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A CHARACTERIZATION OF SINGULAR ELECTROMAGNETIC FIELDS BY AN INDUCTIVE APPROACH

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Abstract. In this article, we are interested in the mathematical modeling of singular electromagnetic fields, in a non-convex polyhedral domain. We first describe the local trace (i. e. defined on a face) of the normal derivative of an L^2 function, with L^2 Laplacian. Among other things, this allows us to describe dual singularities of the Laplace problem with homogeneous Neumann boundary condition. We then provide generalized integration by parts formulae for the Laplace, divergence and curl operators. With the help of these results, one can split electromagnetic fields into regular and singular parts, which are then characterized. We also study the particular case of divergence-free and curl-free fields, and provide non-orthogonal decompositions that are numerically computable.

Key Words. Maxwell's equations, singular geometries, polyhedral domains.

Introduction

When one solves boundary value problems in a bounded polyhedron Ω of $\mathbb{R}^3$ with a Lipschitz boundary, it is well-known that the presence of reentrant corners and/or edges on the boundary deteriorates the smoothness of the solution [30, 28]. This problem is all the more relevant since boundary value problems which arise in practice, are often posed in domains with a simple but non smooth geometry, such as three-dimensional polyhedra.

More specifically, consider Maxwell's equations with perfect conductor boundary conditions and right-hand sides in $L^2(\Omega)$. Then the electromagnetic field $(\mathcal{E}, \mathcal{H})$ always belongs to $H^1(\Omega)^6$ when Ω is convex[1]. On the other hand, it is only guaranteed that it belongs to $H^\sigma(\Omega)^6$, for any $\sigma < \sigma_{max}$, with $\sigma_{max} \in]1/2, 1[$, when Ω is non-convex (see for instance [24]). In the latter case, strong electromagnetic fields can occur, near the reentrant corners and/or edges. For practical applications, we refer for instance to [31]. Nevertheless, one can split (cf. [7]) the field into two parts: a regular one, which belongs to $H^1(\Omega)^6$, and a singular one. According to [28, 7, 12, 24], the subspace of regular fields is closed, so one can choose to define the singular fields by orthogonality. Other approaches are possible and useful, see [23].

In the same way, when solving a problem involving the Laplace operator with data in $L^2(\Omega)$, the solution is in $H^2(\Omega)$ when Ω is either a convex polyhedron or

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[1] As proven in [28, p. 12], if Ω is convex, then its boundary is automatically Lipschitz.

a bounded domain with a smooth boundary. However, it is only guaranteed to be in $H^{1+s}(\Omega)$ for $s < s_{max}$, when Ω is a non-convex polyhedron (one can prove that $s_{max} = \sigma_{max}$, see Section 3). Grisvard showed in [28] that a solution of the Laplace operator can be decomposed into the sum of a regular part and a singular part, the latter being called a *primal singularity*. This decomposition is based on a decomposition of $L^2(\Omega)$ into the sum of the image space of the regular parts and its orthogonal (the latter is the space of *dual singularities*).

As it is well known, the singular part of the electromagnetic field is linked [7, 8, 10] to the primal singularities of the Laplace problem, respectively with

- homogeneous Dirichlet boundary condition for the electric field $\mathcal{E}$;
- homogeneous Neumann boundary condition for the magnetic field $\mathcal{H}$.

For a comprehensive theory on this topic, we refer the reader to the works of Birman and Solomyak [7, 8, 9, 10, 11]. Among other results, they proved a splitting of the space of electromagnetic fields into a two-term *simple* sum. First, the subspace of regular fields. Second, the subspace made of gradients of solutions to the Laplace problem.

During the 1990s, Costabel and Dauge [25, 19, 20, 22, 23] provided new insight into the characterizations of the singularities of the electromagnetic fields, called afterwards *electromagnetic singularities*. In the process, they proved very useful density results.

In [2], we first studied, for L^2 functions with L^2 Laplacians, a possible definition of the trace on the boundary. Actually, it was proven that it can be understood locally – face by face – with values in $H^{-1/2}$-like Sobolev spaces. This being clarified, we inferred a *generalized integration by parts* (gibp) formula. Finally, in [4], we were able to describe precisely the space of all divergence-free singular electric fields. Indeed, starting from the orthogonality relationship with regular fields, the gibp formula allowed us to build a suitable characterization. In the present article, we would like to extend the results first to the case of magnetic fields and second to the case of any electric field, by using the same three step procedure.

The article is organized as follows. We first introduce some notations and define the Sobolev spaces that we will use throughout this paper. In the following Section, we recall some definitions on local traces together with the resulting gibp formula for the Laplace problem with Dirichlet boundary condition. These results are then extended to the Laplace operator with Neumann boundary condition. In Section 3, we transpose (part of) these results to the electromagnetic fields, from which, in Section 4, we can prove characterizations of the singular electromagnetic fields. Section 5 is devoted to the study of the divergence-free case. Then, in Section 6, we relate the regular/singular fields to the vector and scalar potentials. We also give their characterizations, using for this *ad hoc* isomorphisms. Finally, in the last Section, we consider curl-free spaces, that allow us to define non-orthogonal but numerically useful decompositions of the electromagnetic fields.

1. Notations and functional spaces

Let Ω be a bounded open set of $\mathbb{R}^3$, with a Lipschitz polyhedral boundary $\partial\Omega$. For simplicity reasons (cf. Remark 3.1), we assume that Ω is simply connected and that $\partial\Omega$ is connected. The unit outward normal to $\partial\Omega$ is denoted by $\mathbf{n}$. We call

$(S_v)_{1\le v\le N_V}$ the vertices of $\partial\Omega$. Let $(F_i)_{1\le i\le N_F}$ be the faces of $\partial\Omega$: $\forall i$ means that i spans $\{1,\cdots,N_F\}$, whereas i stands for a given index. We also introduce the edges between faces: e_{ij} is the edge between the faces F_i and F_j (when it exists), and $\forall(i,j)$ means that (i,j) spans the subset of $\{1,\cdots,N_F\}^2$ which corresponds to actual edges of $\partial\Omega$. In this case, setting $\mathbf{n}_i = \mathbf{n}_{|F_i}$, $\boldsymbol{\tau}_{ij}$ is a unit vector parallel to e_{ij}, and $\boldsymbol{\tau}_i$ is such that $(\boldsymbol{\tau}_i, \boldsymbol{\tau}_{ij}, \mathbf{n}_i)$ forms an orthonormal basis of $\mathbb{R}^3$. Finally, we denote by Γ_{ij} the reunion of two faces F_i and F_j when there are connected by an edge, i. e. Γ_{ij} is open and such that $\bar\Gamma_{ij} = \bar F_i \cup \bar e_{ij} \cup \bar F_j$.

In the text, names of functional spaces of scalar fields usually begin with an italic letter, whereas they begin with a bold or calligraphic letter for spaces of vector fields (for instance, $\mathbf{L}^2(\Omega) = L^2(\Omega)^3$). The scalar products (respectively the norms) of $L^2(\Omega)$ and of $\mathbf{L}^2(\Omega)$ are both denoted by $(\cdot,\cdot)_0$ (resp. by $\|\cdot\|_0$), and the duality product between X and its dual X' is denoted by $\langle\cdot,\cdot\rangle_X$. We assume that the reader is familiar with the space of smooth functions with compact support on Ω, called $\mathcal{D}(\Omega)$, and with its dual $\mathcal{D}'(\Omega)$, the space of distributions on Ω. Likewise for the Sobolev spaces $H^m(\Omega)$, $m\ge 1$, and the closure of $\mathcal{D}(\Omega)$ in these spaces, denoted by $H_0^m(\Omega)$. Now, let us consider some well-known, but more specialized, Sobolev spaces.

The first specialized functional space is $L_0^2(\Omega)$, i. e. the subspace of $L^2(\Omega)$ made of elements with zero mean value over Ω. Then, to study electromagnetic fields, it is convenient to introduce

$$\mathbf{H}(\mathbf{curl}\,,\Omega) = \{\mathbf{v}\in\mathbf{L}^2(\Omega)\ :\ \mathbf{curl}\,\mathbf{v}\in\mathbf{L}^2(\Omega)\},\ \|\mathbf{v}\|_{\mathbf{H}(\mathbf{curl}\,,\Omega)} = \left[\|\mathbf{v}\|_0^2 + \|\mathbf{curl}\,\mathbf{v}\|_0^2\right]^{1/2}$$
$$\mathbf{H}(\mathrm{div}\,,\Omega) = \{\mathbf{v}\in\mathbf{L}^2(\Omega)\ :\ \mathrm{div}\,\mathbf{v}\in L^2(\Omega)\},\ \|\mathbf{v}\|_{\mathbf{H}(\mathrm{div}\,,\Omega)} = \left[\|\mathbf{v}\|_0^2 + \|\mathrm{div}\,\mathbf{v}\|_0^2\right]^{1/2},$$

and the closure of $\mathcal{D}(\Omega)^3$ in those spaces, respectively denoted by $\mathbf{H}_0(\mathbf{curl}\,,\Omega)$ and $\mathbf{H}_0(\mathrm{div}\,,\Omega)$. Let us also set $\mathbf{H}(\mathrm{div}\,0;\Omega) = \{\mathbf{v}\in\mathbf{H}(\mathrm{div}\,,\Omega)\ :\ \mathrm{div}\,\mathbf{v}=0\}$.

Next, we shall use throughout this paper trace mappings on the boundary $\partial\Omega$, together with functional spaces related to these mappings.

For scalar fields, the trace $\gamma_0\ :\ u\mapsto u_{|\partial\Omega}$, and the trace of the normal derivative $\gamma_1\ :\ u\mapsto \partial_n u_{|\partial\Omega}$. Then, introduce $H^{1/2}(\partial\Omega)$ as the range of γ_0 from $H^1(\Omega)$, that is $\gamma_0(H^1(\Omega))$ (note that $H_0^1(\Omega) = Ker(\gamma_0)$), and its dual $H^{-1/2}(\partial\Omega)$. This allows us to define functions, the support of which is restricted to any given face F_i. Define $L^2(F_i)$ as usual and

$$\begin{array}{l} H^{1/2}(F_i) = \{v\in L^2(F_i)\ :\ \exists w\in H^{1/2}(\partial\Omega),\ v = w_{|F_i}\} \\ \widetilde{H}^{1/2}(F_i) = \{v\in H^{1/2}(F_i)\ :\ \tilde v\in H^{1/2}(\partial\Omega)\} \end{array},$$

where $\tilde v$ is the continuation of v by zero to $\partial\Omega$. Denote by $\widetilde{H}^{-1/2}(F_i)$ the dual space of $\widetilde{H}^{1/2}(F_i)$.
Following [13, 14], we introduce then $H^{3/2}(\partial\Omega) = \gamma_0(H^2(\Omega))$, and, on any given face, successively $H^{3/2}(F_i)$, $\widetilde{H}^{3/2}(F_i)$ and $\widetilde{H}^{-3/2}(F_i)$.
To conclude on the scalar functions defined on a part of $\partial\Omega$, consider finally any given reunion of two faces Γ_{ij}, and let $s\in\{1/2,3/2\}$. We define $H^s(\Gamma_{ij})$, $\widetilde{H}^s(\Gamma_{ij})$ and $\widetilde{H}^{-s}(\Gamma_{ij})$ as previously.

Now, let us introduce the spaces of vector fields. The normal trace $\gamma_n\ :\ \mathbf{u} \mapsto \mathbf{u}\cdot\mathbf{n}_{|\partial\Omega}$ is surjective from $\mathbf{H}(\mathrm{div}\,,\Omega)$ to $H^{-1/2}(\partial\Omega)$ ($\mathbf{H}_0(\mathrm{div}\,,\Omega) = Ker(\gamma_n)$). Following [13], we introduce then two companion vector traces. The first one, the 'tangential trace' $\gamma_\tau\ :\ \mathbf{v} \mapsto \mathbf{v}\times\mathbf{n}_{|\partial\Omega}$, and the second one, the 'tangential components trace' $\pi_\tau\ :\ \mathbf{v}\mapsto \mathbf{n}\times(\mathbf{v}\times\mathbf{n})_{|\partial\Omega}$. One can prove simply that, when considered from $\mathbf{H}(\mathbf{curl}\,,\Omega)$, γ_τ takes its values in $\mathbf{H}^{-1/2}(\partial\Omega)$, without being surjective. The characterization of its range is more delicate, but it can be obtained nonetheless [13]. We do not provide it here, since it is not relevant for our studies. However, we shall need

$$\mathbf{L}^2_t(\partial\Omega) = \{\mathbf{v}\in L^2(\partial\Omega)^3\ :\ \mathbf{v}\cdot\mathbf{n} = 0\}, \tag{1}$$

and the respective ranges of γ_τ and π_τ, when considered from $\mathbf{H}^1(\Omega)$: $\mathbf{H}^{1/2}_\perp(\partial\Omega) = \gamma_\tau(\mathbf{H}^1(\Omega))$, and $\mathbf{H}^{1/2}_\parallel(\partial\Omega) = \pi_\tau(\mathbf{H}^1(\Omega))$. A characterization of these spaces can be obtained, and it will be stated when it is needed. As before, for any given face F_i, we introduce the sequences: $\mathbf{H}^{1/2}_\perp(F_i)$ and $\widetilde{\mathbf{H}}^{1/2}_\perp(F_i)$; $\mathbf{H}^{1/2}_\parallel(F_i)$ and $\widetilde{\mathbf{H}}^{1/2}_\parallel(F_i)$.

2. Local traces and generalized integration by parts formulas for the Laplace problem

We want to characterize singular electromagnetic fields. To achieve our goal, we retrace the three-step procedure of [4], in which divergence-free singular electric fields were scrutinized:

(1) derivation of generalized integration by parts (gibp) formulas;
(2) characterization of dual singularities related to the Laplace problem;
(3) characterization of singular electromagnetic fields.

In this Section, we focus on the first two steps. We begin by introducing *primal scalar fields*, which are the solutions to the Laplace problem set in $H^1(\Omega)$, with right-hand side in $L^2(\Omega)$.
The *dual scalar fields* correspond to the right-hand sides, i. e. either $L^2(\Omega)$ or $L^2_0(\Omega)$.

