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FINITE ELEMENT APPROXIMATION FOR TV REGULARIZATION

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Abstract. In this paper, we will develop the convergence of the solution of TV-regularization equations with regularized parameter $\varepsilon \longrightarrow 0$ in $BV(\Omega)$ for practical purposes. Originated from the effects of regularized parameter ε, the error rate of finite element approximation for TV-regularization equations will be controlled by the regularized parameter ε^{-1} polynomially in the energy norm when using linearization technique and duality argument. And in the L^p-norm, the effect of regularized parameter ε will be more extremely.

Key Words. TV-regularization, Regularized Parameter, Finite Element Method

1. Introduction

We consider the following total variation(TV) regularization equations

$$div\big(\frac{\nabla u^\varepsilon}{\sqrt{|\nabla u^\varepsilon|^2+\varepsilon^2}}\big)-\lambda(u^\varepsilon-g)=0, \quad in\ \Omega, \tag{1}$$

$$\frac{\partial u^\varepsilon}{\partial n}=0, \quad on\ \partial\Omega. \tag{2}$$

Equation (1) is an Euler-Lagrange equation originated from the following unconstraint minimization problem

$$\min_{u^\varepsilon} J_{\lambda,\varepsilon}(u^\varepsilon)=\min_{u^\varepsilon}\{\int_\Omega \sqrt{|\nabla u^\varepsilon|^2+\varepsilon^2}dx+\frac{\lambda}{2}\int_\Omega |u^\varepsilon-g|^2dx\}. \tag{3}$$

Usually, equations (1)-(2) are the numerical regularized approximation of the following equations, respectively

$$div\big(\frac{\nabla u}{|\nabla u|}\big)-\lambda(u-g)=0, \quad in\ \Omega, \tag{4}$$

$$\frac{\partial u}{\partial n}=0, \quad on\ \partial\Omega. \tag{5}$$

which corresponds to an unconstrained minimization problem

$$\min_u J_\lambda(u)=\min_u\{\int_\Omega |\nabla u|dx+\frac{\lambda}{2}\int_\Omega |u-g|^2dx\}. \tag{6}$$

where, especially in image processing, $\lambda>0$ is the penalization parameter which controls the trade-off between goodness of fit-to the data and variability in u, $u:\Omega\subset\mathcal{R}^2\longrightarrow\mathcal{R}$ denote the gray level of an image describing a real scene, and g be the observed image of the same scene, which is a degradation of u. And (6) is usually called the total variation (TV) model or ROF model duo to Rudin, Osher

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and Fatemi [20]. It is one of the best known and most successful noise removal and image restoration model, too.

(1)-(2) can be taken as nonlinear elliptic problem and we will consider finite element method approximation in this paper. The common nonlinear elliptic problem, with Dirichlet boundary conditions , Neumann boundary conditions or mixed boundary conditions, have been studied theoretically and numerically in the past thirty years, see [24, 19, 18, 12, 4, 11, 10, 2, 13, 14]. In [2], the authors proved the existence and uniqueness of the solution of nonlinear elliptic equations of monotone type and also derived error estimates for the finite element approximation in the energy norm as well as assumption that $u \in W_p^1, p > 1$. In [10], they presented Galerkin approximations of a quasi-linear non-potential elliptic problem of a non-monotone type. For the $u \in H^1(\Omega)$, u_h converges to u weakly and for the $u \in W_p^1(\Omega), p > 2$, u_h converges to u strongly in the H^1-norm. In, [11], the existence and uniqueness of the finite element solution of quasi-linear elliptic equations with mixed Dirichlet-Neumann conditions are derived by developing a one-parameter family of $hp-$version discontinuous Galerkin finite element methods in the divergence form on a bounded open set Ω. If $\lambda = 0$, (4)-(5) or (1)(2) can be regards as the mean curvature problems. In [12, 7], numerical approximation for the mean curvature was also set up on finite element error estimations and adaptive algorithm, respectively.

Interestingly, responding to (6), it is usually solved by formulating the steepest descent gradient method, which motivates to consider its gradient flow as well as its numerical form like (1)(2). In [5, 6], they considered the relations between $u(t)$ and $u^\varepsilon(t)$, proved that $u^\varepsilon(t)$ converged to $u(t)$ in $L^1((0,T); BV(\Omega)) \bigcap C^0([0,T], L^2(\Omega))$. Dramatically, the convergence rate of the finite element approximation is depend on the parameter ε by the form $C(\frac{1}{\varepsilon})$. It is important for such a result when we deal with the similar numerical problems because we have to select a proper mesh size to keep the convergence by finite element method or finite difference method.

