_{p,T}$ depends only on p and T.

From the estimates (38) and (39), we get

$$\int_0^t \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2|y^{\Delta t}(s)|^2\right] ds \le C_3(1+\int_0^t \mathbb{E}\left[|y^{\Delta t}(s)|^2\right] ds), \tag{40}$$

where the constant $C_3 > 0$ does not depend on $y^{\Delta t}$ and the time partition. Insert the inequality (40) into the inequality (37), we get

$$\int_0^t \mathbb{E}\left[|g_2^{\Delta t}(y^{\Delta t}(s))|^2\right] ds \le C_4(1+\int_0^t \mathbb{E}\left[|y^{\Delta t}(s)|^2\right] ds), \tag{41}$$

where constant $C_4 > 0$ is also independent of $y^{\Delta t}$ and time partition. From the inequalities (34), (35), (36) and (41), we deduce that there is a constant C_5 independent of $y^{\Delta t}$ and the time partition such that the inequality

$$\mathbb{E}\left[|y^{\Delta t}(t)|^2\right] \leq C_5(1 + \int_0^t \mathbb{E}\left[|y^{\Delta t}(s)|^2\right] ds). \tag{42}$$

holds true for $t \in [0, T]$. Finally applying Gronwall lemma to (42) leads to the inequality (33). □

Now, using the above lemmas, we can obtain the following error estimate.

Proposition 3.1. *Let $y(t)$ and $y^{\Delta t}(t)$ be the solutions of the SDEs (10) and the SDEs (20), respectively. Then under the conditions of the Theorem 3.1, we have the error estimate*

$$\mathbb{E}\left[|y^{\Delta t}(t) - y(t)|^2\right] \leq C\Delta t \tag{43}$$

where the constant C does not depend on $y(t)$, $y^{\Delta t}(t)$, and the time partition.

Proof: From Lemma 3.1, we conclude that there is a function $\phi : (0, \infty) \to (0, \infty)$ and $\phi(\Delta t) \to 0$ as $\Delta t \to 0$, such that

$$\phi(\Delta t) = O(\Delta t), \tag{44}$$

and

$$|f^{\Delta t}(u) - f(u)|^2 \vee |g_1^{\Delta t}(u) - g_1(u)|^2 \leq \phi(\Delta t), \quad \forall\, u \in R^m, \tag{45}$$

provided that Δt is small enough. Using the SDEs (20) and the SDEs (10), the Cauchy-Schwartz inequality, and the Itô isometry formula, we obtain

$$\begin{aligned}
\mathbb{E}\left[|y^{\Delta t}(t) - y(t)|^2\right] &= \mathbb{E}\Bigg[|\int_0^t (f^{\Delta t}(y^{\Delta t}(s)) - f(y(s)))ds \\
&\quad + \int_0^t (g_1^{\Delta t}(y^{\Delta t}(s)) - g_1(y(s)))dW_1(s) \\
&\quad + \int_0^t (g_2^{\Delta t}(y^{\Delta t}(s)) - g_2(y(s)))dW_2(s)|^2\Bigg] \\
&\leq T_1 + T_2 + T_3. \qquad (46)
\end{aligned}$$

where

$$\begin{aligned}
T_1 &= 3T\mathbb{E}\left[\int_0^t |f^{\Delta t}(y^{\Delta t}(s)) - f(y(s))|^2 ds\right], \\
T_2 &= 3\mathbb{E}\left[\int_0^t |g_1^{\Delta t}(y^{\Delta t}(s)) - g_1(y(s))|^2 ds\right], \\
T_3 &= 3\mathbb{E}\left[\int_0^t |g_2^{\Delta t}(y^{\Delta t}(s)) - g_2(y(s))|^2 ds\right].
\end{aligned}$$