Definition 2.1. *Let Ψ and Φ be the spaces of primal fields for the Laplace operator, respectively with homogeneous Dirichlet or homogeneous Neumann boundary conditions*

$$\Psi = \{\psi\in H^1_0(\Omega)\ :\ \Delta\psi\in L^2(\Omega)\}\,,$$
$$\Phi = \{\phi\in H^1(\Omega)\cap L^2_0(\Omega)\ :\ \Delta\phi\in L^2_0(\Omega)\ :\ \partial_n\phi_{|\partial\Omega} = 0\}\,.$$

It is common knowledge (use the Lax-Milgram Theorem) that both spaces Ψ and Φ can be equipped with the equivalent norm $\|v\|_\Delta = \|\Delta v\|_0$. Then, we split the primal fields into H^2-regular fields and singular fields, and study the properties related to the splittings. The latter fields are called *primal singularities.*

For the electric case, we assume that the domain Ω is enclosed in a perfectly conducting material, so that the boundary condition for the electric field is $\mathcal{E}\times\mathbf{n}_{|\partial\Omega} = 0$. Hence, the singular electric fields are related to the singularities of the Laplace operator with Dirichlet boundary condition [7]. As far as primal and dual singularities are concerned, let us thus begin by the case of the Dirichlet boundary conditions, which is addressed in [2]. Following this Ref., it is convenient to introduce (sub)spaces of regular solutions of the Laplace problem, such as the ones below.

Definition 2.2. *Consider the regular subspaces of* Ψ

$$
\begin{array}{l}
H^D(\Omega) = H^2(\Omega) \cap H^1_0(\Omega) \\
H^D_i(\Omega) = \{v \in H^D(\Omega) \ : \ \partial_n v_{|F_j} = 0, \ \forall j \neq i\}, \ 1 \leq i \leq N_F
\end{array} \quad . \tag{2}
$$

Since $H^D(\Omega)$ is closed in Ψ (cf. [28]), we can introduce its orthogonal subspace, called $\underline{H}^D(\Omega)$, the space of primal singularities and get the splitting

$$
\Psi = H^D(\Omega) \overset{\perp_\Delta}{\oplus} \underline{H}^D(\Omega).
$$

Next, we consider $\mathbb{S}_D$, equal to the range of the scalar Laplace operator acting on the space of primal singularities $\underline{H}^D(\Omega)$. Elements of $\mathbb{S}_D$ are called *dual singularities.* We explain below how they can be characterized. To proceed, we define $D(\Delta;\Omega) = \{q \in L^2(\Omega) \ : \ \Delta q \in L^2(\Omega)\}$, endowed with the graph norm $\| \cdot \|_D$. This space is natural for the study of dual singularities of the Laplace problem. Since $H^2(\Omega)$ is dense in $D(\Delta;\Omega)$ [2], one can prove simply some gibp formulas, such as (3) and other formulas afterwards.

Theorem 2.1. *Consider* $i \in \{1, \cdots, N_F\}$.
(*) The mapping $v \mapsto \partial_n v_{|F_i}$ *is linear and continuous from* $H^D(\Omega)$ *to* $\widetilde{H}^{1/2}(F_i)$*;*
(i) The mapping $v \mapsto \partial_n v_{|F_i}$ *is surjective from* $H^D_i(\Omega)$ *to* $\widetilde{H}^{1/2}(F_i)$*;*
(ii) The mapping $p \mapsto p_{|F_i}$ *is linear and continuous from* $D(\Delta;\Omega)$ *to* $\widetilde{H}^{-1/2}(F_i)$*;*
(iii) The following gibp formula holds:

$$
(p, \Delta v)_0 - (v, \Delta p)_0 = \sum_i \langle p_{|F_i}, \partial_n v_{|F_i} \rangle_{\widetilde{H}^{1/2}(F_i)}, \quad \forall (p, v) \in D(\Delta;\Omega) \times H^D(\Omega). \tag{3}
$$

With the help of these results, we can characterize dual singularities of $\mathbb{S}_D$ as follows [2].

Corollary 2.1. *An element* s *of* $L^2(\Omega)$ *belongs to* $\mathbb{S}_D$ *if, and only if, there holds:*

$$
s \in D(\Delta;\Omega), \quad -\Delta s = 0 \ \text{in } \Omega, \quad s_{|F_i} = 0 \ \text{in } \widetilde{H}^{-1/2}(F_i), \ \forall i. \tag{4}
$$

We recall briefly how these results can be established (cf. [2]).
Proof. Consider $s \in \mathbb{S}_D$. By definition, $(s, \Delta v)_0 = 0$, for all $v \in H^D(\Omega)$.
Since $\mathcal{D}(\Omega)$ is a subset of $H^D(\Omega)$, one finds, for any $v \in \mathcal{D}(\Omega)$:

$$
\langle \Delta s, v \rangle = \langle s, \Delta v \rangle = (s, \Delta v)_0 = 0.
$$

So $\Delta s = 0$ and s belongs to $D(\Delta;\Omega)$. According to item (ii) of Theorem 2.1, $s_{|F_i} \in \widetilde{H}^{-1/2}(F_i)$, for all i. Also, given any $\mu \in \widetilde{H}^{1/2}(F_i)$, there exists $v \in H^D_i(\Omega)$ such that $\partial_n v_{|F_i} = \mu$ (item (i)). Then, formula (3) applied to the couple (s, v) yields $0 = \langle s_{|F_i}, \mu \rangle_{\widetilde{H}^{1/2}(F_i)}$. In other words, $s_{|F_i} = 0$ in $\widetilde{H}^{-1/2}(F_i)$.
The reciprocal assertion is an easy consequence of formula (3). ◇

In order to proceed similarly for the magnetic field, recall first that the perfect conductor boundary condition is written here $\mathcal{H} \cdot \mathbf{n}_{|\partial\Omega} = 0$. Hence, the singular magnetic fields are related to the singularities of the Laplace operator with Neumann boundary condition [7, 10]. Compared to the electric case, the idea is then to swap the trace mapping γ_0 with the trace of the normal derivative mapping γ_1.

Definition 2.3. *Consider the regular subspaces of* Φ

$$
\begin{array}{l}
H^N(\Omega) = \{v \in H^2(\Omega) \cap L^2_0(\Omega) \ : \ \partial_n v_{|\partial\Omega} = 0\}, \\
H^N_i(\Omega) = \{v \in H^N(\Omega) \ : \ v_{|F_j} = 0, \ \forall j \neq i\}, \ 1 \leq i \leq N_F.
\end{array} \tag{5}
$$

Since $H^N(\Omega)$ is closed in Φ (cf. [28]), we introduce its orthogonal subspace, $\underline{H}^N(\Omega)$, the space of primal singularities with Neumann boundary conditions and get the splitting

$$\Phi = H^N(\Omega) \overset{\perp_\Delta}{\oplus} \underline{H}^N(\Omega).$$

Next, we define $\mathbb{S}_N$, equal to the range of the Laplace operator acting on the space of primal singularities $\underline{H}^N(\Omega)$. Elements of $\mathbb{S}_N$ are called *dual singularities* (with Neumann boundary conditions). The characterization of the elements of $\mathbb{S}_N$ will prove slightly more complicated then the $\mathbb{S}_D$ counterpart, as we will see below. Following basically the same techniques as in the case of the Dirichlet boundary conditions [26, pp. 175-176], one can first derive the results below (with the important exception of item (*) of Theorem 2.1, see Remark 2.1).

Theorem 2.2. *Consider* $i \in \{1, \cdots, N_F\}$.
(i) The mapping $v \mapsto v_{|F_i}$ *is surjective from* $H_i^N(\Omega)$ *to* $\widetilde{H}^{3/2}(F_i)$*;*
(ii) The mapping $p \mapsto \partial_n p_{|F_i}$ *is linear and continuous from* $D(\Delta;\Omega)$ *to* $\widetilde{H}^{-3/2}(F_i)$*;*
(iii) The following gibp formula holds:

(6) $$(p, \Delta v)_0 - (v, \Delta p)_0 = -\langle \partial_n p_{|F_i}, v_{|F_i} \rangle_{\widetilde{H}^{3/2}(F_i)}, \quad \forall (p,v) \in D(\Delta;\Omega) \times H_i^N(\Omega).$$

Remark 2.1. *One can not transpose item (*) of Theorem 2.1 from the electric to the magnetic case. As a matter of fact, given* $v \in H^N(\Omega)$*, it is true that* $v_{|F_i}$ *belongs to* $H^{3/2}(F_i)$*, but* $v_{|F_i} \in \widetilde{H}^{3/2}(F_i)$ *is not automatically fulfilled. However, the gibp formula (6) is easily extended to* (p,v) *of* $D(\Delta;\Omega) \times H^N(\Omega)$*, such that* $v_{|F_i} \in \widetilde{H}^{3/2}(F_i)$*,* $\forall i$*.*

With the help of Theorem 2.2, one gets easily (transpose the proof of Corollary 2.1):

Corollary 2.2. *An element* s *of* $\mathbb{S}_N$ *satisfies necessarily:*

(7) $$s \in D(\Delta;\Omega), \quad -\Delta s = 0 \text{ in } \Omega, \quad \partial_n s_{|F_i} = 0 \text{ in } \widetilde{H}^{-3/2}(F_i), \ \forall i.$$

The question is: does (7) completely characterizes dual singularities of $\mathbb{S}_N$? The problem here is the absence of item (*) in the magnetic case. However, if $\sum_i H_i^N(\Omega)$ were dense in $H^N(\Omega)$, then the characterization (7) would be complete! But it is not the case. It can be explained simply by contradiction if one recalls the Sobolev imbedding Theorem, which states that $H^2(\Omega)$ is continuously imbedded in the Hölder space $\mathcal{C}^{0,1/2}(\bar{\Omega})$ (cf. for instance [28, p. 27]). In particular,

$$\exists C > 0, \ \sup_{\mathbf{x}\in\bar{\Omega}} |u(\mathbf{x})| \le C \,\|u\|_{H^2(\Omega)}, \ \forall u \in H^2(\Omega).$$

If one assumes that $\sum_i H_i^N(\Omega)$ is dense in $H^N(\Omega)$, one gets that, given any $f \in H^N(\Omega)$ and any $\varepsilon > 0$, there exists $f_\Sigma \in \sum_i H_i^N(\Omega)$ such that $\|f - f_\Sigma\|_{H^2(\Omega)} \le \varepsilon$, so that $\sup_{\mathbf{x}\in\bar{\Omega}} |f(\mathbf{x}) - f_\Sigma(\mathbf{x})| \le C\,\varepsilon$. But, since $f_{\Sigma|F_i}$ belongs to $\widetilde{H}^{3/2}(F_i)$, for all i, one gets in particular that $f_{\Sigma|e_{ij}} = 0$, for all (i,j). Passing to the limit, one finds that $f_{|e_{ij}} = 0$, for all (i,j). But, there exist elements of $H^N(\Omega)$ that do not fulfill this property (cf. the Annex, Subsection A.2). So, as (7) provides an incomplete characterization of the elements of $\mathbb{S}_N$, one has to model them more precisely.

Definition 2.4. *Consider the regular subspaces of* Φ*, for all* (i,j)*,*

(8) $$H_{ij}^N(\Omega) = \{v \in H^N(\Omega) \ : \ v_{|F_k} = 0, \ \forall k \notin \{i,j\}\},$$

and the trace spaces, for all (i,j),

$$\mathbb{A}_{ij} = \{v \in \widetilde{H}^{3/2}(\Gamma_{ij}) \ : \ \nabla_\Gamma v_{|F_i} \cdot \mathbf{n}_j \overset{1/2}{=} 0,\ 0 \overset{1/2}{=} \nabla_\Gamma v_{|F_j} \cdot \mathbf{n}_i\}. \tag{9}$$

$\Big($ Above, ∇_Γ is the surface gradient on $\partial\Omega$, and the notation $f_i \overset{1/2}{=} f_j$ has the following meaning: consider $(f_i, f_j) \in H^{1/2}(F_i) \times H^{1/2}(F_j)$; we write $f_i \overset{1/2}{=} f_j$ if, and only if, the function f_{ij}, equal to f_i on F_i and to f_j on F_j, belongs to $H^{1/2}(\Gamma_{ij})$.$\Big)$

One can prove easily that the $\mathbb{A}_{ij}$s are Hilbert spaces (i. e. they are complete), and moreover that they are the *ad hoc* trace spaces (i. e. the trace mapping is surjective). As a matter of fact, using [6, Theorem 2] for item (iv) below, one can prove enhanced results, for the trace mappings γ_0 and γ_1.

Theorem 2.3. *Consider* (i,j) *associated to an edge* e_{ij}.
(iv) The mapping $v \mapsto v_{|\Gamma_{ij}}$ *is surjective from* $H^N_{ij}(\Omega)$ *to* $\mathbb{A}_{ij}$;
(v) The mapping $p \mapsto \partial_n p_{|\Gamma_{ij}}$ *is linear and continuous from* $D(\Delta;\Omega)$ *to* $(\mathbb{A}_{ij})'$;
(vi) The following gibp formula holds:

$$(p, \Delta v)_0 - (v, \Delta p)_0 = -\langle \partial_n p_{|\Gamma_{ij}}, v_{|\Gamma_{ij}} \rangle_{\mathbb{A}_{ij}}, \quad \forall (p,v) \in D(\Delta;\Omega) \times H^N_{ij}(\Omega). \tag{10}$$

In other words, one has to add another condition to (7) in Corollary 2.2, namely $\partial_n s_{|\Gamma_{ij}} = 0$ in $(\mathbb{A}_{ij})'$, for all (i,j). The question is: is it the only one? The answer is yes: from the above, we can actually provide a full characterization of the elements of $\mathbb{S}_N$.

Corollary 2.3. *An element* s *of* $L^2_0(\Omega)$ *belongs to* $\mathbb{S}_N$ *if, and only if, there holds:*

$$\begin{aligned} & s \in D(\Delta;\Omega),\ -\Delta s = 0 \text{ in } \Omega,\ \partial_n s_{|F_i} = 0 \text{ in } \widetilde{H}^{-3/2}(F_i),\ \forall i, \\ & \partial_n s_{|\Gamma_{ij}} = 0 \text{ in } (\mathbb{A}_{ij})',\ \forall (i,j). \end{aligned} \tag{11}$$

Proof. Following the proof of Corollary 2.1, one obtains easily that any element s of $\mathbb{S}_N$ satisfies all the conditions expressed in (11).

To prove the reciprocal assertion, we consider an s which satisfies (11), together with any $u \in H^N(\Omega)$, and compute $(s, \Delta u)_0$. We split u into four parts, in the following way

$$u = \sum_v u_v + \sum_{(i,j)} u_{ij} + \sum_i u_i + u_\Omega,$$

where the first summation is taken over all vertices, the second one over all edges, and the last one over all faces. Let us explain how each part is built. Introduce $\chi : \mathbb{R}^+ \to \mathbb{R}^+$, a smooth cut-off function, equal to one in a neighborhood of zero, and to zero near $+\infty$.