Numerically, some works have pointed out that the chose of regularized parameter ε is vital in image processing, see [3, 22, 23]. The selection of an appropriate regularized parameter has been one of difficulties in image processing. Some others have pointed out that ε will influence the convergence rate of level set function, for example, in inverse problems, [16, 15, 17]. Therefore, one of the aims of this paper is to construct and analyze a finite element method for approximating the solution of equation (1)-(2) for each $\varepsilon > 0$ and approximating the solution of equation (4)-(5) by taking $\varepsilon \longrightarrow 0$.

Based on the above discussing, in this paper, our presentation follows the frameworks established in [10, 11, 5, 6] in order to develop the convergence relation of u, u^ε in the space $BV(\Omega)$, and error convergence rate of finite element approximation for u^ε. And we also try to demonstrate how ε affects the convergence rate of finite element approximation u^ε.

This paper is organized in the following way. In section §2, we prove that the solution u^ε of problem (1)-(2) will converge to the solution u of problem (4)-(5) in $BV(\Omega)$ space when the regularized parameter $\varepsilon \longrightarrow 0$. In section §3, by introducing the linearization of the nonlinear problem, we give coercion and duality operator of the linearization operator, which are the foundation of studying the nonlinear elliptic partial differential equation for finite element methods. In section §4, firstly, we introduce an operator T which is contract proved by Lemma 3. Then, based on fixed point theorem, we prove that the fixed point of operator T is the solution of finite element approximating for the variation of problem (1)-(2) in the energy norm. Also, the convergence rate will be controlled by the regularization parameter ε^{-1}. In the second subsection, we give the L^p-norm estimation for the finite element method by the duality argument on the linearization operator. And the convergence rate depends on the the regularized parameter ε^{-4} polynomially. In section §5, we conclude that we should pay more attentions to the chose of regularized parameter to be used in the practices.

2. TV Convergence

In this section, we will present a proof that the solution u^ε of (1)-(2) will converge to the solution u of (4)-(5) when $\varepsilon \longrightarrow 0$.

Before our discussion, we need recall that a function $u \in L^1(\Omega)$ is called of bounded variational if all of its first order partial derivatives are measures with finite total variations in Ω. Hence, the gradient of such a function u, still denoted by ∇u, is a bounded vector-valued measure, with the finite total variation

$$\|\nabla u\| = \int_\Omega |\nabla u| dx := \sup\{\int_\Omega u div\ \mathbf{v} dx; \mathbf{v} \in [C_0^1(\Omega)]^2, \|\mathbf{v}\|_{L^\infty} \leq 1\}.$$

The space of functions of bounded variation is denoted by $BV(\Omega)$, endowed with norm

$$\|u\|_{BV} := \|u\|_{L^1} + \|\nabla u\|.$$

and we also refer to [8][9] for definitions of standard space notations.

Also, we want to give some notations of Sobleve spaces and norms used in this paper. Let $W^{k,p}(\Omega)$ be standard Sobolev space

$$W^{k,p}(\Omega) = \{f : \|f\|_{W^{k,p}} < \infty\},$$

where

$$\|f\|_{W^{k,p}} = (\sum_{|\alpha| \leq k} \|D^\alpha f\|^p_{L^p})^{\frac{1}{p}}, k = 0, 1, \cdots.$$

Also denote the dual space of $W^{k,p}(\Omega)$ by $W^{-k,p}(\Omega)$ with norm

$$\|f\|_{W^{-k,p}} = \sup_{\psi \in W^{k,p'}(\Omega), \|\psi\| \neq 0} \frac{|(f, \psi)|}{\|\phi\|_{W^{k,p}}}.$$

where p' is the dual number of p such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 1. *For any $\varepsilon > 0$, the solution sequence $\{u^\varepsilon\}$ of problem (1) (2) satisfy*

$$u^\varepsilon \longrightarrow u, \quad in \ \ BV(\Omega) \bigcap L^2(\Omega). \tag{7}$$

where u satisfies (4) (5).

