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A UNIFORMLY OPTIMAL-ORDER ERROR ESTIMATE OF AN ELLAM SCHEME FOR UNSTEADY-STATE ADVECTION-DIFFUSION EQUATIONS

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Abstract. We prove an optimal-order error estimate in a weighted energy norm for the Eulerian-Lagrangian localized adjoint method (ELLAM) for unsteady-state advection-diffusion equations with general inflow and outflow boundary conditions. It is well known that these problems admit dynamic fronts with interior and boundary layers. The estimate holds uniformly with respect to the vanishing diffusion coefficient.

Key Words. characteristic methods, Eulerian-Lagrangian methods, interpolation of spaces, uniform error estimates

1. Introduction

We consider unsteady-state advection-diffusion equations with general inflow and outflow boundary conditions, which arise in mathematical modeling of petroleum reservoir simulation, environmental modeling, and other applications [1, 7]. It is well known that these problems admit solutions with dynamic fronts and complex structures including interior and boundary layers, and present serious mathematical and numerical difficulties. Classical finite difference or finite element methods tend to generate numerical solutions with nonphysical oscillations, while upwind methods often produce excessive numerical diffusion that smears out fronts and generates spurious grid orientation effects [7].

Eulerian-Lagrangian methods combine the advection and capacity terms in the governing equations to carry out the temporal discretization in a Lagrangian coordinate, and discretize the diffusion term on a fixed mesh in an Eulerian manner [4, 5, 11]. These methods symmetrize the governing equation and stabilize their numerical approximations. They generate accurate numerical solutions and significantly reduce the numerical diffusion and grid-orientation effect present in upwind methods, even if large time steps and coarse spatial meshes are used. Eulerian-Lagrangian methods were shown to be very competitive in terms of accuracy and efficiency [4, 12]. Mathematically, *A priori* optimal-order error estimates were derived for the modified method of characteristics (MMOC) [5] and the modified method of characteristics with adjusted advection [4] for unsteady-state advection-diffusion equations with periodic or noflow boundary conditions and the Eulerian-Lagrangian localized adjoint method (ELLAM) for unsteady-state advection-diffusion equations with general boundary conditions [13, 10]. However, the general constant in this type of estimates may depend inversely on the vanishing diffusion parameter. Consequently, these estimates could blow up as the diffusion coefficient tends to zero. To our best knowledge, there is no *a priori* optimal-order error estimate in a

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weighted energy norm for an Eulerian-Lagrangian method with uniform partition for unsteady-state advection-diffusion equations with general inflow and outflow boundary conditions, which holds uniformly with respect to the vanishing diffusion parameter.

In contrast to the steady-state analogue where a uniform L^∞ error estimate was derived for numerical methods with a Shishkin mesh [9], unsteady-state advection-diffusion equations admit dynamic interior and boundary layers and complicated structures. These boundary and interior layers are dynamic and do not always coincide with the spatial mesh. Consequently, a uniform error estimate in the L^∞-norm is generally impossible, since the true solution could exhibit shock discontinuity in the limiting case of the diffusion parameter vanishes. This is why L^∞ norm is not used in the numerical analysis for hyperbolic conservation laws [8]. The goal of the present paper is to derive an optimal-order error estimate in a weighted energy norm for the ELLAM scheme for unsteady-state advection-diffusion equations with general inflow and outflow boundary conditions. Thus, these results theoretically justify the numerical advantages of Eulerian-Lagrangian methods, which were observed numerically [11, 12, 13].

This paper is organized as follows. Sections 2 and 3 recall preliminary results on Sobolev and interpolation results and revisit the ELLAM scheme, respectively. In this section 4, we prove an ε-uniform optimal-order error estimate in a weighted-energy norm for the ELLAM scheme for unsteady-state advection-diffusion equations with an inflow total flux and an outflow diffusive flux boundary conditions, which admit both interior and boundary layers. In section 5, we prove auxiliary estimates that were used in the proof in section 4.

2. Model Problem and Preliminaries

We consider the unsteady-state advection-diffusion equation in one space dimension with a representative combination of an inflow total flux boundary condition and an outflow diffusive flux boundary condition. The analysis in this paper applies to any combinations of boundary conditions. For the sake of exposition, we restrict ourselves to this representative combination of boundary conditions, which is well known to present mathematical and numerical difficulties in the theoretical analysis of Eulerian-Lagrangian methods [13]

$$
\begin{array}{rclrcl}
u_t + (V(x,t)u - \varepsilon D(x,t)u_x)_x & = & f(x,t), & (x,t) & \in & (a,b)\times(0,T) \\
Vu(a,t) - \varepsilon D u_x(a,t) & = & g(t), & t & \in & (0,T] \\
-\varepsilon D u_x(b,t) & = & h(t), & t & \in & (0,T] \\
u(x,0) & = & u_o(x), & x & \in & [a,b].
\end{array}
\tag{1}
$$

Here $V(x,t)$ is a velocity field, $f(x,t)$ accounts for external sources and sinks, $g(t)$ and $h(t)$ are the prescribed inflow and outflow boundary data, respectively, $u_o(x)$ is the prescribed initial data, and $u(x,t)$ is the ε-dependent unknown function. $D(x,t)$ is a diffusion coefficient with $0 < D_{min} \le D(x,t) \le D_{max} < +\infty$ for any $(x,t) \in [a,b]\times[0,T]$ and $0 < \varepsilon << 1$ is a parameter that scales the diffusion and characterizes the advection-dominance of Eq. (1).

Let $W_p^k(a,b)$ consist of functions whose weak derivatives up to order-k are p-th Lebesgue integrable in (a,b). Let $H^k(a,b) := W_2^k(a,b)$. For any Banach space X, we introduce Sobolev spaces involving time [6]

$$
\begin{aligned}
W_p^k(t_1,t_2;X) \quad &:= \Big\{ f(x,t) \ : \Big\| \frac{\partial^\alpha f}{\partial t^\alpha}(\cdot,t)\Big\|_X \in L^p(t_1,t_2),\ 0 \le \alpha \le k,\ 1 \le p \le \infty \Big\}, \\
\|f\|_{W_p^k(t_1,t_2;X)} \quad &:= \begin{cases} \Big(\displaystyle\sum_{\alpha=0}^{k} \int_{t_1}^{t_2} \Big\| \frac{\partial^\alpha f}{\partial t^\alpha}(\cdot,t)\Big\|_X^p dt \Big)^{1/p}, & 1 \le p < \infty, \\ \displaystyle\max_{0\le\alpha\le k} \ \text{ess}\sup_{(t_1,t_2)} \Big\| \frac{\partial^\alpha f}{\partial t^\alpha}(\cdot,t)\Big\|_X, & p = \infty. \end{cases}
\end{aligned}
$$

When it is clear from the context, we use $\|\cdot\|_{L^2}$, $\|\cdot\|_{H^2}$ and $\|\cdot\|_{W^2_\infty}$ to denote $\|\cdot\|_{L^2(a,b)}$, $\|\cdot\|_{H^2(a,b)}$ and $\|\cdot\|_{W^k_\infty(a,b)}$, respectively.

We define a uniform space-time partition on $[a,b]\times[0,T]$: $x_i := a + ih$ for $0 \le i \le I$ with $h := (b-a)/I$ and $t_n := n\Delta t$ for $0 \le n \le N$ with $\Delta t := T/N$. Then we introduce the following ε-weighted energy norms

$$
\begin{aligned}
\|f\|_{L_\varepsilon(0,T;H^1_D(a,b))} := \quad & \Big(\max_{1\le n\le N} \|f(\cdot,t_n)\|^2_{L^2} + \int_0^T V f^2(b,t)dt \\
& + \sum_1^N \int_a^b \varepsilon \Delta t_h(x) D(x,t_n) f_x^2(x,t_n)dx \Big)^{1/2}.
\end{aligned}
$$

Let $S_h(a,b) \subset H^1(a,b)$ be the finite element space that consists of continuous and piecewise-linear functions with respect to the spatial partition in $[a,b]$ at time t_n and constant on the time interval $[t_{n-1},t_n]$. We let $\Pi_h v \in S_h(a,b)$ be the piecewise-linear interpolation of v for any $v \in H^1(a,b)$. The following estimates hold [3]

$$
\begin{aligned}
\|\Pi_h v - v\|_{H^k(a,b)} &\le C_1 h^{2-k}\ \|v\|_{H^2(a,b)}, \quad \forall v \in H^2(a,b),\ k = 0,1 \\
\|v_h\|_{L^\infty(a,b)} &\le C_2 h^{-1/2} \|v_h\|_{L^2(a,b)}, \quad \forall v_h \in S_h(a,b).
\end{aligned} \tag{2}
$$

Since $(\Pi p - p)(x_{i-1}) = (\Pi p - p)(x_i) = 0$, there exists an $\hat{x}_{i-1/2} \in (x_{i-1},x_i)$ such that $(\Pi p - p)'(\hat{x}_{i-1/2}) = 0$. Thus, for $x \in [x_{i-1},x_i]$ we have

$$
(\Pi p - p)(x) = (\Pi p - p)(x_i) - \int_x^{x_i} (\Pi p - p)_y(y)dy = \int_x^{x_i} \int_{\hat{x}_{i-1/2}}^{y} p_{zz}(z)dzdy. \tag{3}
$$

In this paper, we use C to denote a general positive constant that could assume different values at different occurrences.

