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EXACT DIFFERENCE SCHEMES FOR PARABOLIC EQUATIONS

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Abstract. The Cauchy problem for the parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k\,(x,t)\,\frac{\partial u}{\partial x}\right) + f(u,x,t),\ \ x \in R,\ \ t > 0,$$
$$u(x,0) = u_0(x),\ \ x \in R,$$

is considered. Under conditions $u(x,t) = X(x)T_1(t) + T_2(t)$, $\frac{\partial u}{\partial x} \neq 0$, $k(x,t) = k_1(x)k_2(t)$, $f(u,x,t) = f_1(x,t)f_2(u)$, it is shown that the above problem is equivalent to a system of two first-order ordinary differential equations for which exact difference schemes with special Steklov averaging and difference schemes with any order of approximation are constructed on the moving mesh. On the basis of this approach, the exact difference schemes are constructed also for boundary-value problems and multi-dimensional problems. Presented numerical experiments confirm the theoretical results investigated in the paper.

Key Words. exact difference scheme, difference scheme with an arbitrary order of accuracy, parabolic equation, system of ordinary differential equations.

1. Introduction

Various schemes have been constructed to approximate initial- and boundary-value problems for parabolic equations [17]. One of the main questions in investigating difference schemes is the approximation order, which is desired to be as high as possible.

In the last few years, the exact difference schemes for some partial differential equations have been constructed [3], [5], [6]. It is worth here to mention the papers by R.E. Mickens [7] - [12], in which certain rules for construction of the nonstandard finite difference schemes are given and several such schemes were introduced, for example for the Burgers partial differential equation having no diffusion and a nonlinear logistic reaction term [11]. S. Rucker [16] applied techniques initiated by R. E. Mickens to obtain exact difference scheme for an advection – reaction equation. In the paper [5], under natural conditions, the authors proved existence of a two-point exact difference scheme for systems of first-order boundary value problems. Difference schemes of high order of approximation were also constructed in [15], [19].

The authors earlier established that for problems for parabolic equations with solutions of the separated variables $u(x,t) = X(x)T_1(t) + T_2(t)$ the exact difference scheme may be constructed. The main feature of this paper is to apply the method introduced in [6] for a wider classes of problems. The attention is mainly devoted

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to constructing a difference scheme of arbitrary order of approximation in the case when the integral in special Steklov averaging cannot be evaluated exactly, as well as developing the exact difference schemes in multi-dimensional case by using the presented approach.

Consider the Cauchy problem for the one-dimensional parabolic equation

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(k\,(x,t)\,\frac{\partial u}{\partial x}\right)+f(u,x,t),\ x\in R,\ t>0, \tag{1.1}$$

$$u(x,0)=u_0(x),\ x\in R. \tag{1.2}$$

Under conditions $u(x,t)=X(x)T_1(t)+T_2(t)$, $\frac{\partial u}{\partial x}\neq 0$, $k(x,t)=k_1(x)k_2(t)$, $f(u,x,t)=f_1(x,t)f_2(u)$, we show that problem (1.1) - (1.2) is equivalent to the following system of two ordinary differential equations [6]:

$$\frac{dx}{dt}=c_1(x)k_2(t), \tag{1.3}$$

$$\left.\frac{du}{dt}\right|_{\frac{dx}{dt}=c_1(x)k_2(t)}=f_1(x(t),t)f_2(u),\ u(x(0),0)=u_0(x(0)), \tag{1.4}$$

where $c_1(x)=-\frac{\left(k_1(x)u_0'(x)\right)'}{u_0'(x)}$. ¿From (1.3) we find the curve $x=x(t)$, along which we get from (1.4) the solution $u(x,t)=u(x(t),t)$ of problem (1.1) - (1.2). Here $x(0)=x^0\in R$ is the initial state of the curve $x=x(t)$. Special Steklov averaging [6], [17]

$$c(x(t))\approx\left[\frac{1}{x^{n+1}-x^n}\int\limits_{x^n}^{x^{n+1}}\frac{dx}{c(x)}\right]^{-1},\ t_n\le t\le t_{n+1},\ x^n=x(t_n),\ t_n=n\tau$$

is used to construct exact difference schemes only on the moving mesh. On the basis of this approach, the exact difference schemes are constructed also for boundary-value problems and for multi-dimensional problems. A difference scheme of arbitrary order of approximation is proposed in the case when the integral in the Steklov averaging cannot be evaluated exactly.

2. Exact difference schemes: the Cauchy problem for parabolic equations

In this Section, using the special Steklov averaging the exact difference scheme for the Cauchy problem for parabolic equations is constructed.

Let us consider in the domain $Q_T=R\times[0,\infty)$ the Cauchy problem for the one-dimensional parabolic equation:

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(k(x,t)\frac{\partial u}{\partial x}\right)+f(x,t,u),\ x\in R,\ t>0, \tag{2.5}$$

$$u(x,0)=u_0(x),\ x\in R. \tag{2.6}$$

Assume that the problem (2.5) - (2.6) has an unique solution $u(x,t)\in C_1^2(Q_T)$, $u(x,t)=X(x)T_1(t)+T_2(t)$, $\frac{\partial u}{\partial x}\neq 0$ and that the input data has the following form $k(x,t)=k_1(x)k_2(t)$, $f(x,t,u)=f_1(x,t)f_2(u)$. The coefficient k is bounded from above and below, i.e. $0<k_1\le k(x,t)\le k_2$, for $(x,t)\in R\times(0,\infty)$, where $k_1,\ k_2=const$, and $k(x,t)\in C_1^1(Q_T)$.

Rewriting Equation (2.5) as

$$\frac{\partial u}{\partial t} - \frac{\frac{\partial}{\partial x}\left(k(x,t)\frac{\partial u}{\partial x}\right)}{\frac{\partial u}{\partial x}}\frac{\partial u}{\partial x} = f(x,t,u)$$

and using the notation

$$\frac{dx}{dt} = -\frac{\frac{\partial}{\partial x}\left(k(x,t)\frac{\partial u}{\partial x}\right)}{\frac{\partial u}{\partial x}}$$

yields the following form:

$$\frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} = f(x,t,u).$$

For $\frac{dx}{dt}$ the following holds:

$$\frac{dx}{dt} = -\frac{\frac{\partial}{\partial x}\left(k(x,t)\frac{\partial u}{\partial x}\right)}{\frac{\partial u}{\partial x}} = -\frac{\frac{\partial}{\partial x}\left(k(x,t)X'(x)T_1(t)\right)}{X'(x)T_1(t)}$$

$$= -\frac{\frac{\partial}{\partial x}\left(k(x,t)X'(x)\right)}{X'(x)} = -\frac{\frac{\partial}{\partial x}\left(k(x,t)X'(x)T_1(0)\right)}{X'(x)_1T(0)} = -\frac{\frac{\partial}{\partial x}\left(k_1(x)k_2(t)u_0'(x)\right)}{u_0'(x)}$$