Proof: For any $\varepsilon > 0$, test (1) by u^ε, we can get

$$\int_\Omega \frac{(\nabla u^\varepsilon)^2}{\sqrt{|\nabla u^\varepsilon|^2 + \varepsilon^2}} + \frac{\lambda}{2}\|u^\varepsilon\|^2_{L^2} \leq \frac{\lambda}{2}\|g\|^2_{L^2}.$$

which means

$$\|\nabla u^\varepsilon\| + C(\lambda, |\Omega|)\|u^\varepsilon\|_{L^1(\Omega)} \leq \sqrt{\frac{\lambda}{2}}\|g\|_{L^2}. \tag{8}$$

Then there exists a function $u \in BV(\Omega) \bigcap L^2(\Omega)$ and a subsequence $\{u^\varepsilon\}$ (denotes the same notation) such that as $\varepsilon \longrightarrow 0$

$$u^\varepsilon \longrightarrow u, \quad in \ BV(\Omega) \bigcap L^2(\Omega). \tag{9}$$

Now we can proof that u is the weak solution of (4) (5). Let v is a test function and $g(v) = \int_\Omega \lambda g v dx$, then

$$\begin{aligned}
&|\int_\Omega \frac{\nabla u \cdot \nabla v}{|\nabla u|} dxdy + \int_\Omega \lambda u v dxdy - g(v)| \\
&\leq |\int_\Omega (\frac{\nabla u}{|\nabla u|} - \frac{\nabla u^\varepsilon}{\sqrt{|\nabla u^\varepsilon|^2 + \varepsilon^2}}) \cdot \nabla v dxdy| + |\int_\Omega \lambda(u - u^\varepsilon) v dxdy| \\
&= |\int_\Omega (\frac{\nabla u(\nabla u^\varepsilon + \nabla u)(\nabla u^\varepsilon - \nabla u) + \varepsilon^2 \nabla u}{(\sqrt{|\nabla u^\varepsilon|^2 + \varepsilon^2} + |\nabla u|)\sqrt{|\nabla u^\varepsilon|^2 + \varepsilon^2}|\nabla u|} \\
&+ \frac{|\nabla u|(\nabla u - \nabla u^\varepsilon)}{\sqrt{|\nabla u^\varepsilon|^2 + \varepsilon^2}|\nabla u|}) \cdot \nabla v dxdy| + |\int_\Omega \lambda(u - u^\varepsilon) v dxdy|
\end{aligned} \tag{10}$$

Let

$$\gamma = \sup_{\xi \in (L^2(\Omega))^2} \frac{|\xi|}{\sqrt{|\xi|^2 + \varepsilon^2}},$$

then

$$\begin{aligned}
&|\int_\Omega \frac{\nabla u \cdot \nabla v}{|\nabla u|} dxdy + \int_\Omega \lambda u v dxdy - g(v)| \\
&\leq \int_\Omega \gamma^2 \frac{|\nabla u - \nabla u^\varepsilon|}{|\nabla u|} + \gamma \frac{\varepsilon^2}{|\nabla u|(\sqrt{|\nabla u^\varepsilon|^2 + \varepsilon^2} + |\nabla u|)} \\
&+ \gamma \frac{|\nabla u^\varepsilon - \nabla u|}{|\nabla u|} + |\int_\Omega \lambda(u - u^\varepsilon) v dxdy|
\end{aligned}$$

Altogether with (9) and $\varepsilon \longrightarrow 0$ we can get

$$\int_\Omega \frac{\nabla u \cdot \nabla v}{|\nabla u|} dxdy + \int_\Omega \lambda(u - g) v dxdy = 0, \quad \forall v \in BV(\Omega) \bigcap L^2(\Omega).$$

We note that the above argumentation can be applied to any convergent subsequence in $W_1^1(\Omega)$. And correspondingly, this argumentation can be applied to any convergent subsequence in $BV(\Omega)$ by density argument. Therefore, Theorem 1 follows.

Remark: *Theorem 1 shows that the regularization equation (1) is an reasonable numerical approximation for the nonlinear equation (4) in the $BV(\Omega)$ space. Since in image processing and inverse problems using level set methods, many theories are set up in the $BV(\Omega)$ space, Theorem 1 gives the explanation why we can study the regularization equations directly.*

3. Linearization of the Nonlinear Differential Operator

We begin this section by studying the linearization of the nonlinear differential operator resulted from equation (1). The property of this linearization will play an important role for the error analysis in the following formulations.