3. Revisit of ELLAM

The ELLAM uses a time-marching approach, so we need only to define these methods at the current time interval $[t_{n-1},t_n]$. In the ELLAM formulation, the space-time test functions $w(x,t)$ are chosen to be continuous and piecewise smooth and to vanish outside the space-time strip $[a,b]\times(t_{n-1},t_n]$. In particular, the test functions $w(x,t)$ satisfy that $w(x,t_n) = \lim_{t\to t_n-0} w(x,t)$, but $w(x,t_{n-1}) \ne \lim_{t\to t_{n-1}+0} w(x,t)$ in general. In this case, we use the notation $w(x,t^+_{n-1}) = \lim_{t\to t_{n-1}+0} w(x,t)$ to account for the possible discontinuity of $w(x,t)$ in time at time t_{n-1}. We multiply Eq. (1) by test functions w and integrate the resulting equation on $[a, b] \times (t_{n-1}, t_n]$ by parts to obtain

$$\begin{aligned}
&\int_a^b u(x,t_n)w(x,t_n)dx + \int_{t_{n-1}}^{t_n} (Vu - \varepsilon D u_x)w\Big|_a^b dt \\
&\quad + \int_{t_{n-1}}^{t_n}\int_a^b \varepsilon D(x,t)u_x(x,t)w_x(x,t)dxdt \\
&\quad - \int_{t_{n-1}}^{t_n}\int_a^b u(x,t)\Big(w_t(x,t) + V(x,t)w_x(x,t)\Big)dxdt \\
&= \int_a^b u(x,t_{n-1})w(x,t_{n-1}^+)dx + \int_{t_{n-1}}^{t_n}\int_a^b f(x,t)w(x,t)dxdt.
\end{aligned}$$

In the ELLAM framework [2] the adjoint equation of the hyperbolic part of Eq. (1) is used to define the temporal variation of w

$$w_t + V w_x = 0. \tag{4}$$

This implies the test functions w to be constant along the characteristics $r(t; x, t_n)$, where $r(t; \bar{x}, \bar{t})$ refers to the characteristic curve passing $\bar{x}$ at time $\bar{t}$ defined by

$$\frac{dr}{dt} = V(r,t), \quad r(t;\bar{x},\bar{t})\Big|_{t=\bar{t}} = \bar{x}. \tag{5}$$

Thus, once the test functions $w(x,t)$ are specified in $[a, b]$ at time step t_n and the outflow boundary, they are determined completely in the space-time strip $[a, b] \times (t_{n-1}, t_n]$. So combining with the boundary condition, we obtain the following weak formulation

$$\begin{aligned}
&\int_a^b u(x,t_n)w(x,t_n)dx + \int_{t_{n-1}}^{t_n} Vu(b,t)w(b,t)dt \\
&\quad + \int_{t_{n-1}}^{t_n}\int_a^b \varepsilon D(x,t)u_x(x,t)w_x(x,t)dxdt \\
&= \int_a^b u(x,t_{n-1})w(x,t_{n-1}^+)dx + \int_{t_{n-1}}^{t_n} g(t)w(a,t)dt \\
&\quad - \int_{t_{n-1}}^{t_n} h(t)w(b,t)dt + \int_{t_{n-1}}^{t_n}\int_a^b f(x,t)w(x,t)dxdt.
\end{aligned} \tag{6}$$

3.1. Evaluation of Diffusion and Source Terms. First of all, we use $t_n^*(x)$ to denote the time instant if the characteristic intersects the boundary $x = a$ during the time period $[t_{n-1}, t_n]$, i.e. $a = r(t_n^*(x); x, t_n)$ and $t_n^*(x) = t_{n-1}$ otherwise. We also let $\Delta t(x) = t_n - t_n^*(x)$. We differentiate the characteristic $r(s; x, t)$ respect to x and t, respectively, to get

$$\begin{aligned}
r_x(s;x,t) &= 1 - \int_s^t V_x(r(\theta;x,t),\theta)r_x(\theta;x,t)d\theta, \\
r_t(s;x,t) &= -V(b,t) - \int_s^t V_x(r(\theta;x,t),\theta)r_t(\theta;x,t)d\theta.
\end{aligned} \tag{7}$$

In the evaluation of source and diffusion terms we reserve x for points in $[a, b]$ at time t_n representing the heads of characteristics. We use the variable y to represent the spatial coordinate of an arbitrary point at time $t \in (t_{n-1}, t_n)$. We use (7) and the Euler quadrature to evaluate the source term as follows

$$
\begin{aligned}
&\int_{t_{n-1}}^{t_n}\int_a^b f(y,t)w(y,t)dydt \\
&= \int_a^b \int_{t_n^*(x)}^{t_n} f(r(t;x,t_n),t)w(r(t;x,t_n),t)r_x(t;x,t_n)dtdx \\
&\quad - \int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t} f(r(\theta;b,t),\theta)w(r(\theta;b,t),\theta)r_t(\theta;b,t)d\theta dt \\
&= \int_a^b \Delta t(x) f(x,t_n)w(x,t_n)dx \\
&\quad + \int_{t_{n-1}}^{t_n}(t-t_{n-1})V(b,t)f(b,t)w(b,t)dt + E_1(w),
\end{aligned} \tag{8}
$$

Here $E_1(w)$ is the local truncation error defined by

$$
\begin{aligned}
E_1(w) \quad := &\int_a^b \int_{t_n^*(x)}^{t_n} \Big[f(r(t;x,t_n),t)r_x(t;x,t_n) - f(x,t_n)\Big]dt\ w(x,t_n)dx \\
&- \int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t} \Big[f(r(\theta;b,t),\theta)r_t(\theta;b,t) + V(b,t)f(b,t)\Big]d\theta w(b,t)dt.
\end{aligned} \tag{9}
$$

From (7), we know $\left(\frac{\partial r(\theta;x,t_n)}{\partial x}\right)^{-1} = 1 + O(t_n-\theta)$ and $r_t(s;b,t) = -V(b,t)(1+O(t-s))$. Since $w(b,t)$ is constant on $[t_{n-1},t_n]$, from the adjoint equation we know $V(b,t)w_x(b,t) = -w_t(b,t) = 0$. Then the diffusion term can be evaluated similarly

$$
\begin{aligned}
&\int_{t_{n-1}}^{t_n}\int_a^b \varepsilon D(y,t)u_y(y,t)w_y(y,t)dydt \\
&= \int_a^b \int_{t_n^*(x)}^{t_n} \varepsilon D(r(t;x,t_n),t)u_y(r(t;x,t_n),t)w_y(r(t;x,t_n),t)r_x(t;x,t_n)dtdx \\
&\quad - \int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t} \varepsilon D(r(\theta;b,t),\theta)u_y(r(\theta;b,t),\theta)w_y(r(\theta;b,t),\theta)r_t(\theta;b,t)d\theta dt \\
&= \int_a^b \int_{t_n^*(x)}^{t_n} \varepsilon D(r(t;x,t_n),t)u_y(r(t;x,t_n),t)w_y(r(t;x,t_n),t)r_x(t;x,t_n)dtdx \\
&\quad - \int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t} \varepsilon D(r(\theta;b,t),\theta)u_y(r(\theta;b,t),\theta)w_x(b,t) \\
&\qquad\qquad \cdot\Big(\frac{\partial r(\theta;b,t)}{\partial x}\Big)^{-1} r_t(\theta;b,t)d\theta dt \\
&= \int_a^b \int_{t_n^*(x)}^{t_n} \varepsilon D(r(t;x,t_n),t)u_y(r(t;x,t_n),t)w_y(r(t;x,t_n),t)r_x(t;x,t_n)dtdx \\
&\quad + \int_{t_{n-1}}^{t_n}\int_{t_{n-1}}^{t} \varepsilon D(r(\theta;b,t),\theta)u_y(r(\theta;b,t),\theta)V(b,t)w_x(b,t)(1+O(\Delta t))d\theta dt \\
&= \int_a^b \int_{t_n^*(x)}^{t_n} \varepsilon D(r(t;x,t_n),t)u_y(r(t;x,t_n),t)w_x(x,t_n)dtdx \\
&= \int_a^b \varepsilon \Delta t(x) D(x,t_n)u_x(x,t_n)w_x(x,t_n)dx + E_2(u,w),
\end{aligned} \tag{10}
$$