Around vertex v, consider the spherical coordinates $(\rho_v, \theta_v, \varphi_v)$, and set $u^n_v(\mathbf{x}) = \chi(n\rho_v)u(\mathbf{x})$, for an integer $n \geq 1$. In other words, the support of u^n_v is located around vertex v, and it shrinks as n increases.

The difference $v^n = u - \sum_v u^n_v$ is equal to zero around all vertices. Around edge e_{ij}, consider the cylindrical coordinates $(\rho_{ij}, \theta_{ij}, z_{ij})$, and set $u^n_{ij}(\mathbf{x}) = \chi(n\rho_{ij})v^n(\mathbf{x})$. The support of u^n_{ij} is located around edge e_{ij} and, in addition, for n large enough, one gets that u^n_{ij} belongs to $H^N_{ij}(\Omega)$.

The difference $w^n = u - \sum_v u^n_v - \sum_{ij} u^n_{ij}$ is equal to zero around all edges and vertices. Near face F_i, consider the distance to the face z_i, and set $u^n_i(\mathbf{x}) = \chi(nz_i)w^n(\mathbf{x})$. The support of u^n_i is located near face F_i and, in addition, for n

large enough, one gets that u_i^n belongs to $H_i^N(\Omega)$ (although the distance z_i is not a 'smooth' function, the regularity of u_i^n is not an issue, since it vanishes near the edges surrounding the face, i. e. precisely where z_i is not smooth).

The difference $u_\Omega^n = u - \sum_v u_v^n - \sum_{ij} u_{ij}^n - \sum_i u_i^n$ belongs to $H_0^2(\Omega)$ (by construction, it is equal to zero near the boundary). Summing up, one can thus write, for n large enough

$$(s, \Delta u)_0 = \sum_v (s, \Delta u_v^n)_0,$$

since the three other terms disappear according to the (three) conditions specified in (11). Now, one has simply to prove that the remaining term is actually zero. To that aim, we use Lemma A.1 ((cf. the Annex, Subsection A.1), which states precisely that, for each vertex v, the sequence $(\Delta u_v^n)_n$ converges weakly to zero in $L^2(\Omega)$. This ends the proof. $\diamond$

Remark 2.2. *Once again, using the same counter-example (cf. the Annex, Subsection A.2), one can prove that, with respect to a given edge e_{ij} of $\partial\Omega$, there exists f of $H^N(\Omega)$ such that Δf_{ij}^n does not converge weakly to zero in $L^2(\Omega)$.*

As a conclusion to this Section, we make the following remarks. The full characterization of dual singularities (the elements of $\mathbb{S}_N$) does not only rely on a strict transposition of the arguments developed in [2] for the characterization of dual singularities (the elements of $\mathbb{S}_D$). As a matter of fact, it is more complex to achieve, in the sense that additional tools must be used in the case of elements of $\mathbb{S}_N$. Interestingly, the 3D context is different from the 2D context, in which the characterizations of $\mathbb{S}_D$ and $\mathbb{S}_N$ are completely symmetric [29]. For intermediate situations – $2\frac{1}{2}$D geometries – we refer the reader for instance to [17, 18].

3. Functional spaces, local traces and generalized integration by parts formulas for the electromagnetic field

Assuming that the right-hand sides of Maxwell's equations belong to $L^2(\Omega)$ (or to $\mathbf{L}^2(\Omega)$) amounts to saying that both electric and magnetic fields belong to $\mathbf{H}(\mathbf{curl}\,,\Omega)\cap\mathbf{H}(\mathrm{div}\,,\Omega)$, plus (perfect conductor) boundary conditions. We thus consider the spaces $\mathcal{X}\times\mathcal{Y}$ of electromagnetic fields as below.

Definition 3.1. *Let $\mathcal{X}\times\mathcal{Y}$ be the space of electromagnetic fields, with*

$$\mathcal{X} = \{\mathbf{x}\in\mathbf{H}(\mathbf{curl}\,,\Omega)\cap\mathbf{H}(\mathrm{div}\,,\Omega) \ : \ \mathbf{x}\times\mathbf{n}_{|\partial\Omega} = 0\}; \tag{12}$$

$$\mathcal{Y} = \{\mathbf{y}\in\mathbf{H}(\mathbf{curl}\,,\Omega)\cap\mathbf{H}(\mathrm{div}\,,\Omega) \ : \ \mathbf{y}\cdot\mathbf{n}_{|\partial\Omega} = 0\}. \tag{13}$$

Recall that our aim is to characterize the singular electromagnetic fields by orthogonality (to all regular electromagnetic fields). Therefore, we have to focus on the scalar product. Since both $\mathcal{X}$ and $\mathcal{Y}$ are subsets of $\mathbf{H}(\mathbf{curl}\,,\Omega)\cap\mathbf{H}(\mathrm{div}\,,\Omega)$, a natural choice would be the scalar product induced by the graph norm

$$(\mathbf{u},\mathbf{v})\mapsto(\mathbf{u},\mathbf{v})_0 + (\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v})_0 + (\mathrm{div}\,\mathbf{u},\mathrm{div}\,\mathbf{v})_0. \tag{14}$$

However, we know from Weber [32] that both $\mathcal{X}$ and $\mathcal{Y}$ are compactly imbedded in $\mathbf{L}^2(\Omega)$. As a consequence, one can prove that

$$(\cdot,\cdot)_W \ : \ (\mathbf{u},\mathbf{v})\mapsto(\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v})_0 + (\mathrm{div}\,\mathbf{u},\mathrm{div}\,\mathbf{v})_0, \tag{15}$$

defines a norm, which is equivalent to the graph norm.

Remark 3.1. *Following* [1], *this result holds true in $\mathcal{X}$ provided that the boundary $\partial\Omega$ is connected, and in $\mathcal{Y}$ provided that the domain Ω is simply connected. Otherwise, one has to add another term to (15) (different in each case) that allows to take* *care of the elements of $\mathcal{X}$ and $\mathcal{Y}$ which are curl- and divergence-free, but not equal to zero. The subspaces of $\mathcal{X}$ and $\mathcal{Y}$ made up of such fields are finite dimensional. However, the theory we develop here is not modified.*

Since we are interested in the regular/singular splitting of the fields, we introduce some subspaces of $\mathcal{X}$ and $\mathcal{Y}$.

Definition 3.2. *Consider the regular subspaces of $\mathcal{X}$ and $\mathcal{Y}$*

$$\begin{array}{l} \mathcal{X}^R = \mathcal{X} \cap \mathbf{H}^1(\Omega), \\ \mathcal{X}_i^R = \{\mathbf{x} \in \mathcal{X}^R \ : \ \mathbf{x} \cdot \mathbf{n}_{|F_j} = 0, \ \forall j \neq i\}, \ 1 \leq i \leq N_F \end{array} ; \tag{16}$$

$$\begin{array}{l} \mathcal{Y}^R = \mathcal{Y} \cap \mathbf{H}^1(\Omega), \\ \mathcal{Y}_i^R = \{\mathbf{y} \in \mathcal{Y}^R \ : \ \mathbf{y} \times \mathbf{n}_{|F_j} = 0, \ \forall j \neq i\}, \ 1 \leq i \leq N_F \end{array} . \tag{17}$$

As already mentioned, we have the closedness results

Theorem 3.1. *The following decompositions are direct and continuous*

$$\mathcal{X} = \mathcal{X}^R \overset{c}{\oplus} \nabla \underline{H}^D(\Omega), \qquad \mathcal{Y} = \mathcal{Y}^R \overset{c}{\oplus} \nabla \underline{H}^N(\Omega)\,. \tag{18}$$

In particular, $\mathcal{X}^R$ is closed in $\mathcal{X}$ and $\mathcal{Y}^R$ is closed in $\mathcal{Y}$.

We explain here how these results can be established (cf. [12, 24]).
Proof. We consider mainly the electric case. According to [7], one can split $\mathcal{X}$ *continuously* as

$$\mathcal{X} = \mathcal{X}^R \overset{c}{+} \nabla\Psi.$$

Note however that the sum is not direct. Then, following [28], one knows that Ψ can be split orthogonally as $\Psi = H^D(\Omega) \overset{\perp_\Delta}{\oplus} \underline{H}^D(\Omega)$. Interestingly, this yields

$$\nabla\Psi = \nabla H^D(\Omega) \overset{\perp_W}{\oplus} \nabla\underline{H}^D(\Omega).$$

Since $\nabla H^D(\Omega)$ is itself a subset of $\mathcal{X}^R$, one finds [12, 24] that

$$\mathcal{X} = \mathcal{X}^R \overset{c}{\oplus} \nabla\underline{H}^D(\Omega).$$

Indeed, the sum is direct, because by construction $\mathcal{X}^R \cap \nabla\underline{H}^D(\Omega) = \{0\}$. From this splitting, one infers easily that $\mathcal{X}^R$ is a closed subspace of $\mathcal{X}$.
For the magnetic case, set in $\mathcal{Y}$, one has to replace the initial citation of [7] by a combined Ref. to [7, 10]. The rest of the proof is left unchanged. $\diamond$

The direct and continuous decompositions will be of use in Section 7.

Moreover, we know from [2, 4] that the following is true.

Theorem 3.2. *Consider $i \in \{1, \cdots, N_F\}$.*
(+) The mapping $\mathbf{x} \mapsto \mathbf{x} \cdot \mathbf{n}_{|F_i}$ is linear and continuous from $\mathcal{X}^R$ to $\widetilde{H}^{1/2}(F_i)$;
(i) The mapping $\mathbf{x} \mapsto \mathbf{x} \cdot \mathbf{n}_{|F_i}$ is surjective from $\mathcal{X}_i^R$ to $\widetilde{H}^{1/2}(F_i)$. Its kernel is $\mathbf{H}_0^1(\Omega)$.

A direct consequence is

Corollary 3.1. *The space $\mathcal{X}^R$ can be written as the sum: $\mathcal{X}^R = \mathcal{X}_1^R + \cdots + \mathcal{X}_{N_F}^R$.*

To obtain a direct sum, note that $\mathbf{H}_0^1(\Omega) = \cap_i \mathcal{X}_i^R$, so one can consider the sum of the quotient spaces $\mathcal{X}_i^R/\mathbf{H}_0^1(\Omega)$, plus $\mathbf{H}_0^1(\Omega)$.

As far as gibp formulas involving vector fields of $\mathcal{X}^R$ are concerned, let us provide one formula from [4], whereas the other one is standard, if one recalls that $\mathbf{H}_0(\mathbf{curl}\,,\Omega)$ is – by definition – the closure of $\mathcal{D}(\Omega)^3$ in $\mathbf{H}(\mathbf{curl}\,,\Omega)$.

Theorem 3.3. *The following gibp formulas hold:*

(19) $(p,\operatorname{div}\mathbf{x})_0+\langle\nabla p,\mathbf{x}\rangle_{\mathcal{X}^R}=\sum_i\langle p_{|F_i},\mathbf{x}\cdot\mathbf{n}_{|F_i}\rangle_{\widetilde{H}^{1/2}(F_i)},\ \forall(p,\mathbf{x})\in D(\Delta;\Omega)\times\mathcal{X}^R;$

(20) $\langle\mathbf{curl}\,\mathbf{p},\mathbf{x}\rangle_{\mathbf{H}_0(\mathbf{curl}\,,\Omega)}-(\mathbf{p},\mathbf{curl}\,\mathbf{x})_0=0,\ \forall(\mathbf{p},\mathbf{x})\in\mathbf{L}^2(\Omega)\times\mathbf{H}_0(\mathbf{curl}\,,\Omega).$

In order to transpose the results (when possible) to the magnetic case, we shall use the vector trace spaces related to the tangential trace of fields, that is $\mathbf{H}_\perp^{1/2}(\partial\Omega)$ and the likes. As a matter of fact, the nature of the normal and tangential traces are fundamentally different, since the first one is a scalar and the second one is a vector. It can be proven [26, pp. 177-178] that

Theorem 3.4. *Consider $i\in\{1,\cdots,N_F\}$.*
The mapping $\mathbf{y}\mapsto\mathbf{y}\times\mathbf{n}_{|F_i}$ is surjective from $\mathcal{Y}_i^R$ to $\widetilde{\mathbf{H}}_\perp^{1/2}(F_i)$. Its kernel is $\mathbf{H}_0^1(\Omega)$.

However, since item (+) of Theorem 3.2 has no equivalent in the magnetic case (for reasons similar to those expressed in Section 2), there is no equivalent of Corollary 3.1. Still, one obtains an adequate property in this case. To that aim, one has to use Lemma 2.6 (step 3 of its proof) in [24], which yields a density property:

Lemma 3.1. *The space*

$$\mathcal{Y}_\infty^R=\{\mathbf{v}\in\mathcal{Y}^R\cap\mathcal{C}^\infty(\bar{\Omega})^3\ :\ \mathbf{v}\text{ vanishes in a neighborhood of the edges of }\partial\Omega\}$$

is dense in $\mathcal{Y}^R$.

Clearly, $\mathcal{Y}_\infty^R$ is a subset of the sum $\sum_i\mathcal{Y}_i^R$, so one infers

Corollary 3.2. *The sum $\sum_i\mathcal{Y}_i^R$ is dense in $\mathcal{Y}^R$.*

Transposing (19) is standard, if one considers elements of $L^2(\Omega)\times\mathbf{H}_0(\operatorname{div}\,,\Omega)$. On the other hand, in order to derive a formula similar to (20) for elements of $\mathbf{D}(\boldsymbol{\Delta};\Omega)\times\mathcal{Y}^R$ (or of a suitable subset), we recall the integration by parts formula

$$(\mathbf{curl}\,\mathbf{p},\mathbf{y})_0-(\mathbf{p},\mathbf{curl}\,\mathbf{y})_0=\int_{\partial\Omega}\pi_\tau\mathbf{p}\cdot\mathbf{y}\times\mathbf{n}\,d\Gamma,\tag{21}$$

for smooth vector fields $\mathbf{p}$, $\mathbf{y}$. Then, since the local tangential trace of elements of $\mathcal{Y}_i^R$ belongs to $\widetilde{\mathbf{H}}_\perp^{1/2}(F_i)$, one requires that $\mathbf{p}$ is such that $\pi_\tau\mathbf{p}$ belongs to its dual $(\widetilde{\mathbf{H}}_\perp^{1/2}(F_i))'$. Owing to item (ii) of Theorem 2.1, this is the case of elements of $\mathbf{D}(\boldsymbol{\Delta};\Omega)$. Summing up the results (cf. [26, pp. 179-181] for (23)), one gets

Theorem 3.5. *The following gibp formulas hold:*

$$(p,\operatorname{div}\mathbf{y})_0+\langle\nabla p,\mathbf{y}\rangle_{\mathbf{H}_0(\operatorname{div}\,,\Omega)}=0,\quad\forall(p,\mathbf{y})\in L^2(\Omega)\times\mathbf{H}_0(\operatorname{div}\,,\Omega);\tag{22}$$

$$\langle\mathbf{curl}\,\mathbf{p},\mathbf{y}\rangle_{\mathcal{Y}_i^R}-(\mathbf{p},\mathbf{curl}\,\mathbf{y})_0=\left\langle\pi_\tau\mathbf{p}_{|F_i},\mathbf{y}\times\mathbf{n}_{|F_i}\right\rangle_{\widetilde{\mathbf{H}}_\perp^{1/2}(F_i)},\quad\forall(\mathbf{p},\mathbf{y})\in\mathbf{D}(\boldsymbol{\Delta};\Omega)\times\mathcal{Y}_i^R.\tag{23}$$

4. Characterizations of the singular electromagnetic fields

From Theorem 3.1, one recalls that $\mathcal{X}^R$ (resp. $\mathcal{Y}^R$) is closed in $\mathcal{X}$ (resp. $\mathcal{Y}$). Then one can define the singular spaces by orthogonality.