Let $M_\varepsilon : W^{1,2}(\Omega) \longrightarrow W^{-1,2}(\Omega)$ denote the differential operator,

$$M_\varepsilon(u) = -div\big(\frac{\nabla u^\varepsilon}{\sqrt{|\nabla u^\varepsilon|^2 + \varepsilon^2}}\big) + \lambda u^\varepsilon. \tag{11}$$

The linearization of M_ε at the solution of (1)-(2) is defined as

$$L_{u^\varepsilon}(\phi) = -div\big(D^2 f_\varepsilon(\nabla u^\varepsilon)\nabla\phi\big) + \lambda\phi \tag{12}$$

where $f_\varepsilon(x)=\sqrt{|x|^2+\varepsilon^2}$ and $D^2 f_\varepsilon(x)$ denote the Hession of $f_\varepsilon(x)$ with respect to x, that is

$$D^2 f_\varepsilon(x)=\frac{(|x|^2+\varepsilon^2)I-x^t x}{(\sqrt{|x|^2+\varepsilon^2})^3},\quad \forall x\in \mathbf{R}^2. \tag{13}$$

where x^t denotes the transpose of x.

Since

$$D^2 f(x)\xi\cdot\xi=\frac{\varepsilon^2|\xi|^2+(|\xi|^2|x|^2-|x\cdot\xi|^2)}{(\sqrt{|x|^2+\varepsilon^2})^3}\geq\frac{\varepsilon^2|\xi|^2}{(\sqrt{|x|^2+\varepsilon^2})^3},\quad \forall\ \xi\in\mathbf{R}^2, \tag{14}$$

then L_{u^ε} is elliptic for $\varepsilon\geq 0$ and uniform elliptic for $\varepsilon>0$. So is $L^*_{u^\varepsilon}$, the adjoint operator of L_{u^ε} with respect to L^2-inner product, that is

$$L^*_{u^\varepsilon}(\phi):=-div(D^2 f_\varepsilon(\nabla u^\varepsilon)\nabla\phi)+\lambda\phi. \tag{15}$$

It is easy to check that there holds the following inequality

$$(L_{u^\varepsilon}(\phi),\phi)\geq c_0\varepsilon^2\|\nabla\phi\|^2_{L^2}+\lambda\|\phi\|^2_{L^2},\quad \forall\ \phi\in V. \tag{16}$$

where

$$V=\{v\in W^{1,2}(\Omega),\int_\Omega v dx=0\},$$

and some positive constant c_0 independent of ε.

4. Finite Element Approximation

4.1. Formulations, and Error Estimations. Before discussing the finite element method, we give the variational formulation for (1)-(2): Seek $u^\varepsilon\in V$ such that

$$\int_\Omega\Big(\frac{\nabla u^\varepsilon\cdot\nabla v}{\sqrt{|\nabla u^\varepsilon|^2+\varepsilon^2}}+\lambda u^\varepsilon v\Big)dx=\int_\Omega\lambda g v dx,\quad \forall v\in V. \tag{17}$$

The polygonal nature of $(\bar\Omega)$ enables us to establish a regular family of partition $\{\mathcal{T}^h\}$, that is parameterized by $h\in(0,1)$. Thus $\bar\Omega=\bigcup_{e\in\mathcal{T}^h} e$. With partition $\mathcal{T}^h$ we may construct a conforming finite element space $V_h\subset V$ such that each element is affine equivalent to some reference element which does not depend on the parameter h. For examples, if the use of linear element is desired, we set

$$V_h=\{v_h\in V, v_h\big|_e\in P_1(e),\quad \forall e\in\mathcal{T}^h\}$$

where $P_1(e)$ is the space of all polynomials of degree one on e.

Denote $I_h : C^0(\bar\Omega)\longrightarrow V_h$ the standard Lagrangian interpolation operator. Based on the variational equation (17), finite element method approximating for (17) is defined as seeking $u_h^\varepsilon\in V_h$ such that

$$\int_\Omega\frac{\nabla u_h^\varepsilon\cdot\nabla v}{\sqrt{|\nabla u_h^\varepsilon|^2+\varepsilon^2}}+\lambda(u_h^\varepsilon-g)v dx=0,\quad \forall v\in V_h. \tag{18}$$

Lemma 1. *Let $\phi\in W^{2,p}(\Omega)$ denote the unique solution of the following equations*

$$L_{I_h u^\varepsilon}(\phi)=g,\quad in\ \Omega, \tag{19}$$

$$\frac{\partial\phi}{\partial n}=0,\qquad in\ \partial\Omega. \tag{20}$$

Then, there exists a unique solution $\phi_h\in V_h$ to the problem

$$(L_{I_h u^\varepsilon}(\phi_h),v_h)=(g,v_h),\quad \forall v_h\in V_h. \tag{21}$$

Moreover,

$$\|\phi - \phi_h\|_{L^2} + h\|\nabla(\phi - \phi_h)\|_{L^2} \le C_1 h^2 \|\phi\|_{W^{2,2}}. \tag{22}$$

$$\|\phi - \phi_h\|_{W^{1,p}} \le Ch(\|g\|_{W^{-1,p}} + \|\phi\|_{W^{2,p}}), \quad \forall 1 < p \le \infty. \tag{23}$$

Proof: The existence and uniqueness, as well as estimate (22) immediately from (14)-(16), and an application of [1], whose proof is known as the Schatz argument [21]. And (23) is a general result of [21].