Here $E_2(u,w)$ is the local truncation error defined by

$$
E_2(u,w) := \int_a^b \int_{t_n^*(x)}^{t_n} \varepsilon\Big[(Du_x)(r(t;x,t_n),t) - (Du_x)(x,t_n)\Big]dt\ w_x(x,t_n)dx. \tag{11}
$$

We substitute Eqs. (8) and (10) into Eq. (6) to obtain an ELLAM formulation for problem (1)

$$
\begin{aligned}
&\int_a^b u(x,t_n)w(x,t_n)dx + \int_{t_{n-1}}^{t_n} Vu(b,t)w(b,t)dt \\
&\quad + \int_a^b \varepsilon\Delta t(x)D(x,t)u_x(x,t)w_x(x,t)dxdt \\
&\quad = \int_{\tilde{a}}^b u(x^*,t_{n-1})w(x,t_n)r_x(t_{n-1};x,t_n)dx \\
&\quad - \int_{t_{n-1}}^{t_n} u(b^*(t),t_{n-1})w(b,t)r_t(t_{n-1};b,t)dt + \int_{t_{n-1}}^{t_n} g(t)w(a,t)dt \\
&\quad - \int_{t_{n-1}}^{t_n} h(t)w(b,t)dt + \int_a^b \Delta t(x)f(x,t_n)w(x,t_n)dx \\
&\quad + \int_{t_{n-1}}^{t_n} (t-t_{n-1})V(b,t)f(b,t)w(b,t)dt + E_1(w) - E_2(u,w).
\end{aligned} \tag{12}
$$

In Equation (12) we have used the notations x^*, $b^*(t)$, and $\tilde{x}$ defined by

$$
x^* = r(t_{n-1};x,t_n), \quad , \quad b^*(t) = r(t_{n-1};b,t), \quad x = r(t_{n-1};\tilde{x},t_n). \tag{13}
$$

We have also used the fact that w is constant along the characteristics to rewrite the first integral at time t_{n-1} on the right-hand side of (6) as an integral at time t_n and the outflow boundary.

3.2. Approximate Characteristics and Numerical Scheme. We note that the initial-value problem (5) can not be solved in a closed form to define true characteristics $r(s;x,t_n)$ or $r(s;b,t)$, for a general velocity field $V(x,t)$. Thus, numerical means have to be used in practice. In order to retain the order of approximation of the ELLAM scheme, we use a second-order Heun's method with a micro-time stepping to define approximate characteristics, which extends backward from $x \in [a,b]$ at time step t_n as follows

$$
\begin{aligned}
\check{x}_{n,k} &= x_{n,k-1} - \Delta t_f V(x_{n,k-1},t_{n,k-1}), \\
x_{n,k} &= x_{n,k-1} - \frac{\Delta t_f}{2}\big[V(x_{n,k-1},t_{n,k-1}) + V(\check{x}_{n,k},t_{n,k})\big], \quad 1 \le k \le IC,
\end{aligned} \tag{14}
$$

with $x_{n,0} = x$ and $x_h^* = x_{n,IC}$. Here Δt_f and $t_{n,k}$ are defined as

$$
\mathrm{Cr} = \max_{(x,t)\in[a,b]\times[t_{n-1},t_n]} \frac{V(x,t)\Delta t}{h},
$$

and IC to be the ceiling of Cr. Then, we partition the space-time outflow boundary $\{(b,t) : t \in [t_{n-1},t_n]\}$ by

$$
t_{n,k} = t_n - k\Delta t_f, \quad k = 0,1,\cdots,IC, \qquad \text{with } \Delta t_f = \frac{\Delta t}{IC}.
$$

It is obvious that $t_{n,0} = t_n$ and $t_{n,IC} = t_{n-1}$.

In Eq. (14), $r_h(s;x_{n,k-1},t_{n,k-1})$ is linear between $(x_{n,k-1},t_{n,k-1})$ and $(x_{n,k},t_{n,k})$ for $s \in [t_{n,k},t_{n,k-1}]$. By the semigroup property of the characteristic tracking, we see that $r_h(s;x_{n,k-1},t_{n,k-1}) = r_h(s;x,t_n)$ for $s \in [t_{n,k},t_{n,k-1}]$. In case $\check{x}_{n,k} < a$ for some $1 \le k \le IC$, we define $t_h^*(x) \in [t_{n,k},t_{n,k-1}]$ to be the time instant when the approximate characteristic $r_h(s;x,t_n)$ backtracks to the inflow boundary $x = a$.

Namely, we solve $t_h^*(x)$ from the following equation

$$\begin{aligned}(15)\qquad a &= x_{n,k-1} - \frac{t_{n,k-1}-t_h^*(x)}{2}\Big[V(x_{n,k-1},t_{n,k-1})\\ &\quad + V\big(x_{n,k-1}-(t_{n,k-1}-t_h^*(x))V(x_{n,k-1},t_{n,k-1}),t_h^*(x)\big)\Big],\end{aligned}$$

and $t_h^*(x) = t_{n-1}$ otherwise.

We similarly define an approximate characteristics $r_h(s;b,t)$ which backtracks from the outflow boundary $x = b$ at time $t \in [t_{n,k}, t_{n,k-1}]$ for $1 \le k \le IC$ as follows

$$(16)\qquad \begin{aligned}\check{x}_{n,k} &= b - (t-t_{n,k})V(b,t),\\ x_{n,k} &= b - \frac{t-t_{n,k}}{2}\big[V(b,t)+V(\check{x}_{n,k},t_{n,k})\big].\end{aligned}$$

$$(17)\qquad \begin{aligned}\check{x}_{n,j} &= x_{n,j-1} - \Delta t_f V(x_{n,j-1},t_{n,j-1}), \quad k+1 \le j \le IC,\\ x_{n,j} &= x_{n,j-1} - \frac{\Delta t_f}{2}\big[V(x_{n,j-1},t_{n,j-1})+V(\check{x}_{n,j},t_{n,j})\big],\end{aligned}$$

and set $b_h^*(t) = x_{n,IC}$. Finally we define $\Delta t_h(x)$ and $\tilde{x}_h$ by

$$(18)\qquad \Delta t_h(x) = t_n - t_h^*(x) \quad \text{and} \quad x = r_h(t_{n-1};\tilde{x}_h,t_n).$$

The following lemma gives several error bounds between the approximate and true characteristics, which can be proved as the numerical approximations to the initial-value problem of an ordinary differential equation (5) and so is omitted here.

Lemma 3.1. *Let $r(s;x,t)$ and $r_h(s;x,t)$ be the true and approximate characteristics defined in (5) and in section 3.2, respectively. Assume that $V \in L^\infty(0,T;W^3_\infty(a,b))$, $\frac{dV}{dt}, \frac{d^2V}{dt^2} \in L^\infty(0,T;W^1_\infty(a,b))$. Then the following estimates hold*

$$(19)\qquad \begin{aligned}&|x_h^* - x^*| + |b^*(t) - b_h^*(t)| + |t_h^*(x) - t^*(x)| = O((\Delta t)^2),\\ &|r_{h,x}(t_{n-1};x,t_n) - r_x(t_{n-1};x,t_n)| = O((\Delta t)^2)\\ &|r_{h,t}(t_{n-1};b,t) - r_t(t_{n-1};b,t)| = O(\Delta t)\\ &|\tilde{x}_h - \tilde{x}| = O(\Delta t(\Delta t_f)^2).\end{aligned}$$