Using the Cauchy-Schwartz inequality, Lemma 3.1 and the inequality (45), we obtain

$$\begin{aligned}
T_1 &\leq 6T\mathbb{E}\left[\int_0^t |f^{\Delta t}(y^{\Delta t}(s)) - f(y^{\Delta t}(s))|^2 ds\right] \\
&\quad + 6T\mathbb{E}\left[\int_0^t |f(y^{\Delta t}(s)) - f(y(s))|^2 ds\right] \\
&\leq 6T\mathbb{E}\left[\int_0^t \phi(\Delta t) ds\right] + 6TK\int_0^t \mathbb{E}\left[|y^{\Delta t}(s) - y(s)|^2\right] ds,
\end{aligned}$$

$$\begin{aligned}
T_2 &\le 6\mathbb{E}\left[\int_0^t |g_1^{\Delta t}(y^{\Delta t}(s)) - g_1(y^{\Delta t}(s))|^2 ds\right] \\
&\quad +6\mathbb{E}\left[\int_0^t |g_1(y^{\Delta t}(s)) - g_1(y(s))|^2 ds\right] \\
&\le 6\mathbb{E}\left[\int_0^t \phi(\Delta t) ds\right] + 6K\int_0^t \mathbb{E}\left[|y^{\Delta t}(s) - y(s)|^2\right] ds.
\end{aligned}$$

For the term T_3, we have

$$\begin{aligned}
T_3 &\le 6\mathbb{E}\left[\int_0^t |g_2^{\Delta t}(y^{\Delta t}(s)) - g_2(y^{\Delta t}(s))|^2 ds\right] \\
&\quad +6\mathbb{E}\left[\int_0^t |g_2(y^{\Delta t}(s)) - g_2(y(s))|^2 ds\right].
\end{aligned}$$

By using Lemma 3.1 and Assumption 3.1, we have

$$\begin{aligned}
T_3 &\le 6\mathbb{E}\Big[\int_0^t |g_2(y^{\Delta t}(s) + \Delta t f^{\Delta t}(y^{\Delta t}(s)) \\
&\qquad + \Delta\hat{W}_1(s) g_1^{\Delta t}(y^{\Delta t}(s))) - g_2(y^{\Delta t}(s))|^2 ds\Big] \\
&\quad +6K\int_0^t \mathbb{E}\left[|y^{\Delta t}(s) - y(s)|^2\right] ds \\
&\le 12K(\Delta t)^2 \int_0^t \mathbb{E}\left[|f^{\Delta t}(y^{\Delta t}(s))|^2\right] ds \\
&\quad +12K\int_0^t \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2 |g_1^{\Delta t}(y^{\Delta t}(s))|^2\right] ds + 6K\int_0^t \mathbb{E}\left[|y^{\Delta t}(s) - y(s)|^2\right] ds \\
&\le 12K(\Delta t)^2 \int_0^t (\alpha + \beta\mathbb{E}\left[|y^{\Delta t}(s)|^2\right]) ds \\
&\quad +12K\int_0^t \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2 (\alpha + \beta|y^{\Delta t}(s)|^2)\right] ds \\
&\quad +6K\int_0^t \mathbb{E}\left[|y^{\Delta t}(s) - y(s)|^2\right] ds \\
&\le 12K(\Delta t)^2 \int_0^t (\alpha + \beta\mathbb{E}\left[|y^{\Delta t}(s)|^2\right]) ds + 12K\alpha\Delta t\int_0^t ds \\
&\quad +12K\beta\int_0^t \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2 |y^{\Delta t}(s)|^2\right] ds + 6K\int_0^t \mathbb{E}\left[|y^{\Delta t}(s) - y(s)|^2\right] ds. \qquad (47)
\end{aligned}$$

From the Lemma 3.5, we know that the first two terms on the right-hand side of the above inequality (47) is bounded by $C_1\Delta t$ with a constant C_1 independent of $y(t)$, $y^{\Delta t}(t)$ and Δt. Let us rewrite the third term on the right-hand side of the inequality (47) as

$$\int_0^t \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2 |y^{\Delta t}(s)|^2\right] ds = T_4 + T_5 \qquad (48)$$

where

$$T_4 = \int_0^{\Delta t} \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2 |y^{\Delta t}(s)|^2\right] ds, \quad T_5 = \int_{\Delta t}^t \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2 |y^{\Delta t}(s)|^2\right] ds.$$

