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A POSTERIORI ERROR ESTIMATION FOR A DUAL MIXED FINITE ELEMENT APPROXIMATION OF NON–NEWTONIAN FLUID FLOW PROBLEMS

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Abstract. A dual mixed finite element method, for quasi–Newtonian fluid flow obeying to the power law, is constructed and analyzed in [8]. This mixed formulation possesses local (i.e., at element level) conservation properties (conservation of the momentum and the mass) as in the finite volume methods. We propose here an *a posteriori* error analysis for this mixed formulation.

Key Words. mixed finite element method, quasi–Newtonian fluid flow, *a posteriori* error analysis.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^2$ with a Lipschitz boundary Γ. Given $\boldsymbol{f}$, $\eta_0 > 0$ and r a real constant verifying $1 < r < \infty$, we consider the following boundary value problem : Find $(\boldsymbol{u}, p)$ such that

$$(1)\qquad \begin{aligned} &-2\eta_0\, \boldsymbol{div} \Big(|\boldsymbol{d}(\boldsymbol{u})|^{r-2}\, \boldsymbol{d}(\boldsymbol{u}) \Big) + \nabla p = \boldsymbol{f} && \text{in } \Omega \\ &div\, \boldsymbol{u} = 0 && \text{in } \Omega \\ &\boldsymbol{u} = 0 && \text{on } \Gamma \end{aligned}$$

where $\boldsymbol{d}(\boldsymbol{u})$ is the rate of strain tensor, $\boldsymbol{d}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^t)$, $\nabla \boldsymbol{u}$ is the tensor gradient of $\boldsymbol{u}$.

Throughout $|\cdot|$ denotes the Euclidian matrix norm, that is for $\boldsymbol{\tau}$, a $d \times d$ real matrix, $|\boldsymbol{\tau}| := \left[\sum_{i,j=1}^{d} (\tau_{ij})^2\right]^{1/2}$. The above system models the steady isothermal flow of an incompressible quasi-Newtonian fluid, $\boldsymbol{f}$ denotes the body force, $\boldsymbol{u}$ the velocity and p the pressure.

The well–posedness of the above nonlinear problem and its standard finite element approximation are well established in Baranger–Najib [1]. Extentions and improvements on the error bounds of [1] have appeared in Sandri [11] and in Barrett–Liu [2, 3].

In the framework of standard finite element method, an *a posteriori* error analysis is developed in Sandri [12]. A mixed finite element method has been introduced and analyzed in Farhloul–Zine [8]. Due to the introduction of the Cauchy stress tensor as a new variable, this new formulation possesses local (i.e., at element level) conservation properties (conservation of the momentum and the mass) as in the finite volume methods. Furthermore, it allows the approximations of all the physical variables (stress, velocity and pressure).

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The aim of this work is to give an *a posteriori* error estimates for the mixed formulation developed in [8]. In the next section we recall the mixed formulation developed in [8] and then we give the *a posteriori* error estimates in section 3.

2. Mixed formulations

For the ease of the presentation, we take $\eta_0 = \frac{1}{2}$. Introducing $\boldsymbol{\sigma} = |\boldsymbol{d}(\boldsymbol{u})|^{r-2}\,\boldsymbol{d}(\boldsymbol{u})$ the extra–stress tensor, and using the fact that

$$|\boldsymbol{\sigma}|^{r'-2}\,\boldsymbol{\sigma} = \boldsymbol{d}(\boldsymbol{u}), \text{ where } r' \text{ is the conjugate of } r, \ i.e., \ \frac{1}{r} + \frac{1}{r'} = 1$$

problem (1) can be formulated as

$$\begin{array}{ll} -\,\boldsymbol{div}(\boldsymbol{\sigma} - p\mathbb{I}) = \boldsymbol{f} & \text{in } \Omega \\ div\,\boldsymbol{u} = 0 & \text{in } \Omega \\ A(\boldsymbol{\sigma}) := |\boldsymbol{\sigma}|^{r'-2}\,\boldsymbol{\sigma} = \boldsymbol{d}(\boldsymbol{u}) & \text{in } \Omega \\ \boldsymbol{u} = 0 & \text{on } \Gamma \end{array} \tag{2}$$

where $\boldsymbol{f} \in [L^{r'}(\Omega)]^2$, $\mathbb{I}$ is the identity tensor and for a given tensor $\boldsymbol{\tau} = (\tau_{ij})_{1\leq i,j\leq 2}$, $(\boldsymbol{div}\,\boldsymbol{\tau})_i = \sum_{j=1}^{2} \frac{\partial \tau_{ij}}{\partial x_j}$.

Note that for all $(\boldsymbol{\tau}, q) \in [L^{r'}(\Omega)]^{2\times 2} \times L_0^{r'}(\Omega)$ such that $\boldsymbol{div}(\boldsymbol{\tau} - q\mathbb{I}) \in [L^{r'}(\Omega)]^2$, as $div\,\boldsymbol{u} = 0$, one has

$$(A(\boldsymbol{\sigma}), \boldsymbol{\tau}) = (\boldsymbol{d}(\boldsymbol{u}), \boldsymbol{\tau}) = -(\boldsymbol{div}(\boldsymbol{\tau} - q\mathbb{I}), \boldsymbol{u}) - (\boldsymbol{\omega}, \boldsymbol{\tau}),$$

where $\boldsymbol{\omega} = \boldsymbol{\omega}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} - \nabla \boldsymbol{u}^t) \in [L^r(\Omega)]^{2\times 2}$ is the vorticity tensor and

$$L_0^{r'}(\Omega) = \left\{ q \in L^{r'}(\Omega); \ \int_\Omega q = 0\, dx \right\}.$$

In order to derive the mixed formulation of problem (2), we define the following spaces

$$\begin{aligned} \Sigma &= \left\{ \underset{\sim}{\tau} = (\boldsymbol{\tau}, q) \in [L^{r'}(\Omega)]^{2\times 2} \times L_0^{r'}(\Omega); \ \boldsymbol{div}(\boldsymbol{\tau} - q\mathbb{I}) \in [L^{r'}(\Omega)]^2 \right\}, \\ M &= \left\{ \underset{\sim}{v} = (\boldsymbol{v}, \boldsymbol{\eta}) \in [L^r(\Omega)]^2 \times [L^r(\Omega)]^{2\times 2}; \ \ \boldsymbol{\eta} + \boldsymbol{\eta}^t = 0 \right\}, \end{aligned}$$

equipped with their respective norms:

$$\| \underset{\sim}{\tau} \|_\Sigma = \left(\|\boldsymbol{\tau}\|_{0,r',\Omega}^{r'} + \|q\|_{0,r',\Omega}^{r'} + \|\boldsymbol{div}(\boldsymbol{\tau} - q\mathbb{I})\|_{0,r',\Omega}^{r'} \right)^{\frac{1}{r'}}, \ \| \underset{\sim}{v} \|_M = \left(\|\boldsymbol{v}\|_{0,r,\Omega}^{r} + \|\boldsymbol{\eta}\|_{0,r,\Omega}^{r} \right)^{\frac{1}{r}}.$$

The mixed formulation of (2) reads as follows: Find $\underset{\sim}{\sigma} = (\boldsymbol{\sigma}, p) \in \Sigma$ and $\boldsymbol{u} \in M$ such that

$$\begin{array}{ll} (A(\boldsymbol{\sigma}), \boldsymbol{\tau}) + (\boldsymbol{div}(\boldsymbol{\tau} - q\mathbb{I}), \boldsymbol{u}) + (\boldsymbol{\tau}, \boldsymbol{\omega}) = 0 & \forall\, \underset{\sim}{\tau} = (\boldsymbol{\tau}, q) \in \Sigma, \\ (\boldsymbol{div}(\boldsymbol{\sigma} - p\mathbb{I}), \boldsymbol{v}) + (\boldsymbol{\sigma}, \boldsymbol{\eta}) + (\boldsymbol{f}, \boldsymbol{v}) = 0 & \forall\, \underset{\sim}{v} = (\boldsymbol{v}, \boldsymbol{\eta}) \in M. \end{array} \tag{3}$$

The results concerning the existence, uniqueness and stability condition of the solution of (3) are developed in Farhloul-Zine [8]. However, we recall some results obtained in [8] that we need in the following section.