In order to demonstrate the property of the solution u_h^ε of problem (18) approximating to the solution u^ε of problem (1)-(2), and also to show the effects of the regularized parameter ε, we need a new technique different from the traditional method. For a given $w_h \in V_h$, we define $T(w_h) \in V_h$ by

$$(L_{I_h u^\varepsilon}(w_h - T(w_h)), \psi_h) = (\frac{\nabla w_h}{\sqrt{|\nabla w_h|^2 + \varepsilon^2}}, \nabla\psi_h) + (\lambda(w_h - g), \psi_h), \forall \psi_h \in V_h. \tag{24}$$

Clearly, T is a mapping from V_h into itself, and from Lemma 1 we can make a conclusion that $T(w_h)$ is well defined and one-to-one.

Also, it is easy to find that the right-hand side of (24) is the left-hand side of (18) with (w_h, ψ_h) in the position of (u_h^ε, v), and it is obvious to see that any fixed point u_h of the mapping $T(i.e., u_h = T(u_h))$ is the solution of problems (18). In the following we will show that the mapping T has a unique fixed point, hence (18) has a unique solution, in a small neighborhood of $I_h u^\varepsilon$. To this end, we need to define such a neighborhood

$$B_h(\rho) = B(I_h u^\varepsilon, \rho) := \{v_h \in V_h : \|v_h - I_h u^\varepsilon\|_{W^{1,p}} \le \rho\}.$$

Then, we have

Lemma 2. *There exists a positive constant $C_2 = C_2(\varepsilon)$ and a sufficiently small number h_1 such that $h \le h_1$, there holds*

$$\|I_h u^\varepsilon - T(I_h u^\varepsilon)\|_{W^{1,p}} \le C_2 h \|u^\varepsilon\|_{W^{1,p}}, \quad \forall 1 < p \le \infty. \tag{25}$$

Proof: Based on the definition of the $T(I_h u^\varepsilon)$ we know that $I_h u^\varepsilon - T(I_h u^\varepsilon)$ is the unique solution of (21) with zero value of Neumann boundary and

$$g = -\frac{1}{1+\lambda} div \frac{\nabla I_h u^\varepsilon}{\sqrt{|I_h u^\varepsilon|^2 + \varepsilon^2}} + \frac{\lambda}{1+\lambda} I_h u^\varepsilon.$$

And by the regularization of the solution of problem (19), it is easy to establish

$$\|I_h u^\varepsilon - T(I_h u^\varepsilon)\|_{W^{1,p}} \le C_3 \|g\|_{W^{-1,p}}. \tag{26}$$

Now, let $\eta_h^\varepsilon = I_h u^\varepsilon - u^\varepsilon$ and using (17) we can then get

$$\begin{aligned}(g, \psi_h) &= (\frac{1}{1+\lambda} \frac{\nabla I_h u^\varepsilon}{\sqrt{|I_h u^\varepsilon|^2 + \varepsilon^2}}, \nabla\psi_h) + \frac{\lambda}{1+\lambda}(I_h u^\varepsilon, \psi_h) \\ &= \frac{1}{1+\lambda}(\frac{\nabla I_h u^\varepsilon}{\sqrt{|I_h u^\varepsilon|^2 + \varepsilon^2}} - \frac{\nabla u^\varepsilon}{\sqrt{|u^\varepsilon|^2 + \varepsilon^2}}, \nabla\psi_h) \\ &\quad + \frac{\lambda}{1+\lambda}(I_h u^\varepsilon - u^\varepsilon, \psi_h) + \frac{\lambda}{1+\lambda}(g, \psi_h).\end{aligned}$$

Therefore,

$$(g,\psi_h) = (\frac{\nabla I_h u^\varepsilon}{\sqrt{|I_h u^\varepsilon|^2+\varepsilon^2}} - \frac{\nabla u^\varepsilon}{\sqrt{|u^\varepsilon|^2+\varepsilon^2}}, \nabla\psi_h) + \lambda(I_h u^\varepsilon - u^\varepsilon, \psi_h).$$
$$= (A_h^\varepsilon \nabla\eta_h^\varepsilon, \nabla\psi_h)) + \lambda(I_h u^\varepsilon - u^\varepsilon, \psi_h).$$

where

$$A_h^\varepsilon := \int_0^1 D^2 f_\varepsilon(\nabla u^\varepsilon + t\nabla(I_h u\varepsilon - u^\varepsilon))dt.$$