$$= -\frac{k_2(t)\left(k_1(x)u_0'(x)\right)'}{u_0'(x)} = c_1(x)k_2(t),$$

where $c_1(x) = -\frac{\left(k_1(x)u_0'(x)\right)'}{u_0'(x)}$. It follows that, instead of the differential problem (2.5) - (2.6), we have

$$\frac{dx}{dt} = c_1(x)k_2(t), \tag{2.7}$$

$$\left.\frac{du}{dt}\right|_{\frac{dx}{dt}=c_1(x)k_2(t)} = f_1(x(t),t)f_2(u),\ u(x(0),0) = u_0(x(0)), \tag{2.8}$$

where $x(0) = x^0 \in R$ is the initial state of the curve $x = x(t)$. Solving this problem, we obtain the following integral equations

$$\int \frac{dx}{c_1(x)} = \int k_2(t)dt, \tag{2.9}$$

$$\int \frac{du}{f_2(u)} = \int f_1(x(t),t)dt. \tag{2.10}$$

Let

$$\overline{\omega}_h^0 = \left\{x_i^0 = ih_i^0,\ i = 0, \pm 1, \pm 2, \dots\right\},$$

$$\overline{\omega}_{hL}^0 = \left\{x_i^0 = -L + ih_i^0,\ h_i^0 = \frac{2L}{N},\ i = \overline{0,N}\right\}$$

be uniform grids in space at $t = 0$ and

$$\overline{\omega}_\tau = \left\{t_n = n\tau,\ n = 0, 1, 2, \dots\right\},$$

$$\overline{\omega}_{\tau T} = \left\{t_n = n\tau,\ n = \overline{0, N_0},\ \tau = \frac{T}{N_0}\right\}$$

be uniform grids in time. Here $h_i^n = x_{i+1}^n - x_i^n$ is the space step at time $t = t_n$.

Definition 2.1. *A difference scheme is exact if the truncation error is equal to zero, i.e., the exact solution agrees with the numerical solution at the grid nodes.*

Let us approximate Problem (2.7) - (2.8) by the difference scheme

$$\frac{x_i^{n+1}-x_i^n}{\tau} = \left(\frac{1}{x_i^{n+1}-x_i^n} \int\limits_{x_i^n}^{x_i^{n+1}} \frac{dx}{c_1(x)} \right)^{-1} \frac{1}{\tau} \int\limits_{t_n}^{t_{n+1}} k_2(t)dt, \tag{2.11}$$
$$x_i^0 \in \overline{\omega}_h^0,\ i=0,\pm1,\pm2,\dots,\ n=0,1,\dots,$$

$$\frac{y_i^{n+1}-y_i^n}{\tau} = \left(\frac{1}{y_i^{n+1}-y_i^n} \int\limits_{y_i^n}^{y_i^{n+1}} \frac{du}{f_2(u)} \right)^{-1} \frac{1}{\tau} \int\limits_{t_n}^{t_{n+1}} f_1(x(t),t)dt, \tag{2.12}$$
$$y_i^0 = u_0(x_i^0),\ i=0,\pm1,\pm2,\dots,\ n=0,1,\dots.$$

Here, Equation (2.11) represents the space-grid as a moving mesh, where $x_i^0 \in \overline{\omega}_h^0$ is the initial partitioning. Then, the following theorem holds:

Theorem 2.1. *The difference scheme (2.11) - (2.12) is exact.*

Proof. We show that the difference scheme (2.11) approximates the differential problem (2.7) exactly [6]. The truncation error $\psi = \psi_i^n$ is

$$\psi_i^n = \frac{x_i^{n+1}-x_i^n}{\tau} - \left(\frac{1}{x_i^{n+1}-x_i^n} \int\limits_{x_i^n}^{x_i^{n+1}} \frac{dx}{c_1(x)} \right)^{-1} \frac{1}{\tau} \int\limits_{t_n}^{t_{n+1}} k_2(t)dt$$
$$= \frac{x_i^{n+1}-x_i^n}{\int\limits_{x_i^n}^{x_i^{n+1}} \frac{dx}{c_1(x)}} \left(\frac{\int\limits_{x^0}^{x_i^{n+1}} \frac{dx}{c_1(x)} - \int\limits_{x^0}^{x_i^n} \frac{dx}{c_1(x)}}{\tau} - \frac{1}{\tau} \int\limits_{t_n}^{t_{n+1}} k_2(t)dt \right). \tag{2.13}$$

On the basis of (2.7), we obtain

$$\frac{dx}{dt} = c_1(x)k_2(t),\ \frac{d}{dt}\left(\int\limits_{x^0}^{x} \frac{dx}{c_1(x)} \right) = k_2(t),\ \int\limits_{x^0}^{x} \frac{dx}{c_1(x)} = \int\limits_{0}^{t} k_2(t)dt,$$
$$\frac{1}{\tau}\left(\int\limits_{x^0}^{x_i^{n+1}} \frac{dx}{c_1(x)} - \int\limits_{x^0}^{x_i^n} \frac{dx}{c_1(x)} \right) = \frac{1}{\tau} \int\limits_{t_n}^{t_{n+1}} k_2(t)dt. \tag{2.14}$$

Substituting (2.14) into (2.13), we obtain $\psi_i^n = 0$, for $n = 0,1,\dots$ and $i = 0,\pm1,\pm2,\dots$. Similarly, we can show that the difference scheme (2.12) approximates the differential problem (2.8) exactly. Hence, Scheme (2.11) - (2.12) is exact. □

Example 2.1. Let us consider the following Cauchy problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - An_1(n_1-1)x^{n_1-2} + Bn_2 t^{n_2-1},\ t>0,\ u(x,0) = Ax^{n_1},\ x \in R, \tag{2.15}$$

where $n_1, n_2 > 2$. The solution of this problem exists and equals $u(x,t) = Ax^{n_1} + Bt^{n_2}$, where $A,\ B = const$. Using the above technique, Equation (2.15) is replaced by the problem

$$\frac{dx}{dt} = -\frac{u_0''(x)}{u_0'(x)} = -\frac{n_1-1}{x}, \tag{2.16}$$

$$\left.\frac{du}{dt}\right|_{\frac{dx}{dt}=-\frac{n_1-1}{x}} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} = f(x(t),t),\ u(x(0),0) = u_0(x(0)) = A\left(x^0\right)^{n_1}, \tag{2.17}$$

where $x(t)$ is the solution of Equation (2.16) and $f(x,t) = -An_1(n_1-1)x^{n_1-2} + Bn_2t^{n_2-1}$.