Proposition 1. *There exists a positive constant β such that*

$$\inf_{\underset{\sim}{v} \in M} \sup_{\underset{\sim}{\tau} \in \Sigma} \frac{(\boldsymbol{div}(\boldsymbol{\tau} - q\mathbb{I}), \boldsymbol{v}) + (\boldsymbol{\tau}, \boldsymbol{\eta})}{\| \underset{\sim}{v} \|_M \| \underset{\sim}{\tau} \|_\Sigma} \geq \beta. \tag{4}$$

Theorem 1. *Problem (3) admits a unique solution satisfying*

$$\| \underset{\sim}{\boldsymbol{u}} \|_M + \| \underset{\sim}{\boldsymbol{\sigma}} \|_\Sigma \leq C,$$

where C is a positive constant depending on $\boldsymbol{f}$.

We assume that the boundary Γ of Ω is polygonal and we consider a regular family of triangulations T_h (triangulation of $\overline{\Omega}$ into closed triangles K). We assume that the triangulation T_h is regular in the classical sense.

Let h_K be the diameter of K and E any edge of K. Let $P_k(K)$ denote the space of polynomials of degree less than or equal to k on K. We set

$$\boldsymbol{\chi} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad R(K) = [P_1(K)]^2 + \alpha\, curl\, \boldsymbol{b}_K,$$

where α is a constant, and $\boldsymbol{b}_K$ the "bubble function", i.e. $\boldsymbol{b}_K(x) = \lambda_1(x)\lambda_2(x)\lambda_3(x)$ with λ_1, λ_2 and λ_3 the barycentric coordinates in K. We define the finite element spaces (see Farhloul–Fortin [7])

$$\Sigma_h = \left\{ \underset{\sim}{\boldsymbol{\tau}}_h = (\boldsymbol{\tau}_h, q_h) \in \Sigma; \quad q_h|_K \in P_1(K) \text{ and } \boldsymbol{\tau}_h|_K \in [R(K)]^2, \ \forall K \in T_h \right\}$$

$$M_h = \left\{ \underset{\sim}{\boldsymbol{v}}_h = (\boldsymbol{v}_h, \boldsymbol{\eta}_h) \in M; \ \boldsymbol{v}_h|_K \in [P_0(K)]^2, \boldsymbol{\eta}_h = \theta_h \boldsymbol{\chi} \text{ with } \theta_h|_K \in P_1(K), \ \forall K \in T_h \right\},$$

and our finite element approximation of problem (3): Find $\underset{\sim}{\boldsymbol{\sigma}}_h = (\boldsymbol{\sigma}_h, p_h) \in \Sigma_h$ and $\underset{\sim}{\boldsymbol{u}}_h = (\boldsymbol{u}_h, \boldsymbol{\omega}_h) \in M_h$ such that

$$(5) \quad \begin{aligned} &(A(\boldsymbol{\sigma}_h), \boldsymbol{\tau}_h) + (\boldsymbol{div}(\boldsymbol{\tau}_h - q_h\mathbb{I}), \boldsymbol{u}_h) + (\boldsymbol{\tau}_h, \boldsymbol{\omega}_h) = 0 && \forall\, \underset{\sim}{\boldsymbol{\tau}}_h = (\boldsymbol{\tau}_h, q_h) \in \Sigma_h, \\ &(\boldsymbol{div}(\boldsymbol{\sigma}_h - p_h\mathbb{I}), \boldsymbol{v}_h) + (\boldsymbol{\sigma}_h, \boldsymbol{\eta}_h) + (\boldsymbol{f}, \boldsymbol{v}_h) = 0 && \forall\, \underset{\sim}{\boldsymbol{v}}_h = (\boldsymbol{v}_h, \boldsymbol{\eta}_h) \in M_h. \end{aligned}$$

The analysis of the problem (5) is performed in [8]. For the same reasons stated above, we recall the following result.

Theorem 2. *Problem (5) admits a unique solution, $\left(\underset{\sim}{\boldsymbol{\sigma}}_h, \underset{\sim}{\boldsymbol{u}}_h\right) \in \Sigma_h \times M_h$, satisfying*

$$\| \underset{\sim}{\boldsymbol{u}}_h \|_M + \| \underset{\sim}{\boldsymbol{\sigma}}_h \|_\Sigma \leq C,$$

where C is a positive constant independent of h.

Finally, for the *a priori* error estimates, we refer to Theorem 3.4 and Theorem 3.5 in [8].

3. *A posteriori* error estimates

Let $(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{u}}) = \big((\boldsymbol{\sigma}, p); (\boldsymbol{u}, \boldsymbol{\omega})\big)$ and $(\underset{\sim}{\boldsymbol{\sigma}}_h, \underset{\sim}{\boldsymbol{u}}_h) = \big((\boldsymbol{\sigma}_h, p_h); (\boldsymbol{u}_h, \boldsymbol{\omega}_h)\big)$ be the solutions of (3) and (5) respectively. On Σ and M, one define the residues $\boldsymbol{R}$ and $\boldsymbol{S}$:

$$(6) \quad < \boldsymbol{R}, \underset{\sim}{\boldsymbol{\tau}} > \ = \ (A(\boldsymbol{\sigma}_h), \boldsymbol{\tau}) + (\boldsymbol{div}(\boldsymbol{\tau} - q\mathbb{I}), \boldsymbol{u}_h) + (\boldsymbol{\tau}, \boldsymbol{\omega}_h), \forall\, \underset{\sim}{\boldsymbol{\tau}} = (\boldsymbol{\tau}, q) \in \Sigma,$$

$$(7) \quad < \boldsymbol{S}, \underset{\sim}{\boldsymbol{v}} > \ = \ (\boldsymbol{div}(\boldsymbol{\sigma}_h - p_h\mathbb{I}), \boldsymbol{v}) + (\boldsymbol{\sigma}_h, \boldsymbol{\eta}) + (\boldsymbol{f}, \boldsymbol{v}), \forall\, \underset{\sim}{\boldsymbol{v}} = (\boldsymbol{v}, \boldsymbol{\eta}) \in M.$$