Since

$$|A_h^\varepsilon| \le 2\varepsilon^{-1},$$

a density augment for $V_h \subset H^1$ yields that there exists a sufficiently small number $h_1 > 0$ such that for $h \le h_1$

(27)

$$\|g\|_{W^{-1,p}} \le 2 \sup_{\psi_h \in W^{1,p'}(\Omega)\bigcap V_h, \|\psi_h\|_{W^{1,p'}} \le 1} |(g,\psi_h)| \le 6\varepsilon^{-1}\|\eta_h^\varepsilon\|_{L^p} \le C\varepsilon^{-1} h |u|_{W^{2,p}}.$$

where $p^{'}$ is the dual number of p such that $\frac{1}{p} + \frac{1}{p'} = 1$. The proof is completed after setting $C_2 = C_3 C\varepsilon^{-1}$.

Next lemma establishes the contracting property of the mapping T.

Lemma 3. *Let h_1 be the same as in Lemma 2, then for $h \le h_1$, let $\rho_0 = \frac{1}{18} C_2^{-1} h^{\frac{2}{p}} \varepsilon^2$, the mapping T is a contracting mapping in the ball $B_h(\rho_0)$ with the contracting factor $\frac{1}{2}$, that is, for any $v_h, w_h \in B_h(\rho_0)$*

$$\|T(v_h) - T(w_h)\|_{W^{1,p}} \le \frac{1}{2}\|v_h - w_h\|_{W^{1,p}}, \quad \forall p \in (2,\infty]. \tag{28}$$

Proof: For any $v_h, w_h \in B_h(\rho_0)$, let $\eta_h = v_h - w_h$, subtracting the two copies of equation (24) which define $T(v_h)$ and $T(w_h)$ and using the Mean Value Theorem of integration yield that for any $\psi_h \in V_h$

$$(L_{I_h u^\varepsilon}(T(v_h) - T(w_h)), \psi_h) = ([D^2 f_\varepsilon(I_h u^\varepsilon) - A_h^\varepsilon]\nabla\xi_h, \nabla\psi_h) + \lambda(\xi_h, \psi_h), \tag{29}$$

where

$$A_h^\varepsilon = \int_0^1 D^2 f_\varepsilon(\nabla w_h + t\nabla(v_h - w_h))dt.$$

And here we have abused the notation A_h^ε to denote the different expression in the different proof.

For $h \le h_0$, (29) implies that $T(v_h) - T(w_h)$ is the solution to (21) with zero value of Neumann boundary and

$$g = -div \cdot ([D^2 f_\varepsilon(\nabla I_h u^\varepsilon) - A_h^\varepsilon]\nabla\eta_h) + \lambda\eta_h.$$

Then, by the regularization of the solution to (21), it follows that

$$\|T(v_h) - T(w_h)\|_1 \le C_2\|g\|_{-1}. \tag{30}$$

From

$$\frac{\partial(D^2(f_\varepsilon(x)))_{ij}}{\partial x_k} = -\frac{x_k\delta_{ij} + x_j\delta_{ik} + x_i\delta_{jk}}{(\sqrt{|x|^2+\varepsilon^2})^3} + \frac{3x_i x_j x_k}{(\sqrt{|x|^2+\varepsilon^2})^4} \tag{31}$$

and the Mean Value Theorem we get

$$|D^2 f(I_h u^\varepsilon) - A_h^\varepsilon| \le 9\varepsilon^{-2}(|\nabla(I_h u^\varepsilon - v_h)| + |\nabla(I_h u^\varepsilon - w_h)|)$$

which together with the Schwarz inequality and an inverse inequality imply that

(32)
$$\begin{aligned}|(g,\psi_h)| &\le 9\varepsilon^{-2}(\|\nabla(I_hu^\varepsilon - v_h)\|_{L^\infty} + \|\nabla(I_hu^\varepsilon - w_h)\|_{L^\infty})\|\nabla\eta_h\|_{L^p}\|\nabla\psi_h\|_{L^{p'}}\\ &\quad + \lambda\|\eta_h\|_{L^p}\|\psi_h\|_{L^{p'}}.\\ &\le 9\varepsilon^{-2}\rho_0 h^{-\frac{2}{p}}\|\eta_h\|_{W^{1,p}}\|\psi_h\|_{L^{p'}}\end{aligned}$$

Hence, for $h \le h_1$

$$\|g\|_{W^{-1,p}} \le 9\varepsilon^{-2}\rho_0 h^{-\frac{2}{p}}\|\eta_h\|_{W^{1,p}}. \tag{33}$$

It follows from (30)(33), and the definition of ρ_0 that

$$\|T(v_h) - T(w_h)\|_{W^{1,p}} \le 9C_2\varepsilon^{-2}\rho_0 h^{-\frac{2}{p}}\|v_h - w_g\|_{W^{1,p}} = \frac{1}{2}\|v_h - w_h\|_{W^{1,p}}.$$

The proof is completed.

