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SOLVING SINGULARLY PERTURBED REACTION DIFFUSION PROBLEMS USING WAVELET OPTIMIZED FINITE DIFFERENCE AND CUBIC SPLINE ADAPTIVE WAVELET SCHEME

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Abstract. In this paper singularly perturbed reaction diffusion equations of elliptic and parabolic types have been discussed using wavelet optimized finite difference (WOFD) method based on an interpolating wavelet transform using cubic spline on dyadic points as discussed in [1]. Adaptive feature is performed automatically by thresholding the wavelet coefficients. WOFD [2] works by using adaptive wavelet to generate an irregular grid which is then exploited for the finite difference method. Numerical examples are presented for elliptic and parabolic problems and comparisons have been made using cubic spline and WOFD. The proposed adaptive method is very effective for studying singular perturbation problems in term of adaptive grid generation and CPU time.

Key Words. Singularly perturbed reaction diffusion problems, WOFD, Splines wavelets, Multiresolution analysis, Fast discrete wavelet transform, Lagrangian finite difference.

1. Introduction

In this paper cubic spline and WOFD have been applied for solving singularly perturbed reaction diffusion problems of elliptic and parabolic types. Problems in which a small parameter is multiplied to the highest derivative arise in various fields of science and engineering, for instance fluid mechanics, elasticity, hydrodynamics, etc. The main concern with such problems is the rapid growth or decay of the solution in one or more narrow "layer region(s)". The specific problems under consideration in this paper is called dissipative because the rapid varying component of the solution decays exponentially (dissipates) away from a localized breakdown in the layer region(s) as $\epsilon \to 0$.

Here the author is trying to use cubic spline adaptive wavelet features to solve singularly perturbed reaction diffusion problems belonging to sobolev space $\mathcal{H}_0^2(\mathcal{I})$. Cai and Wang's [1] wavelets have been chosen because of their interpolating property. They were considered by the authors as "a semi-orthogonal cubic spline wavelet basis of homogenous sobolev space $\mathcal{H}_0^2(\mathcal{I})$".

Two semilinear singularly perturbed reaction diffusion problem of elliptic and parabolic types have been discussed. The first one is the elliptic problem defined as

$$-\epsilon u_\epsilon''(x) = f(x, u_\epsilon) \text{ where } 0 \leq x \leq 1, \tag{1a}$$

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with boundary conditions

$$u_\epsilon(0) = u_a, \;\; u_\epsilon(1) = u_b, \;\; \frac{\partial f}{\partial u_\epsilon} \geq c^2, \; c = \text{const.} > 0, \tag{1b}$$

where ϵ is a small parameter and $f(x, u)$ is sufficiently smooth. For $\epsilon \ll 1$, the solution has boundary layers at $x = 0$ and $x = 1$ [3].

The second problem is the one dimensional parabolic problem

$$-\epsilon \, u_{xx}(x,t) + a(x,t)u(x,t) + b(x,t)u_t(x,t) = f(x,t,u); \tag{2a}$$

where $(x, t) \in Q = (0, 1) \times (0, T]$ and

$$u(x, 0) = s(x) \text{ on } S_x = (x, 0) : \; 0 \leq x \leq 1, \tag{2b}$$

$$u(0, t) = q_0(t) \text{ on } S_0 = (0, t) : 0 < t \leq T, \tag{2c}$$

$$u(1, t) = q_1(t) \text{ on } S_1 = (1, t) : 0 < t \leq T. \tag{2d}$$

Here ϵ is a parameter satisfying $0 < \epsilon \ll 1$. We assume that $a(x, t) \geq a_0 > 0$ and $b(x, t) \geq b_0 > 0$ on $\overline{Q}$, where a_0, b_0 are some constants and $\overline{Q} = [0, 1] \times [0, T]$ denotes the closure of Q. The solution u has in general boundary layers of parabolic type along the sides $x = 0$ and $x = 1$ of $\overline{Q}$ [4].

Singular perturbation problems in consideration have shocks as boundary layers. For such kinds of problems the solution can be smooth in most of the solution domain with small area where the solution changes quickly. When solving such problems numerically, one would like to adjust the discretization to the solution. In term of mesh generation, we want to have many points in area where the solution have strong variations and few points in area where the solution is smooth. Elliptic problem earlier has been discussed [5] using cubic B-spline with Shishkin mesh. One should have the pre-knowledge about the locations of the boundary layers for working with Shishkin meshes, therefore, also motivates us for adaptive methods for effective grid generation.