Definition 4.1. *Let $\mathcal{X}^S$ be the space of singular electric fields so that one has $\mathcal{X} = \mathcal{X}^R \overset{\perp_W}{\oplus} \mathcal{X}^S$.*
Let $\mathcal{Y}^S$ be the space of singular magnetic fields so that one has $\mathcal{Y} = \mathcal{Y}^R \overset{\perp_W}{\oplus} \mathcal{Y}^S$.

Now, with the help of the generalized integration by parts and the surjectivity results (or Corollary 3.2 in the magnetic case), we are able to establish the following characterizations of the singular electromagnetic fields, that involve the vector Laplace operator, standardly defined by $-\boldsymbol{\Delta} = \mathbf{curl}\,\mathbf{curl} - \nabla\mathrm{div}$.

Theorem 4.1. *An element $\underline{\mathbf{x}}$ of $\mathcal{X}$ is singular if, and only if, the condition (24) below is fulfilled:*

$$-\boldsymbol{\Delta}\,\underline{\mathbf{x}} = 0 \ \text{in}\ \Omega, \quad \mathrm{div}\,\underline{\mathbf{x}}_{|F_i} = 0 \ \text{in}\ \widetilde{H}^{-1/2}(F_i),\ \forall i. \tag{24}$$

An element $\underline{\mathbf{y}}$ of $\mathcal{Y}$ is singular if, and only if, the condition (25) below is fulfilled:

$$-\boldsymbol{\Delta}\,\underline{\mathbf{y}} = 0 \ \text{in}\ \Omega, \quad \pi_\tau(\mathbf{curl}\,\underline{\mathbf{y}})_{|F_i} = 0 \ \text{in}\ (\widetilde{\mathbf{H}}_\perp^{1/2}(F_i))',\ \forall i. \tag{25}$$

Proof. The case of the singular electric fields. Let $\underline{\mathbf{x}} \in \mathcal{X}$.

(I) $\underline{\mathbf{x}} \in \mathcal{X}^S \implies \underline{\mathbf{x}}$ satisfies (24).

Any $\mathbf{x}$ of $\mathcal{D}(\Omega)^3$ belongs to $\mathcal{X}^R$. Thus, the orthogonality with respect to the Weber scalar product for smooth vector fields yields

$$0 = (\underline{\mathbf{x}}, \mathbf{x})_W = \langle \mathbf{curl}\,\mathbf{curl}\,\underline{\mathbf{x}} - \nabla\mathrm{div}\,\underline{\mathbf{x}}, \mathbf{x}\rangle, \quad \forall \mathbf{x} \in \mathcal{D}(\Omega)^3.$$

But one has $-\boldsymbol{\Delta} = \mathbf{curl}\,\mathbf{curl} - \nabla\mathrm{div}$, so the first part of (24) follows (in particular, in the dual of $\mathcal{X}^R$).

To prove the second part of (24), let us consider the gibp formulas of Theorem 3.3. The first one, (19), is used with $p = \mathrm{div}\,\underline{\mathbf{x}}$. As a matter of fact, p belongs to $D(\Delta;\Omega)$, according to the remark that, for any v in $\mathcal{D}(\Omega)$, its gradient is in $\mathcal{X}^R$, so the orthogonality with respect to the Weber scalar product once again yields

$$0 = (p, \Delta v)_0 = \langle \Delta p, v\rangle = 0, \quad \forall v \in \mathcal{D}(\Omega),$$

and $\Delta p = 0$. In particular, Δp is in $L^2(\Omega)$. The second gibp formula, (20), is simply used with $\mathbf{p} = \mathbf{curl}\,\underline{\mathbf{x}}$. One finds, for any $\mathbf{x}$ in $\mathcal{X}^R$:

$$\begin{aligned}
0 = (\underline{\mathbf{x}}, \mathbf{x})_W &= -\langle \nabla p, \mathbf{x}\rangle_{\mathcal{X}^R} + \langle \mathbf{curl}\,\mathbf{p}, \mathbf{x}\rangle_{\mathbf{H}_0(\mathbf{curl}\,,\Omega)} + \sum_i \langle p_{|F_i}, \mathbf{x}\cdot\mathbf{n}_{|F_i}\rangle_{\widetilde{H}^{1/2}(F_i)} \\
&= \langle \mathbf{curl}\,\mathbf{curl}\,\underline{\mathbf{x}} - \nabla\mathrm{div}\,\underline{\mathbf{x}}, \mathbf{x}\rangle_{\mathcal{X}^R} + \sum_i \langle \mathrm{div}\,\underline{\mathbf{x}}_{|F_i}, \mathbf{x}\cdot\mathbf{n}_{|F_i}\rangle_{\widetilde{H}^{1/2}(F_i)} \\
&= \sum_i \langle \mathrm{div}\,\underline{\mathbf{x}}_{|F_i}, \mathbf{x}\cdot\mathbf{n}_{|F_i}\rangle_{\widetilde{H}^{1/2}(F_i)}.
\end{aligned}$$

Above, we took advantage of the fact that $\mathcal{X}^R$ is a dense subset of $\mathbf{H}_0(\mathbf{curl}\,,\Omega)$, so one can replace $\langle \mathbf{curl}\,\mathbf{p}, \mathbf{x}\rangle_{\mathbf{H}_0(\mathbf{curl}\,,\Omega)}$ by $\langle \mathbf{curl}\,\mathbf{p}, \mathbf{x}\rangle_{\mathcal{X}^R}$. The conclusion follows from the surjectivity property (i) stated in Theorem 3.2.

(II) $\underline{\mathbf{x}}$ satisfies (24) $\implies \underline{\mathbf{x}} \in \mathcal{X}^S$.

Since $-\mathbf{\Delta}\,\underline{\mathbf{x}} = 0$, one finds that, in the sense of distributions,

$$\Delta(\operatorname{div}\underline{\mathbf{x}}) = \operatorname{div}\nabla(\operatorname{div}\underline{\mathbf{x}}) = \operatorname{div}(\nabla\operatorname{div}\underline{\mathbf{x}}) = -\operatorname{div}(\mathbf{\Delta}\,\underline{\mathbf{x}}) = 0.$$

Therefore, $p = \operatorname{div}\underline{\mathbf{x}}$ belongs to $D(\Delta;\Omega)$, and (19) can be used with p. Evidently, (20) can also be used, with $\mathbf{p} = \mathbf{curl}\,\underline{\mathbf{x}}$.

To check the orthogonality condition for $\underline{\mathbf{x}}$, which satisfies (24), let us write, for any $\mathbf{x}$ in $\mathcal{X}^R$:

$$\begin{aligned}(\underline{\mathbf{x}},\mathbf{x})_W &= \langle \mathbf{curl}\,\mathbf{p},\mathbf{x}\rangle_{\mathbf{H}_0(\mathbf{curl}\,,\Omega)} - \langle \nabla\operatorname{div}\underline{\mathbf{x}},\mathbf{x}\rangle_{\mathcal{X}^R}\\ &= \langle \mathbf{curl}\,\mathbf{curl}\,\underline{\mathbf{x}} - \nabla\operatorname{div}\underline{\mathbf{x}},\mathbf{x}\rangle_{\mathcal{X}^R} = -\langle \mathbf{\Delta}\,\underline{\mathbf{x}},\mathbf{x}\rangle_{\mathcal{X}^R} = 0.\end{aligned}$$

This ends the proof for the characterization of singular electric fields.

The case of the singular magnetic fields. Let $\underline{\mathbf{y}} \in \mathcal{Y}$.

(III) $\underline{\mathbf{y}} \in \mathcal{Y}^S \Longrightarrow \underline{\mathbf{y}}$ satisfies (25).

As in (I), one finds that $-\mathbf{\Delta}\,\underline{\mathbf{y}} = 0$, i. e. the first part of (25) is proved.

To prove the second part of (25), let us use the gibp formulas of Theorem 3.5, with regular fields $\mathbf{y}$ of $\mathcal{Y}_i^R$.
For (22), choose $p = \operatorname{div}\underline{\mathbf{y}}$. For (23), one can consider $\mathbf{p} = \mathbf{curl}\,\underline{\mathbf{y}}$. Indeed, one has in the sense of distributions,

$$\mathbf{\Delta}(\mathbf{curl}\,\underline{\mathbf{y}}) = -\mathbf{curl}\,\mathbf{curl}\,(\mathbf{curl}\,\underline{\mathbf{y}}) = -\mathbf{curl}\,(\mathbf{curl}\,\mathbf{curl}\,\underline{\mathbf{y}}) = \mathbf{curl}\,(\mathbf{\Delta}\,\underline{\mathbf{y}}) = 0,$$

so $\mathbf{p}$ is an element of $\mathbf{D}(\mathbf{\Delta};\Omega)$.
Then, for any $\mathbf{y}$ in $\mathcal{Y}_i^R$:

$$\begin{aligned}0 = (\underline{\mathbf{y}},\mathbf{y})_W &= -\langle \nabla p,\mathbf{y}\rangle_{\mathbf{H}_0(\operatorname{div}\,,\Omega)} + \langle \mathbf{curl}\,\mathbf{p},\mathbf{y}\rangle_{\mathcal{Y}_i^R} + \langle \pi_\tau \mathbf{p}_{|F_i}, \mathbf{y}\times\mathbf{n}_{|F_i}\rangle_{\tilde{\mathbf{H}}_\perp^{1/2}(F_i)}\\ &= \langle \pi_\tau(\mathbf{curl}\,\underline{\mathbf{y}})_{|F_i}, \mathbf{y}\times\mathbf{n}_{|F_i}\rangle_{\tilde{\mathbf{H}}_\perp^{1/2}(F_i)}.\end{aligned}$$

Above, we used the fact that $\mathcal{Y}_i^R$ is a dense subset of $\mathbf{H}_0(\operatorname{div}\,,\Omega)$. The conclusion follows from the surjectivity property stated in Theorem 3.4.

(IV) $\underline{\mathbf{y}}$ satisfies (25) $\Longrightarrow \underline{\mathbf{y}} \in \mathcal{Y}^S$.

Since $-\mathbf{\Delta}\,\underline{\mathbf{y}} = 0$, one finds in the sense of distributions

$$\mathbf{\Delta}(\mathbf{curl}\,\underline{\mathbf{y}}) = -\mathbf{curl}\,\mathbf{curl}\,(\mathbf{curl}\,\underline{\mathbf{y}}) = -\mathbf{curl}\,(\mathbf{curl}\,\mathbf{curl}\,\underline{\mathbf{y}}) = \mathbf{curl}\,(\mathbf{\Delta}\,\underline{\mathbf{y}}) = 0.$$

Therefore, $\mathbf{p} = \mathbf{curl}\,\underline{\mathbf{y}}$ belongs to $\mathbf{D}(\mathbf{\Delta};\Omega)$, and (23) can be used. On the other hand, (20) holds with $p = \operatorname{div}\underline{\mathbf{y}}$.

To check the orthogonality condition for $\underline{\mathbf{y}}$, which satisfies (25), let us recall Corollary 3.2, which states that $\sum_i \mathcal{Y}_i^R$ is a dense subspace of $\mathcal{Y}^R$. So, it is enough to check the orthogonality condition for all i, and for all $\mathbf{y} \in \mathcal{Y}_i^R$:

$$\begin{aligned}(\underline{\mathbf{y}},\mathbf{y})_W &= \langle \mathbf{curl}\,\mathbf{p},\mathbf{y}\rangle_{\mathcal{Y}_i^R} - \langle \nabla\operatorname{div}\underline{\mathbf{y}},\mathbf{y}\rangle_{\mathbf{H}_0(\operatorname{div}\,,\Omega)}\\ &= \langle \mathbf{curl}\,\mathbf{curl}\,\underline{\mathbf{y}} - \nabla\operatorname{div}\underline{\mathbf{y}},\mathbf{y}\rangle_{\mathcal{Y}_i^R} = -\langle \mathbf{\Delta}\,\underline{\mathbf{y}},\mathbf{y}\rangle_{\mathcal{Y}_i^R} = 0.\end{aligned}$$

This concludes the proof for the characterization of singular magnetic fields. $\diamond$

5. Divergence-free space characterizations

Since magnetic fields are actually divergence-free, it is relevant to introduce the following subspace of $\mathcal{Y}$

Definition 5.1. *Let $\mathcal{W}$ be the space of "real" magnetic fields, with*

$$\mathcal{W} = \{\mathbf{w} \in \mathcal{Y}\ :\ \operatorname{div}\mathbf{w} = 0\}. \tag{26}$$

On $\mathcal{W}$, the Weber scalar product reduces to

$$(\mathbf{u},\mathbf{v}) \mapsto (\mathbf{curl}\,\mathbf{u},\mathbf{curl}\,\mathbf{v})_0. \tag{27}$$

Let us then consider the subspace of regular fields.

Definition 5.2. *Consider the regular subspace of* $\mathcal{W}$

$$\mathcal{W}^R = \mathcal{W} \cap \mathbf{H}^1(\Omega). \tag{28}$$

It is clear that $\mathcal{W}^R$ is closed in $\mathcal{W}$, because one can also write $\mathcal{W}^R = \{\mathbf{w} \in \mathcal{Y}^R : \operatorname{div}\mathbf{w} = 0\}$. Indeed, let $(\mathbf{w}_n)_n$ be a sequence of elements of $\mathcal{W}^R$, which converges to $\mathbf{w}$ in $\mathcal{W}$. In $\mathcal{Y}$, this amounts to saying that $(\mathbf{w}_n)_n$ is a sequence of divergence-free elements of $\mathcal{Y}^R$. As $\mathcal{Y}^R$ is closed in $\mathcal{Y}$, $\mathbf{w}$ belongs to $\mathcal{Y}^R$, and therefore to $\mathcal{W}^R$, as it is divergence-free.