Theorem 2. *Let u^ε denote the solution to problem (1)-(2) and let h_1 be same as in Lemma 2. For $h \le h_1$, let $\rho_1 = 2C_2h \le \rho_0$, then the finite element approximation (18) has a unique solution u_h^ε in the ball B_h, Moreover, there hold the estimates*

$$\|u^\varepsilon - u_h^\varepsilon\|_{W^{1,p}} \le C_4h, \quad \forall p \in (2,\infty]. \tag{34}$$

where C_4 is some positive constant which depends on ε^{-1} polynomially.

Proof: Since $B_h(\rho_1) \subset B_h(\rho_0)$, Lemma 3 implies that the mapping T is also a contracting mapping in $B_h(\rho_1)$ with the contraction factor $\frac{1}{2}$. We shall show that T also maps $B_h(\rho_1)$.

For any $v_h \in B_h(\rho_1)$, it follows from Lemma 2,3 and the triangle inequality that

$$\begin{aligned}\|I_hu^\varepsilon - T(v_h)\|_{W^{1,p}} &\le \|I_hu^\varepsilon - T(I_hu^\varepsilon)\|_{W^{1,p}} + \|T(I_hu^\varepsilon) - T(v_h)\|_{W^{1,p}}\\ &\le C_2h + \frac{1}{2}\|I_hu^\varepsilon - v_h\|_1\\ &\le \frac{\rho_1}{2} + \frac{\rho_1}{2} = \rho_1.\end{aligned}$$

Hence, $T(v_h) \in B_h(\rho_1)$. Consequently, T has a unique fixed point in $u_h^\varepsilon \in B_h(\rho_1)$. With the estimation

$$\|u^\varepsilon - I_hu^\varepsilon\|_{W^{1,p}} \le C_5h$$

for some positive constant $C_5 = C_5(\varepsilon)$. The proof is completed.

4.2. L^p-norm estimation.

In this section, we will derive an error estimation for finite element method (18) in the L^p-norm. One of the main techniques is to use duality argument on the linearization operator L_{u^ε} to handle the nonlinearity.

Theorem 3. *Let u^ε and u_h^ε denote the solution of (1)-(2) and (18), respectively. Let h_1 be same as in Lemma 2, then for $h \le h_1$, there holds*

$$\|u^\varepsilon - u_h^\varepsilon\|_{L^p} \le C_6h^2. \tag{35}$$

where $C_6 = C_6(\varepsilon)$ is a positive constant depending on ε^{-4} polynomially.

Proof: Subtracting (18) from (17) yields the error equation

(36)
$$\int_\Omega \left(\frac{\nabla u^\varepsilon}{\sqrt{|\nabla u^\varepsilon|^2+\varepsilon^2}} - \frac{\nabla u_h^\varepsilon}{\sqrt{|\nabla u_h^\varepsilon|^2+\varepsilon^2}}\right)\cdot\nabla\phi_h + \lambda(u^\varepsilon - u_h^\varepsilon)\cdot\phi_h dx = 0, \quad \forall\phi_h \in V_h.$$

Using the Mean Value Theorem we get

$$\int_\Omega A_h^\varepsilon\nabla(u^\varepsilon - u_h^\varepsilon)\cdot\nabla\phi_h dx + \int_\Omega \lambda(u^\varepsilon - u_h^\varepsilon)\phi_h dx = 0, \tag{37}$$

where

$$A_h^\varepsilon = \int_0^1 D^2 f_\varepsilon(\nabla u^\varepsilon + t\nabla(u_h^\varepsilon - u^\varepsilon))dt.$$