We denote by $\boldsymbol{R}_*$ and $\boldsymbol{S}_*$ the dual norms of $\boldsymbol{R}$ and $\boldsymbol{S}$

$$\boldsymbol{R}_* = \sup_{\underset{\sim}{\boldsymbol{\tau}} \in \Sigma} \frac{| < \boldsymbol{R}, \underset{\sim}{\boldsymbol{\tau}} > |}{\| \underset{\sim}{\boldsymbol{\tau}} \|_\Sigma} \quad \text{and} \quad \boldsymbol{S}_* = \sup_{\underset{\sim}{\boldsymbol{v}} \in M} \frac{| < \boldsymbol{S}, \underset{\sim}{\boldsymbol{v}} > |}{\| \underset{\sim}{\boldsymbol{v}} \|_M}.$$

Our goal is to bound the errors $\|\underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\boldsymbol{\sigma}}_h\|_\Sigma$ and $\|\underset{\sim}{\boldsymbol{u}} - \underset{\sim}{\boldsymbol{u}}_h\|_M$ by functions of two error estimators whose expressions involve only the data of the problem and the computed quantities. To this end, we firstly bound the errors in terms of $\boldsymbol{R}_*$ and $\boldsymbol{S}_*$. Afterwards, we bound $\boldsymbol{R}_*$ and $\boldsymbol{S}_*$ in terms of the data of the problem and the computed quantities. As we will see later, these results depend on the parameter r. In fact, we have to distinguish two cases: $r \geq 2$ and $1 < r < 2$. However, we have the following estimate of $(A(\boldsymbol{\sigma}_h) - A(\boldsymbol{\sigma}), \boldsymbol{\sigma}_h - \boldsymbol{\sigma})$ in terms of $\boldsymbol{R}_*$ and $\boldsymbol{S}_*$.

Proposition 2. *There exists a constant C independent of h such that*

(8)
$$\begin{aligned}(A(\boldsymbol{\sigma}_h) - A(\boldsymbol{\sigma}), \boldsymbol{\sigma}_h - \boldsymbol{\sigma}) \quad \leq \quad & C\Big\{ \boldsymbol{R}_* \left(\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}\|_{0,r',\Omega} + \|\boldsymbol{f} - P_h^0 \boldsymbol{f}\|_{0,r',\Omega}\right) + \boldsymbol{S}_*\, \boldsymbol{R}_* \\ & + \boldsymbol{S}_* \sup_{\underset{\sim}{\boldsymbol{\tau}} \in \Sigma} \frac{(A(\boldsymbol{\sigma}) - A(\boldsymbol{\sigma}_h), \boldsymbol{\tau})}{\|\underset{\sim}{\boldsymbol{\tau}}\|_\Sigma}\Big\},\end{aligned}$$

where $P_h^0 \boldsymbol{f}$ is the L^2–projection of $\boldsymbol{f}$ onto $\Big[\prod_{K \in T_h} P_0(K)\Big]^2$.

Proof. Using (3), (6) and (7), we obtain

$$(9)\quad (A(\boldsymbol{\sigma}_h) - A(\boldsymbol{\sigma}), \boldsymbol{\tau}) + (\boldsymbol{div}(\boldsymbol{\tau} - q\mathbb{I}), \boldsymbol{u}_h - \boldsymbol{u}) + (\boldsymbol{\tau}, \boldsymbol{\omega}_h - \boldsymbol{\omega}) = < \boldsymbol{R}, \underset{\sim}{\boldsymbol{\tau}} >,\ \forall\, \underset{\sim}{\boldsymbol{\tau}} \in \Sigma,$$

$$(10)\qquad (\boldsymbol{div}\left[(\boldsymbol{\sigma}_h - p_h\mathbb{I}) - (\boldsymbol{\sigma} - p\mathbb{I})\right], \boldsymbol{v}) + (\boldsymbol{\sigma}_h - \boldsymbol{\sigma}, \boldsymbol{\eta}) = < \boldsymbol{S}, \underset{\sim}{\boldsymbol{v}} >,\ \forall\, \underset{\sim}{\boldsymbol{v}} \in M.$$

Taking $\underset{\sim}{\boldsymbol{\tau}} = (\boldsymbol{\sigma}_h - \boldsymbol{\sigma}, p_h - p)$ in (9) and $\underset{\sim}{\boldsymbol{v}} = (\boldsymbol{u}_h - \boldsymbol{u}, \boldsymbol{\omega}_h - \boldsymbol{\omega})$ in (10), we get

$$(11)\qquad (A(\boldsymbol{\sigma}_h) - A(\boldsymbol{\sigma}), \boldsymbol{\sigma}_h - \boldsymbol{\sigma}) = < \boldsymbol{R}, \underset{\sim}{\boldsymbol{\sigma}}_h - \underset{\sim}{\boldsymbol{\sigma}} > - < \boldsymbol{S}, \underset{\sim}{\boldsymbol{u}}_h - \underset{\sim}{\boldsymbol{u}} > .$$

By the inf–sup condition (4) and (9) it follows

$$\begin{aligned}\beta \|\underset{\sim}{\boldsymbol{u}}_h - \underset{\sim}{\boldsymbol{u}}\|_M \quad & \leq \quad \sup_{\underset{\sim}{\boldsymbol{\tau}} \in \Sigma} \frac{(\boldsymbol{div}(\boldsymbol{\tau} - q\mathbb{I}), \boldsymbol{u}_h - \boldsymbol{u}) + (\boldsymbol{\tau}, \boldsymbol{\omega}_h - \boldsymbol{\omega})}{\|\underset{\sim}{\boldsymbol{\tau}}\|_\Sigma} \\ & \leq \quad \sup_{\underset{\sim}{\boldsymbol{\tau}} \in \Sigma} \frac{< \boldsymbol{R}, \underset{\sim}{\boldsymbol{\tau}} >}{\|\underset{\sim}{\boldsymbol{\tau}}\|_\Sigma} + \sup_{\underset{\sim}{\boldsymbol{\tau}} \in \Sigma} \frac{(A(\boldsymbol{\sigma}) - A(\boldsymbol{\sigma}_h), \boldsymbol{\tau})}{\|\underset{\sim}{\boldsymbol{\tau}}\|_\Sigma}.\end{aligned}$$

Thus,

$$(12)\qquad \|\underset{\sim}{\boldsymbol{u}}_h - \underset{\sim}{\boldsymbol{u}}\|_M \quad \leq \quad C\Big(\boldsymbol{R}_* + \sup_{\underset{\sim}{\boldsymbol{\tau}} \in \Sigma} \frac{(A(\boldsymbol{\sigma}) - A(\boldsymbol{\sigma}_h), \boldsymbol{\tau})}{\|\underset{\sim}{\boldsymbol{\tau}}\|_\Sigma}\Big).$$

Using (11) and (12), we get

(13)
$$(A(\boldsymbol{\sigma}_h) - A(\boldsymbol{\sigma}), \boldsymbol{\sigma}_h - \boldsymbol{\sigma}) \leq C \left\{ \boldsymbol{R}_* \|\underset{\sim}{\boldsymbol{\sigma}}_h - \underset{\sim}{\boldsymbol{\sigma}}\|_\Sigma + \boldsymbol{S}_*\, \boldsymbol{R}_* + \boldsymbol{S}_* \sup_{\underset{\sim}{\boldsymbol{\tau}} \in \Sigma} \frac{(A(\boldsymbol{\sigma}) - A(\boldsymbol{\sigma}_h), \boldsymbol{\tau})}{\|\underset{\sim}{\boldsymbol{\tau}}\|_\Sigma} \right\}.$$