Next we derive the ELLAM scheme based on Eq. (12). Because the characteristics $r(t;x,t_n)$ and $r(\theta;b,t)$ cannot be tracked exactly in general, the test functions w_h in the ELLAM scheme are defined to be constant along the approximate characteristics $r_h(t;x,t_n)$ and $r_h(\theta;b,t)$. Consequently, the ELLAM scheme states as follows: Find $u_h(x,t_n) \in S_h(a,b)$ for $n = 1,\ldots,N$, such that for any $w_h(x,t_n) \in S_h(a,b)$

$$(20)\qquad \begin{aligned}&\int_a^b u_h(x,t_n)w_h(x,t_n)dx + \int_{t_{n-1}}^{t_n} Vu_h(b,t)w_h(b,t)dt\\ &\quad + \int_a^b \varepsilon\Delta t_h(x)D(x,t)u_{h,x}(x,t)w_{h,x}(x,t)dxdt\\ &= \int_{\tilde{a}_h}^b u_h(x_h^*,t_{n-1})w_h(x,t_n)r_{h,x}(t_{n-1};x,t_n)dx\\ &\quad - \int_{t_{n-1}}^{t_n} u_h(b_h^*(t),t_{n-1})w_h(b,t)r_{h,t}(t_{n-1};b,t)dt + \int_{t_{n-1}}^{t_n} g(t)w_h(a,t)dt\\ &\quad - \int_{t_{n-1}}^{t_n} h(t)w_h(b,t)dt + \int_a^b \Delta t_h(x)f(x,t_n)w_h(x,t_n)dx\\ &\quad + \int_{t_{n-1}}^{t_n} (t-t_{n-1})V(b,t)f(b,t)w_h(b,t)dt.\end{aligned}$$

4. An ε-Uniform Error Estimate for the ELLAM Scheme

In this section we prove an *a priori* optimal-order error estimates for the ELLAM schemes in an ε-weighted energy norm for problem (1), which holds uniformly with respect to ε. Let $\lambda = 1$ if $\mathrm{Cr} < 1$ or $= 0$ otherwise. The main result is given in the theorem below.

Theorem 4.1. *Let u be the true solution to problem (1) and u_h be the numerical solution of scheme* (20). *The following optimal-order error estimate holds uniformly with respect to ε*

$$
\begin{aligned}
&\|u_h - u\|_{L_\varepsilon(0,T;H^1_D(a,b))} \\
&\le C\Delta t\Big(\|u(b,\cdot)\|_{H^1(0,T)} + \Big\|\frac{du}{dt}\Big\|_{L^2(0,T;H^1)} + \Big\|\frac{df}{dt}\Big\|_{L^\infty(0,T;L^2)} \\
&\quad + \|f\|_{L^\infty(0,T;L^2)}\Big) + C(\Delta t + h^2)\|u\|_{L^\infty(0,T;H^2)} \\
&\quad + \lambda C h^2(\|u\|_{H^1(0,T;H^2)} + \|u\|_{L^\infty(0,T;H^3)}).
\end{aligned} \tag{21}
$$

Here the constant C is independent of u or the parameter ε.

Remark 4.1. *The Sobolev norms of u can be bounded by the corresponding Sobolev norms of the initial and boundary conditions of u as well as the right-hand side source term f, which are independent of ε* [6].

Proof. We let $e = u_h - u$ and choose the test function $w(x,t_n)$ in (12) to be $w_h(x,t_n) \in S_h(a,b)$. We then subtract Eq. (12) from the ELLAM reference equation (20) to obtain an ELLAM error equation for any $w_h(x,t_n) \in S_h(a,b)$

$$
\begin{aligned}
&\int_a^b e(x,t_n)w_h(x,t_n)dx + \int_{t_{n-1}}^{t_n} Ve(b,t)w_h(b,t)dt \\
&\quad + \int_a^b \varepsilon\Delta t_h(x)D(x,t_n)e_x(x,t_n)w_{h,x}(x,t_n)dx \\
&= \int_{\tilde{a}_h}^b e(x_h^*(x),t_{n-1})w_h(x,t_n)r_{h,x}(t_{n-1};x,t_n)dx \\
&\quad - \int_{t_{n-1}}^{t_n} e(b_h^*(t),t_{n-1})w_h(b,t)r_{h,t}(t_{n-1};b,t)dt \\
&\quad + \int_{\tilde{a}_h}^b u(x_h^*,t_{n-1})w_h(x,t_n)r_{h,x}(t_{n-1};x,t_n)dx \\
&\quad - \int_{\tilde{a}}^b u(x^*,t_{n-1})w_h(x,t_n)r_x(t_{n-1};x,t_n)dx \\
&\quad - \int_{t_{n-1}}^{t_n} u(b_h^*(t),t_{n-1})w_h(b,t)r_{h,t}(t_{n-1};b,t)dt \\
&\quad + \int_{t_{n-1}}^{t_n} u(b^*(t),t_{n-1})w_h(b,t)r_t(t_{n-1};b,t)dt \\
&\quad - \int_a^b \varepsilon(\Delta t_h(x) - \Delta t(x))D(x,t_n)u_x(x,t_n)w_{h,x}(x,t_n)dx \\
&\quad + \int_a^b (\Delta t_h(x) - \Delta t(x))f(x,t_n)w_h(x,t_n)dx - E_1(w_h) + E_2(u,w_h).
\end{aligned} \tag{22}
$$

Let $\Pi_h u \in S_h(a,b)$ be the interpolation of the true solution u, $\xi_h = u_h - \Pi_h u \in S_h(a,b)$, and $\eta = \Pi_h u - u$. The error estimates for η are given in (2), so we need only to estimate ξ_h. We choose $w_h(x,t_n) = \xi_h(x,t_n)$ in Eq. (22) and rewrite the error equation in terms of ξ_h and η as follows

$$
\begin{aligned}
&\int_a^b \xi_h^2(x,t_n)dx + \int_{t_{n-1}}^{t_n} V\xi_h^2(b,t)dt + \int_a^b \varepsilon\Delta t_h(x)D(x,t_n)\xi_{h,x}^2(x,t_n)dx \\
&\quad = \int_{\tilde{a}_h}^b \xi_h(x_h^*,t_{n-1})\xi_h(x,t_n)r_{h,x}(t_{n-1};x,t_n)dx \\
&\qquad - \int_{t_{n-1}}^{t_n} \xi_h(b_h^*(t),t_{n-1})\xi_h(b,t)r_{h,t}(t_{n-1};b,t)dt \\
&\qquad + \int_{\tilde{a}_h}^b \eta(x_h^*,t_{n-1})\xi_h(x,t_n)r_{h,x}(t_{n-1};x,t_n)dx - \int_a^b \eta(x,t_n)\xi_h(x,t_n)dx \\
&\qquad - \int_{t_{n-1}}^{t_n} \eta(b_h^*(t),t_{n-1})\xi_h(b,t)r_{h,t}(t_{n-1};b,t)dt - \int_{t_{n-1}}^{t_n} V\eta(b,t)\xi_h(b,t)dt \\
&\qquad - \int_a^b \varepsilon\Delta t_h(x)D(x,t_n)\eta_x(x,t_n)\xi_{h,x}(x,t_n)dx \\
&\qquad + \int_{\tilde{a}_h}^b u(x_h^*,t_{n-1})\xi_h(x,t_n)r_{h,x}(t_{n-1};x,t_n)dx \\
&\qquad - \int_{\tilde{a}}^b u(x^*,t_{n-1})\xi_h(x,t_n)r_x(t_{n-1};x,t_n)dx \\
&\qquad - \int_{t_{n-1}}^{t_n} u(b_h^*(t),t_{n-1})\xi_h(b,t)r_{h,t}(t_{n-1};b,t)dt \\
&\qquad + \int_{t_{n-1}}^{t_n} u(b^*(t),t_{n-1})\xi_h(b,t)r_t(t_{n-1};b,t)dt \\
&\qquad - \int_a^b \varepsilon(\Delta t_h(x)-\Delta t(x))D(x,t_n)u_x(x,t_n)\xi_{h,x}(x,t_n)dx \\
&\qquad + \int_a^b (\Delta t_h(x)-\Delta t(x))f(x,t_n)\xi_h(x,t_n)dx - E_1(\xi_h) + E_2(u,\xi_h).
\end{aligned} \tag{23}
$$

The first two terms on the right side are bounded by

$$
\begin{aligned}
&\Big|\int_{\tilde{a}_h}^b \xi_h(x_h^*,t_{n-1})\xi_h(x,t_n)r_{h,x}(t_{n-1};x,t_n)dx \\
&\qquad - \int_{t_{n-1}}^{t_n} \xi_h(b_h^*(t),t_{n-1})\xi_h(b,t)r_{h,t}(t_{n-1};b,t)dt\Big| \\
&\quad \le \frac{1+C\Delta t}{2}\|\xi_h(\cdot,t_n)\|_{L^2(\tilde{a}_h,b)}^2 + \frac{1+C\Delta t}{2}\|\xi_h(\cdot,t_{n-1})\|_{L^2}^2 \\
&\qquad + \frac{1+C\Delta t}{2}\int_{t_{n-1}}^{t_n} V\xi_h^2(b,t)dt.
\end{aligned} \tag{24}
$$