Let $e_h^\varepsilon := u^\varepsilon - u_h^\varepsilon$, it follows Lemma 1 that there exists a unique $\phi \in W^{2,\infty}(\Omega)$ such that

$$\begin{aligned} -div(D^2 f_\varepsilon(\nabla u^\varepsilon)\nabla\phi) + \lambda\phi &= sign(e_h^\varepsilon)|e_h^\varepsilon|^{p-1}, \quad in \ \ \Omega, \\ \frac{\partial u^\varepsilon}{\partial n} &= 0, \qquad\qquad\qquad\qquad on \ \ \partial\Omega. \end{aligned}$$

and by the regularization of the solution,

$$\|\phi\|_{W^{2,p'}} \le C_7 \|e_h^\varepsilon\|_{L^{p'}}^{p-1}. \tag{38}$$

Testing the above equation with e_h^ε and using (37) with $\phi_h = I_h\phi$ we have

$$\begin{aligned} \|e_h^\varepsilon\|_{L^p}^p &= \int_\Omega D^2 f_\varepsilon(\nabla u^\varepsilon)\nabla e_h^\varepsilon \cdot \nabla\phi dx + \int_\Omega \lambda e_h^\varepsilon \cdot \phi dx \\ &= \int_\Omega D^2 f_\varepsilon(\nabla u^\varepsilon)\nabla e_h^\varepsilon \cdot \nabla(\phi - I_h\phi)dx + \int_\Omega \lambda e_h^\varepsilon(\phi - I_h\phi)dx \\ &+ \int_\Omega [D^2 f_\varepsilon(\nabla u^\varepsilon) - A_h^\varepsilon]\nabla e_h^\varepsilon \cdot \nabla I_h\phi dx + \int_\Omega \lambda e_h^\varepsilon \cdot I_h\phi dx. \end{aligned}$$

Also we need notice that $D^2 f_\varepsilon(\nabla u^\varepsilon)$ is controlled by

$$|D^2 f_\varepsilon(\nabla u^\varepsilon)| \le \varepsilon^{-1}.$$

So it can be easy to bound the above equation as following

$$\begin{aligned} |\int_\Omega D^2 f_\varepsilon(\nabla u^\varepsilon)\nabla e_h^\varepsilon \cdot \nabla(\phi - I_h\phi)dx| &\le \varepsilon^{-1}\|\nabla e_h^\varepsilon\|_{L^p}\|\nabla(\phi - I_h\phi)\|_{L^{p'}} \\ &\le C\varepsilon^{-1}h\|\nabla e_h^\varepsilon\|_{L^p}\|\phi\|_{W^{2,p'}}, \end{aligned}$$

$$|\int_\Omega \lambda e_h^\varepsilon(\phi - I_h\phi)dx| \le Ch^2\|e_h^\varepsilon\|_{L^p}\|\phi\|_{W^{2,p'}}.$$

With the Mean Value Theorem, we can get

$$|D^2 f_\varepsilon(\nabla u^\varepsilon) - A_h^\varepsilon| \le 9\varepsilon^{-2}|\nabla e_h^\varepsilon|$$

Using approximation properties of I_h we get

$$\begin{aligned} |\int_\Omega [D^2 f_\varepsilon(\nabla u^\varepsilon) - A_h^\varepsilon]\nabla e_h^\varepsilon \cdot \nabla I_h\phi dx| &\le 9\varepsilon^{-2}\int_\Omega |\nabla e_h^\varepsilon|^2|\nabla I_h\phi|dx \\ &\le C\varepsilon^{-2}\|\nabla e_h^\varepsilon\|_{L^p}^2\|\phi\|_{W^{2,p'}}. \end{aligned} \tag{39}$$

$$|\int_\Omega \lambda e_h^\varepsilon \cdot I_h\phi dx| \le \lambda\|e_h^\varepsilon\|_{L^p}\|\phi\|_{L^{p'}} \le \lambda\|e_h^\varepsilon\|_{L^p}\|\phi\|_{W^{2,p'}}.$$

With (38), we can get

$$\|e_h^\varepsilon\|_{L^p} \le C_7 C(\varepsilon^{-1}h\|\nabla e_h^\varepsilon\|_{L^p} + \varepsilon^{-2}\|\nabla e_h^\varepsilon\|_{L^p}^2). \tag{40}$$

Which together with Theorem 2 leads to the desired estimation. The proof is completed.

5. Conclusion

We derive rates of convergence for regularization procedures(characterized by parameter ε) and finite element approximations of the TV regularization equation, which arises in image processing, geometric analysis and material science. We try to end this paper with some suggestion when we use finite element method approximation of TV-regularization equations. In this paper, we only give the controlling upper bound of the influence of regularized parameter ε. Therefore, for practical proposes, for example in image segmentation, the selection of regularization parameter ε can be taken by the measurement of mesh size.

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