Now, from (3) and (5), we have

$$(14)\qquad (\boldsymbol{div}\left[(\boldsymbol{\sigma} - p\mathbb{I}) - (\boldsymbol{\sigma}_h - p_h\mathbb{I})\right], \boldsymbol{v}) + (\boldsymbol{f} - P_h^0 \boldsymbol{f}, \boldsymbol{v}) = 0,\ \forall \boldsymbol{v} \in [L^r(\Omega)]^2.$$

On the other hand, since $\boldsymbol{f} - P_h^0 \boldsymbol{f} \in [L_0^{r'}(\Omega)]^2$, there exists (see Galdi [9])

$$\boldsymbol{\xi} \in \{\boldsymbol{\tau} \in [L^{r'}(\Omega)]^{2\times 2};\ \boldsymbol{div}\, \boldsymbol{\tau} \in [L^{r'}(\Omega)]^2\}$$

such that

$$\boldsymbol{div}\,\boldsymbol{\xi} = \boldsymbol{f} - P_h^0\boldsymbol{f} \;\text{ in } \Omega, \;\text{ and } \|\boldsymbol{\xi}\|_{0,r',\Omega} + \|\,\boldsymbol{div}\,\boldsymbol{\xi}\|_{0,r',\Omega} \le C\|\boldsymbol{f} - P_h^0\boldsymbol{f}|_{0,r',\Omega}.$$

Thus, from these last relations and (14), we get

$$\Big(\boldsymbol{div}\,\big[(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) + \boldsymbol{\xi} - (p - p_h)\mathbb{I}\big], \boldsymbol{v}\Big) = 0,\; \forall \boldsymbol{v} \in [L^r(\Omega)]^2,$$

and (by Lemma 4 in [10])

$$\|p - p_h\|_{0,r',\Omega} \le C\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h \;+ \boldsymbol{\xi}\|_{0,r',\Omega},$$

which implies

$$\|p - p_h\|_{0,r',\Omega} \le C\left(\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,r',\Omega} + \|\boldsymbol{f} - P_h^0\boldsymbol{f}\|_{0,r',\Omega}\right). \tag{15}$$

Owing to (14), we also have

$$\|\,\boldsymbol{div}\,\big[(\boldsymbol{\sigma} - p\mathbb{I}) - (\boldsymbol{\sigma}_h - p_h\mathbb{I})\big]\|_{0,r',\Omega} = \|\boldsymbol{f} - P_h^0\boldsymbol{f}\|_{0,r',\Omega}. \tag{16}$$

Therefore, the estimate (8) follows from (13), (15) and (16). □

Our purpose now is to bound $\|\,\underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\boldsymbol{\sigma}}_h\,\|_\Sigma$ and $\|\,\underset{\sim}{\boldsymbol{u}} - \underset{\sim}{\boldsymbol{u}}_h\,\|_M$ in terms of $\boldsymbol{R}_*$ and $\boldsymbol{S}_*$. We will have to distinguish two cases: $r \ge 2$ and $1 < r < 2$. We begin with the case $r \ge 2$.

Theorem 3. *Let $(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{u}})$ and $(\underset{\sim}{\boldsymbol{\sigma}}_h, \underset{\sim}{\boldsymbol{u}}_h)$ be the solutions of problems (3) and (5), respectively. Suppose that $r \ge 2$, then there exists a constant C independent of h such that*

$$\|\,\underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\boldsymbol{\sigma}}_h\,\|_\Sigma \le C\Big(\,\boldsymbol{R}_* + \boldsymbol{S}_* + {\boldsymbol{S}_*}^{r'/2} + \|\boldsymbol{f} - P_h^0\boldsymbol{f}\|_{0,r',\Omega}\Big), \tag{17}$$

$$\|\,\underset{\sim}{\boldsymbol{u}} - \underset{\sim}{\boldsymbol{u}}_h\,\|_M \le C\Big(\,\boldsymbol{R}_* + {\boldsymbol{R}_*}^{2/r} + {\boldsymbol{S}_*}^{r'/r} + {\boldsymbol{R}_*}^{1/r}\big(\,\boldsymbol{S}_* + \|\boldsymbol{f} - P_h^0\boldsymbol{f}\|_{0,r',\Omega}\big)^{1/r}\Big). \tag{18}$$

Proof. Following Sandri [11], we have

$$\big(A(\boldsymbol{\sigma}_h) - A(\boldsymbol{\sigma}), \boldsymbol{\sigma}_h - \boldsymbol{\sigma}\big) \ge C\left\{\frac{\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}\|^2_{0,r',\Omega}}{\|\boldsymbol{\sigma}_h\|^{2-r'}_{0,r',\Omega} + \|\boldsymbol{\sigma}\|^{2-r'}_{0,r',\Omega}} + \int_\Omega |A(\boldsymbol{\sigma}_h) - A(\boldsymbol{\sigma})|\,|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}|\,dx\right\},$$

and, $\forall \boldsymbol{\tau} \in \big[L^{r'}(\Omega)\big]^{2\times 2}$,

$$\big(A(\boldsymbol{\sigma}_h) - A(\boldsymbol{\sigma}), \boldsymbol{\tau}\big) \le C\left[\int_\Omega |A(\boldsymbol{\sigma}_h) - A(\boldsymbol{\sigma})|\,|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}|\,dx\right]^{1/r} \|\boldsymbol{\tau}\|_{0,r',\Omega}. \tag{19}$$

Then, from (8), we get

$$\begin{aligned}
&\frac{\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}\|^2_{0,r',\Omega}}{\|\boldsymbol{\sigma}_h\|^{2-r'}_{0,r',\Omega} + \|\boldsymbol{\sigma}\|^{2-r'}_{0,r',\Omega}} + \int_\Omega |A(\boldsymbol{\sigma}_h) - A(\boldsymbol{\sigma})|\,|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}|\,dx \\
&\le C\Big\{\boldsymbol{R}_*\left(\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}\|_{0,r',\Omega} + \|\boldsymbol{f} - P_h^0\boldsymbol{f}\|_{0,r',\Omega}\right) \\
&\quad + \boldsymbol{S}_*\,\boldsymbol{R}_* + \boldsymbol{S}_*\left[\int_\Omega |A(\boldsymbol{\sigma}_h) - A(\boldsymbol{\sigma})|\,|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}|\,dx\right]^{1/r}\Big\}.
\end{aligned}$$

Using the Young inequality, *i.e.* $\forall a \geq 0,\ \forall b \geq 0,\ ab \leq \dfrac{1}{r}a^r + \dfrac{1}{r'}b^{r'}$, we obtain $\forall\, \epsilon > 0$ and $\forall\, \bar{\epsilon} > 0$,

$$\begin{aligned}
&\frac{\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}\|^2_{0,r',\Omega}}{\|\boldsymbol{\sigma}_h\|^{2-r'}_{0,r',\Omega} + \|\boldsymbol{\sigma}\|^{2-r'}_{0,r',\Omega}} + \int_\Omega |A(\boldsymbol{\sigma}_h) - A(\boldsymbol{\sigma})|\, |\boldsymbol{\sigma}_h - \boldsymbol{\sigma}|\ dx \\
&\leq C\Big[\epsilon^{-1}\left(\|\boldsymbol{\sigma}_h\|^{2-r'}_{0,r',\Omega} + \|\boldsymbol{\sigma}\|^{2-r'}_{0,r',\Omega}\right)\boldsymbol{R}^2_* \\
&+\epsilon\frac{\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}\|^2_{0,r',\Omega}}{\|\boldsymbol{\sigma}_h\|^{2-r'}_{0,r',\Omega} + \|\boldsymbol{\sigma}\|^{2-r'}_{0,r',\Omega}} + \boldsymbol{R}_*\, \|\boldsymbol{f} - P^0_h\boldsymbol{f}\|_{0,r',\Omega} + \boldsymbol{S}_*\, \boldsymbol{R}_* + (\bar{\epsilon})^{-r'}\ \boldsymbol{S}^{r'}_* \\
&+(\bar{\epsilon})^r \int_\Omega |A(\boldsymbol{\sigma}_h) - A(\boldsymbol{\sigma})|\, |\boldsymbol{\sigma}_h - \boldsymbol{\sigma}|\ dx\Big].
\end{aligned}$$