The singular space can thus be defined by orthogonality. To that aim, we shall use

$$V = \mathbf{H}^1_0(\Omega) \cap \mathbf{H}(\operatorname{div} 0;\Omega).$$

Definition 5.3. *Let* $\mathcal{W}^S$ *be the space of "real" singular magnetic fields so that one has* $\mathcal{W} = \mathcal{W}^R \overset{\perp_W}{\oplus} \mathcal{W}^S$.

Following the process described in [4], one can prove along the same lines the

Theorem 5.1. *An element* $\underline{\mathbf{w}}$ *of* $\mathcal{W}$ *is singular if, and only if, the condition (29) below is fulfilled:*

$$\exists p \in L^2_0(\Omega) \ \text{s.t.}\ (\mathbf{curl}\,\underline{\mathbf{w}},\mathbf{curl}\,\mathbf{y})_0 + (p,\operatorname{div}\mathbf{y})_0 = 0,\ \forall \mathbf{y} \in \mathcal{Y}^R. \tag{29}$$

Proof. Let $\underline{\mathbf{w}}$ be an element of $\mathcal{W}$, we have $\underline{\mathbf{w}} \in \mathcal{W}^S$ if and only if

$$(\mathbf{curl}\,\underline{\mathbf{w}},\mathbf{curl}\,\mathbf{z})_0 = 0 \ \forall \mathbf{z} \in \mathcal{W}^R\,.$$

For $\underline{\mathbf{w}} \in \mathcal{W}^S$, consider then the linear form l defined by

$$l \ : \ \mathbf{y} \mapsto \langle l,\mathbf{y}\rangle = (\mathbf{curl}\,\underline{\mathbf{w}},\mathbf{curl}\,\mathbf{y})_0$$

defined and continuous on $\mathcal{Y}^R$ which cancels on $\mathcal{W}^R$. In particular, it is a continuous linear form on $\mathbf{H}^1_0(\Omega) \subset \mathcal{Y}^R$, which cancels on V. Due to the de Rham Theorem [27, Theorem 2.3], there exists $p \in L^2(\Omega)$, defined up to a constant, such that

$$\langle l,\mathbf{y}\rangle = -(p,\operatorname{div}\mathbf{y})_0 \ \forall \mathbf{y} \in \mathbf{H}^1_0(\Omega)\,.$$

Hence, one can choose $p \in L^2_0(\Omega)$, and we have

$$(\mathbf{curl}\,\underline{\mathbf{w}},\mathbf{curl}\,\mathbf{y})_0 + (p,\operatorname{div}\mathbf{y})_0 = 0 \ \forall \mathbf{y} \in \mathbf{H}^1_0(\Omega)\,.$$

Now let $\mathbf{y} \in \mathcal{Y}^R$. In particular, Green's formula yields $(\operatorname{div}\mathbf{y},1)_0 = 0$, i. e. $\operatorname{div}\mathbf{y} \in L^2_0(\Omega)$. After [27, Corollary 2.4], there exists a function $\mathbf{v} \in \mathbf{H}^1_0(\Omega)$ such that $\operatorname{div}\mathbf{v} = -\operatorname{div}\mathbf{y}$.

Then the function $\mathbf{y}+\mathbf{v}$ of $\mathcal{Y}^R$ verifies $\operatorname{div}(\mathbf{y}+\mathbf{v}) = 0$, so that $\mathbf{y}+\mathbf{v} \in \mathcal{W}^R$. This implies

$$(\mathbf{curl}\,\underline{\mathbf{w}},\mathbf{curl}\,(\mathbf{y}+\mathbf{v}))_0 = 0,$$

that is

$$(\mathbf{curl}\,\underline{\mathbf{w}},\mathbf{curl}\,\mathbf{y})_0 = -(\mathbf{curl}\,\underline{\mathbf{w}},\mathbf{curl}\,\mathbf{v})_0 = (p,\operatorname{div}\mathbf{v})_0 = -(p,\operatorname{div}\mathbf{y})_0\,,$$

or

$$(\mathbf{curl}\,\underline{\mathbf{w}},\mathbf{curl}\,\mathbf{y})_0 + (p,\operatorname{div}\mathbf{y})_0 = 0\,, \quad \forall \mathbf{y} \in \mathcal{Y}^R.$$

Conversely if $\underline{\mathbf{w}} \in \mathcal{W}$ satisfies this last relation, we have straightforwardly

$$(\mathbf{curl}\,\underline{\mathbf{w}},\mathbf{curl}\,\mathbf{z})_0 = 0 \ \forall \mathbf{z} \in \mathcal{W}^R,$$

so that $\underline{\mathbf{w}} \in \mathcal{W}^S$. $\diamond$

Then, we relate fields of $\mathcal{W}^S$ to $\mathbb{S}_N$. Recall that we provided in Corollary 2.3 a characterization of the elements of $\mathbb{S}_N$, the dual singularities with Neumann boundary condition.

First, one gets easily the intermediate characterization hereunder.

Proposition 5.1. *Let* $(\mathbf{v}, p) \in \mathbf{H}(\mathbf{curl}\,, \Omega) \times L^2_0(\Omega)$. *The couple* $(\mathbf{v}, p)$ *satisfies*

$$(\mathbf{curl}\,\mathbf{v}, \mathbf{curl}\,\mathbf{y})_0 + (p, \operatorname{div}\mathbf{y})_0 = 0,\ \forall \mathbf{y} \in \mathcal{Y}^R,$$

if, and only if, p *belongs to* $\mathbb{S}_N$*, and the conditions below hold*

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} = \nabla p \text{ in } (\mathbf{H}_0(\operatorname{div}\,, \Omega))', \quad \pi_\tau(\mathbf{curl}\,\mathbf{v})_{|F_i} = 0 \text{ in } (\widetilde{\mathbf{H}}^{1/2}_{\perp}(F_i))',\ \forall i.$$

Proof. Choosing $\mathbf{y} = \nabla v$, for any element $v \in H^N(\Omega)$, we find $(p, \Delta v)_0 = 0$, so that $p \in (\Delta H^N(\Omega))^{\perp} = \mathbb{S}_N$.
Taking $\mathbf{y} \in \mathcal{D}(\Omega)^3$, we immediately have $\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} = \nabla p$ in the sense of distributions. With the help of the gibp (22), we see that this relation actually holds in $(\mathbf{H}_0(\operatorname{div}\,, \Omega))'$, as p belongs to $L^2(\Omega)$.

To prove the second part, we consider successively the two gibp formulas of Theorem 3.5. For the first one, we take advantage of the fact that $\mathcal{Y}^R_i$ is a dense subset of $\mathbf{H}_0(\operatorname{div}\,, \Omega)$, so one can replace $\langle \nabla p, \mathbf{y}\rangle_{\mathbf{H}_0(\operatorname{div}\,,\Omega)}$ by $\langle \nabla p, \mathbf{y}\rangle_{\mathcal{Y}^R_i}$, when $\mathbf{y} \in \mathcal{Y}^R_i$.

The second gibp formula is used with $\mathbf{curl}\,\mathbf{v} = \mathbf{p}$. Indeed, since $\mathbf{curl}\,\mathbf{p} = \nabla p$, such a $\mathbf{p}$ is an element of $\mathbf{D}(\boldsymbol{\Delta}; \Omega)$: in the sense of distributions, there holds

$$\boldsymbol{\Delta}\,\mathbf{p} = \boldsymbol{\Delta}(\mathbf{curl}\,\mathbf{v}) = -\mathbf{curl}\,\mathbf{curl}\,(\mathbf{curl}\,\mathbf{v}) = -\mathbf{curl}\,(\mathbf{curl}\,\mathbf{curl}\,\mathbf{v}) = -\mathbf{curl}\,(\nabla p) = 0.$$

One then finds, for any $\mathbf{y}$ in $\mathcal{Y}^R_i$:

$$\begin{aligned} 0 = (\mathbf{p}, \mathbf{curl}\,\mathbf{y})_0 + (p, \operatorname{div}\mathbf{y})_0 &= \langle \mathbf{curl}\,\mathbf{p} - \nabla p, \mathbf{y}\rangle_{\mathcal{Y}^R_i} - \sum_i \langle \pi_\tau \mathbf{p}_{|F_i}, \mathbf{y} \times \mathbf{n}_{|F_i}\rangle_{\widetilde{\mathbf{H}}^{1/2}_{\perp}(F_i)} \\ &= \sum_i \langle \pi_\tau \mathbf{p}_{|F_i}, \mathbf{y} \times \mathbf{n}_{|F_i}\rangle_{\widetilde{\mathbf{H}}^{1/2}_{\perp}(F_i)}\,. \end{aligned}$$

The conclusion follows from the surjectivity property stated in Theorem 3.4:
$\pi_\tau(\mathbf{curl}\,\mathbf{v})_{|F_i} = 0$ holds in $(\widetilde{\mathbf{H}}^{1/2}_{\perp}(F_i))'$.

Conversely, if $p \in \mathbb{S}_N$ verifies $\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} = \nabla p$ in $H_0(\operatorname{div}\,, \Omega)'$, let us use the gibp formulas of Theorem 3.5, with regular fields $\mathbf{y}$ of $\mathcal{Y}^R_i$.
For (23), one can still consider $\mathbf{p} = \mathbf{curl}\,\mathbf{v}$. Then, on the one hand, for any $\mathbf{y}$ in $\mathcal{Y}^R_i$:

$$\langle \mathbf{curl}\,\mathbf{curl}\,\mathbf{v}, \mathbf{y}\rangle_{\mathcal{Y}^R_i} = (\mathbf{curl}\,\mathbf{v}, \mathbf{curl}\,\mathbf{y})_0 + \langle \pi_\tau \mathbf{curl}\,\mathbf{v}_{|F_i}, \mathbf{y} \times \mathbf{n}_{|F_i}\rangle_{\widetilde{\mathbf{H}}^{1/2}_{\perp}(F_i)}\,,$$

that is, using the above boundary condition

$$\langle \mathbf{curl}\,\mathbf{curl}\,\mathbf{v}, \mathbf{y}\rangle_{\mathcal{Y}^R_i} = (\mathbf{curl}\,\mathbf{v}, \mathbf{curl}\,\mathbf{y})_0,\ \forall \mathbf{y} \in \mathcal{Y}^R_i.$$

On the other hand, one gets from (22)

$$-\langle \nabla p, \mathbf{y}\rangle_{\mathcal{Y}^R_i} = (p, \operatorname{div}\mathbf{y})_0,\ \forall \mathbf{y} \in \mathcal{Y}^R_i.$$

Hence, the conclusion follows since $\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} = \nabla p$ holds in $(\mathcal{Y}^R_i)'$. $\diamond$

These two results can be aggregated.

Theorem 5.2. *An element $\underline{\mathbf{w}}$ of $\mathcal{W}$ is singular if, and only if, the condition (30) below is fulfilled:*

$$\begin{aligned} &\mathbf{curl}\,\mathbf{curl}\,\underline{\mathbf{w}} = \nabla p \ \textit{in}\ (\mathbf{H}_0(\mathrm{div}\,,\Omega))',\ \textit{with}\ p \in \mathbb{S}_N;\\ &\pi_\tau(\mathbf{curl}\,\underline{\mathbf{w}})_{|F_i} = 0 \ \textit{in}\ (\widetilde{\mathbf{H}}_\perp^{1/2}(F_i))',\ \forall i. \end{aligned} \tag{30}$$

We recall here without proofs (cf. [4]) the analogous results for the electric fields, in the case when they are divergence-free. Let us introduce the following subspace of $\mathcal{X}$

Definition 5.4. *Let $\mathcal{V}$ be the space of divergence-free electric fields, with*

$$\mathcal{V} = \{\mathbf{v} \in \mathcal{X}\ :\ \mathrm{div}\,\mathbf{v} = 0\}. \tag{31}$$

On $\mathcal{V}$, the Weber scalar product reduces to the same as the one on $\mathcal{W}$, i. e. $(\mathbf{u}, \mathbf{v}) \mapsto (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_0$. The subspace of regular fields is defined here as

Definition 5.5.

$$\mathcal{V}^R = \mathcal{V} \cap \mathbf{H}^1(\Omega), \tag{32}$$

and it is closed in $\mathcal{V}$. The singular space can still be defined by orthogonality.

Definition 5.6. *Let $\mathcal{V}^S$ be the space of divergence-free singular electric fields: $\mathcal{V} = \mathcal{V}^R \overset{\perp_W}{\oplus} \mathcal{V}^S$.*

Then it is proved [4, Theorem 3.3]

Theorem 5.3. *An element $\underline{\mathbf{v}}$ of $\mathcal{V}$ is singular if, and only if, the condition (33) below is fulfilled:*

$$\exists p \in L^2(\Omega)\ \textit{s.t.}\ (\mathbf{curl}\,\underline{\mathbf{v}}, \mathbf{curl}\,\mathbf{x})_0 + (p, \mathrm{div}\,\mathbf{x})_0 = 0,\ \forall \mathbf{x} \in \mathcal{X}^R. \tag{33}$$

Next, it is proved [4, Lemma 3.2] that p is a dual singularity (with Dirichlet boundary condition).

Proposition 5.2. *Let $(\mathbf{v}, p) \in \mathbf{H}_0(\mathbf{curl}\,,\Omega) \times L^2(\Omega)$. The couple $(\mathbf{v}, p)$ satisfies*

$$(\mathbf{curl}\,\mathbf{v}, \mathbf{curl}\,\mathbf{x})_0 + (p, \mathrm{div}\,\mathbf{x})_0 = 0,\ \forall \mathbf{x} \in \mathcal{X}^R,$$

if, and only if, p belongs to $\mathbb{S}_D$, and the condition below holds

$$\mathbf{curl}\,\mathbf{curl}\,\mathbf{v} = \nabla p \ \textit{in}\ (\mathbf{H}_0(\mathbf{curl}\,,\Omega))'.$$

These two results can be aggregated (cf. [4, Theorem 3.2]).