About the term T_4, we have the estimate

$$\begin{aligned} T_4 &\le 2\int_0^{\Delta t} \mathbb{E}\left[|\Delta \hat{W}_1(s)|^2 |y^{\Delta t}(s) - y^{\Delta t}(0)|^2\right] ds \\ &\quad +2\int_0^{\Delta t} \mathbb{E}\left[|\Delta \hat{W}_1(s)|^2 |y^{\Delta t}(0)|^2\right] ds \\ &\le \int_0^{\Delta t} \mathbb{E}\left[|\Delta \hat{W}_1(s)|^4\right] ds + \int_0^{\Delta t} \mathbb{E}\left[|y^{\Delta t}(s) - y^{\Delta t}(0)|^4\right] ds \\ &\quad +2\Delta t\int_0^{\Delta t} \mathbb{E}\left[|Y_0|^2\right] ds \\ &\le (\Delta t)^3 + \int_0^{\Delta t} \mathbb{E}\left[|y^{\Delta t}(s) - y^{\Delta t}(0)|^4\right] ds + 2(\Delta t)^2 |Y_0|^2, \end{aligned}$$

About the term T_5, we have the estimate

$$\begin{aligned} T_5 &\le \int_{\Delta t}^t \Big\{ \mathbb{E}\left[|\Delta \hat{W}_1(s)|^4\right] \\ &\qquad +\mathbb{E}\left[|y^{\Delta t}(s) - y^{\Delta t}(s-\Delta t)|^4\right] + 2\Delta t \mathbb{E}\left[|y^{\Delta t}(s-\Delta t)|^2\right] \Big\} ds \\ &\le T(\Delta t)^2 + \int_{\Delta t}^t \mathbb{E}\left[|y^{\Delta t}(s) - y^{\Delta t}(s-\Delta t)|^4\right] ds \\ &\quad +2\Delta t\int_{\Delta t}^t \mathbb{E}\left[|y^{\Delta t}(s-\Delta t)|^2\right] ds \end{aligned}$$

From the inequality (39), we deduce that there is a positive constant C_2 independent of $y(t)$, $y^{\Delta t}(t)$ and Δt such that

$$\int_0^{\Delta t} \mathbb{E}\left[|y^{\Delta t}(s) - y^{\Delta t}(0)|^4\right] ds \le C_2 \int_0^{\Delta t} |s|^2 ds \le C_2(\Delta t)^3, \tag{49}$$

$$\int_{\Delta t}^t \mathbb{E}\left[|y^{\Delta t}(s) - y^{\Delta t}(s-\Delta t)|^4\right] ds \le C_2 \int_{\Delta t}^t \Delta t^2 ds \le C_2 T(\Delta t)^2. \tag{50}$$

Together with Lemma 3.5, we have

$$\int_{\Delta t}^t \mathbb{E}\left[|y^{\Delta t}(s-\Delta t)|^2\right] ds \le C_3 \tag{51}$$

where the constant C_3 is independent of $y(t)$, $y^{\Delta t}(t)$ and Δt.

Now by the estimates of T_3, T_4, T_5 and the inequalities (48), (49), (50) and (51), we deduce

$$T_3 \le C_4\left(\Delta t + \int_0^t \mathbb{E}\left[|y^{\Delta t}(s) - y(s)|^2\right] ds\right), \tag{52}$$

where $C_4 > 0$ is a constant independent of $y(t)$, $y^{\Delta t}(t)$ and Δt. Inserting the estimates of T_1, T_2, T_3 into the inequality (46), we get

$$\mathbb{E}\left[|y^{\Delta t}(t) - y(t)|^2\right] \le C_5\left(\phi(\Delta t) + \Delta t + \int_0^t \mathbb{E}\left[|y^{\Delta t}(s) - y(s)|^2\right] ds\right) \tag{53}$$

with the constant C_5 independent of $y(t)$, $y^{\Delta t}(t)$ and Δt. Finally using Gronwall lemma again, we obtain

$$\mathbb{E}\left[|y^{\Delta t}(t) - y(t)|^2\right] \le C_6(\phi(\Delta t) + \Delta t) \tag{54}$$

for $t \in [0,T]$, where the constant C_6 is independent of $y(t)$, $y^{\Delta t}(t)$ and Δt. Combining (44) and (54), we complete the proof. □