Wavelets have been making an appearance in many pure and applied area of science and engineering [2]. Wavelets detect information at different scales and at different locations throughout a computational domain. Wavelets can provide a basis set in which the basis functions are constructed by dilating and translating a fixed function known as the mother wavelet. The mother wavelet can be seen as a high pass filter in the frequency domain. One of the key strength of the wavelet methods is data compression. An efficient basis is one in which a given set of data can be represented with as few basis elements as possible. Given a wavelet representation of a function

$$\sum_k c_{j,k} \varphi_{j,k}(x) + \sum_{j,k} d_{j,k} \psi_{j,k}(x),$$

where $\varphi_{j,k}(x)$ are scaling functions and $\psi_{j,k}(x)$ are wavelets, the scaling function coefficients $c_{j,k}$, essentially encode the smooth part of the function, while the wavelet coefficient $d_{j,k}$ contains information of the function behavior on successive finer scales. The most common way of compressing such a representation is thresholding. We delete all wavelet coefficients of magnitude less than some threshold, say τ. If the total no. of coefficients in the original representation was N, we have N_a significant coefficients left after the thresholding. Note that by thresholding a wavelet representation we have a way to find an adaptive feature and we can also use this representation to compute function values at any point.

Cubic splines have several useful properties: they are compactly supported, smooth and symmetric, which is an advantage when approximating shocks [6]. Moreover they have an analytic expression which makes their values and the values of their derivatives at any point easily available. There is no system to solve or data vectors to store as it was the case with Daubechies scaling functions [7]. The main restriction imposed by the use of spline is the fact that it is not possible to build with them a basis which is both orthogonal and compectly supported: only biorthogonal systems are possible, for the resolution of a PDE, the interpolating wavelets fit easily in a collocaton method which presents the advantage that no evaluation of inner product is necessary.

This paper has been divided into the following sections. In section 2, we introduce the cubic scaling functions $\phi(x), \phi_b(x)$ and their wavelet functions $\psi(x), \psi_b(x)$. An multiresolution analysis (MRA) and its corresponding wavelet decomposition of the sobolev space $\mathcal{H}_0^2(\mathcal{I})$ are constructed using $\phi(x), \phi_b(x)$ and $\psi(x), \psi_b(x)$. In section 3, we introduce the fast DWT (discrete wavelet transform) between functions and their wavelet coefficients. In 4, we discuss the adaptive collocation method using cubic spline and WOFD. In section 5, numerical experiments are presented and all numerical results using Lagrangian finite difference (WOFD) have been generated for $p = 3$, where p is the degree of the spline.

2. Basis functions

Let $\mathcal{I}$ denotes a finite interval, $\mathcal{I} = [0, L]$, and $\mathcal{H}^2(\mathcal{I})$ and $\mathcal{H}_0^2(\mathcal{I})$ denote the following two subspaces of $\mathcal{L}_2(\mathcal{R})$:

$$\mathcal{H}^2(\mathcal{I}) := \{f(x), x \in \mathcal{I} | \int_I (|f(x)|^2 + |f'(x)|^2 + |f''(x)|^2)dx < \infty\}, \tag{3}$$

$$\mathcal{H}_0^2(\mathcal{I}) := \{f(x) \in \mathcal{H}^2(\mathcal{I}) | f(0) = f'(0) = f(L) = f'(L) = 0\}. \tag{4}$$

The space $\mathcal{H}_0^2(\mathcal{I})$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_I f''(x) g''(x) dx, \tag{5}$$

thus

$$\|f\| = \sqrt{\langle f, f \rangle}, \tag{6}$$

provides a norm for $\mathcal{H}_0^2(\mathcal{I})$.

We are going to use a set of basis functions for the space $\mathcal{H}_0^2(\mathcal{I})$ to generate a multiresolution analysis (MRA). To deal with the boundary conditions, we consider two scaling functions $\phi(x)$ as an interior scaling function and $\phi_b(x)$ a boundary scaling function,

$$\phi(x) = \frac{1}{6} \sum_{j=0}^{4} \begin{pmatrix} 4 \\ j \end{pmatrix} (-1)^j (x - j)_+^3, \tag{7}$$

$$\phi_b(x) = \frac{3}{2} x_+^2 - \frac{11}{12} x_+^3 + \frac{3}{2}(x-1)_+^3 - \frac{3}{4}(x-2)_+^3, \tag{8}$$

where $\phi(x)$ is a cubic spline and for any real number n

$$x_+^n(x) = \begin{cases} x^n, & \text{if } x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Scaling function satisfies the following relations.

Lemma:- 1.

$$\phi(x) = \sum_{k=0}^{4} 2^{-3} \begin{pmatrix} 4 \\ k \end{pmatrix} \phi(2x-k),$$

(9)

$$\phi_b(x) = \beta_{-1}\phi_b(2x) + \sum_{k=0}^{2} \beta_k \phi(2x-k),$$

where $\beta_{-1} = 1/4, \beta_0 = 11/16, \beta_1 = 1/2$ and $\beta_2 = 1/8$.