Solving problem (2.16) - (2.17) analytically, we obtain

$$x(t) = \begin{cases} \sqrt{-2(n_1-1)t + (x^0)^2}, & \text{if } x^0 \geq 0, \\ -\sqrt{-2(n_1-1)t + (x^0)^2}, & \text{if } x^0 < 0, \end{cases}$$

$$u(x(t),t) = \begin{cases} Bt^{n_2} + A\left(\left(x^0\right)^2 - 2(n_1-1)t\right)^{\frac{n_1}{2}}, & \text{if } x^0 \geq 0, \\ Bt^{n_2} + (-1)^{n_1-2}A\left(\left(x^0\right)^2 - 2(n_1-1)t\right)^{\frac{n_1}{2}}, & \text{if } x^0 < 0, \end{cases}$$

where $x^0 \in R$ and $0 < t < \frac{(x^0)^2}{2(n_1-1)}$. Substituting $\left(x^0\right)^2 = x^2 + 2\left(n_1-1\right)t$ in the above equation, we find the solution of the Cauchy problem (2.15) in the explicit form:

$$u(x(t),t) = \begin{cases} Bt^{n_2} + A\left|x\right|^{n_1}, & \text{if } x \geq 0, \\ Bt^{n_2} + (-1)^{n_1-2}A\left|x\right|^{n_1}, & \text{if } x < 0, \end{cases}$$
$$= Bt^{n_2} + Ax^{n_1}.$$

Applying the exact difference scheme (2.11) - (2.12) and evaluating exactly the integrals in it, we get the following formulas (See Fig. 2.1 and Fig. 2.2):

$$x_i^{n+1} = \begin{cases} \sqrt{(x_i^n)^2 - 2\tau(n_1-1)}, & \text{if } x_i^n \geq 0, \\ -\sqrt{(x_i^n)^2 - 2\tau(n_1-1)}, & \text{if } x_i^n < 0, \end{cases},\ x_i^0 \in \overline{\omega}_h^0,\ n = 0,1,\ldots, \tag{2.18}$$

$$y_i^{n+1} = \begin{cases} y_i^n + B\left(t_{n+1}^{n_2} - t_n^{n_2}\right) \\ \quad +A\left(\left(x_i^0\right)^2 - 2(n_1-1)t_{n+1}\right)^{\frac{n_1}{2}} - A\left(\left(x_i^0\right)^2 - 2(n_1-1)t_n\right)^{\frac{n_1}{2}}, & \text{if } x_i^0 \geq 0, \\ y_i^n + B\left(t_{n+1}^{n_2} - t_n^{n_2}\right) \\ \quad +(-1)^{n_1-2}A\left[\left(\left(x_i^0\right)^2 - 2(n_1-1)t_{n+1}\right)^{\frac{n_1}{2}} - \left(\left(x_i^0\right)^2 - 2(n_1-1)t_n\right)^{\frac{n_1}{2}}\right], & \text{if } x_i^0 < 0, \end{cases}$$

$$y_i^0 = u_0(x_i^0),\ i = 0, \pm 1, \pm 2, \ldots,\ n = 0,1,\ldots, \tag{2.19}$$

where $t_n = n\tau$ and $0 < t_n < \frac{(x^0)^2}{2(n_1-1)}$. From the above equations we obtain the formulas:

$$x_i^{n+1} = \begin{cases} \sqrt{(x_i^0)^2 - 2t_{n+1}(n_1-1)}, & \text{if } x_i^0 \geq 0, \\ -\sqrt{(x_i^0)^2 - 2t_{n+1}(n_1-1)}, & \text{if } x_i^0 < 0, \end{cases},\ x_i^0 \in \overline{\omega}_h^0,\ n = 0,1,\ldots,$$

$$y_i^{n+1} = \begin{cases} Bt_{n+1}^{n_2} + A\left(\left(x_i^0\right)^2 - 2(n_1-1)t_{n+1}\right)^{\frac{n_1}{2}}, & \text{if } x_i^0 \geq 0, \\ Bt_{n+1}^{n_2} + (-1)^{n_1-2}A\left(\left(x_i^0\right)^2 - 2(n_1-1)t_{n+1}\right)^{\frac{n_1}{2}}, & \text{if } x_i^0 < 0, \end{cases} \tag{2.20}$$

which coincide with the analytical solution of System (2.16) - (2.17).

Tables 2.1, 2.2, display the results of numerical experiments for different parameter values and confirm the theoretical results stated in Theorem 2.1.

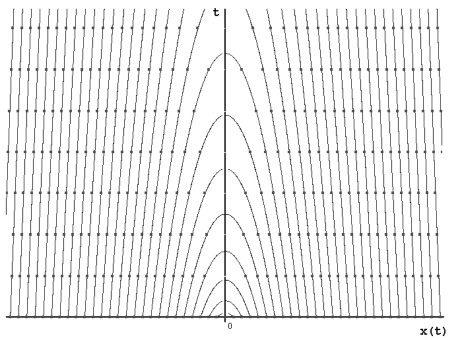

Fig. 2.1: Moving mesh given by (2.18)

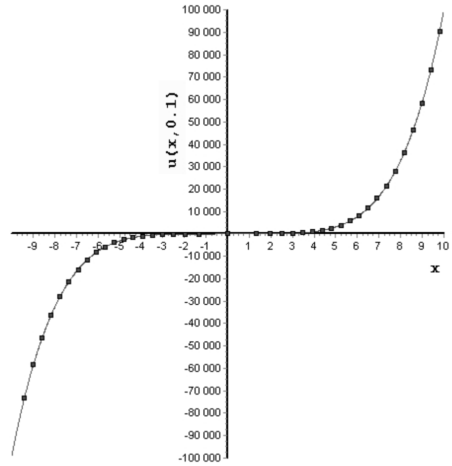

Fig. 2.2: Exact and approximate solution given by (2.19) with parameters $n_1 = 5$, $n_2 = 4$, $A = 1$, $B = 1$.

h_i^0	τ	$\max\limits_{0\le n\le N_0} \|y^n - u(t_n)\|_{\overline{C}}$
1.0	1.0	$6.66E-16$
0.2	0.2	$1.55E-15$
0.1	0.1	$2.11E-15$
0.02	0.02	$6.61E-15$
0.01	0.01	$7.99E-15$

Table 2.1: $L = 5$, $T = 10$

h_i^0	τ	$\max\limits_{0\le n\le N_0} \|y^n - u(t_n)\|_{\overline{C}}$
0.2	0.02	$1.90E-19$
0.04	0.004	$3.79E-19$
0.02	0.002	$6.51E-19$
0.004	0.0004	$2.33E-18$
0.002	0.0002	$8.10E-18$

Table 2.2: $L = 1$, $T = 0.2$

The boundary-value problem for parabolic equation was investigated in [6] by using the same approach. Interested readers are referred to this work for studying examples of the exact difference schemes in this case.