We use the following estimate for the interpolation $\eta(x,t_n)$ on $[b_h^*,b]$

$$
\int_{b_h^*}^b \eta^2(x,t_n)dx = \int_{b_h^*}^b \Big\{\int_x^b \eta_y(y,t_n)dy\Big\}^2 dx \le C(\Delta t)^2h^2\|u\|_{L^\infty(0,T;H^2)}^2
$$

to bound the fifth and sixth terms on the right side of Eq. (23) to conclude that

$$
\begin{aligned}
&\Big|\int_{t_{n-1}}^{t_n} \eta(b_h^*(t),t_{n-1})\xi_h(b,t)r_{h,t}(t_{n-1};b,t)dt + \int_{t_{n-1}}^{t_n} V\eta(b,t)\xi_h(b,t)dt\Big| \\
&\le \varepsilon_1\int_{t_{n-1}}^{t_n} V\xi_h^2(b,t)dt + C\|\eta(b,\cdot)\|_{L^2(t_{n-1},t_n)}^2 + C\int_{b_h^*}^b \eta^2(x,t_n)dx \\
&\le \varepsilon_1\int_{t_{n-1}}^{t_n} V\xi_h^2(b,t)dt + C(\Delta t)^2\|u(b,\cdot)\|_{H^1(t_{n-1},t_n)}^2 \\
&\quad + C(\Delta t)^2h^2\|u\|_{L^\infty(0,T;H^2)}^2.
\end{aligned} \tag{25}
$$

The estimates of the third, fourth, and seventh terms on the right side of Eq. (23) present major difficulties. For clarity of exposition, the proofs are presented in Lemma 5.1 and Lemma 5.2, respectively; there we obtain

$$
\begin{aligned}
&\Big|\int_{\tilde a_h}^b \eta(x_h^*,t_{n-1})\xi_h(x,t_n)r_{h,x}(t_{n-1};x,t_n)dx-\int_a^b \eta(x,t_n)\xi_h(x,t_n)dx\Big|\\
&\le \frac12\|\xi(\cdot,t_n)\|^2_{L^2(a,\tilde a_h)}+C\Delta t\|\xi_h(\cdot,t_n)\|^2_{L^2}+C(\Delta t)^3\|u\|^2_{L^\infty(0,T;H^2)}\\
&\quad +C\Delta t(\min\{h^2,(\Delta t)^2\}+h^4)\|u\|^2_{L^\infty(0,T;H^2)}+\lambda\varepsilon_1\int_{t_{n-1}}^{t_n}V\xi^2(b,t)dt\\
&\quad +\lambda Ch^4(\|u\|^2_{H^1(t_{n-1},t_n;H^2)}+\Delta t\|u\|^2_{L^\infty(0,T;H^3)}),
\end{aligned}
\tag{26}
$$

and

$$
\begin{aligned}
&\Big|\int_a^b \varepsilon\Delta t_h(x)D(x,t_n)\eta_x(x,t_n)\xi_{h,x}(x,t_n)dx\Big|\\
&\le \varepsilon_2\int_a^b \varepsilon\Delta t_h(x)D(x,t_n)\xi^2_{h,x}(x,t_n)dx+C\Delta t((\Delta t)^2+h^4)\|u\|^2_{L^\infty(0,T;H^2)}.
\end{aligned}
\tag{27}
$$

To bound the eighth and ninth terms on the right side of Eq. (23), we assume that $\tilde a_h\le\tilde a$ without loss of generality (otherwise, we switch $\tilde a_h$ and $\tilde a$ by symmetry). We decompose these two terms as

$$
\begin{aligned}
&\int_{\tilde a_h}^b u(x_h^*,t_{n-1})\xi_h(x,t_n)r_{h,x}(t_{n-1};x,t_n)dx\\
&\qquad -\int_{\tilde a}^b u(x^*,t_{n-1})\xi_h(x,t_n)r_x(t_{n-1};x,t_n)dx\\
&\quad =\int_{\tilde a_h}^{\tilde a} u(x_h^*,t_{n-1})\xi_h(x,t_n)r_{h,x}(t_{n-1};x,t_n)dx\\
&\qquad +\int_{\tilde a}^b \big(u(x_h^*,t_{n-1})-u(x^*,t_{n-1})\big)\xi_h(x,t_n)r_{h,x}(t_{n-1};x,t_n)dx\\
&\qquad +\int_{\tilde a}^b u(x^*,t_{n-1})\xi_h(x,t_n)\big(r_{h,x}(t_{n-1};x,t_n)-r_x(t_{n-1};x,t_n)\big)dx.
\end{aligned}
\tag{28}
$$

We use estimates of (19) to bound the three terms on the right side by

$$
\begin{aligned}
&\Big|\int_{\tilde a_h}^{\tilde a} u(x_h^*,t_{n-1})\xi_h(x,t_n)r_{h,x}(t_{n-1};x,t_n)dx\Big|\\
&\le C\Delta t(\Delta t_f)^2\|u\|_{L^\infty(0,T;L^\infty)}\|\xi(\cdot,t_n)\|_{L^\infty}\\
&\le C\Delta t\|\xi(\cdot,t_n)\|^2_{L^2}+C(\Delta t)^3\|u\|^2_{L^\infty(0,T;L^\infty)},
\end{aligned}
\tag{29}
$$

here we have used the fact that $\Delta t_f\le Ch$ and the inverse inequality,

$$
\begin{aligned}
&\Big|\int_{\tilde a}^b u(x^*,t_{n-1})\xi_h(x,t_n)(r_{h,x}(t_{n-1};x,t_n)-r_x(t_{n-1};x,t_n))dx\Big|\\
&\le C(\Delta t)^2\Big|\int_{\tilde a}^b u(x^*,t_{n-1})\xi_h(x,t_n)dx\Big|\\
&\le C\Delta t\|\xi_h(\cdot,t_n)\|_{L^2}+C(\Delta t)^3\|u\|_{L^\infty(0,T;L^2)},
\end{aligned}
\tag{30}
$$

and

$$
\begin{aligned}
&\Big|\int_{\tilde a}^b (u(x_h^*,t_{n-1})-u(x^*,t_{n-1}))r_{h,x}(t_{n-1};x,t_n)\xi_h(x,t_n)dx\Big|\\
&\le C(\Delta t)^2\|\xi_h(x,t_n)\|_{L^2}\|u\|_{L^\infty(0,T;W^1_\infty)}\\
&\le C\Delta t\|\xi_h(\cdot,t_n)\|^2_{L^2}+C(\Delta t)^3\|u\|^2_{L^\infty(0,T;W^1_\infty)}.
\end{aligned}
\tag{31}
$$

We use Lemma 3.1 to bound the tenth to eleventh terms, the twelfth to thirteenth terms, and the last two terms on the right side of Eq. (23) by

$$
\begin{aligned}
&\Big|\int_{t_{n-1}}^{t_n} u(b_h^*(t),t_{n-1})\xi_h(b,t)r_{h,t}(t_{n-1};b,t)dt \\
&\quad -\int_{t_{n-1}}^{t_n} u(b^*(t),t_{n-1})\xi_h(b,t)r_t(t_{n-1};b,t)dt\Big| \\
&\quad =\Big|\int_{t_{n-1}}^{t_n} \big(u(b_h^*(t),t_{n-1})-u(b^*(t),t_{n-1})\big)\xi_h(b,t)r_{h,t}(t_{n-1};b,t)dt \\
&\quad +\int_{t_{n-1}}^{t_n} u(b^*(t),t_{n-1})\xi_h(b,t)\big(r_{h,t}(t_{n-1};b,t)-r_t(t_{n-1};b,t)\big)dt\Big| \\
&\quad \le (\varepsilon_1 + C\Delta t)\int_{t_{n-1}}^{t_n} V\xi^2(b,t)dt + C(\Delta t)^3\|u\|^2_{L^\infty(0,T;W^1_\infty)},
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
&\Big|\int_a^b \varepsilon(\Delta t_h(x)-\Delta t(x))D(x,t_n)u_x(x,t_n)\xi_{h,x}(x,t_n)dx \\
&\quad +\int_a^b (\Delta t_h(x)-\Delta t(x))f(x,t_n)\xi_h(x,t_n)dx\Big| \\
&\quad \le \varepsilon_2\int_a^b \varepsilon\Delta t_h(x)D(x,t_n)\xi^2_{h,x}(x,t_n)dx + C\Delta t\|\xi(\cdot,t_n)\|^2_{L^2} \\
&\qquad +C(\Delta t)^3\|u\|^2_{L^\infty(0,T;H^1)} + C(\Delta t)^3\|f\|^2_{L^\infty(0,T;L^2)},
\end{aligned} \tag{33}
$$