Theorem 5.4. *An element $\underline{\mathbf{v}}$ of $\mathcal{V}$ is singular if, and only if, the condition (34) below is fulfilled:*

$$\mathbf{curl}\,\mathbf{curl}\,\underline{\mathbf{v}} = \nabla p \ \textit{in}\ (\mathbf{H}_0(\mathbf{curl}\,,\Omega))',\ \textit{with}\ p \in \mathbb{S}_D. \tag{34}$$

As far as numerical computations are concerned, the above characterizations of $\mathcal{W}$ and $\mathcal{V}$ are not very useful as they stand. However, in 2D geometries (cf. [5]) and in $2\frac{1}{2}$D geometries (cf. [16, 3]), a fruitful approach consists in introducing scalar potentials, which are then primal fields of the Laplace operator (with *ad hoc* boundary conditions). Here, in 3D geometries, we shall also introduce potentials of the elements of $\mathcal{V}$ and $\mathcal{W}$. Contrarily to the 2D and $2\frac{1}{2}$D cases, these are vector potentials.

6. The potentials and their characterizations

Our aim is to link the electromagnetic subspaces $\mathcal{W}$ and $\mathcal{V}$ (and more generally $\mathcal{Y}$ and $\mathcal{X}$) to the primal or dual spaces related to the vector Laplace operator. We have already introduced the scalar primal or dual fields (related to the scalar Laplace operator). Let us consider now the vector fields. For this purpose, one can still use the relation $-\boldsymbol{\Delta} = \mathbf{curl}\ \mathbf{curl} - \nabla \operatorname{div}$. First, we choose the gauge condition corresponding to divergence-free primal vector fields. Therefore, the previous operator identity reduces to $\boldsymbol{\Delta}\,\mathbf{u} = -\mathbf{curl}\ \mathbf{curl}\ \mathbf{u}$ when applied to such fields. As a consequence, the dual vector fields are also divergence-free: indeed, $\operatorname{div}(\boldsymbol{\Delta}\,\mathbf{u}) = 0$. In-between, we introduce the electromagnetic fields as follows. According to [27, Theorems 3.5 and 3.6], it is established that given a divergence-free field $\mathbf{u}_e \in \mathbf{H}(\operatorname{div}, \Omega)$ (resp. $\mathbf{u}_m \in \mathbf{H}_0(\operatorname{div}, \Omega)$), there exists $\mathbf{w} \in \mathcal{W}$ (resp. $\mathbf{v} \in \mathcal{V}$) such that $\mathbf{curl}\,\mathbf{w} = \mathbf{u}_e$ (resp. $\mathbf{curl}\,\mathbf{v} = \mathbf{u}_m$). In particular, this result concerning the existence of a potential applies to any $\mathbf{u}_e \in \mathcal{V}$ (resp. $\mathbf{u}_m \in \mathcal{W}$). Hence, it is relevant to introduce the following spaces

Definition 6.1. *Let $\boldsymbol{\Psi}(\mathcal{V})$ and $\boldsymbol{\Phi}(\mathcal{W})$ be the spaces of primal fields of $\mathcal{W}$ and $\mathcal{V}$ respectively, with*

$$\boldsymbol{\Psi}(\mathcal{V}) = \{\boldsymbol{\psi} \in \mathcal{V} \,:\, \boldsymbol{\Delta}\,\boldsymbol{\psi} \in \mathbf{L}^2(\Omega)\}\,,$$
$$\boldsymbol{\Phi}(\mathcal{W}) = \{\boldsymbol{\phi} \in \mathcal{W} \,:\, \boldsymbol{\Delta}\,\boldsymbol{\phi} \in \mathbf{L}^2(\Omega) \,:\, (\mathbf{curl}\,\boldsymbol{\phi}) \times \mathbf{n}_{|\partial\Omega} = 0\}\,.$$

The additional boundary condition in the definition of $\boldsymbol{\Phi}(\mathcal{W})$ allows us to obtain a one-to-one mapping between potentials and fields.

The spaces $\boldsymbol{\Psi}(\mathcal{V})$ and $\boldsymbol{\Phi}(\mathcal{W})$ are equipped with the following equivalent norm (the equivalence result is once more obtained as a consequence of the use of the Weber norm on $\mathcal{W}$ and $\mathcal{V}$)

$$\|\cdot\|_{\boldsymbol{\Delta}} = \|\mathbf{curl}\ \mathbf{curl}\ \cdot\|_0 = \|\,\boldsymbol{\Delta}\,\cdot\|_0 \tag{35}$$

This definition allows us to prove isomorphisms between the relevant spaces. These isomorphisms are linked by $\boldsymbol{\Delta} = -\mathbf{curl}\ \mathbf{curl}$ and can be summarized in the graphs below (see [26, pp. 185-186] for detailed proofs)

$$\begin{tikzcd}
(\boldsymbol{\Psi}(\mathcal{V}); \|\cdot\|_{\boldsymbol{\Delta}}) \arrow[r, "\mathbf{curl}"] \arrow[drr, phantom, "\xrightarrow{\boldsymbol{\Delta}=-\mathbf{curl}\,\mathbf{curl}}"] & (\mathcal{W}; \|\cdot\|_W) \arrow[r, "\mathbf{curl}"] & (\mathbf{H}(\operatorname{div} 0; \Omega); \|\cdot\|_0) \\
(\boldsymbol{\Phi}(\mathcal{W}); \|\cdot\|_{\boldsymbol{\Delta}}) \arrow[r, "\mathbf{curl}"] & (\mathcal{V}; \|\cdot\|_W) \arrow[r, "\mathbf{curl}"] & (\mathbf{H}_0(\operatorname{div} 0; \Omega); \|\cdot\|_0)
\end{tikzcd}$$

In addition, since norms are preserved[2], these isomorphisms are *isometries*, so orthogonality is also preserved, thanks to the well-known identity $(a, b)_X = \frac{1}{4}(\|a + b\|_X^2 - \|a - b\|_X^2)$.

Remark 6.1. *As a matter of fact, one gets*

$$\boldsymbol{\Psi}(\mathcal{V}) = \{\boldsymbol{\psi} \in \mathcal{V} \,:\, \boldsymbol{\Delta}\,\boldsymbol{\psi} \in \mathbf{H}(\operatorname{div} 0; \Omega)\}\,,$$
$$\boldsymbol{\Phi}(\mathcal{W}) = \{\boldsymbol{\phi} \in \mathcal{W} \,:\, \boldsymbol{\Delta}\,\boldsymbol{\phi} \in \mathbf{H}_0(\operatorname{div} 0; \Omega) \,:\, (\mathbf{curl}\,\boldsymbol{\phi}) \times \mathbf{n}_{|\partial\Omega} = 0\}\,.$$

Moreover, it is possible to rewrite the boundary conditions differently for elements of $\boldsymbol{\Phi}(\mathcal{W})$ (see [26, pp. 186-189] *for more details). First, for a smooth field, one has the identity, $\partial_n \boldsymbol{\phi} = \nabla(\boldsymbol{\phi} \cdot \mathbf{n}_i) + (\mathbf{curl}\,\boldsymbol{\phi}) \times \mathbf{n}$ on the face F_i. Then, one proves by density of $\mathbf{H}^2(\Omega)$ in $\mathbf{D}(\boldsymbol{\Delta}; \Omega)$ that this holds true for any element $\boldsymbol{\phi}$ of $\boldsymbol{\Phi}(\mathcal{W})$,*

[2]For instance, for any $\boldsymbol{\psi} \in \boldsymbol{\Psi}(\mathcal{V})$, one has $\|\boldsymbol{\psi}\|_{\boldsymbol{\Delta}} = \|\mathbf{curl}\,\boldsymbol{\psi}\|_W = \|\,\boldsymbol{\Delta}\,\boldsymbol{\psi}\|_0$.

in the dual space $\widetilde{\mathbf{H}}^{-3/2}(F_i)$*. In particular, since* $\boldsymbol{\phi} \cdot \mathbf{n}_{|F_i} = 0$*, by restricting the identity to the tangential components, one finds actually* $\pi_\tau(\partial_n \boldsymbol{\phi}) = \gamma_\tau(\mathbf{curl}\,\boldsymbol{\phi})$ *in* $(\widetilde{\mathbf{H}}_\perp^{3/2}(F_i))'$*. This allows us to replace the boundary condition* $(\mathbf{curl}\,\boldsymbol{\phi}) \times \mathbf{n}_{|\partial\Omega} = 0$ *in the definition of* $\boldsymbol{\Phi}(\mathcal{W})$ *by the equivalent*

$$\pi_\tau(\partial_n \boldsymbol{\phi}) = 0 \ \textit{in} \ (\widetilde{\mathbf{H}}_\perp^{3/2}(F_i))',\ \forall\ i.$$

As the domain Ω is non-convex, we have now to consider the subspaces of regular primal fields.

Remark 6.2. *When the domain is convex,* $\boldsymbol{\Psi}(\mathcal{V})$ *and* $\boldsymbol{\Phi}(\mathcal{W})$ *are both included in*

$$\mathbf{H}^1(\mathbf{curl}\,,\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathbf{curl}\,\mathbf{u} \in \mathbf{H}^1(\Omega)\}.$$

Unfortunately, it is not worthwhile to define the intersection with the space $\mathbf{H}^1(\mathbf{curl}\,,\Omega)$ *when* Ω *is non-convex. Indeed, there is no guarantee that the inverse image* $\boldsymbol{\psi} \in \boldsymbol{\Psi}(\mathcal{V})$ *of an element of* $\mathbf{w} \in \mathcal{W}^R$ *belongs to* $\mathbf{H}^1(\Omega)$*. The same is true for* $\boldsymbol{\Phi}(\mathcal{W})$*.*

We thus introduce the subspaces of regular fields.

Definition 6.2. *Consider the regular subspaces of* $\boldsymbol{\Psi}(\mathcal{V})$ *and* $\boldsymbol{\Phi}(\mathcal{W})$

$$\boldsymbol{\Psi}^R(\mathcal{V}) = \{\boldsymbol{\psi} \in \boldsymbol{\Psi}(\mathcal{V}) :\ \mathbf{curl}\,\boldsymbol{\psi} \in \mathcal{W}^R\}\,, \qquad \boldsymbol{\Phi}^R(\mathcal{W}) = \{\boldsymbol{\phi} \in \boldsymbol{\Phi}(\mathcal{W}) :\ \mathbf{curl}\,\boldsymbol{\phi} \in \mathcal{V}^R\}\,.$$

It is clear that $\boldsymbol{\Psi}^R(\mathcal{V})$ and $\boldsymbol{\Phi}^R(\mathcal{W})$ are closed in $\boldsymbol{\Psi}(\mathcal{V})$ and $\boldsymbol{\Phi}(\mathcal{W})$ respectively. The singular spaces can thus be defined by orthogonality.

Definition 6.3. *Let* $\boldsymbol{\Psi}^S(\mathcal{V})$ *and* $\boldsymbol{\Phi}^S(\mathcal{W})$ *be the spaces of singular primal fields:*

$$\boldsymbol{\Psi}(\mathcal{V}) = \boldsymbol{\Psi}^R(\mathcal{V}) \overset{\perp_{\boldsymbol{\Delta}}}{\oplus} \boldsymbol{\Psi}^S(\mathcal{V}), \qquad \boldsymbol{\Phi}(\mathcal{W}) = \boldsymbol{\Phi}^R(\mathcal{W}) \overset{\perp_{\boldsymbol{\Delta}}}{\oplus} \boldsymbol{\Phi}^S(\mathcal{W}).$$

As we are interested in the regular/singular splitting of the fields, we link the regular (respectively singular) parts of the potentials to the regular (resp. singular) parts of the divergence-free fields. Since we have isometries at our disposal, this is straightforward. In other words, **curl** is an isomorphism between $\boldsymbol{\Psi}^R(\mathcal{V})$ and $\mathcal{W}^R$, and also between $\boldsymbol{\Psi}^S(\mathcal{V})$ and $\mathcal{W}^S$. Similarly, it is an isomorphism between $\boldsymbol{\Phi}^R(\mathcal{W})$ and $\mathcal{V}^R$, and between $\boldsymbol{\Phi}^S(\mathcal{W})$ and $\mathcal{V}^S$.

Next, let us consider the spaces $\mathbf{H}(\mathrm{div}\,0;\Omega)$ and $\mathbf{H}_0(\mathrm{div}\,0;\Omega)$ of dual vector fields. In order to define the subspaces of regular dual fields, a natural choice induced by the above mappings and the previous results is

Definition 6.4. *Introduce the regular subspaces of* $\mathbf{H}(\mathrm{div}\,0;\Omega)$ *and* $\mathbf{H}_0(\mathrm{div}\,0;\Omega)$*, respectively equal to* $\boldsymbol{\Delta}\,\boldsymbol{\Psi}^R(\mathcal{V})$ *and* $\boldsymbol{\Delta}\,\boldsymbol{\Phi}^R(\mathcal{W})$*.*

Now, $\boldsymbol{\Delta}\,\boldsymbol{\Psi}^R(\mathcal{V})$ is closed in $\mathbf{H}(\mathrm{div}\,0;\Omega)$, and $\boldsymbol{\Delta}\,\boldsymbol{\Phi}^R(\mathcal{W})$ is also closed in $\mathbf{H}_0(\mathrm{div}\,0;\Omega)$. Indeed, $\boldsymbol{\Psi}^R(\mathcal{V})$ is closed in $\boldsymbol{\Psi}(\mathcal{V})$, and $\boldsymbol{\Delta}$ is an isometry from $\boldsymbol{\Psi}(\mathcal{V})$ to $\mathbf{H}(\mathrm{div}\,0;\Omega)$ (the same for $\boldsymbol{\Delta}\,\boldsymbol{\Phi}^R(\mathcal{W})$). Again the singular subspaces can be defined by orthogonality.