Proposition 3.2. *Let $y^{\Delta t}(t)$ and $\bar{Y}(t)$ be the solutions of the equations (20) and (22), respectively. Then under the conditions of the Theorem 3.1, we have the estimate*

$$\mathbb{E}\left[|\bar{Y}(t) - y^{\Delta t}(t)|^2\right] \leq C\Delta t, \tag{55}$$

where the constant C does not depend on $y^{\Delta t}(t)$, $\bar{Y}(t)$, and the time partition.

Proof: Similar to the proof of Proposition 3.1, using the Cauchy-Schwartz inequality, Itô isometry formula, Assumption 3.1 and Lemma 3.5, we obtain

$$\begin{aligned}
&\mathbb{E}\left[|\bar{Y}(t) - y^{\Delta t}(t)|^2\right] \\
&= \mathbb{E}\Big[\Big|\int_0^t (f^{\Delta t}(Y(s)) - f^{\Delta t}(y^{\Delta t}(s)))ds \\
&\qquad + \int_0^t (g_1^{\Delta t}(Y(s)) - g_1^{\Delta t}(y^{\Delta t}(s)))dW_1(s) \\
&\qquad + \int_0^t (g_2^{\Delta t}(Y(s)) - g_2^{\Delta t}(y^{\Delta t}(s)))dW_2(s)\Big|^2\Big] \\
&\leq 3T\mathbb{E}\left[\int_0^t |f^{\Delta t}(Y(s)) - f^{\Delta t}(y^{\Delta t}(s))|^2 ds\right] \\
&\quad +3\mathbb{E}\left[\int_0^t |g_1^{\Delta t}(Y(s)) - g_1^{\Delta t}(y^{\Delta t}(s))|^2 ds\right] \\
&\quad +3\mathbb{E}\left[\int_0^t |g_2^{\Delta t}(Y(s)) - g_2^{\Delta t}(y^{\Delta t}(s))|^2 ds\right] \\
&\leq 3L(T+1)\int_0^t \mathbb{E}\left[|Y(s) - y^{\Delta t}(s)|^2\right] ds \\
&\quad +9K\int_0^t \mathbb{E}\left[|Y(s) - y^{\Delta t}(s)|^2\right] ds + 9KL(\Delta t)^2\int_0^t \mathbb{E}\left[|Y(s) - y^{\Delta t}(s)|^2\right] ds \\
&\quad +9KL\int_0^t \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2|Y(s) - y^{\Delta t}(s)|^2\right] ds \\
&= 3[L(T+1) + 3K(L\Delta t^2 + 1)]\int_0^t \mathbb{E}\left[|Y(s) - y^{\Delta t}(s)|^2\right] ds \\
&\quad +9KL\int_0^t \mathbb{E}\left[|\Delta\hat{W}_1|^2|Y(s) - y^{\Delta t}(s)|^2\right] ds.
\end{aligned} \tag{56}$$