3. Arbitrary-order difference schemes: the boundary-value problem for parabolic equations

In this Section, the difference scheme of arbitrary order of approximation is considered in the case when the integral in it can not be evaluated exactly. The trapezoid rule is applied to approximate the integral and an iteration method is used for finding the solution of the difference scheme.

Consider the boundary-value problem:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right),\ 0 < x < L,\ 0 < t \le T,$$
$$u(x,0) = u_0(x),\ 0 \le x \le L,\ u(0,t) = \mu_1(t),\ u(L,t) = \mu_2(t),\ 0 < t \le T,$$

where $0 < k_1 \le k(x) \le k_2$ for $0 < x < L$. This problem may be rewritten in the form:

$$\frac{dx}{dt} = -\frac{(k(x)u_0'(x))'}{u_0'(x)} = c(x), \tag{3.21}$$

$$\left.\frac{du}{dt}\right|_{\frac{dx}{dt}=c(x)} = 0,\ u(x(0),0) = u_0(x(0)), \tag{3.22}$$

where $c(x)$ is some function of the spatial variable x. We assume that $c(x) \neq 0$ for $x \in [0, L]$ and $c(x) \in C^2[0, L]$. Applying the trapezoid rule

(3.23)
$$\int\limits_{x_i^n}^{x_i^{n+1}} \frac{dx}{c(x)} \approx \frac{x_i^{n+1} - x_i^n}{m}\left(\frac{1}{2c\left(x_i^{n+1}\right)} + \sum_{j=1}^{m-1}\frac{1}{c\left(x_i^n + j\frac{x_i^{n+1}-x_i^n}{m}\right)} + \frac{1}{2c\left(x_i^n\right)}\right),$$

Equation (3.21) is approximated by the difference scheme

(3.24)
$$\frac{x_{hi}^{n+1} - x_{hi}^n}{\tau} = \left[\frac{1}{2m}\left(\frac{1}{c(x_{hi}^n)} + \frac{1}{c\left(x_{hi}^{n+1}\right)}\right) + \frac{1}{m}\sum_{j=1}^{m-1}\frac{1}{c\left(x_{hi}^n + j\frac{x_{hi}^{n+1}-x_{hi}^n}{m}\right)}\right]^{-1},$$
$$x_{hi}^0 = x_i^0 \in \overline{\omega}_h^0,\ i = \overline{0, N},\ n = \overline{0, N_0 - 1},$$
$$x_{hi}^{i-N} = L,\ i = \overline{N+1, N+N_0-1},\ n = \overline{i-N, N_0-1},$$

where $\overline{\omega}_h^0 = \left\{x_i^0 = ih_i^0,\ h_i^0 = \frac{L}{N},\ i = \overline{0, N}\right\}$. The error of the approximation equals $O((\frac{h}{m})^2)$, where $h = \max\limits_{\substack{0\le i\le N+N_0-1\\ 0\le n\le N_0}} \left|x_{hi}^{n+1} - x_{hi}^n\right|$.

However, Equation (3.24) is nonlinear, so we use the iteration method

(3.25)
$$\frac{\overset{s+1}{x}{}_{hi}^{n+1} - x_{hi}^n}{\tau} = \left[\frac{1}{2m}\left(\frac{1}{c(x_{hi}^n)} + \frac{1}{c\left(\overset{s}{x}{}_{hi}^{n+1}\right)}\right) + \frac{1}{m}\sum_{j=1}^{m-1}\frac{1}{c\left(x_{hi}^n + j\frac{\overset{s}{x}{}_{hi}^{n+1}-x_{hi}^n}{m}\right)}\right]^{-1},$$

where the initial approximation $\overset{0}{x}{}_{hi}^{n+1}$ is calculated from the equation

$$\frac{\overset{0}{x}{}_{hi}^{n+1} - x_{hi}^n}{\tau} = c(x_{hi}^n).$$

The stopping criterion in the iteration method is $\max\limits_{0\le i\le N+N_0-1}\left|\overset{s+1}{x}{}_{hi}^{n+1} - \overset{s}{x}{}_{hi}^{n+1}\right| \le \epsilon$, where ϵ is a previously given tolerance. When the above condition is satisfied, we advance to the next level with $x_{hi}^{n+1} = \overset{s+1}{x}{}_{hi}^{n+1}$. Thus, Problem (3.21) - (3.22) is approximated by the difference scheme:

(3.26)

$$\frac{\overset{s+1}{x}{}^{n+1}_{hi} - x^n_{hi}}{\tau} = \left[\frac{1}{2m}\left(\frac{1}{c(x^n_{hi})} + \frac{1}{c\left(\overset{s}{x}{}^{n+1}_{hi}\right)}\right) + \frac{1}{m}\sum_{j=1}^{m-1}\frac{1}{c\left(x^n_{hi} + j\frac{\overset{s}{x}{}^{n+1}_{hi} - x^n_{hi}}{m}\right)}\right]^{-1},$$

$$x^0_{hi} = x^0_i \in \overline{\omega}^0_h,\ i = \overline{0,N},\ n = \overline{0,N_0-1},$$

$$x^{i-N}_{hi} = L,\ i = \overline{N+1,N+N_0-1},\ n = \overline{i-N,N_0-1},$$

$$y^{n+1}_i = y^n_i,\ y^0_i = u_0(x^0_{hi}),\ i = \overline{0,N},\ n = \overline{0,N_0-1},$$

(3.27)

$$y^{i-N}_i = \mu_2(t_{i-N}),\ i = \overline{N+1,N+N_0-1},\ n = \overline{i-N,N_0-1}.$$

Example 3.1. Consider the boundary-value problem:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{6}(x+1)^2\frac{\partial u}{\partial x}\right),\ 0 < x < 1,\ 0 < t \le 1,$$

$$u(x,0) = (x+1)^2,\ 0 \le x \le 1,\ u(0,t) = e^t,\ u(l,t) = 4e^t,\ 0 < t \le 1,$$

with an exact solution $u(x,t) = e^t(x+1)^2$. The corresponding difference scheme is

$$\frac{\overset{s+1}{x}{}^{n+1}_{hi} - x^n_{hi}}{\tau} = \left[\frac{1}{2m}\left(\frac{-2}{x^n_{hi}+1} + \frac{-2}{\overset{s}{x}{}^{n+1}_{hi}+1}\right) + \frac{1}{m}\sum_{j=1}^{m-1}\frac{-2}{x^n_{hi} + j\frac{\overset{s}{x}{}^{n+1}_{hi} - x^n_{hi}}{m} + 1}\right]^{-1},$$

$$\overset{0}{x}{}^{n+1}_{hi} = x^n_{hi} - \tau\frac{x^n_{hi}+1}{2},\ x^0_{hi} = x^0_i \in \overline{\omega}^0_h,\ i = \overline{0,N},\ n = \overline{0,N_0-1},$$

$$x^{i-N}_{hi} = L,\ i = \overline{N+1,N+N_0-1},\ n = \overline{i-N,N_0-1},$$

(3.28)

$$y^{n+1}_i = y^n_i,\ y^0_i = (x^0_i+1)^2,\ i = \overline{0,N},\ n = \overline{0,N_0-1},$$

$$y^{i-N}_i = \mu_2(t_{i-N}),\ i = \overline{N+1,N+N_0-1},\ n = \overline{i-N,N_0-1}.$$

Here, we use an iteration scheme, despite the fact that the integral in special Steklov averaging can be easily evaluated, since we want to compare the exact solution $x(t)$ of the equation $\frac{dx}{dt} = -\frac{x+1}{2}$ with its numerical approximation (See Fig. 4.5).