Definition 6.5. *Let* $\mathbf{S}$ *and* $\mathbf{S}_0$ *be the spaces of singular dual fields:*

$$\mathbf{H}(\mathrm{div}\,0;\Omega) = \boldsymbol{\Delta}\,\boldsymbol{\Psi}^R(\mathcal{V}) \overset{\perp_0}{\oplus} \mathbf{S}, \qquad \mathbf{H}_0(\mathrm{div}\,0;\Omega) = \boldsymbol{\Delta}\,\boldsymbol{\Phi}^R(\mathcal{W}) \overset{\perp_0}{\oplus} \mathbf{S}_0.$$

Remark 6.3. *The spaces* $\mathbf{S}$ *and* $\mathbf{S}_0$ *can be equivalently defined by*

$$\mathbf{H}(\mathrm{div}\,0;\Omega) = \mathbf{curl}\,\mathcal{W}^R \overset{\perp_0}{\oplus} \mathbf{S}, \qquad \mathbf{H}_0(\mathrm{div}\,0;\Omega) = \mathbf{curl}\,\mathcal{V}^R \overset{\perp_0}{\oplus} \mathbf{S}_0.$$

The one-to-one, and surjective, mappings, considered above, are still valid for regular and singular subspaces, respectively. We summarize these results for the singular spaces in the graph hereunder. The same remains true between the regular spaces.

$$\begin{array}{ccccc} (\mathbf{\Psi}^S(\mathcal{V}); \|\cdot\|_{\mathbf{\Delta}}) & \xrightarrow{\mathbf{curl}} & (\mathcal{W}^S; \|\cdot\|_W) & \xrightarrow{\mathbf{curl}} & (\mathbf{S}; \|\cdot\|_0) \\ & & \xrightarrow{\mathbf{\Delta}=-\mathbf{curl}\,\mathbf{curl}} & & \\ (\mathbf{\Phi}^S(\mathcal{W}); \|\cdot\|_{\mathbf{\Delta}}) & \xrightarrow{\mathbf{curl}} & (\mathcal{V}^S; \|\cdot\|_W) & \xrightarrow{\mathbf{curl}} & (\mathbf{S}_0; \|\cdot\|_0) \end{array}$$

We have already characterized the spaces $\mathbb{S}_N$ and $\mathbb{S}_D$ of dual singularities of the scalar Laplace problem, with homogeneous Neumann or Dirichlet boundary condition. This is readily extended to the vector Laplace operator $\mathbf{\Delta}$ with vector homogeneous boundary conditions by introducing $\mathbf{S}_N = (\mathbb{S}_N)^3$ and $\mathbf{S}_D = (\mathbb{S}_D)^3$. Now with $\mathbf{S}$ (and $\mathbf{S}_0$), we are interested in divergence-free fields, *a priori* without explicit (vector) boundary conditions. With the help of the generalized integration by parts formulas of Theorem 3.5, and a surjectivity result, one gets elements for the characterization of $\mathbf{S}$. Define $\mathcal{W}_i^R = \mathcal{W}^R \cap \mathcal{Y}_i^R$.

Lemma 6.1. *Consider $i \in \{1, \cdots, N_F\}$.*
The mapping $\mathbf{w} \mapsto \mathbf{w} \times \mathbf{n}_{|F_i}$ is surjective from $\mathcal{W}_i^R$ to $\widetilde{\mathbf{H}}_{\perp}^{1/2}(F_i)$. Its kernel is $\mathbf{H}_0^1(\Omega)$.

Proof. One has to construct a divergence-free lifting of any element of $\widetilde{\mathbf{H}}_{\perp}^{1/2}(F_i)$. Now, for any $\mathbf{h} \in \widetilde{\mathbf{H}}_{\perp}^{1/2}(F_i)$, there exists (cf. Theorem 3.4) $\mathbf{y} \in \mathcal{Y}_i^R$ such that $\mathbf{h} = \mathbf{y} \times \mathbf{n}_{|F_i}$. Using the fact that $\mathbf{y} \cdot \mathbf{n}_{|\partial\Omega} = 0$, we have $\operatorname{div} \mathbf{y} \in L_0^2(\Omega)$ and so there exists $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ such that $\operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{y}$. Hence, we set $\mathbf{w} = \mathbf{y} - \mathbf{u}$: this field belongs to $\mathcal{W}_i^R$, and one has $\mathbf{h} = \mathbf{w} \times \mathbf{n}_{|F_i}$. This proves that the mapping $\mathbf{w} \mapsto \mathbf{w} \times \mathbf{n}_{|F_i}$ is indeed surjective from $\mathcal{W}_i^R$ to $\widetilde{\mathbf{H}}_{\perp}^{1/2}(F_i)$. The characterization of the kernel is straightforward. $\diamond$

Theorem 6.1. *Any element $\mathbf{s}$ of $\mathbf{H}(\operatorname{div} 0; \Omega)$ which is singular, verifies the condition (36) below:*

$$\mathbf{s} \in \mathbf{D}(\mathbf{\Delta}, \Omega), \quad -\mathbf{\Delta}\,\mathbf{s} = 0, \quad \pi_\tau \mathbf{s}_{|F_i} = 0 \text{ in } (\widetilde{\mathbf{H}}_{\perp}^{1/2}(F_i))', \ \forall i. \tag{36}$$

Proof. Following the previous isomorphisms, $\mathbf{curl}\,\mathcal{W}^S = \mathbf{S}$, and any $\mathbf{s} \in \mathbf{S}$ is orthogonal to $\mathbf{curl}\,\mathbf{w}$, with $\mathbf{w} \in \mathcal{W}^R$ for the *ad hoc* scalar product: it reads

$$(\mathbf{s}, \mathbf{curl}\,\mathbf{w})_0 = 0, \qquad \forall \mathbf{s} \in \mathbf{S}, \quad \forall \mathbf{w} \in \mathcal{W}^R\,.$$

Using the fact that V is a subset of $\mathcal{W}^R$, and due to the de Rham Theorem [27, Theorem 2.3], there exists $p \in L^2(\Omega)$ such that

$$\mathbf{curl}\,\mathbf{s} = \nabla p \quad \text{ in the dual of } \mathcal{Y}^R\,. \tag{37}$$

But since $-\mathbf{\Delta}\,\mathbf{s} = \mathbf{curl}\,\mathbf{curl}\,\mathbf{s}$, one has $\mathbf{s} \in \mathbf{D}(\mathbf{\Delta}, \Omega)$ with $-\mathbf{\Delta}\,\mathbf{s} = 0$.

To prove the second part of (36), let us successively consider the gibp formulas of Theorem 3.5. The second one, (23), is used with $\mathbf{p} = \mathbf{s}$. According to (37), we find

$$\langle \nabla p, \mathbf{y} \rangle_{\mathcal{Y}_i^R} - (\mathbf{s}, \mathbf{curl}\,\mathbf{y})_0 = \left\langle \pi_\tau \mathbf{s}_{|F_i}, \mathbf{y} \times \mathbf{n}_{|F_i} \right\rangle_{\widetilde{\mathbf{H}}_{\perp}^{1/2}(F_i)}, \ \forall \mathbf{y} \in \mathcal{Y}_i^R.$$

As p belongs to $L^2(\Omega)$, the other gibp formula (22) can be used to replace the first term. Then, we reach

$$(p, \operatorname{div} \mathbf{y})_0 + (\mathbf{s}, \mathbf{curl}\, \mathbf{y})_0 = -\langle \pi_\tau \mathbf{s}_{|F_i}, \mathbf{y} \times \mathbf{n}_{|F_i} \rangle_{\widetilde{\mathbf{H}}_\perp^{1/2}(F_i)}, \ \forall \mathbf{y} \in \mathcal{Y}_i^R.$$

Choose $\mathbf{y} \in \mathcal{W}_i^R$. The previous relation, together with the orthogonality between $\mathbf{S}$ and $\mathbf{curl}\, \mathcal{W}^R$ leads to

$$\langle \pi_\tau \mathbf{s}_{|F_i}, \mathbf{w} \times \mathbf{n}_{|F_i} \rangle_{\widetilde{\mathbf{H}}_\perp^{1/2}(F_i)} = 0, \quad \forall \mathbf{w} \in \mathcal{W}_i^R.$$

The conclusion follows from the previous Lemma. $\diamond$

Remark 6.4. *Any element* $\mathbf{s}$ *of* $\mathbf{H}(\operatorname{div} 0; \Omega)$ *that verifies the condition (36) belongs to the orthogonal of* $\sum_i \mathbf{curl}\, \mathcal{W}_i^R$*. Then, one could prove the reciprocal assertion if* $\sum_i \mathcal{W}_i^R$ *were a dense subset of* $\mathcal{W}^R$*. Alternately, if the density result were not true, one would have to add a supplementary condition to (36) to fully characterize the singular elements.*

Remark 6.5. *We did not prove that* $\mathbf{S} = \mathbf{S}_D \cap \mathbf{H}(\operatorname{div} 0; \Omega)$*! Actually, the (third) boundary condition* $\mathbf{s} \cdot \mathbf{n}_{|F_i} = 0$ *in* $\widetilde{H}^{-1/2}(F_i)$*, for all* i*, is missing in (36).*

Remark 6.6. *Observe that* $\mathbf{S}_0$, $\mathbf{\Phi}(\mathcal{W})$ *and* $\mathbf{\Phi}^R(\mathcal{W})$ *differ one from the other by their respective curl regularity. Indeed, the elements of* $\mathbf{S}_0$ *have their curl in* $(\mathbf{H}_0(\mathbf{curl}\,, \Omega))'$*, those of* $\mathbf{\Phi}(\mathcal{W})$ *have their curl in* $\mathbf{L}^2(\Omega)$*, whereas the curl of elements of* $\mathbf{\Phi}^R(\mathcal{W})$ *belongs to* $\mathbf{H}^1(\Omega)$*.*

7. curl-free spaces and non-orthogonal decompositions

Based on a Helmholtz decomposition [1], one infers easily that the electromagnetic fields can be split into a divergence-free and a curl-free part. In the previous section, we have focused on the divergence-free fields. In order to proceed similarly for the curl-free part of the fields, it is relevant to introduce the following subspaces of $\mathcal{Y}$ and $\mathcal{X}$.

Definition 7.1. *Let* $\mathcal{M}, \mathcal{L}$ *be the spaces of the "curl-free" part of the magnetic and electric fields, with*

$$\mathcal{M} = \{\mathbf{m} \in \mathcal{Y} \ : \ \mathbf{curl}\ \mathbf{m} = 0\}, \qquad \mathcal{L} = \{\mathbf{l} \in \mathcal{X} \ : \ \mathbf{curl}\ \mathbf{l} = 0\}. \tag{38}$$

Remark 7.1. *Evidently, one has* $\mathcal{X} = \mathcal{V} \overset{\perp_W}{\oplus} \mathcal{L}$ *and* $\mathcal{Y} = \mathcal{W} \overset{\perp_W}{\oplus} \mathcal{M}$*.*

On $\mathcal{M}$ and $\mathcal{L}$, the Weber scalar product reduces to

$$(\mathbf{u}, \mathbf{v}) \mapsto (\operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v})_0. \tag{39}$$

Let us then consider the subspace of regular fields.

Definition 7.2. *Consider the regular subspaces of* $\mathcal{M}$ *and* $\mathcal{L}$ *respectively*

$$\mathcal{M}^R = \mathcal{M} \cap \mathbf{H}^1(\Omega), \qquad \mathcal{L}^R = \mathcal{L} \cap \mathbf{H}^1(\Omega). \tag{40}$$

It is clear that $\mathcal{M}^R$ is closed in $\mathcal{M}$. The same is true for $\mathcal{L}^R$. The singular spaces can thus be defined by orthogonality.

Definition 7.3. *Let* $\mathcal{M}^S$ *and* $\mathcal{L}^S$ *be the spaces of "curl-free" singular magnetic and electric fields:*

$$\mathcal{M} = \mathcal{M}^R \overset{\perp_W}{\oplus} \mathcal{M}^S, \qquad \mathcal{L} = \mathcal{L}^R \overset{\perp_W}{\oplus} \mathcal{L}^S.$$

Our aim is now to characterize the singular curl-free spaces $\mathcal{L}^S$ and $\mathcal{M}^S$. As for the divergence-free spaces $\mathcal{W}$ and $\mathcal{V}$, we link the subspaces $\mathcal{M}, \mathcal{L}$ to the primal or dual spaces of the scalar Laplace operator. So, they correspond *precisely* to Ψ and Φ, the spaces of scalar primal fields. One can prove isomorphisms between the relevant spaces; these isomorphisms can be established by standart arguments, and are summarized in the graph below

$$\begin{array}{ccccc} & \xrightarrow{\nabla} & (\mathcal{M}; \|\cdot\|_W) & \xrightarrow{\mathrm{div}} & \\ (\Phi; \|\cdot\|_\Delta) & & & & (L^2_0(\Omega); \|\cdot\|_0) \\ & & \overset{\Delta = \mathrm{div}\,\nabla}{\longrightarrow} & & \\ (\Psi; \|\cdot\|_\Delta) & & & & (L^2(\Omega); \|\cdot\|_0) \\ & \xrightarrow{\nabla} & (\mathcal{L}; \|\cdot\|_W) & \xrightarrow{\mathrm{div}} & \end{array}$$

As before, one notices that these isomorphisms preserve norms, i. e. they are *isometries*. Let us then consider the subspaces of regular fields.

Definition 7.4. *Consider the regular subspaces of* Ψ *and* Φ

$$\Psi^R = \Psi \cap H^2(\Omega)\,, \qquad \Phi^R = \Phi \cap H^2(\Omega)\,.$$

The regular spaces Ψ^R and Φ^R coincide respectively with the spaces $H^D(\Omega)$ and $H^N(\Omega)$ originally introduced in (2) and (5).

Definition 7.5. *Let* Ψ^S *and* Φ^S *be the spaces of singular scalar primal fields:*

$$\Psi = \Psi^R \overset{\perp_\Delta}{\oplus} \Psi^S, \qquad \Phi = \Phi^R \overset{\perp_\Delta}{\oplus} \Phi^S.$$

The singular spaces Ψ^S and Φ^S respectively coincide with $\underline{H}^D(\Omega)$ and $\underline{H}^N(\Omega)$.

We now relate the regular (resp. singular) parts of the primal fields to the regular (resp. singular) curl-free parts of the electromagnetic fields.

Proposition 7.1. *The following mappings are isomorphisms, linked by* $\Delta = \mathrm{div}\ \nabla$*,*

$$\begin{array}{ccccc} & \xrightarrow{\nabla} & (\mathcal{M}^S; \|\cdot\|_W) & \xrightarrow{\mathrm{div}} & \\ (\Phi^S; \|\cdot\|_\Delta) & & & & (\mathbb{S}_N; \|\cdot\|_0) \\ & & \overset{\Delta = \mathrm{div}\,\nabla}{\longrightarrow} & & \\ (\Psi^S; \|\cdot\|_\Delta) & & & & (\mathbb{S}_D; \|\cdot\|_0) \\ & \xrightarrow{\nabla} & (\mathcal{L}^S; \|\cdot\|_W) & \xrightarrow{\mathrm{div}} & \end{array}$$

Similar results hold for regular spaces.
Finally, with the help of this proposition, one establishes simply characterizations of the singular curl-free part $\mathcal{M}^S$ and $\mathcal{L}^S$, by using the results of Section 2 on dual scalar fields.