Lemma:- 2. *$\phi(x)$ and $\phi_b(x)$ as defined above satisfy the following properties:*

(10) $$\phi(x), \phi_b(x) \ \in \mathcal{H}_0^2(\mathcal{I});$$

(11) $$supp(\phi(x)) = [0,4]; \ \ supp(\phi_b(x)) = [0,3],$$

(12) $$\phi'(1) = -\phi'(3) = \frac{1}{2}, \ \phi'(2) = 0, \ \phi_b'(1) = \frac{1}{4}, \ \phi_b'(2) = -\frac{1}{2};$$

(13) $$\phi(1) = \phi(3) = \frac{1}{6}, \ \phi(2) = \frac{2}{3}, \ \phi_b(1) = \frac{7}{12}, \ \phi_b(2) = \frac{1}{6}.$$

For any $j, k \in \mathbb{Z}$, we define

(14) $$\phi_{j,k}(x) = \phi(2^j x - k), \ \ \phi_{b,j}(x) = \phi_b(2^j x),$$

and V_j can be written as

(15) $$V_j = \text{span}\{\phi_{j,k}(x) | 0 \leq k \leq n_j - 4; \phi_{b,j}(x), \phi_{b,j}(L-x)\},$$

where $n_j = 2^j L$.

Theorem:- 1. *Let $V_j, j \in \mathbb{Z}^+$ be a linear span as defined in (15). Then V_j forms an MRA for $\mathcal{H}_0^2(\mathcal{I})$ having norm (6) in the following sense:*
(i) $V_0 \subset V_1 \subset V_2 \subset ....$;
(ii) $clos_{\mathcal{H}_0^2(\mathcal{I})}(\cup_{j\in\mathbb{Z}^+} V_j) = \mathcal{H}_0^2(\mathcal{I})$; and
(iii) $\cap_{j\in\mathbb{Z}^+} V_j = V_0$;

We consider the following two wavelet functions $\psi(x), \psi_b(x)$ as

(16) $$\psi(x) = -\frac{3}{7}\phi(2x) + \frac{12}{7}\phi(2x-1) - \frac{3}{7}\phi(2x-2) \in V_1,$$

(17) $$\psi_b(x) = \frac{24}{13}\phi_b(2x) - \frac{6}{13}\phi(2x) \in V_1,$$

and

(18) $$\psi(n) = \psi_b(n) = 0 \text{ for all } n \in \mathbb{Z}.$$

Property (18) will be used in construction of a fast DWT.
For each $j \geq 0$, we define

(19) $$W_j = \text{span}\{\psi_{j,k}(x) | k = -1, 0, ...n_j - 2\},$$

where

(20) $$\psi_{j,k}(x) = \psi(2^j x - k), \ \ j \geq 0, k = 0, 1, 2...n_j - 3,$$

(21) $$\psi_{b,j}^l(x) = \psi_b(2^j x), \ \ \psi_{b,j}^r(x) = \psi_b(2^j(L-x)),$$

for the sake of simplicity, we use the following notation:

(22) $$\psi_{j,-1}(x) = \psi_{b,j}^l(x), \ \ \psi_{j,n_j-2} = \psi_{b,j}^r(x).$$

So, when $k = -1$ and $k = n_j - 2$, wavelet functions $\psi_{j,k}(x)$ will denote the two boundary wavelet functions, which can't be obtained by translating and dilating $\psi(x)$.

Theorem:- 2. *The $W_j, j \geq 0$, defined in (19) is the orthogonal compliment of V_j in V_{j+1} under the inner product (6),i.e*
(i) $V_{j+1} = V_j \bigoplus W_j$ for $j \in \mathbb{Z}^+$, where $\bigoplus$ stands for $V_j \perp W_j$ under the defined inner product and $V_{j+1} = V_j + W_j$, therefore,
(ii) $W_j \perp W_{j+1}, j \in \mathbb{Z}^+$, and
(iii) $\mathcal{H}_0^2(\mathcal{I}) = V_0 \bigoplus_{j \in \mathbb{Z}^+} W_j$

Proof. For proof see [1]. □

We can see that $\dim V_j = n_j - 1$ and $\dim W_j = n_j$. As a consequence of above theorem 2, any function $u(x) \in \mathcal{H}_0^2(\mathcal{I})$ can be approximated as closely as needed by a function $u_j(x) \in V_j = V_0 \bigoplus W_0 \bigoplus W_1 \bigoplus ... \bigoplus W_{j-1}$ for a sufficiently large j and $u_j(x)$ has a unique orthogonal decomposition

$$u_j(x) = u_0 + g_0 + g_1 + ... + g_{j-1}, \tag{23}$$

where $u_0 \in V_0, g_i \in W_i$.