The following table presents the numerical results obtained by our simulation, where S is the number of iterations. We can easily observe that the error of the difference approximation of parabolic equation have nearly the same order as the error of the approximation of moving mesh $x(t)$.

m	τ	$\max_{0\le n\le N_0}\|x^n_h - x(t_n)\|_{\overline{C}}$	$\max_{0\le n\le N_0}\|y^n - u(t_n)\|_{\overline{C}}$	S
10	0.1	$2.53E-06$	$1.67E-05$	9
100	0.1	$2.53E-08$	$1.67E-07$	9
1000	0.1	$2.53E-10$	$1.67E-09$	9
10000	0.1	$2.53E-12$	$1.67E-11$	9
10	0.01	$2.53E-08$	$1.67E-07$	5
100	0.01	$2.53E-10$	$1.67E-09$	5
1000	0.01	$2.53E-12$	$1.67E-11$	5
10000	0.01	$2.54E-14$	$1.68E-13$	5

Table 3.3: $\epsilon = 1.0E-15$, $h^0_i = 0.1$.

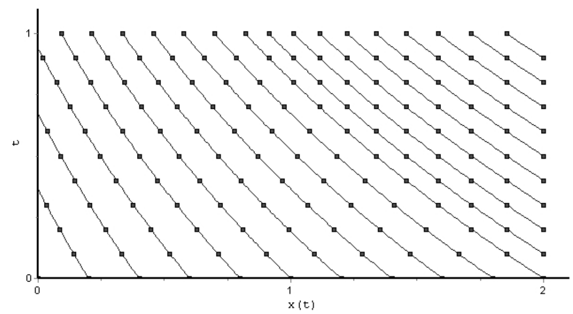

Fig. 3.3: The exact solution and its numerical approximation of the equation $\frac{dx}{dt} = -\frac{x+1}{2}$ for $h_i^0 = 0.2$, $\tau = 0.1$ and $m = 10$

Next, we compare our scheme (3.26) - (3.27) with the following well known scheme with weights [17]:

$$y_t^n = (ay_{\overline{x}})^{\sigma}_{x,i}\,, \ i = \overline{1, N-1}, \ n = \overline{0, N_0 - 1}, \tag{3.29}$$

$$y_i^0 = u_0(x_i), \ i = \overline{0, N}, \tag{3.30}$$

$$y_0^{n+1} = \mu_1(t_{n+1}), \ y_N^{n+1} = \mu_2(t_{n+1}), \ n = \overline{0, N_0 - 1}, \tag{3.31}$$

where the functional stencil is equal $a_i = \frac{k(x_{i-1})+k(x_i)}{2}$. Scheme (3.29) - (3.31) is considered on the product $\overline{\omega}_h^0 \times \overline{\omega}_\tau$, while scheme (3.26) - (3.27) is on the moving mesh. Numerical comparison results are presented in the table below, where Q is the number of arithmetic operations.

scheme (3.29) -(3.31)				scheme (3.26) -(3.27)					
$\max\limits_{t_n \in \overline{\omega}_\tau} \|y^n - u^n\|_{\overline{C}}$	h	τ	Q	$\max\limits_{t_n \in \overline{\omega}_\tau} \|y^n - u^n\|_{\overline{C}}$	h_i^0	τ	m	S	Q
$2.52E-03$	0.1	0.1	3315	$1.67E-03$	0.1	0.1	1	9	11193
$1.01E-04$	0.02	0.02	88555	$1.04E-04$	0.1	0.1	4	9	36465
$2.53E-05$	0.01	0.01	357105	$2.61E-05$	0.1	0.1	8	9	70161
$1.01E-06$	0.002	0.002	8985505	$1.36E-06$	0.1	0.1	35	9	297609
$2.53E-07$	0.001	0.001	35971005	$2.61E-07$	0.1	0.1	80	9	676689

Table 3.4: $\sigma = 0.5$, $\epsilon = 1.0E-15$.

Table 4.5 demonstrates numerically that usual scheme with weight requires significantly smaller time and space steps and more arithmetic operations then our scheme to obtain the same error of the method.

To obtain better numerical results, under condition $c(x) \in C^{2M+2}[0,L]$, where $M = const$, we use the Euler-MacLaurin formula in place of the trapezoid rule [4]:

$$\int\limits_{x_i^n}^{x_i^{n+1}} \frac{dx}{c(x)} \approx \frac{x_i^{n+1}-x_i^n}{m}\left(\frac{1}{2c\left(x_i^{n+1}\right)} + \sum_{j=1}^{m-1} \frac{1}{c\left(x_i^n + j\frac{x_i^{n+1}-x_i^n}{m}\right)} + \frac{1}{2c\left(x_i^n\right)}\right)$$
$$+\sum_{j=1}^{M}(-1)^j a_j \left(\frac{x_i^{n+1}-x_i^n}{m}\right)^{2j}\left[\left(\frac{1}{c\left(x_i^{n+1}\right)}\right)^{(2j-1)} - \left(\frac{1}{c\left(x_i^n\right)}\right)^{(2j-1)}\right], \quad (3.32)$$

where a_j is calculated from $\frac{1}{2M+1} = \frac{1}{2} + \sum\limits_{j=1}^{M}(-1)^j \frac{(2M)!}{(2M-2j+1)!}a_j$. For $M = 0$ Formula (3.32) equals the trapezoid rule. Here $\left(\frac{1}{c(x)}\right)^{(j)}$ is the $j-th$ derivative of $\frac{1}{c(x)}$ with respect to x. In this case the error of the integral approximation is equal $O((\frac{h}{m})^{2M+2})$, where $h = \max\limits_{\substack{0\le i\le N+N_0-1\\ 0\le n\le N_0}} \left|x_{hi}^{n+1} - x_{hi}^n\right|$. The following table presents the obtained numerical results and confirms our theoretical results.