To simplify the notations, we set

$$\boldsymbol{\Lambda} = \left(\|\boldsymbol{\sigma}_h\|^{2-r'}_{0,r',\Omega} + \|\boldsymbol{\sigma}\|^{2-r'}_{0,r',\Omega}\right)\boldsymbol{R}^2_* + \boldsymbol{R}_*\, \|\boldsymbol{f} - P^0_h\boldsymbol{f}\|_{0,r',\Omega} + \boldsymbol{S}_*\, \boldsymbol{R}_* + \boldsymbol{S}^{r'}_*\ .$$

We then have

$$\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}\|^2_{0,r',\Omega} \quad\leq\quad C\Big(\|\boldsymbol{\sigma}_h\|^{2-r'}_{0,r',\Omega} + \|\boldsymbol{\sigma}\|^{2-r'}_{0,r',\Omega}\Big)\boldsymbol{\Lambda}$$

and

$$\int_\Omega |A(\boldsymbol{\sigma}_h) - A(\boldsymbol{\sigma})|\, |\boldsymbol{\sigma}_h - \boldsymbol{\sigma}|\ dx \quad\leq\quad C\boldsymbol{\Lambda}$$

Using the fact that $\|\boldsymbol{\sigma}\|_{0,r',\Omega}$ and $\|\boldsymbol{\sigma}_h\|_{0,r',\Omega}$ are bounded, (see Theorem 1 and Theorem 2), it follows from the previous inequalities :

$$\begin{aligned}
\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}\|^2_{0,r',\Omega} &\leq C\left(\boldsymbol{R}^2_* + \boldsymbol{R}_*\, \|\boldsymbol{f} - P^0_h\boldsymbol{f}\|_{0,r',\Omega} + \boldsymbol{S}_*\, \boldsymbol{R}_* + \boldsymbol{S}^{r'}_*\right), \\
\int_\Omega |A(\boldsymbol{\sigma}_h) - A(\boldsymbol{\sigma})|\, |\boldsymbol{\sigma}_h - \boldsymbol{\sigma}|\ dx &\leq C\left(\boldsymbol{R}^2_* + \boldsymbol{R}_*\, \|\boldsymbol{f} - P^0_h\boldsymbol{f}\|_{0,r',\Omega} + \boldsymbol{S}_*\, \boldsymbol{R}_* + \boldsymbol{S}^{r'}_*\right),
\end{aligned}$$

which implies

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,r',\Omega} \leq C\left(\boldsymbol{R}_* + \|\boldsymbol{f} - P^0_h\boldsymbol{f}\|_{0,r',\Omega} + \boldsymbol{S}_* + \boldsymbol{S}^{r'/2}_*\right) \tag{20}$$

and

$$\int_\Omega |A(\boldsymbol{\sigma}_h) - A(\boldsymbol{\sigma})|\, |\boldsymbol{\sigma}_h - \boldsymbol{\sigma}|\ dx \leq C\left(\boldsymbol{R}^2_* + \boldsymbol{S}^{r'}_* + \boldsymbol{R}_*\left(\boldsymbol{S}_* + \|\boldsymbol{f} - P^0_h\boldsymbol{f}\|_{0,r',\Omega}\right)\right). \tag{21}$$

Thus, the estimate (17) is a consequence of (20), (15) and (16). On the other hand, from (12) and (19) we have

$$\|\,\underset{\sim}{\boldsymbol{u}} - \underset{\sim}{\boldsymbol{u}_h}\,\|_M \leq C\Big(\boldsymbol{R}_* + \Big[\int_\Omega |A(\boldsymbol{\sigma}_h) - A(\boldsymbol{\sigma})|\, |\boldsymbol{\sigma}_h - \boldsymbol{\sigma}|\ dx\Big]^{1/r}\Big).$$

Therefore, the estimate (18) follows from this last one and (21). □

We now turn to the case $1 < r < 2$.

Theorem 4. *Let* $(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{u}})$ *and* $(\underset{\sim}{\boldsymbol{\sigma}_h}, \underset{\sim}{\boldsymbol{u}_h})$ *be the solutions of problems (3) and (5), respectively. Suppose that* $1 < r < 2$*, then there exists a constant* C *independent of* h *such that*

$$\|\,\underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\boldsymbol{\sigma}_h}\,\|_\Sigma \quad\leq\quad C\Big(\boldsymbol{R}^{r/r'}_* + \boldsymbol{S}_* + \boldsymbol{S}^{2/r'}_* + \|\boldsymbol{f} - P^0_h\boldsymbol{f}\|_{0,r',\Omega}\Big), \tag{22}$$

$$\|\,\underset{\sim}{\boldsymbol{u}} - \underset{\sim}{\boldsymbol{u}_h}\,\|_M \quad\leq\quad C\Big(\boldsymbol{R}_* + \boldsymbol{R}^{r/2}_* + \boldsymbol{S}_* + \|\boldsymbol{f} - P^0_h\boldsymbol{f}\|_{0,r',\Omega}\Big). \tag{23}$$

Proof. As in the proof of Theorem 3, we have, by using the fact that (cf. Sandri [11])

$$\big(A(\boldsymbol{\sigma}_h)-A(\boldsymbol{\sigma}),\boldsymbol{\sigma}_h-\boldsymbol{\sigma}\big) \geq C\left\{\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}\|_{0,r',\Omega}^{r'}+\int_\Omega (|\boldsymbol{\sigma}_h|+|\boldsymbol{\sigma}|)^{r'-2}|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}|^2\,dx\right\},$$

$$\|A(\boldsymbol{\sigma}_h)-A(\boldsymbol{\sigma})\|_{0,r,\Omega} \leq C\left[\int_\Omega (|\boldsymbol{\sigma}_h|+|\boldsymbol{\sigma}|)^{r'-2}|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}|^2\,dx\right]^{1/2}$$

$$[\|\boldsymbol{\sigma}_h\|_{0,r',\Omega}+\|\boldsymbol{\sigma}\|_{0,r',\Omega}]^{(r'-2)/2}, \tag{24}$$

and (8)

$$\begin{aligned}&\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}\|_{0,r',\Omega}^{r'}+\int_\Omega (|\boldsymbol{\sigma}_h|+|\boldsymbol{\sigma}|)^{r'-2}|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}|^2\,dx\\ &\leq C\Big\{\boldsymbol{R}_*\left(\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}\|_{0,r',\Omega}+\|\boldsymbol{f}-P_h^0\boldsymbol{f}\|_{0,r',\Omega}\right)\\ &+\boldsymbol{S}_*\,\boldsymbol{R}_*+\boldsymbol{S}_*\left[\int_\Omega (|\boldsymbol{\sigma}_h|+|\boldsymbol{\sigma}|)^{r'-2}|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}|^2\,dx\right]^{1/2}[\|\boldsymbol{\sigma}_h\|_{0,r',\Omega}+\|\boldsymbol{\sigma}\|_{0,r',\Omega}]^{(r'-2)/2}\Big\}.\end{aligned}$$