and

$$
\begin{aligned}
&\big|E_1(\xi_h)+E_2(u,\xi_h)\big| \\
&\quad \le C\Delta t\|\xi_h(\cdot,t_n)\|^2_{L^2} + \varepsilon_2\int_a^b \varepsilon\Delta t_h(x)D(x,t_n)\xi^2_{h,x}(x,t_n)dx \\
&\qquad +C\Delta t\int_{t_{n-1}}^{t_n} V\xi^2(b,t)dt + C(\Delta t)^2\Big(\Big\|\frac{du}{dt}\Big\|^2_{L^2(t_{n-1},t_n;H^1)} \\
&\qquad +\Big\|\frac{df}{dt}\Big\|^2_{L^2(t_{n-1},t_n;L^2)} + \|f\|^2_{L^2(t_{n-1},t_n;L^2)}\Big).
\end{aligned} \tag{34}
$$

We substitute the estimates (24)–(34) for the corresponding terms in Eq. (23) to obtain the following estimate

$$
\begin{aligned}
&\|\xi_h(\cdot,t_n)\|^2_{L^2} + \int_{t_{n-1}}^{t_n} V\xi^2(b,t)dt + \int_a^b \varepsilon\Delta t_h(x)D(x,t_n)\xi^2_{h,x}(x,t_n)dx \\
&\le \frac{1+C\Delta t}{2}\Big(\|\xi_h(\cdot,t_n)\|^2_{L^2} + \|\xi_h(\cdot,t_{n-1})\|^2_{L^2}\Big) + \Big(\frac{1}{2}+3\varepsilon_1 + C\Delta t\Big)\int_{t_{n-1}}^{t_n} V\xi^2(b,t)dt \\
&\quad +3\varepsilon_2\int_a^b \varepsilon\Delta t_h(x)D(x,t_n)\xi^2_{h,x}(x,t_n)dx + C\Delta t((\Delta t)^2+h^4)\|u\|^2_{L^\infty(0,T;H^2)} \\
&\quad +C(\Delta t)^2\Big(\|u(b,\cdot)\|^2_{H^1(t_{n-1},t_n)} + \|f\|^2_{L^2(t_{n-1},t_n;L^2)} + \Big\|\frac{du}{dt}\Big\|^2_{L^2(t_{n-1},t_n;H^1)} \\
&\quad +\Big\|\frac{df}{dt}\Big\|^2_{L^2(t_{n-1},t_n;L^2)} + \Delta t\big(\|u\|^2_{L^\infty(0,T;H^2)} + \|f\|^2_{L^\infty(0,T;L^2)}\big)\Big) \\
&\quad +\lambda Ch^4\big(\|u\|^2_{H^1(t_{n-1},t_n;H^2)} + \Delta t\|u\|^2_{L^\infty(0,T;H^3)}\big).
\end{aligned}
$$

Choosing $\varepsilon_1 = \frac{1}{16}$ and $\varepsilon_2 = \frac{1}{4}$, we sum the estimate for $n = 1, \ldots, N_1(\leq N)$ and cancel like terms to obtain

$$\begin{aligned}
&\|\xi_h(\cdot, t_{N_1})\|^2_{L^2} + \int_0^{t_{N_1}} V\xi^2(b,t)dt + \sum_{n=1}^{N_1}\int_a^b \varepsilon\Delta t_h(x)D(x,t_n)\xi^2_{h,x}(x,t_n)dx \\
&\leq C\Delta t \sum_{n=0}^{N_1-1} \|\xi_h(\cdot,t_n)\|^2_{L^2} + C((\Delta t)^2 + h^4)\|u\|^2_{L^\infty(0,T;H^2)} \\
&+C(\Delta t)^2\Big(\|u(b,\cdot)\|^2_{H^1(0,T)} + \Big\|\frac{du}{dt}\Big\|^2_{L^2(0,T;H^1)} + \Big\|\frac{df}{dt}\Big\|^2_{L^2(0,T;L^2)} \\
&+\|f\|^2_{L^\infty(0,T;L^2)}\Big) + \lambda Ch^4(\|u\|^2_{H^1(0,T;H^2)} + \|u\|^2_{L^\infty(0,T;H^3)}).
\end{aligned}$$

We then apply Gronwall inequality to conclude

$$\begin{aligned}
&\|\xi_h\|_{L_\varepsilon(0,T;H^1_D(a,b))} \\
&\leq C\Delta t\Big(\|u(b,\cdot)\|_{H^1(0,T)} + \Big\|\frac{du}{dt}\Big\|_{L^2(0,T;H^1)} + \Big\|\frac{df}{dt}\Big\|_{L^2(0,T;L^2)} + \|f\|_{L^\infty(0,T;L^2)}\Big) \\
&+C(\Delta t + h^2)\|u\|_{L^\infty(0,T;H^2)} + \lambda Ch^2(\|u\|_{H^1(0,T;H^2)} + \|u\|_{L^\infty(0,T;H^3)}).
\end{aligned}$$

The general constant C depends exponentially on the final time T in problem (1) due to the application of Gronwall inequality, but does not depend on the parameter ε. Combining this result with the well-known estimate for η, we finish the proof. ☐

5. Auxiliary Lemmas

In this section, we prove two auxiliary lemmas which address the estimates in (26) and (27), respectively .

5.1. A Superconvergent Estimate on Interpolation.

We prove the following superconvergence estimate on the interpolation error.

Lemma 5.1. *Assume $u \in L^\infty(0,T;H^3(a,b)) \cap H^1(0,T;H^2(a,b))$. Let $\Pi_h u \in S_h(a,b)$ be the interpolation of u and $\eta = \Pi_h u - u$. Let λ be the parameter defined in Theorem 4.1. Then the superconvergence estimate (26) holds.*

Proof. When $\mathrm{Cr} \geq 1$ that implies $h \leq C\Delta t$, the left side of (26) can be bounded as

$$\begin{aligned}
&\Big|\int_{\tilde{a}_h}^b \eta(x^*_h, t_{n-1})\xi_h(x,t_n)r_{h,x}(t_{n-1};x,t_n)dx - \int_a^b \eta(x,t_n)\xi_h(x,t_n)dx\Big| \\
&\leq C\|\xi_h(\cdot,t_n)\|_{L^2}(\|\eta(\cdot,t_n)\|_{L^2} + \|\eta(\cdot,t_{n-1})\|_{L^2}) \\
&\leq C\Delta t\|\xi_h(\cdot,t_n)\|^2_{L^2} + C\Delta t\min\{h^2,(\Delta t)^2\}\|u\|^2_{L^\infty(0,T;H^2)}.
\end{aligned} \tag{35}$$

For $\mathrm{Cr} < 1$, we decompose the left-hand side of (26) as follows:

$$\begin{aligned}
&\int_a^b \eta(x,t_n)\xi_h(x,t_n)dx - \int_{\tilde{a}_h}^b \eta(x^*_h,t_{n-1})\xi_h(x,t_n)r_{h,x}(t_{n-1};x,t_n)dx \\
&= \int_a^{\tilde{a}_h} \eta(x,t_n)\xi_h(x,t_n)dx + \int_{\tilde{a}_h}^b (\eta(x,t_{n-1}) - \eta(x^*_h,t_{n-1}))\xi_h(x,t_n)dx \\
&+\int_{\tilde{a}_h}^b\int_{t_{n-1}}^{t_n} \eta_t(x,t)dt\xi_h(x,t_n)dx + \int_{\tilde{a}_h}^b O(\Delta t)\eta(x^*_h,t_{n-1})\xi_h(x,t_n)dx.
\end{aligned} \tag{36}$$