m	M	τ	$\max\limits_{0\le n\le N_0} \|x_h^n - x(t_n)\|_{\overline{C}}$	$\max\limits_{0\le n\le N_0} \|y^n - u(t_n)\|_{\overline{C}}$	S
10	0	0.1	$2.53E-06$	$1.67E-05$	12
10	1	0.1	$1.27E-11$	$8.35E-11$	12
10	2	0.1	$2.27E-16$	$1.49E-15$	12
10	3	0.1	$1.22E-19$	$1.73E-18$	12
100	0	0.1	$2.53E-08$	$1.67E-07$	12
100	1	0.1	$1.27E-15$	$8.35E-15$	12
100	2	0.1	$1.22E-19$	$1.73E-18$	12
100	3	0.1	$1.22E-19$	$1.73E-18$	12

Table 3.5: Numerical comparison of the trapezoid rule ($M = 0$) and the Euler-MacLaurin formula with parameters $\epsilon = 1.0E-19$, $h_i^0 = 0.1$

4. Exact difference schemes: the boundary-value problem for parabolic equations with small parameter

In the paper [6], the boundary-value problem for parabolic equations with small parameter is considered on a coarse mesh. However, in practice nonuniform grids, for example Shiskin meshes, Bakhvalov meshes, etc., are used in the domain, where solution has singularities. In this Section, the exact difference scheme is constructed on an arbitrary nonuniform grid.

Consider the following boundary-value problem [2]

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \epsilon\frac{\partial^2 u}{\partial x^2},\ 0 < x < L,\ 0 < t \le T, \quad (4.33)$$

$$u(x,0) = \frac{1-\exp\{\frac{1-x}{\epsilon}\}}{1-\exp\{\frac{1}{\epsilon}\}},\ 0 \le x \le L, \quad (4.34)$$

$$u(0,t) = 1,\ u(l,t) = \frac{1-\exp\{\frac{1-l+2t}{\epsilon}\}}{1-\exp\{\frac{1}{\epsilon}\}},\ 0 < t \le T. \quad (4.35)$$

Assume that the solution of the problem (4.33) - (4.35) exists and has the form $u(x,t) = X(x)T(t) + C$, where $C = const$. Rewrite problem (4.33) - (4.35) in the form:

$$\frac{dx}{dt} = 1 - \frac{\epsilon \frac{\partial^2 u}{\partial x^2}}{\frac{\partial u}{\partial x}} = 2, \tag{4.36}$$

$$\left.\frac{du}{dt}\right|_{\frac{dx}{dt}=2} = \frac{\partial u}{\partial t} + \frac{dx}{dt}\frac{\partial u}{\partial x} = 0,\ u(x(0),0) = \frac{1-\exp\{\frac{1-x^0}{\epsilon}\}}{1-\exp\{\frac{1}{\epsilon}\}}. \tag{4.37}$$

For equations (4.36) - (4.37) we obtain the analytical results:

$$x(t) = 2t + x^0,$$

$$u(x(t),t) = \frac{1-\exp\{\frac{1-x^0}{\epsilon}\}}{1-\exp\{\frac{1}{\epsilon}\}},$$

and the explicit form of the solution:

$$u(x(t),t) = \frac{1-\exp\{\frac{1-x+2t}{\epsilon}\}}{1-\exp\{\frac{1}{\epsilon}\}}.$$

Consider a nonuniform grid $\hat{\overline{\omega}}_h^0 = \left\{x_i^0 = x_{i-1}^0 + h_i^0,\ i = \overline{1,N},\ x_0^0 = 0,\ \sum_{i=1}^{N} h_i^0 = L\right\}$, where h_i^0 is the space step. Problem (4.36) - (4.37) is approximated by the following scheme:

$$\frac{x_i^{n+1} - x_i^n}{\tau} = \left(\frac{1}{x_i^{n+1} - x_i^n}\int_{x_i^n}^{x_i^{n+1}} \frac{dx}{2}\right)^{-1},\ x_i^0 \in \hat{\overline{\omega}}_h^0,\ i = \overline{0,N},\ n = 0,1,\ldots,$$

$$\frac{y_i^{n+1} - y_i^n}{\tau} = 0,\ y_i^0 = \frac{1-\exp\{\frac{1-x_i^0}{\epsilon}\}}{1-\exp\{\frac{1}{\epsilon}\}},\ i = \overline{0,N},\ n = 0,1,\ldots.$$

¿From the above equations we obtain

$$x_i^{n+1} = x_i^n + 2\tau,\ x_i^0 \in \hat{\overline{\omega}}_h^0,\ i = \overline{0,N},\ n = 0,1,\ldots, \tag{4.38}$$

$$y_i^{n+1} = y_i^n,\ y_i^0 = \frac{1-\exp\{\frac{1-x_i^0}{\epsilon}\}}{1-\exp\{\frac{1}{\epsilon}\}},\ i = \overline{0,N},\ n = 0,1,\ldots. \tag{4.39}$$

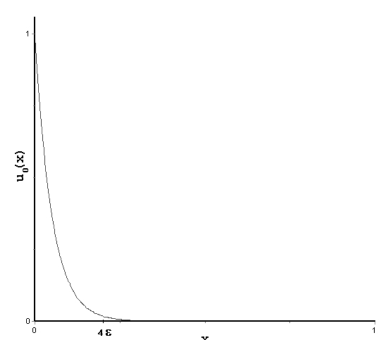

Fig. 4.4: Initial condition $u_0(x)$

In our experiments, based on the function $u_0(x)$ (See Fig. 4.4), for the initial partitioning we use an almost uniform grid, i.e. $h_i = h_1$ for $0 < x_i^0 \le 4\epsilon$ and $h_i = h_2$ for $4\epsilon < x_i^0 \le L$. The exact solution of the problem equals:

$$u(x,t) = \begin{cases} 1, & 0 \le x \le 2t, \\ \frac{1-\exp\{\frac{1-x+2t}{\epsilon}\}}{1-\exp\{\frac{1}{\epsilon}\}}, & 2t < x \le L. \end{cases}$$

Figures 5.7, 5.8 present the numerical results with $\epsilon = 0.05$, $L = 1$, $T = 0.2$, $\tau = 0.02$, while Tables 5.7, 5.8 present the obtained results for different time and space steps, which confirms that the difference scheme (4.38) - (4.39) is exact.