About the last term on the right hand side of the inequality (56), we easily get

$$
\begin{aligned}
&\int_0^t \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2|Y(s)-y^{\Delta t}(s)|^2\right]ds \\
&\quad\le \int_0^{\Delta t} \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2|Y(s)-y^{\Delta t}(s)|^2\right]ds \\
&\qquad+\int_{\Delta t}^t \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2|Y(s)-y^{\Delta t}(s)|^2\right]ds \\
&\quad\le 2\int_0^{\Delta t} \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2|Y(s)-y^{\Delta t}(0)|^2\right]ds \\
&\qquad+2\int_0^{\Delta t} \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2|y^{\Delta t}(s)-y^{\Delta t}(0)|^2\right]ds \\
&\qquad+2\int_{\Delta t}^t \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2|Y(s)-y^{\Delta t}(s-\Delta t)|^2\right]ds \\
&\qquad+2\int_{\Delta t}^t \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2|y^{\Delta t}(s)-y^{\Delta t}(s-\Delta t)|^2\right]ds.
\end{aligned}
$$

and so we have

$$
\begin{aligned}
&\int_0^t \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2|Y(s)-y^{\Delta t}(s)|^2\right]ds \\
&\quad\le \int_0^{\Delta t} \mathbb{E}\left[|\Delta\hat{W}_1(s)|^4\right]ds+\int_0^{\Delta t} \mathbb{E}\left[|y^{\Delta t}(s)-y^{\Delta t}(0)|^4\right]ds \\
&\qquad+2\Delta t\int_{\Delta t}^t \mathbb{E}\left[|Y(s)-y^{\Delta t}(s-\Delta t)|^2\right]ds \\
&\qquad+\int_{\Delta t}^t \mathbb{E}\left[|\Delta\hat{W}_1(s)|^4\right]ds+\int_{\Delta t}^t \mathbb{E}\left[|y^{\Delta t}(s)-y^{\Delta t}(s-\Delta t)|^4\right]ds \\
&\quad\le T(\Delta t)^2+\int_0^{\Delta t} \mathbb{E}\left[|y^{\Delta t}(s)-y^{\Delta t}(0)|^4\right]ds \\
&\qquad+2\Delta t\int_{\Delta t}^t \mathbb{E}\left[|Y(s)-y^{\Delta t}(s-\Delta t)|^2\right]ds \\
&\qquad+\int_{\Delta t}^t \mathbb{E}\left[|y^{\Delta t}(s)-y^{\Delta t}(s-\Delta t)|^4\right]ds. \qquad (57)
\end{aligned}
$$

In the above estimation (57), we used the facts that $Y(s)=y^{\Delta t}(0)$ for $s\in[0,\Delta t)$ from the definitions of $Y(t)$ and $y^{\Delta t}(t)$, and the two random processes $\hat{W}_1(s)$ and $Y(s)-y^{\Delta t}(s-\Delta t)$ are independent. Now we estimate the third term on the right hand side of the inequality (57). From (39), we get

$$
\begin{aligned}
&\int_{\Delta t}^t \mathbb{E}\left[|Y(s)-y^{\Delta t}(s-\Delta t)|^2\right]ds \\
&\quad\le 2\int_{\Delta t}^t \mathbb{E}\left[|Y(s)-y^{\Delta t}(s)|^2\right]ds \\
&\qquad+2\int_{\Delta t}^t \mathbb{E}\left[|y^{\Delta t}(s)-y^{\Delta t}(s-\Delta t)|^2\right]ds \\
&\quad\le 2\int_0^t \mathbb{E}\left[|Y(s)-y^{\Delta t}(s)|^2\right]ds+2C_{2,T}T\Delta t \qquad (58)
\end{aligned}
$$

Inserting the estimates (49), (50) and (58) into the estimate (57), we get

$$\begin{aligned}\int_0^t \mathbb{E}\left[|\Delta\hat{W}_1(s)|^2|Y(s)-y^{\Delta t}(s)|^2\right] ds \\ \leq [T+C_2(T+\Delta t)+4C_{2,T}T](\Delta t)^2 \\ +4\Delta t\int_0^t \mathbb{E}\left[|Y(s)-y^{\Delta t}(s)|^2\right] ds.\end{aligned} \tag{59}$$