We use the property (3) to estimate the first term as follows

$$
\begin{aligned}
&\Big|\int_a^{\tilde a_h} \eta(x,t_n)\xi_h(x,t_n)dx\Big| \\
&= \Big|\int_a^{\tilde a_h}\int_a^x\int_{\hat x_{1/2}}^y u_{zz}(z,t_n)dzdy\xi_h(x,t_n)dx\Big| \\
&\le C\Delta t\|\xi_h(\cdot,t_n)\|_{L^2}^2 + C(\Delta t)^3\|u\|^2_{L^\infty(0,T;H^2)}.
\end{aligned}\tag{37}
$$

We apply the estimate (2) to bound the third term on the right side by

$$
\begin{aligned}
&\Big|\int_{\tilde a_h}^b\int_{t_{n-1}}^{t_n}\eta_t(x,t)dt\ \xi_h(x,t_n)dx\Big| \\
&\le C\Delta t\|\xi_h(\cdot,t_n)\|^2_{L^2} + Ch^4\|u\|^2_{H^1(t_{n-1},t_n;H^2)}.
\end{aligned}\tag{38}
$$

We bound the fourth term on the right-hand side in a similar way to (24)

$$
\begin{aligned}
&\Big|\int_{\tilde a_h}^b O(\Delta t)\eta(x_h^*,t_{n-1})\xi_h(x,t_n)dx\Big| \\
&\le C\Delta t\|\xi_h(\cdot,t_n)\|^2_{L^2} + C\Delta t h^4\|u\|^2_{L^\infty(0,T;H^2)}.
\end{aligned}\tag{39}
$$

We decompose the second term on the right side of (36) as follows

$$
\begin{aligned}
&\int_{\tilde a_h}^b(\eta(x,t_{n-1})-\eta(x_h^*,t_{n-1}))\xi_h(x,t_n)dx \\
&= \sum_{i=1}^I\int_{\tilde x_{i-1,h}}^{x_i}\int_{x_h^*}^x\eta_z(z,t_{n-1})dz\xi_h(x,t_n)dx \\
&\quad + \sum_{i=2}^I\int_{x_{i-1}}^{\tilde x_{i-1,h}}\int_{x_h^*}^x\eta_z(z,t_{n-1})dz\xi_h(x,t_n)dx.
\end{aligned}\tag{40}
$$

For $x\in[x_{i-1},\tilde x_{i-1,h}]$, $x_h^*\in[x_{i-2},x_{i-1}]$. Thus, we can estimate the second term on the right side by

$$
\begin{aligned}
&\Big|\sum_{i=2}^I\int_{x_{i-1}}^{\tilde x_{i-1,h}}\int_{x_h^*}^x\eta_z(z,t_{n-1})dz\xi_h(x,t_n)dx\Big| \\
&= \Big|\sum_{i=2}^I\int_{x_{i-1}}^{\tilde x_{i-1,h}}\Big[\int_{x_h^*}^{x_{i-1}}\eta_z(z,t_{n-1})dz+\int_{x_{i-1}}^x\eta_z(z,t_{n-1})dz\Big]\xi_h(x,t_n)dx\Big| \\
&\le C(\Delta t)^2\sum_{i=2}^I\max\{|\xi(\mathrm{x_{i-1},t_n})|,|\xi(\mathrm{x_i,t_n})|\}\int_{\mathrm{x_{i-2}}}^{\mathrm{x_i}}|\mathrm{u_{xx}(x,t_{n-1})}|\mathrm{dx} \\
&\le C\Delta t\|\xi_h(\cdot,t_n)\|^2_{L^2}+C(\Delta t)^3\|u\|^2_{L^\infty(0,T;H^2)}.
\end{aligned}\tag{41}
$$

To bound the first term on the right side of (40), we decompose it as follows

$$
\begin{aligned}
&\sum_{i=1}^I\int_{\tilde x_{i-1,h}}^{x_i}\int_{x_h^*}^x\eta_z(z,t_{n-1})dz\xi_h(x,t_n)dx \\
&= \sum_{i=1}^I\int_{x_{i-1}}^{x_i}V(x,t_n)\Delta t\eta_x(x,t_{n-1})\xi_h(x,t_n)dx \\
&\quad -\sum_{i=1}^I\int_{x_{i-1}}^{\tilde x_{i-1,h}}V(x,t_n)\Delta t\eta_x(x,t_{n-1})\xi_h(x,t_n)dx \\
&\quad -\sum_{i=1}^I\int_{\tilde x_{i-1,h}}^{x_i}\int_{x_h^*}^x\int_z^x u_{yy}(y,t_{n-1})dydz\xi_h(x,t_n)dx.
\end{aligned}\tag{42}
$$

Let $\chi_{[a,b]}$ be the indicator function of x over $[a,b]$. We interchange of the order of integrations in the third term on the right side to get

$$
\begin{aligned}
&\Big|\sum_{i=1}^{I}\int_{\tilde{x}_{i-1,h}}^{x_i}\int_{x_h^*}^{x}\int_{z}^{x} u_{yy}(y,t_{n-1})dydz\xi_h(x,t_n)dx\Big| \\
&\quad = \Big|\sum_{i=1}^{I}\int_{\tilde{x}_{i-1,h}}^{x_i}\int_{x_h^*}^{x} u_{yy}(y,t_{n-1})(y-x_h^*)dy\xi_h(x,t_n)dx\Big| \\
&\quad \le C\Delta t\|\xi_h(\cdot,t_n)\|_{L^2}^2 + C(\Delta t)^2\Big|\sum_{i=1}^{I}\int_{\tilde{x}_{i-1,h}}^{x_i}\int_{x_{i-1}}^{x_i} u_{yy}^2(y,t_{n-1})\chi_{[x_h^*,x]}dydx\Big| \\
&\quad \le C\Delta t\|\xi_h(\cdot,t_n)\|_{L^2}^2 + C(\Delta t)^3\|u\|_{L^\infty(0,T;H^2)}^2.
\end{aligned}
\tag{43}
$$

We bound the second term on the right side of (42) by

$$
\begin{aligned}
&\Big|\sum_{i=1}^{I}\int_{x_{i-1}}^{\tilde{x}_{i-1,h}} V(x,t_n)\Delta t\eta_x(x,t_{n-1})\xi_h(x,t_n)dx\Big| \\
&\quad = \Big|\sum_{i=1}^{I}\int_{x_{i-1}}^{\tilde{x}_{i-1,h}} V(x,t_n)\Delta t\int_{\hat{x}_{i-1/2}}^{x} u_{zz}(z,t_{n-1})dz\xi_h(x,t_n)dx\Big| \\
&\quad \le C\Delta t\|\xi_h(\cdot,t_n)\|_{L^2}^2 + C(\Delta t)^3\|u\|_{L^\infty(0,T;H^2)}^2.
\end{aligned}
\tag{44}
$$

Finally, we decompose the first term on the right side of (42) as

$$
\begin{aligned}
&\sum_{i=1}^{I}\int_{x_{i-1}}^{x_i} V(x,t_n)\Delta t\eta_x(x,t_{n-1})\xi_h(x,t_n)dx \\
&\quad = \sum_{i=1}^{I} V(x_{i-1/2},t_n)\Delta t\int_{x_{i-1}}^{x_i}\eta_x(x,t_{n-1})\xi_h(x,t_n)dx \\
&\qquad + \sum_{i=1}^{I}\int_{x_{i-1}}^{x_i}(V(x,t_n)-V(x_{i-1/2},t_n))\Delta t\eta_x(x,t_{n-1})\xi_h(x,t_n)dx.
\end{aligned}
\tag{45}
$$

Here $x_{i-1/2}$ is the middle point of the interval $[x_{i-1},x_i]$. The second term can be bounded in a standard way as follows:

$$
\begin{aligned}
&\Big|\sum_{i=1}^{I}\int_{x_{i-1}}^{x_i}(V(x,t_n)-V(x_{i-1/2},t_n))\Delta t\eta_x(x,t_{n-1})\xi_h(x,t_n)dx\Big| \\
&\quad \le C\Delta t\|\xi_h(\cdot,t_n)\|_{L^2}^2 + C\Delta t h^4\|u\|_{L^\infty(0,T;H^2)}^2.
\end{aligned}
\tag{46}
$$

Next, we rewrite the integral in the first term on the right side of (45) by

$$
\begin{aligned}
&\int_{x_{i-1}}^{x_i}\eta_x(x,t_{n-1})\xi_h(x,t_n)dx \\
&= \int_{x_{i-1}}^{x_i}\eta_x(x,t_{n-1})\xi_{h,x}(x_{i-1/2},t_n)(x-x_{i-1/2})dx \\
&= -\xi_{h,x}(x_{i-1/2},t_n)\int_{x_{i-1}}^{x_i}\int_{x_{i-1/2}}^{x}\int_{x_{i-1/2}}^{z} u_{yyy}(y,t_{n-1})dydz(x-x_{i-1/2})dx \\
&\quad -\frac{h^3}{12}\xi_{h,x}(x_{i-1/2},t_n)u_{xx}(x_{i-1/2},t_n).
\end{aligned}
\tag{47}
$$