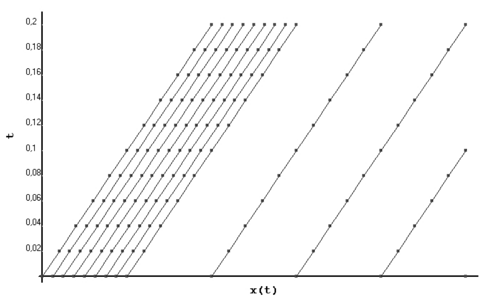

Fig. 4.5: Moving mesh given by (4.38)

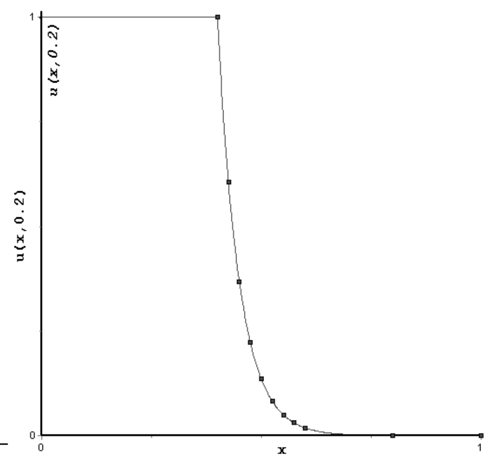

Fig. 4.6: Exact and approximate solution given by (4.39).

h_1	h_2	τ	$\max\limits_{0\le n\le N_0} \|y^n - u(t_n)\|_{\overline{C}}$
0.02	0.16	0.02	$8.13E-19$
0.01	0.08	0.01	$1.95E-18$
0.004	0.032	0.004	$3.58E-18$
0.002	0.016	0.002	$3.58E-18$
0.0004	0.0032	0.0004	$3.29E-17$

Table 4.6: $\epsilon = 0.05$, $L = 1$, $T = 0.2$

h_1	h_2	τ	$\max\limits_{0\le n\le N_0} \|y^n - u(t_n)\|_{\overline{C}}$
0.0004	0.1992	0.02	$3.46E-17$
0.0002	0.0996	0.01	$1.26E-16$
0.00008	0.03984	0.004	$1.54E-16$
0.00004	0.01992	0.002	$1.97E-16$
0.000008	0.003984	0.0004	$1.62E-15$

Table 4.7: $\epsilon = 0.001$, $L = 1$, $T = 0.2$

5. Exact difference schemes: the Cauchy problem for multi-dimensional parabolic equations

In this Section, on the basis of the previous method, the Cauchy problem for two-dimensional parabolic equations is investigated. Analytical and numerical results are derived similar to those for the one dimensional case.

Consider the Cauchy problem for the following two-dimensional parabolic equation [18]:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + u \ln u, \ x_1, \ x_2 \in R, \ t > 0, \tag{5.40}$$

$$u(x_1, x_2, 0) = \exp\{\beta_0 + 1 - \frac{x_1^2}{4} - \frac{x_2^2}{4}\}, \ x_1 \in R, \ x_2 \in R. \tag{5.41}$$

Assume that the solution of the above problem exists and has the form $u(x,t) = X_1(x_1)X_2(x_2)T(t)$. Rewrite Equations (5.40) - (5.41) as

$$\frac{dx_1}{dt} = -\frac{\frac{\partial^2 u}{\partial x_1^2}}{\frac{\partial u}{\partial x_1}} = -\frac{X_1''(x_1)}{X_1'(x_1)} = \frac{x_1^2 - 2}{2x_1}, \tag{5.42}$$

$$\frac{dx_2}{dt} = -\frac{\frac{\partial^2 u}{\partial x_2^2}}{\frac{\partial u}{\partial x_2}} = -\frac{X_2''(x_2)}{X_2'(x_2)} = \frac{x_2^2 - 2}{2x_2}, \tag{5.43}$$

$$\begin{aligned} \left.\frac{du}{dt}\right|_{\frac{dx_1}{dt} = \frac{x_1^2-2}{2x_1}, \ \frac{dx_2}{dt} = \frac{x_2^2-2}{2x_2}} &= \frac{\partial u}{\partial t} + \frac{dx_1}{dt}\frac{\partial u}{\partial x_1} + \frac{dx_2}{dt}\frac{\partial u}{\partial x_2} = u \ln u, \\ u(x_1(0), x_2(0), 0) &= \exp\{\beta_0 + 1 - \frac{\left(x_1^0\right)^2}{4} - \frac{\left(x_2^0\right)^2}{4}\}, \end{aligned} \tag{5.44}$$

where $x_1(0) = x_1^0, \ x_2(0) = x_2^0 \in R$ are the initial states of the curves $x_1 = x_1(t)$ and $x_2 = x_2(t)$, respectively. ¿From the analytical results for Equations (5.42) - (5.44):

$$x_1(t) = \begin{cases} \sqrt{2 + e^t((x_1^0)^2 - 2)}, & \text{if } x_1^0 \geq 0, \\ -\sqrt{2 + e^t((x_1^0)^2 - 2)}, & \text{if } x_1^0 < 0, \end{cases}$$

$$x_2(t) = \begin{cases} \sqrt{2 + e^t((x_2^0)^2 - 2)}, & \text{if } x_2^0 \geq 0, \\ -\sqrt{2 + e^t((x_2^0)^2 - 2)}, & \text{if } x_2^0 < 0, \end{cases}$$

$$u(x(t), t) = \left(u_0(x^0)\right)^{e^t},$$

we can determined the explicit form of the solution (See Fig. 5.8):

$$u(x(t), t) = \exp\left\{\beta_0 e^t + 1 - \frac{x_1^2 + x_2^2}{4}\right\},$$

where $t \leq \ln \frac{2}{2-(x_1^0)^2}$ for $0 < x_1^0 < \sqrt{2}$ and $t \leq \ln \frac{2}{2-(x_2^0)^2}$ for $0 < x_2^0 < \sqrt{2}$.

Next, let us approximate Problem (5.42) - (5.44) by the scheme:

$$\frac{x_{1i}^{n+1}-x_{1i}^{n}}{\tau}=\left(\frac{1}{x_{1i}^{n+1}-x_{1i}^{n}}\int\limits_{x_{1i}^{n}}^{x_{1i}^{n+1}}\frac{2xdx}{x^2-2}\right)^{-1},\ x_{1i}^0\in\overline{\omega}_{h_1}^0,\ i=\overline{0,N_1},\ n=0,1,\dots,$$

$$\frac{x_{2j}^{n+1}-x_{2j}^{n}}{\tau}=\left(\frac{1}{x_{2j}^{n+1}-x_{2j}^{n}}\int\limits_{x_{2j}^{n}}^{x_{2j}^{n+1}}\frac{2xdx}{x^2-2}\right)^{-1},\ x_{2j}^0\in\overline{\omega}_{h_2}^0,\ j=\overline{0,N_2},\ n=0,1,\dots,$$

$$\frac{y_{ij}^{n+1}-y_{ij}^{n}}{\tau}=\left(\frac{1}{y_{ij}^{n+1}-y_{ij}^{n}}\int\limits_{y_{ij}^{n}}^{y_{ij}^{n+1}}\frac{du}{u\ln u}\right)^{-1},$$

$$y_{ij}^0=u(x_{1i}^0,x_{2j}^0,0),\ i=\overline{0,N_1},\ j=\overline{0,N_2},\ n=0,1,\dots,$$

where

$$\overline{\omega}_{h_1}^0=\left\{x_{1,i}^0=-L_1+ih_{1i}^0,\ h_{1i}^0=\frac{2L_1}{N_1},\ i=\overline{0,N_1}\right\},$$

$$\overline{\omega}_{h_2}^0=\left\{x_{2,j}^0=-L_2+jh_{2j}^0,\ h_{2j}^0=\frac{2L_2}{N_2},\ j=\overline{0,N_2}\right\},$$

$$\overline{\omega}_{\tau}=\left\{t_n=n\tau,n=\overline{0,N_0},\ \tau=\frac{T}{N_0}\right\}$$

are the uniform grids in space and time, respectively. After evaluating the integrals, we obtain (See Fig. 5.7):