Combining the two estimates (56) and (59) leads to

$$\begin{aligned}\mathbb{E}\left[|\bar{Y}(t)-y^{\Delta t}(t)|^2\right] &\leq C_7(\Delta t)^2+C_8\int_0^t \mathbb{E}\left[|Y(s)-y^{\Delta t}(s)|^2\right] ds \\ &\leq C_7(\Delta t)^2+2C_8\int_0^t \mathbb{E}\left[|\bar{Y}(s)-Y(s)|^2\right] ds \\ &\quad +2C_8\int_0^t \mathbb{E}\left[|\bar{Y}(s)-y^{\Delta t}(s)|^2\right] ds,\end{aligned} \tag{60}$$

where

$$\begin{aligned}C_7 &= 9KL\,[T+C_2(T+\Delta t)+4C_{2,T}T] \\ C_8 &= 3\left[L(T+1)+3K(L\Delta t^2+1)+12KL\Delta t\right].\end{aligned}$$

Now let us estimate the third term on the right-hand side of the inequality (60). For each $s\in[0,T)$, let k_s be the integer such that $s\in[t_{k_s},t_{k_s+1})$. Then by using the Lemma 3.3 and the estimates (28), (29) and (30) for the vector functions $f^{\Delta t}$, $g_1^{\Delta t}$ and $g_2^{\Delta t}$, we obtain

$$\begin{aligned}&\mathbb{E}\left[|\bar{Y}(s)-Y(s)|^2\right] \\ &\quad = \mathbb{E}\left[|f^{\Delta t}(Y_{k_s})(s-t_{k_s})-g_1^{\Delta t}(Y_{k_s})(W_1(s)-W_1(t_{k_s}))\right. \\ &\qquad \left.-g_2^{\Delta t}(Y_{k_s})(W_2(s)-W_2(t_{k_s}))|^2\right] \\ &\quad \leq 3\mathbb{E}\left[|f^{\Delta t}(Y_{k_s})|^2|s-t_{k_s}|^2\right]+3\mathbb{E}\left[|g_1^{\Delta t}(Y_{k_s})|^2|W_1(s)-W_1(t_{k_s})|^2\right] \\ &\qquad +3\mathbb{E}\left[|g_2^{\Delta t}(Y_{k_s})|^2|W_2(s)-W_2(t_{k_s})|^2\right] \\ &\quad \leq 3(\Delta t)^2\mathbb{E}\left[|f^{\Delta t}(Y_{k_s})|^2\right]+3\Delta t\mathbb{E}\left[|g_1^{\Delta t}(Y_{k_s})|^2\right]+3\Delta t\mathbb{E}\left[|g_2^{\Delta t}(Y_{k_s})|^2\right] \\ &\quad \leq C_9\Delta t\end{aligned} \tag{61}$$

where the constant C_9 depends on the constant A in the Lemma 3.3, and the constants α and β in the inequalities (28) and (29). From the estimate (61), we easily get

$$\int_0^t \mathbb{E}\left[|\bar{Y}(s)-Y(s)|^2\right] ds \leq C_9T\Delta t. \tag{62}$$

With the use of the estimates (60) and (62) we obtain

$$\mathbb{E}\left[|\bar{Y}(t)-y^{\Delta t}(t)|^2\right] \leq C_{10}\Delta t+C_{11}\int_0^t \mathbb{E}\left[|\bar{Y}(s)-y^{\Delta t}(s)|^2\right] ds, \tag{63}$$

where $C_{10}=C_7\Delta t+2C_8C_9T$ and $C_{11}=2C_8$. Finally the use of Gronwall lemma to the inequality (63) leads the conclusion of this proposition. □

Now let us turn to the proof of the Theorem 3.1.

Proof: (**Proof of Theorem 3.1**) Noting the fact

$$\bar{Y}(t)-y(t)=[\bar{Y}(t)-y^{\Delta t}(t)]+[y^{\Delta t}(t))-y(t)],$$

Table 1: The average sample errors of the E-M method and the splitting method for the linear problem (64).