Thus, using this last relation and the Young inequality, we get

$$\begin{aligned}&\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}\|_{0,r',\Omega}^{r'}+\int_\Omega (|\boldsymbol{\sigma}_h|+|\boldsymbol{\sigma}|)^{r'-2}|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}|^2\,dx\\ &\leq C\left\{\epsilon^{r'}\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}\|_{0,r',\Omega}^{r'}+\epsilon^{-r}\boldsymbol{R}_*^r+\boldsymbol{R}_*\|\boldsymbol{f}-P_h^0\boldsymbol{f}\|_{0,r',\Omega}+\boldsymbol{S}_*\,\boldsymbol{R}_*\right\}\\ &+C\left\{\bar{\epsilon}\int_\Omega (|\boldsymbol{\sigma}_h|+|\boldsymbol{\sigma}|)^{r'-2}|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}|^2\,dx+(\bar{\epsilon})^{-1}[\|\boldsymbol{\sigma}_h\|_{0,r',\Omega}+\|\boldsymbol{\sigma}\|_{0,r',\Omega}]^{(r'-2)}\boldsymbol{S}_*^2\right\},\end{aligned}$$

and then (using the fact that $\|\boldsymbol{\sigma}\|_{0,r',\Omega}$ and $\|\boldsymbol{\sigma}_h\|_{0,r',\Omega}$ are bounded)

$$\begin{aligned}\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}\|_{0,r',\Omega}^{r'}&+\int_\Omega (|\boldsymbol{\sigma}_h|+|\boldsymbol{\sigma}|)^{r'-2}|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}|^2\,dx\\ &\leq C\left\{\boldsymbol{R}_*^r+\boldsymbol{R}_*\|\boldsymbol{f}-P_h^0\boldsymbol{f}\|_{0,r',\Omega}+\boldsymbol{S}_*\,\boldsymbol{R}_*+\boldsymbol{S}_*^2\right\}\\ &\leq C\left\{\boldsymbol{R}_*^r+\|\boldsymbol{f}-P_h^0\boldsymbol{f}\|_{0,r',\Omega}^{r'}+\boldsymbol{S}_*^{r'}+\boldsymbol{S}_*^2\right\}.\end{aligned}$$

Thus

$$\|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}\|_{0,r',\Omega} \leq C\left\{\boldsymbol{R}_*^{r/r'}+\boldsymbol{S}_*+\boldsymbol{S}_*^{2/r'}+\|\boldsymbol{f}-P_h^0\boldsymbol{f}\|_{0,r',\Omega}\right\} \tag{25}$$

and

$$\int_\Omega (|\boldsymbol{\sigma}_h|+|\boldsymbol{\sigma}|)^{r'-2}|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}|^2\,dx \leq C\left\{\boldsymbol{R}_*^r+\boldsymbol{R}_*^2+\|\boldsymbol{f}-P_h^0\boldsymbol{f}\|_{0,r',\Omega}^2+\boldsymbol{S}_*^2\right\}. \tag{26}$$

Thus the estimate (22) is a consequence of (25), (15) and (16).

On the other hand from (12), (24) and the fact that $\|\boldsymbol{\sigma}\|_{0,r',\Omega}$ and $\|\boldsymbol{\sigma}_h\|_{0,r',\Omega}$ are bounded, we have

$$\|\underset{\sim}{\boldsymbol{u}}-\underset{\sim}{\boldsymbol{u}_h}\|_M \quad\leq\quad C\left(\boldsymbol{R}_*+\left[\int_\Omega (|\boldsymbol{\sigma}_h|+|\boldsymbol{\sigma}|)^{r'-2}|\boldsymbol{\sigma}_h-\boldsymbol{\sigma}|^2\,dx\right]^{1/2}\right).$$

Therefore, the estimate (23) follows from this last one and (26) □

Owing to the results of Theorem 3 and Theorem 4 it is sufficient to estimate $\boldsymbol{R}_*$ and $\boldsymbol{S}_*$. To this end, we first precise some notations: for a tensor field $\boldsymbol{\tau}$, and for a vector field $\boldsymbol{v}=(v_1,v_2)$,

$$tr(\boldsymbol{\tau})=\tau_{11}+\tau_{22},\ as(\boldsymbol{\tau})=\tau_{21}-\tau_{12},\ rot(\boldsymbol{\tau})=\left(\frac{\partial\tau_{12}}{\partial x_1}-\frac{\partial\tau_{11}}{\partial x_2},\frac{\partial\tau_{22}}{\partial x_1}-\frac{\partial\tau_{21}}{\partial x_2}\right),$$

$$Curl(\boldsymbol{v}) = \begin{pmatrix} \dfrac{\partial v_1}{\partial x_2} & -\dfrac{\partial v_1}{\partial x_1} \\ \dfrac{\partial v_2}{\partial x_2} & -\dfrac{\partial v_2}{\partial x_1} \end{pmatrix},$$

and $[\![g]\!]_E$ is the jump of g across an edge E.

The key for the estimation of $\boldsymbol{R}_*$ is the following Helmholtz decomposition of a tensor field in Σ.

Proposition 3. *Let* $\underset{\sim}{\tau} \in \Sigma$*. Then there exist* $\boldsymbol{z} \in \left[W^{2,r'}(\Omega)\right]^2$ *and* $\boldsymbol{\psi} \in \left[W^{1,r'}(\Omega)\right]^2$ *such that*

$$\boldsymbol{\tau} - q\mathbb{I} = \nabla \boldsymbol{z} + Curl\boldsymbol{\psi}, \tag{27}$$

with the estimate

$$\|\boldsymbol{z}\|_{2,r',\Omega} + \|\boldsymbol{\psi}\|_{1,r',\Omega} \leq C \,\|\underset{\sim}{\tau}\|_{\Sigma}. \tag{28}$$

Proof. To prove this result it is sufficient to apply Theorem 1.1 of [6] to each row of the tensor $\boldsymbol{\tau} - q\mathbb{I}$, i.e. the two vector fields $(\tau_{11} - q, \tau_{12})$ and $(\tau_{21}, \tau_{22} - q)$. □

We then have the following result.