As far as numerical computations are concerned, one can also introduce direct, albeit non-orthogonal two-part sums of the spaces $\mathcal{X}$ and $\mathcal{Y}$. For instance, whenever the electric field is concerned, it appears more convenient to solve a scalar Laplace problem to determine $\mathcal{L}^S$, than a vector one for the $\mathcal{X}^S$ characterization (cf. [3] for a practical implementation). To that aim, it is interesting to use non-orthogonal decompositions introduced in (18) that we rewrite here as

$$\mathcal{X} = \mathcal{X}^R \overset{c}{\oplus} \mathcal{L}^S, \qquad \mathcal{Y} = \mathcal{Y}^R \overset{c}{\oplus} \mathcal{M}^S\,.$$

These results can be reinterpreted as follows in the framework of curl-free fields: the electric (resp. magnetic) singular fields are one-to-one with the gradients of the primal singularities of the scalar Laplace operator with Dirichlet boundary condition (resp. with Neumann boundary condition).

Remark 7.2. *With the help of those non-orthogonal decompositions, one can prove that the divergence mapping is an isomorphism from $\mathcal{X}_S$ to $\mathbb{S}_D$, resp. from $\mathcal{Y}_S$ to $\mathbb{S}_N$. The first result was proven in* [21], *whereas the second one is established in* [26, p. 198].

Alternately, one can obtain non-orthogonal decompositions involving divergence-free fields.

Proposition 7.2. *The following decompositions are direct and continuous:*

$$\mathcal{X} = \mathcal{X}^R \overset{c}{\oplus} \mathcal{V}^S, \qquad \mathcal{Y} = \mathcal{Y}^R \overset{c}{\oplus} \mathcal{W}^S\,.$$

Below, we provide the main steps of the proof in the electric case (details can be found in Chapter 11 of [26]).

Proof. To begin with, one remarks that $\mathcal{L}^R \overset{\perp_W}{\oplus} \mathcal{V}^R$ is a strict (closed) subset of $\mathcal{X}^R$. The idea is to introduce its orthogonal in $\mathcal{X}^R$, called $\mathcal{X}_\perp$. Interestingly, $\mathcal{X}_\perp$ is also the (orthogonal) 'missing part' of $\mathcal{X}^S$ in $\mathcal{L}^S \overset{\perp_W}{\oplus} \mathcal{V}^S$. As a consequence, one proves simply that both $\mathbf{x}_\perp \mapsto \|\mathrm{div}\,\mathbf{x}_\perp\|_0$ and $\mathbf{x}_\perp \mapsto \|\mathbf{curl}\,\mathbf{x}_\perp\|_0$ define norms on $\mathcal{X}_\perp$.
Then, one recalls that the divergence mapping is surjective from $\mathcal{X}^R$ to $L^2(\Omega)$ [4, Proposition 3.5]. Also, the curl mapping is surjective from $\mathbf{H}^1_0(\Omega)$ (and so from $\mathcal{X}^R$) to $\mathbf{H}(\mathrm{div}\,0;\Omega) \cap \mathbf{H}_0(\mathrm{div}\,;\Omega)$ [15].
From the orthogonal decompositions and the surjectivity results above, it stems that the divergence mapping is an isomorphism from $\mathcal{X}_\perp$ to $\mathbb{S}_D$, while the curl mapping is an isomorphism from $\mathcal{X}_\perp$ to $\mathbf{S}_0$. These last two properties allow us to conclude that $\mathcal{X}$ can indeed be identified with the sum $\mathcal{X}^R \oplus \mathcal{V}^S$. Moreover, the decomposition is continuous: $\mathcal{X} = \mathcal{X}^R \overset{c}{\oplus} \mathcal{V}^S$.

As far as the continuous decomposition of $\mathcal{Y}$ is concerned, one uses similar tools. In particular, that the divergence mapping is surjective from $\mathbf{H}^1_0(\Omega)$ (and so from $\mathcal{Y}^R$) to $L^2_0(\Omega)$ [27, Corollary 2.4], and that the curl mapping is surjective from $\mathcal{Y}^R$ to $\mathbf{H}(\mathrm{div}\,0;\Omega)$, thanks to the combination of [27, Theorem 3.4] and [10]. $\diamond$

Remark 7.3. *The proofs are completely different than the one provided for establishing the non-orthogonal decomposition involving curl-free singular fields (18). As a matter of fact, the Birman and Solomyak splitting has no equivalent in the current case: even though it is used as a starting point, the process still requires some caution.*

To conclude, we note that, at least for 3D problems, these decompositions relying on divergence-free singular fields are less useful, due to the difficulty in approximating (vector) elements of $\mathcal{V}^S$ or $\mathcal{W}^S$, or their vector potentials. The situation is evidently completely different in 2D geometries [5].

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A. Annex

In this Section, we consider $\chi : \mathbb{R}^+ \to [0,1]$, a smooth cut-off function, equal to one in a neighborhood of zero, and to zero near $+\infty$. Select $\rho_0, \rho_1 > 0$ such that χ is equal to one on $[0, \rho_0]$ and to zero on $[\rho_1, +\infty[$. Recall the Sobolev imbedding Theorem, which states that $H^2(\Omega)$ is continuously imbedded in $\mathcal{C}^{0,1/2}(\bar{\Omega})$ (cf. for instance [28, p. 27]).

A.1. A convergence result around corners. Let u be an element of $H^N(\Omega)$. Around a given vertex S_v, consider the spherical coordinates $(\rho_v, \theta_v, \varphi_v)$ (with associated orthonormal basis $(\mathbf{e}_{\rho_v}, \mathbf{e}_{\theta_v}, \mathbf{e}_{\varphi_v})$), and define Σ implicitly by $\Omega \cap B(S, \rho_\star) = \{(\rho, \theta, \varphi) \ : \ \rho \in [0, \rho_\star], \ (\theta, \varphi) \in \Sigma\}$, for small $\rho_\star$. Finally, set $\chi^n(\mathbf{x}) = \chi(n\rho_v)$ and $u_v^n = \chi^n\, u$, for any integer $n \geq 1$. Then one has the weak convergence result below.

Lemma A.1. *The sequence $(\Delta u_v^n)_n$ converges weakly to zero in $L^2(\Omega)$.*

In the proof, we drop the index $_v$ to lighten the notations.

Proof. Note that since $\Delta u^n(\mathbf{x}) \to 0$ pointwise a. e. when $n \to +\infty$, we simply have to prove that the sequence $(\Delta u^n)_n$ is bounded in $L^2(\Omega)$. Also, since we are interested in 2nd order derivatives and since u is continuous up to the boundary, we can assume that $u(S) = 0$.

Then, using the chain rule, we find that

$$\Delta u^n = \chi^n \Delta u + 2\nabla\chi^n \cdot \nabla u + u\Delta\chi^n.$$

Let us consider the three terms one after the other.

According to the bounded convergence Theorem, $(\chi^n \Delta u)_n$ converges to zero in $L^2(\Omega)$. As far as the second term is concerned, we note that $\nabla\chi^n = n\chi'(n\rho)\mathbf{e}_\rho$. Then, the product $\nabla\chi^n \cdot \nabla u$ reduces to $n\chi'(n\rho)\, g$, with $g = \partial_\rho f$ an element of $H^1(\Omega)$. According to step 1 of the proof of Lemma 2.6 in [24], one gets that $\nabla\chi^n \cdot \nabla u$ also converges to zero in $L^2(\Omega)$.

Therefore, we only have to tackle the third term. To that aim, let us carry out some elementary computations:

$$\Delta\chi^n = n^2\chi''(n\rho) + \frac{2n}{\rho}\chi'(n\rho).$$

One has $\|\chi''(n\cdot)\|_\infty = \|\chi''\|_\infty$, which is independent of n. Also, $\chi'(n\cdot)$ vanishes on $[0, \rho_0/n]$, so

$$\|\frac{1}{\cdot}\chi'(n\cdot)\|_\infty = \|\frac{1}{\cdot}\chi'(n\cdot)\|_{\infty,[\rho_0/n,+\infty[} \leq \frac{n}{\rho_0}\|\chi'\|_\infty.$$

Accreting the two yields

$$\|\Delta\chi^n\|_{\infty,\Omega} \leq C\, n^2, \text{ with } C = \|\chi''\|_\infty + \frac{2}{\rho_0}\|\chi'\|_\infty.$$

We can now compute the quantity of interest

$$\|u\Delta\chi^n\|_0^2 = \int_{\Omega\cap supp(\chi^n)} |u|^2 |\Delta\chi^n|^2\, d\Omega \leq C^2\, n^4 \int_{\Omega\cap supp(\chi^n)} |u|^2\, d\Omega.$$

By definition, we know that the support of χ^n is included in the ball of center S, with radius ρ_1/n. Also, thanks to the Sobolev imbedding Theorem, u belongs to $\mathcal{C}^{0,1/2}(\bar{\Omega})$. Thus, there exists a constant C_u (which depends only on u), such that $|u(\mathbf{x}) - u(S)| \leq C_u \|\mathbf{x} - S\|^{1/2}$, for all $\mathbf{x} \in \bar{\Omega}$. Since $u(S) = 0$, this reads, $|u(\mathbf{x})| \leq C_u \rho^{1/2}$. Summing up all the results, we find finally

$$\|u\Delta\chi^n\|_0^2 \leq C^2 C_u^2\, n^4 \int_{\Omega\cap B(S,\rho_1/n)} \rho\, d\Omega \leq C^2 C_u^2 C_\Sigma\, n^4 \int_{\rho=0}^{\rho=\rho_1/n} \rho^3\, d\rho = \frac{C^2 C_u^2 C_\Sigma \rho_1^4}{4}.$$

$\left(\text{Above, } C_\Sigma \text{ is equal to } \int_{(\theta,\varphi)\in\Sigma} \sin\theta\, d\theta d\varphi \text{ for sufficiently large } n. \right)$

This is exactly the desired result. ◇

A.2. A counter-example around edges. Consider a domain Ω, such that the geometry of its boundary $\partial\Omega$ locally coincides with that of a wedge, with two plane faces Γ_1 and Γ_2 such that $\mathbf{n}_{|\Gamma_1} = \mathbf{e}_1$ and $\mathbf{n}_{|\Gamma_2} = \mathbf{e}_2$, with an edge e_{12} parallel to $\mathbf{e}_3$ of unit length. Let us choose $f(\mathbf{x}) = \chi(\rho_{12}) x_3^2 (1-x_3)^2$, where ρ_{12} is the distance to the edge e_{12}. Clearly, f belongs to $H^2(\Omega)$. In addition (at least locally, that is for small ρ_{12}), $\partial_n f_{|\Gamma_1} = \nabla f_{|\Gamma_1} \cdot \mathbf{e}_1 = 0$ and $\partial_n f_{|\Gamma_2} = \nabla f_{|\Gamma_2} \cdot \mathbf{e}_2 = 0$. Then f belongs[3] to $H^N(\Omega)$.

On e_{12}, one has $f_{|e_{12}} = x_3^2(1-x_3)^2$, so f does not vanish on the edge!

Also, if one defines $f^n = \chi_{12}^n f$ with $\chi_{12}^n(\mathbf{x}) = \chi(n\rho_{12})$, Δf^n does not converge weakly to zero in $L^2(\Omega)$! To prove this second (negative) result, we basically follow the proof of Lemma A.1. Let us drop the double index $_{12}$. We have

$$\Delta f^n = \chi^n \Delta f + 2\nabla\chi^n \cdot \nabla f + f\Delta\chi^n.$$

As before, $\chi^n \Delta f$ converges to zero in $L^2(\Omega)$.

Then, for sufficiently large n, one notices that $\nabla\chi^n$ and ∇f are pointwise orthogonal over Ω: $\nabla\chi^n$ is different from zero only near the edge, and it is orthogonal to $\mathbf{e}_3$ there, whereas in this region ∇f is parallel to $\mathbf{e}_3$. The second term is zero. As far as the third term is concerned, let us pick a suitable test-function g of $L^2(\Omega)$, and prove that $(f\Delta\chi^n, g)_0$ does not converge to zero. We choose $g(\mathbf{x}) = \rho$.

To begin with, one finds, in cylindrical coordinates, $\Delta\chi^n = n^2\chi''(n\rho) + n/\rho\chi'(n\rho)$. Then, one gets (with the change of variables $\eta = n\rho$ from the line before last to the last line):

$$\begin{aligned}
(f\Delta\chi^n, g)_0 &= \int_{\Omega\cap supp(\chi^n)} g\, f\, \Delta\chi^n\, d\Omega \\
&= \int_{\rho=\rho_0/n}^{\rho=\rho_1/n} \left(n^2\rho^2\chi''(n\rho) + n\rho\chi'(n\rho)\right) d\rho \int_{\theta=0}^{\theta=\pi/2} d\theta \int_{x_3=0}^{x_3=1} x_3^2(1-x_3)^2\, dx_3 \\
&= \frac{C_e}{n} \int_{\rho_0}^{\rho_1} \left(\eta^2\chi''(\eta) + \eta\chi'(\eta)\right) d\eta.
\end{aligned}$$

$\left(\text{Above, } C_e \text{ is equal to } \frac{\pi}{2}\int_0^1 x_3^2(1-x_3)^2\, dx_3, \text{ so } C_e > 0. \right)$

To conclude, we evaluate the integral in η by performing elementary integration by

[3]According to the definition of f, one could add faces perpendicular to $\mathbf{e}_3$ to 'close' the domain Ω at the endpoints of the edge, and find that $\partial_n f$ also vanishes on those faces.

parts:

$$\begin{aligned}\int_{\rho_0}^{\rho_1} \left(\eta^2\chi''(\eta) + \eta\chi'(\eta)\right)\, d\eta &= \int_{\rho_0}^{\rho_1} \eta^2\chi''(\eta)\, d\eta + \int_{\rho_0}^{\rho_1} \eta\chi'(\eta)\, d\eta \\ &= [\eta^2\chi'(\eta)]_{\rho_0}^{\rho_1} - 2\int_{\rho_0}^{\rho_1} \eta\chi'(\eta)\, d\eta + \int_{\rho_0}^{\rho_1} \eta\chi'(\eta)\, d\eta \\ &= -\int_{\rho_0}^{\rho_1} \eta\chi'(\eta)\, d\eta = -[\eta\chi(\eta)]_{\rho_0}^{\rho_1} + \int_{\rho_0}^{\rho_1} \chi(\eta)\, d\eta \\ &= \rho_0 + \int_{\rho_0}^{\rho_1} \chi(\eta)\, d\eta > 0.\end{aligned}$$

Therefore, $(f\Delta\chi^n, g)_0$ is proportional to n^{-1}: we conclude negatively!

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