We bound the first term on the right side of (47) by

$$
\begin{aligned}
&\Big|\sum_{i=1}^{I} V(x_{i-1/2},t_n)\Delta t\xi_{h,x}(x,t_n) \\
&\qquad\qquad \times \int_{x_{i-1}}^{x_i}\int_{x_{i-1/2}}^{x}\int_{x_{i-1/2}}^{z} u_{yyy}(y,t_{n-1})dydz(x-x_{i-1/2})dx\Big| \\
&\quad \le C\Delta t h^2 \sum_{i=1}^{I} |\xi_h(x_i,t_n)-\xi_h(x_{i-1},t_n)| \int_{x_{i-1}}^{x_i} |u_{yyy}(y,t_{n-1})|dy \\
&\quad \le C\Delta t\|\xi_h(\cdot,t_n)\|_{L^2}^2 + C\Delta t h^4 \|u\|^2_{L^\infty(0,T;H^3)}.
\end{aligned}
\tag{48}
$$

As for the second term on the right side of (47), we decompose it as follows first

$$
\begin{aligned}
&\sum_{i=1}^{I} \frac{\Delta t h^3}{12} V(x_{i-1/2},t_n)\xi_{h,x}(x,t_n)u_{xx}(x_{i-1/2},t_n) \\
&\quad = \frac{\Delta t h^2}{12}\Big[\sum_{1}^{I-1} \xi_h(x_i,t_n)\int_{x_{i-1/2}}^{x_{i+1/2}} (V_x(x,t_n)u_{xx}(x_{i-1/2},t_n) \\
&\qquad +V(x_{i+1/2},t_n)u_{xxx}(x,t_{n-1}))dx + V(x_{1/2},t_n)\xi_h(a,t_n)u_{xx}(x_{1/2},t_n) \\
&\qquad -V(x_{I-1/2},t_n)\xi_h(b,t_n)u_{xx}(x_{I-1/2},t_n)\Big].
\end{aligned}
\tag{49}
$$

By the inverse inequality and the fact that $\xi_h(b,t)=\xi_h(b,t_n)$, we can estimate the first and the third terms on the right side of (49) by

$$
\begin{aligned}
&\Big|\frac{\Delta t h^2}{12}\Big[\sum_{1}^{I-1} \xi_h(x_i,t_n)\int_{x_{i-1/2}}^{x_{i+1/2}} (V_x(x,t_n)u_{xx}(x_{i-1/2},t_n) \\
&\quad +V(x_{i+1/2},t_n)u_{xxx}(x,t_{n-1}))dx - V(x_{I-1/2},t_n)\xi_h(b,t_n)u_{xx}(x_{I-1/2},t_n)\Big]\Big| \\
&\le C\Delta t\|\xi_h(\cdot,t_n)\|_{L^2}^2 + C\Delta t h^4\|u\|^2_{L^\infty(0,T;H^3)} + \varepsilon_1\int_{t_{n-1}}^{t_n} V(x,t_n)\xi_h^2(b,t)dt.
\end{aligned}
$$

Note that in the current context of Cr<1, $\xi(x,t_n)$ is linear on $[a,\tilde{a}_h]\subset[a,x_1]$ and so can be expressed as

$$
\xi(x,t_n) = \frac{1}{\tilde{a}_h - a}\big(\xi(a,t_n)(\tilde{a}_h - x) + \xi(\tilde{a}_h,t_n)(x-a)\big).
$$

Then we can get

$$
\begin{aligned}
\int_a^{\tilde{a}_h} \xi^2(x,t_n)dx &= \frac{\tilde{a}_h - a}{3}\big(\xi^2(a,t_n)+\xi(a,t_n)\xi(\tilde{a}_h,t_n)+\xi^2(\tilde{a}_h,t_n)\big) \\
&\ge \frac{\tilde{a}_h - a}{6}\xi^2(a,t_n).
\end{aligned}
$$

Therefore the second term on the right side of (49) is bounded by

$$
\begin{aligned}
&\Big|\frac{\Delta t h^2}{12} V(x_{\frac12},t_n)u_{xx}(x_{\frac12},t_{n-1})\xi(a,t_n)\Big| \\
&\quad \le \frac{1}{2}\|\xi(\cdot,t_n)\|^2_{L^2(a,\tilde{a}_h)} + C\Delta t h^4\|u\|^2_{L^\infty(0,T;W^{2,\infty})}.
\end{aligned}
\tag{50}
$$

We combine the estimates (35)–(50) to finish the proof. □

5.2. Proof of the estimate (27).

Lemma 5.2. *Let $\Pi_h u \in S_h(a,b)$ be the interpolation of u, $\eta = \Pi_h u - u$, and λ be defined in Theorem 4.1. Then the superconvergence estimate (27) holds.*

Proof. In the case $Cr \geq 1$ which implies $h \leq C\Delta t$, we have

$$\begin{aligned}&\Big|\int_a^b \varepsilon \Delta t_h(x) D(x,t_n)\eta_x(x,t_n)\xi_{h,x}(x,t_n)dx\Big| \\ &\quad \leq \varepsilon_2 \int_a^b \varepsilon \Delta t_h(x) D(x,t_n)\xi_{h,x}^2(x,t_n)dx + C(\Delta t)^3 \|u\|^2_{L^\infty(0,T;H^2)}.\end{aligned} \tag{51}$$

When $Cr < 1$, $\tilde{a}_h - a \leq h$. Thus, $\Delta t_h(x) = \Delta t$ on $[\tilde{a}_h, b]$. Hence, we can rewrite the left side of (27) as

$$\begin{aligned}&\int_a^b \varepsilon \Delta t_h(x) D(x,t_n)\eta_x(x,t_n)\xi_{h,x}(x,t_n)dx \\ &= \Delta t \int_a^b \varepsilon D(x,t_n)\eta_x(x,t_n)\xi_{h,x}(x,t_n)dx \\ &\quad -\xi_{h,x}(x_{1/2},t_n)\int_a^{\tilde{a}_h} \varepsilon(t_h^*(x) - t_{n-1})D(x,t_n)\eta_x(x,t_n)dx.\end{aligned} \tag{52}$$

We use the inverse estimate (2) to bound the second term by

$$\begin{aligned}&\Big|\xi_{h,x}(x_{1/2},t_n)\int_a^{\tilde{a}_h} \varepsilon(t_h^*(x) - t_{n-1})D(x,t_n)\eta_x(x,t_n)dx\Big| \\ &\leq C\varepsilon(\Delta t)^2|\xi_{h,x}(x_{1/2},t_n)| \int_a^{x_1} |u_{xx}(x,t_n)|dx \\ &\leq \frac{\varepsilon_2}{2}\int_a^{x_1} \varepsilon \Delta t_h(x) D(x,t_n)\xi_{h,x}^2(x,t_n)dx + C(\Delta t)^3 \|u\|^2_{L^\infty(0,T;H^2)}.\end{aligned} \tag{53}$$

Note that $\xi_{h,x}(x,t_n)$ is constant on each interval $[x_{i-1}, x_i]$ and that η satisfies $\eta(x_{i-1},t_n) = \eta(x_i,t_n) = 0$ for $i = 1,\ldots,I$, we bound the first term on the right-hand side of Eq. (52)

$$\begin{aligned}&\Big|\Delta t \int_a^b \varepsilon D(x,t_n)\eta_x(x,t_n)\xi_{h,x}(x,t_n)dx\Big| \\ &= \Big|\Delta t \sum_{i=1}^{I} \varepsilon \xi_{h,x}(x_{i-1/2},t_n)\int_{x_{i-1}}^{x_i} (D(x,t_n) - D(x_{i-1/2},t_n))\eta_x(x,t_n)dx\Big| \\ &\leq \frac{\varepsilon_2}{2}\int_a^b \varepsilon \Delta t(x) D(x,t_n)\xi_{h,x}^2(x,t_n)dt + C\Delta t h^4 \|u\|^2_{L^\infty(0,T;H^2)}.\end{aligned} \tag{54}$$

We combine the estimates (51)–(54) to finish the proof. □

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