Lemma 1. *For every* $\underset{\sim}{\tau} \in \Sigma$*, we have*

(29)
$$\begin{aligned} < \boldsymbol{R}, \underset{\sim}{\tau} > \;&=\; \textstyle\sum_{K\in T_h} (A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h, \nabla \boldsymbol{z} - \Pi_h(\nabla \boldsymbol{z})) + \sum_{K\in T_h} (tr(A(\boldsymbol{\sigma}_h)), q) \\ &+\; \textstyle\sum_{K\in T_h} (rot(A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h), \boldsymbol{\psi} - I_{cl}(\boldsymbol{\psi})) \\ &-\; \textstyle\sum_{E\in E_h} < [\![(A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h)\boldsymbol{t}]\!]_E, \boldsymbol{\psi} - I_{cl}(\boldsymbol{\psi}) >_E \end{aligned}$$

where

- $(\boldsymbol{z}, \boldsymbol{\psi}) \in [W^{2,r'}(\Omega)]^2 \times [W^{1,r'}(\Omega)]^2$ *denotes the Helmholtz decomposition of* $\underset{\sim}{\tau} \in \Sigma$,
- $I_{cl}(\boldsymbol{\psi})$ *is the Clément interpolate of* $\boldsymbol{\psi}$ *(see* [5]*),*
- E_h *denotes the set of all edges of the triangulation* T_h*,*
- $[\![(A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h)\boldsymbol{t}]\!]_E$ *denotes the tangential jump of* $A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h$*,*
- $\Pi_h(\nabla \boldsymbol{z})$ *is the Brezzi-Douglas-Marini interpolate of the lowest degree of* $\nabla \boldsymbol{z}$ *(see* [4]*).*

Proof. By (6) for every $\underset{\sim}{\tau} \in \Sigma$,

$$< \boldsymbol{R}, \underset{\sim}{\tau} >= (A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h, \boldsymbol{\tau}) + (\boldsymbol{div}(\boldsymbol{\tau} - q\mathbb{I}), \boldsymbol{u}_h).$$

Then, using the Helmholtz decomposition (27), we get

$$\begin{aligned} < \boldsymbol{R}, \underset{\sim}{\tau} > \;&=\; (A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h, \nabla \boldsymbol{z}) + (A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h, q\mathbb{I}) \\ &+\; (A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h, Curl\boldsymbol{\psi}) + (\boldsymbol{div}(\nabla \boldsymbol{z}), \boldsymbol{u}_h). \end{aligned} \tag{30}$$

Let $\Pi_h(\nabla \boldsymbol{z})$ denote the Brezzi-Douglas-Marini interpolate of the lowest degree of $\nabla \boldsymbol{z}$. We have (see [4])

$$(\boldsymbol{div}(\Pi_h(\nabla \boldsymbol{z})), \boldsymbol{v}_h) = (\boldsymbol{div}(\nabla \boldsymbol{z}), \boldsymbol{v}_h), \;\forall \boldsymbol{v}_h \in \left[\prod_{K\in T_h} P_0(K)\right]^2.$$

Thus, using this last relation and the fact that $tr(\boldsymbol{\omega}_h) = 0$, (30) may be rewritten:

$$\begin{aligned} < \boldsymbol{R}, \underset{\sim}{\tau} > \;&=\; (A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h, \nabla \boldsymbol{z}) + (tr(A(\boldsymbol{\sigma}_h)), q) \\ &+\; (A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h, Curl\boldsymbol{\psi}) + (\boldsymbol{div}(\Pi_h(\nabla \boldsymbol{z})), \boldsymbol{u}_h). \end{aligned} \tag{31}$$

Taking successively $\underset{\sim}{\boldsymbol{\tau}}_h = (\Pi_h(\nabla \boldsymbol{z}), 0) \in \Sigma_h$ and $\underset{\sim}{\boldsymbol{\tau}}_h = (Curl(I_{cl}(\boldsymbol{\psi})), 0) \in \Sigma_h$ in the first equation of the discrete problem (5), we obtain

$$(A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h, \Pi_h(\nabla \boldsymbol{z})) + (\boldsymbol{div}(\Pi_h(\nabla \boldsymbol{z})), \boldsymbol{u}_h) = 0$$

and

$$(A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h, Curl(I_{cl}(\boldsymbol{\psi}))) = 0.$$

Injecting these two last relations in the right-hand side of (31), we get

$$\begin{aligned} < \boldsymbol{R}, \underset{\sim}{\boldsymbol{\tau}} > &= (A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h, \nabla \boldsymbol{z} - \Pi_h(\nabla \boldsymbol{z})) + (tr(A(\boldsymbol{\sigma}_h)), q) \\ &+ (A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h, Curl(\boldsymbol{\psi} - I_{cl}(\boldsymbol{\psi}))). \end{aligned}$$

Thus, using Green's formula, we obtain

$$\begin{aligned} < \boldsymbol{R}, \underset{\sim}{\boldsymbol{\tau}} > &= (A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h, \nabla \boldsymbol{z} - \Pi_h(\nabla \boldsymbol{z})) + (tr(A(\boldsymbol{\sigma}_h)), q) \\ &+ \sum_{K \in T_h} \{(rot(A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h), \boldsymbol{\psi} - I_{cl}(\boldsymbol{\psi})) \\ &- < (A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h)\boldsymbol{t}, \boldsymbol{\psi} - I_{cl}(\boldsymbol{\psi}) >_{\partial K}\} \\ &= \sum_{K \in T_h} (A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h, \nabla \boldsymbol{z} - \Pi_h(\nabla \boldsymbol{z})) + \sum_{K \in T_h} (tr(A(\boldsymbol{\sigma}_h)), q) \\ &+ \sum_{K \in T_h} (rot(A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h), \boldsymbol{\psi} - I_{cl}(\boldsymbol{\psi})) \\ &- \sum_{E \in E_h} < [\![(A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h)\boldsymbol{t}]\!]_E, \boldsymbol{\psi} - I_{cl}(\boldsymbol{\psi}) >_E . \end{aligned}$$

□

We are now in a position to bound $\boldsymbol{R}_*$ and $\boldsymbol{S}_*$ by two error estimators η_1 and η_2.

Theorem 5. *There exists a constant C independent of h such that*

$$\boldsymbol{R}_* \leq C\,\boldsymbol{R}_1, \text{ where } \boldsymbol{R}_1 \text{ is given by } \boldsymbol{R}_1 = \Big(\sum_{K \in T_h} \eta_1(K)^r\Big)^{1/r}, \tag{32}$$

$$\boldsymbol{S}_* \leq C\,\boldsymbol{S}_1, \text{ where } \boldsymbol{S}_1 \text{ is given by } \boldsymbol{S}_1 = \Big(\sum_{K \in T_h} \eta_2(K)^{r'}\Big)^{1/r'}, \tag{33}$$

where $\eta_1(K)$ and $\eta_2(K)$ are the local estimators given by

$$\begin{aligned} \eta_1(K)^r &= h_K^r \|A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h\|_{0,r,K}^r + \|tr(A(\boldsymbol{\sigma}_h))\|_{0,r,K}^r \\ &+ h_K^r \|\, rot(A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h)\|_{0,r,K}^r + \sum_{E \in \partial K} h_E \|\, [\![(A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h)\boldsymbol{t}]\!]_E\|_{0,r,E}^r \\ \eta_2(K)^{r'} &= \|\boldsymbol{f} - P_h^0 \boldsymbol{f}\|_{0,r',K}^{r'} + \|as(\boldsymbol{\sigma}_h)\|_{0,r',K}^{r'}. \end{aligned}$$