(5.45)

$$x_{1i}^{n+1}=\begin{cases}\sqrt{2+e^{\tau}((x_{1i}^n)^2-2)}, & \text{if } x_{1i}^n\ge 0,\\ -\sqrt{2+e^{\tau}((x_{1i}^n)^2-2)}, & \text{if } x_{1i}^n<0,\end{cases}\quad x_{1i}^0\in\overline{\omega}_{h_1}^0,\ i=\overline{0,N_1},\ n=0,1,\dots,$$

(5.46)

$$x_{2j}^{n+1}=\begin{cases}\sqrt{2+e^{\tau}((x_{2j}^n)^2-2)}, & \text{if } x_{2i}^n\ge 0,\\ -\sqrt{2+e^{\tau}((x_{2j}^n)^2-2)}, & \text{if } x_{2i}^n<0,\end{cases}\quad x_{2j}^0\in\overline{\omega}_{h_2}^0,\ j=\overline{0,N_2},\ n=0,1,\dots,$$

(5.47) $y_{ij}^{n+1}=(y_{ij}^n)^{\exp\{\tau\}},\ y_{ij}^0=u(x_{1i}^0,x_{2j}^0,0),\ i=\overline{0,N_1},\ j=\overline{0,N_2},\ n=0,1,\dots.$

Tables 6.9, 6.10 present the numerical results for different time and space steps, where $\|y\|_{\overline{C}}=\max\limits_{\substack{0\le i\le N_1\\0\le j\le N_2}}|y_{i,j}|$, and demonstrate that the difference scheme (5.45) - (5.47) is exact.

h_{1i}^0	h_{2j}^0	τ	$\max\limits_{0\le n\le N_0}\|y^n-u(t_n)\|_{\overline{C}}$
1	1	0.1	$1.39E-16$
0.5	0.5	0.05	$6.38E-16$
0.2	0.2	0.02	$3.25E-15$
0.1	0.1	0.01	$1.04E-14$

Table 5.8: $\beta_0=2,\ L_1=5,\ L_2=5,\ T=1$

h_{1i}^0	h_{2j}^0	τ	$\max\limits_{0\le n\le N_0}\|y^n-u(t_n)\|_{\overline{C}}$
0.1	0.1	0.1	$4.34E-19$
0.05	0.05	0.05	$8.67E-19$
0.02	0.02	0.02	$1.52E-18$
0.01	0.01	0.01	$2.60E-18$

Table 5.9: $\beta_0=0.1,\ L_1=0.5,\ L_2=0.5,\ T=1$

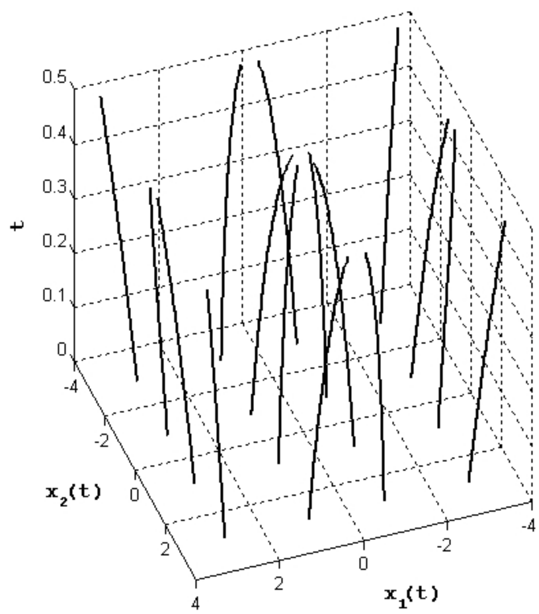

Fig. 5.7: Moving mesh given by formulas (5.45) - (5.46)

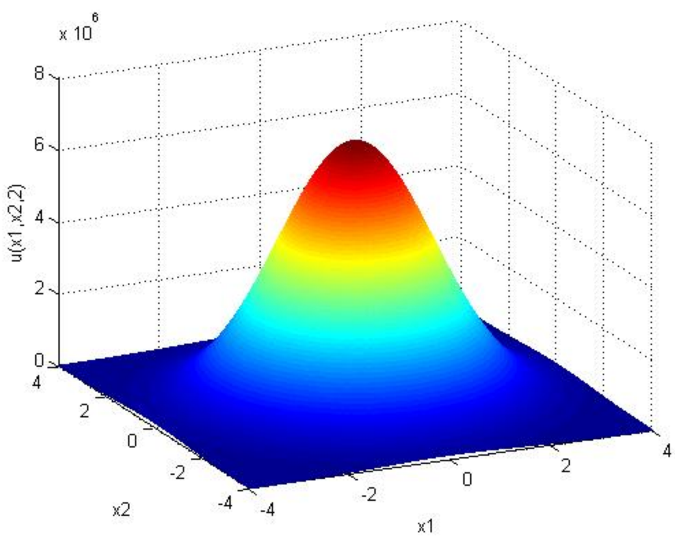

Fig. 5.8: Exact solution of problem (5.40) - (5.41) for $t = 2$ and $\beta_0 = 2$

The technique for numerically solving multi-dimensional parabolic problems (in space $\mathbb{R}^p$, $p = 2, 3, \dots$) is a natural extension of the above technique for solving two-dimensional problems.

6. Conclusions

In this paper, under condition $u(x,t) = X(x)T_1(t) + T_2(t)$, using the special Steklov averaging we have constructed on the moving mesh the exact difference schemes for the Cauchy problem for parabolic equations. The multi-dimensional problems and boundary-value problems also have been considered. The difference scheme of arbitrary order of approximation have been constructed in the case when interval in it cannot be evaluated exactly.

Numerical results have been presented to confirm theoretical results stated in the paper.

Future research will focus on two-dimensional nonlinear problems for parabolic equations with wave propagation solution $u(x_1, x_2, t) = f_1(x_1 - at) + f_2(x_2 - at)$ as well as on nonlinear problems for hyperbolic equations of the second order. Finally, to generalize presented idea in the case when no explicit solution is available remains to be a very challenging task.

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