N	2^5	2^6	2^7	2^8	2^9
$E-M$	0.0488	0.0284	0.0174	0.0120	0.0081
$Splitting$	0.0206	0.0146	0.0093	0.0069	0.0045

and using the triangle inequality, Proposition 3.1 and Proposition 3.2, we obtain the estimate (23). □

4. Numerical Experiments

To see how well the splitting scheme (9) works, we compare its performance with the standard E-M scheme (5) by solving two sample stochastic differential equations.

The first example that we are going to test is the following linear SDEs:

$$\begin{cases} dy(t) = y(t)\left[\lambda + \sigma dW(t)\right], & 0 < t \leq T \\ y(0) = y_0, \end{cases} \tag{64}$$

where λ is a parameter, $\sigma = (\sigma_1, \cdots, \sigma_d)$ is a vector, and $W(t)$ is a standard d-dimensional Brownian motion. It is easy to find that the exact solution of the equations (64) is given by

$$y(t) = y_0 \exp\left(((\lambda - \tfrac{1}{2}\sigma\sigma^T)t + \sigma W(t)\right). \tag{65}$$

The second example to test is the following nonlinear SDEs:

$$\begin{cases} dy(t) = [(2\lambda + \sigma\sigma^T)y(t) - 2(\lambda + \sigma\sigma^T)\sqrt{y(t)+1} + 2(\lambda + \sigma\sigma^T)]dt \\ \qquad\qquad +2(y(t) + 1 - \sqrt{y(t)+1})\sigma dW(t), \qquad 0 < t \leq T \\ y(0) = y_0 \end{cases} \tag{66}$$

where λ, σ and $W(t)$ are defined as before. It is also easy to verify that the analytic solution of the equations (66) is given by

$$\begin{aligned} y(t) = {} & (y_0 - 2\sqrt{y_0+1} + 2)\exp((2\lambda - \sigma\sigma^T)t + 2\sigma W(t)) \\ & +2(\sqrt{y_0+1} - 1)\exp((\lambda - \tfrac{1}{2}\sigma\sigma^T)t + \sigma W(t)). \end{aligned} \tag{67}$$

In our numerical simulations, we set $T = 1$, $\lambda = -2$, and $\sigma = (1, 1, 1)$ for both examples. To compute the expectations of the approximate solution Y_n, we repeat the solution process 1000 times for each method so that the errors caused by sampling is much smaller than that by time discretization.

The initial condition for the linear SDE (64) was set to be $y_0 = 1$. In Figure 1, we plot the analytic solution of the SDEs (64), the approximate solutions of the linear problem (64) by the splitting scheme (9) and the E-M scheme (6) respectively with the time step $\Delta t = \frac{1}{2^5}$. In order to clearly demonstrate the convergence rate of the splitting method, we also present the average sample errors at the terminal time T (i.e., $[|y(T) - Y_N|]$) for the E-M scheme (6) and the splitting scheme (9) with Δt $=\frac{1}{2^5}$, $\frac{1}{2^6}$, $\frac{1}{2^7}$, $\frac{1}{2^8}$ and $\frac{1}{2^9}$ in Table 1, and plot the corresponding convergence curve for each method in Figure 2. The initial condition for the nonlinear SDE (66) was set to be $y_0 = 3$. We then repeated same simulations for this nonlinear problem as the first example and corresponding figures and table are given in Figure 3, Table 2 and Figure 4.

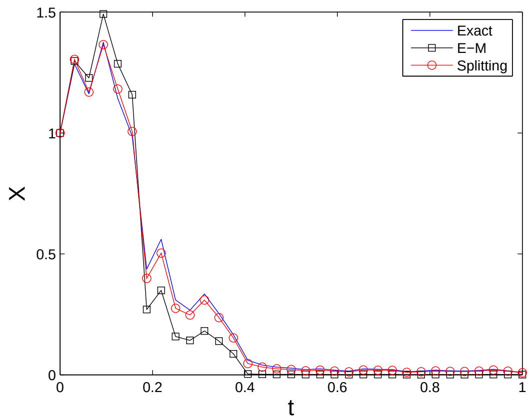

Figure 1: The plots of the exact solution (65), the approximate solutions for the linear problem (64) by the E-M method and the splitting method respectively with $\Delta t = \frac{1}{2^5}$.