Proof. It follows from (29) that for every $\underset{\sim}{\boldsymbol{\tau}} \in \Sigma$

$$\begin{aligned} | < \boldsymbol{R}, \underset{\sim}{\boldsymbol{\tau}} > | &\leq \textstyle\sum_{K \in T_h} \|A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h\|_{0,r,K} \|\nabla \boldsymbol{z} - \Pi_h(\nabla \boldsymbol{z})\|_{0,r',K} \\ &+ \textstyle\sum_{K \in T_h} \|tr(A(\boldsymbol{\sigma}_h))\|_{0,r,K} \|q\|_{0,r',K} \\ &+ \textstyle\sum_{K \in T_h} \|\, rot(A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h)\|_{0,r,K} \|\boldsymbol{\psi} - I_{cl}(\boldsymbol{\psi})\|_{0,r',K} \\ &+ \textstyle\sum_{E \in E_h} \|\, [\![(A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h)\boldsymbol{t}]\!]_E\|_{0,r,E} \|\boldsymbol{\psi} - I_{cl}(\boldsymbol{\psi})\|_{0,r',E}. \end{aligned} \tag{34}$$

Now, by Lemma 3.1 of [13], we have

$$\|\boldsymbol{\psi} - I_{cl}(\boldsymbol{\psi})\|_{0,r',K} \leq C h_K |\boldsymbol{\psi}|_{1,r',\omega_K}$$

and

$$\|\boldsymbol{\psi} - I_{cl}(\boldsymbol{\psi})\|_{0,r',E} \leq C h_E^{1/r} |\boldsymbol{\psi}|_{1,r',\omega_E}$$

where ω_K denotes the union of K with all the triangles from the triangulation T_h adjacent to the triangle K and ω_E denotes the union of at most two triangles of T_h admitting E as an edge. Thus, using these two last estimates and the fact that $\|\nabla \boldsymbol{z} - \Pi_h(\nabla \boldsymbol{z})\|_{0,r',K} \leq C h_K |\nabla \boldsymbol{z}|_{1,r',K}$, (34) yield

$$\begin{aligned}
| < \boldsymbol{R}, \underset{\sim}{\boldsymbol{\tau}} > | \quad & \leq \quad C \sum_{K \in T_h} h_K \|A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h\|_{0,r,K} |\nabla \boldsymbol{z}|_{1,r',K} \\
& + \quad C \sum_{K \in T_h} \|tr(A(\boldsymbol{\sigma}_h))\|_{0,r,K} \|q\|_{0,r',K} \\
& + \quad C \sum_{K \in T_h} h_K \| \, rot(A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h)\|_{0,r,K} |\boldsymbol{\psi}|_{1,r',\omega_K} \\
& + \quad C \sum_{E \in E_h} h_E^{1/r} \| \left[\!\left[(A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h)\boldsymbol{t} \right]\!\right]_E \|_{0,r,E} |\boldsymbol{\psi}|_{1,r',\omega_E} \\
& \leq \quad C(\sum_{K \in T_h} h_K^r \|A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h\|_{0,r,K}^r)^{1/r} |\nabla \boldsymbol{z}|_{1,r',\Omega} \\
& + \quad C(\sum_{K \in T_h} \|tr(A(\boldsymbol{\sigma}_h))\|_{0,r,K}^r)^{1/r} \|q\|_{0,r',\Omega} \\
& + \quad C(\sum_{K \in T_h} h_K^r \| \, rot(A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h)\|_{0,r,K}^r)^{1/r} |\boldsymbol{\psi}|_{1,r',\Omega} \\
& + \quad C(\sum_{E \in E_h} h_E \| \left[\!\left[(A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h)\boldsymbol{t} \right]\!\right]_E \|_{0,r,E}^r)^{1/r} |\boldsymbol{\psi}|_{1,r',\Omega}
\end{aligned}$$

and so

$$\begin{aligned}
| < \boldsymbol{R}, \underset{\sim}{\boldsymbol{\tau}} > | \quad & \leq \quad C \Bigg\{ \sum_{K \in T_h} (h_K^r \|A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h\|_{0,r,K}^r + \|tr(A(\boldsymbol{\sigma}_h))\|_{0,r,K}^r \\
& + \quad h_K^r \| \, rot(A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h)\|_{0,r,K}^r + \sum_{E \subset \partial K} h_E \| \left[\!\left[(A(\boldsymbol{\sigma}_h) + \boldsymbol{\omega}_h)\boldsymbol{t} \right]\!\right]_E \|_{0,r,E}^r) \Bigg\}^{1/r} \\
& \times \quad \left\{ |\nabla \boldsymbol{z}|_{1,r',\Omega}^{r'} + \|q\|_{0,r',\Omega}^{r'} + |\boldsymbol{\psi}|_{1,r',\Omega}^{r'} \right\}^{1/r'} .
\end{aligned}$$

Therefore, using (28), we obtain

$$| < \boldsymbol{R}, \underset{\sim}{\boldsymbol{\tau}} > | \leq C(\sum_{K \in T_h} \eta_1(K)^r)^{1/r} \| \underset{\sim}{\boldsymbol{\tau}} \|_\Sigma \leq C \, \boldsymbol{R}_1 \, \| \underset{\sim}{\boldsymbol{\tau}} \|_\Sigma$$

and (32) follows immediately.

It remains to prove (33). By (7), we have for every $\underset{\sim}{\boldsymbol{v}} \in M$

$$\begin{aligned}
| < \boldsymbol{S}, \underset{\sim}{\boldsymbol{v}} > | \quad & \leq \quad (\sum_{K \in T_h} \| \, \boldsymbol{div}(\boldsymbol{\sigma}_h - p_h \mathbb{I}) + \boldsymbol{f}\|_{0,r',K}^{r'})^{1/r'} \|\boldsymbol{v}\|_{0,r,\Omega} \\
& + \quad C(\sum_{K \in T_h} \|as(\boldsymbol{\sigma}_h)\|_{0,r',K}^{r'})^{1/r'} \|\boldsymbol{\eta}\|_{0,r,\Omega} \\
& \leq \quad C(\sum_{K \in T_h} \| \, \boldsymbol{div}(\boldsymbol{\sigma}_h - p_h \mathbb{I}) + \boldsymbol{f}\|_{0,r',K}^{r'} + \|as(\boldsymbol{\sigma}_h)\|_{0,r',K}^{r'})^{1/r'} \\
& \quad\ \times (\|\boldsymbol{v}\|_{0,r,\Omega}^r + \|\boldsymbol{\eta}\|_{0,r,\Omega}^r)^{1/r}.
\end{aligned}$$

But by the second equation of the discrete problem (5), we have $\boldsymbol{div}(\boldsymbol{\sigma}_h - p_h\mathbb{I}) = -P_h^0 \boldsymbol{f}$. Therefore, for every $\underset{\sim}{\boldsymbol{v}} \in M$,

$$|<\boldsymbol{S}, \underset{\sim}{\boldsymbol{v}}>| \leq C(\sum_{K\in T_h} \|\boldsymbol{f} - P_h^0\boldsymbol{f}\|_{0,r',K}^{r'} + \|as(\boldsymbol{\sigma}_h)\|_{0,r',K}^{r'})^{1/r'} \| \underset{\sim}{\boldsymbol{v}} \|_M$$

which implies $\boldsymbol{S}_* \leq C\,\boldsymbol{S}_1$. □

4. Conclusion

A new a posteriori error estimator for a mixed finite element approximation of non-Newtonian fluid flow problems is introduced and analyzed. The estimator justifies an adaptive finite element scheme which refines a given grid only in regions where the error is relatively large. Finally, the technique developed to establish this estimator can be extended to the three dimensional case.

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