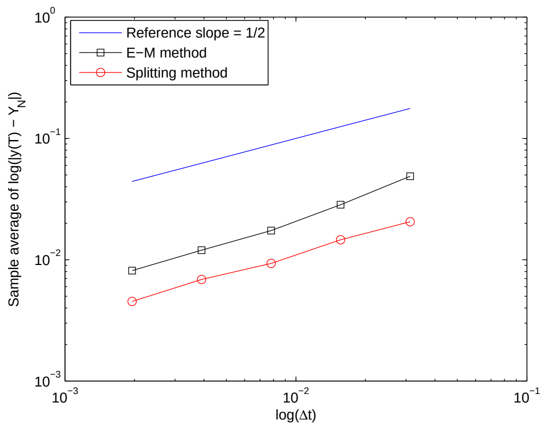

Figure 2: The convergence rates of the E-M method and the splitting method for the linear problem (64).

Table 2: The average sample errors of the E-M method and the splitting method for the nonlinear problem (65).

N	2^5	2^6	2^7	2^8	2^9
$E-M$	0.3837	0.1730	0.0989	0.0703	0.0461
$Splitting$	0.1289	0.0898	0.0517	0.0388	0.0229

From the above figures and tables, it is easy to see that both the E-M scheme and the splitting scheme have the half order convergence, but the splitting scheme obtains better approximate solutions for both problems.

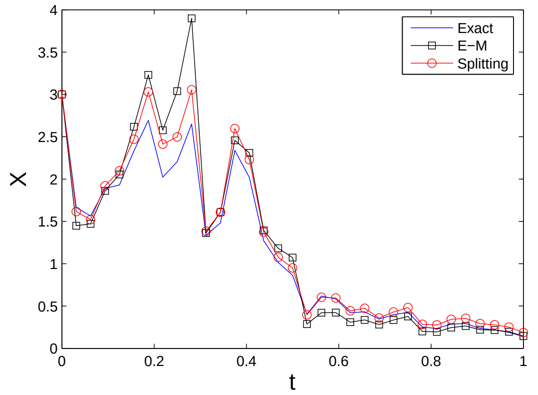

Figure 3: The plots of the exact solution (67), the approximate solutions for the nonlinear problem (65) by the E-M method and the splitting method respectively with $\Delta t = \frac{1}{2^5}$.

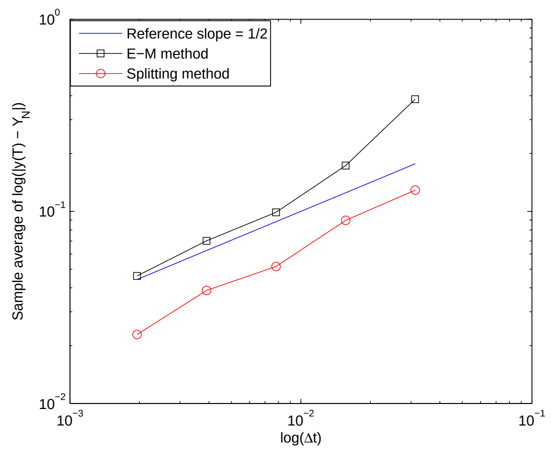

Figure 4: The convergence rates of the E-M method and the splitting method for the nonlinear problem (65).

5. Conclusions

In this paper, we proposed and discussed a fully drift-implicit splitting method for numerical solution of the stochastic differential equations driven by the d-dimensional Brownian motion. We proved its strong convergence to be of half order and compared its performance with the standard Euler-Maruyama method. Although the strong convergence rate of our splitting method is of the same order as that of the E-M method, this scheme allows us to use the latest information inside each iteration in the E-M method and makes it possible to obtain better approximate solutions than the standard approach.

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