When we deal with the functions belonging to $\mathcal{H}^2(\mathcal{I})$ having non-homogeneous boundary conditions we use the following interpolation near the boundaries for $j \geq 0$:

$$\Im_{b,j} u(x) = \alpha_1 \eta_1(2^j x) + \alpha_2 \eta_2(2^j x) + \alpha_3 \eta_2(2^j (L - x)) + \alpha_4 \eta_1(2^j (L - x)), \tag{24}$$

where η_1, η_2 are given as

$$\eta_1(x) = (1 - x)_+^3, \tag{25}$$

$$\eta_2(x) = 2x_+ - 3x_+^2 + \frac{7}{6}x_+^3 - \frac{4}{3}(x-1)_+^3 + \frac{1}{6}(x-2)_+^3, \tag{26}$$

and coefficients $\alpha_1, \alpha_2, \alpha_3$ and α_4 are determined by certain interpolating conditions. Since spline $\Im_{b,j} u(x)$ is expected to approximate the non-homogeneities of the function $u(x)$ at the boundaries, these interpolating conditions is called end conditions. Since we have shocks at boundary layers, therefore, we use not-a-knot conditions.

2.1. Not-a-Knot Conditions. In our case, solution changes dramatically near the boundary so we use the so called not-a-knot end conditions [8], which amounts to requiring that the spline $\Im_{b,j} u(x)$ agrees with function $u(x)$ at one additional point near each boundary. So we have the following equations for $\alpha_k, 1 \leq k \leq 4$:

$$\begin{aligned} \Im_{b,j} u(0) = u(0), \;\; \Im_{b,j} u(L) = u(L), \\ \Im_{b,j} u(\gamma_1) = u(\gamma_1), \;\; \Im_{b,j} u(\gamma_2) = u(\gamma_2). \end{aligned} \tag{27}$$

In our case we take $\gamma_1 = \frac{1}{2^{j+1}}, \; \gamma_2 = L - \frac{1}{2^{j+1}}$, we have

$$\begin{aligned} \alpha_1 = u(0), \;\; \alpha_2 = 6u(\gamma_1) - \frac{3u(0)}{4}, \\ \alpha_3 = 6u(\gamma_2) - \frac{3u(L)}{4}, \;\; \alpha_4 = u(L). \end{aligned} \tag{28}$$

Although in this case $u(x) - \Im_{b,j} u(x)$ is no longer in the space $\mathcal{H}_0^2(\mathcal{I})$, an interpolating spline

$$u_j(x) = \Im_{b,j} u(x) + u_0 + w_0 + w_1 + ... + w_{j-1}, \; u_0 \in V_0, w_i \in W_i, \; 0 \leq i \leq j-1,$$

with $\Im_{b,j} u(x)$ will still have an approximation of $u(x)$ of order $O(2^{-4j})$[9].

3. Discrete wavelet transform (DWT)

In this section, we introduce a DWT which directly decompose the sampling data of a function to its wavelet coefficients.

3.1. Interpolation operator $\Im_{V_j}$ in V_j. Consider any function $u(x) \in \mathcal{H}_0^2(\mathcal{I})$ and we define the interpolation operator $\Im_{V_j} u(x)$ for the data $\{u_k^j\}$ as,

$$\Im_{V_j} u(x) = c_{-1}\phi_b(x) + \sum_{k=0}^{n_j-4} c_k \phi_{j,k}(x) + c_{n_j-3}\phi_b(1-x), \tag{29}$$

where

$$u_k^j = u(x_k^j) \text{ for } x_k^j = \frac{k}{2^j},\ k = 1,..,n_j - 1. \tag{30}$$

$\Im_{V_j} u(x)$ interpolates data $u_k^j,\ k = 1,..,n_j - 1$, namely

$$\Im_{V_j} u(x_k^j) = u_k^j,\ \ k = 1, 2.., n_j - 1. \tag{31}$$

The transformation between $\mathbf{u}^{(j)} = (u_1^j, ..., u_{n_j-1}^j)^T$ and the coefficient $\mathbf{c} = (c_{-1}, c_0, ..., c_{n_j-3})^T$ is given by

$$\mathbf{u}^{(j)} = \mathbf{Bc}, \tag{32}$$

where

$$\mathbf{B} = \begin{bmatrix} \frac{7}{12} & \frac{1}{6} & & & & \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & & \\ & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ & & & & \frac{1}{6} & \frac{7}{12} \end{bmatrix}, \tag{33}$$

for our conveniences, interpolation in V_0 is considered for $j = -1$.

The interpolation operator $\Im_{W_j}$ in $W_j,\ j = 0, 1, 2..., J-1$ can be similarly defined. We choose the grid points in $\mathcal{I}$:

$$x_k^j = \frac{k + 1.5}{2^j},\ \ -1 \le k \le n_j - 2, \tag{34}$$

where $n_j = \text{Dim} W_j = 2^j L$.

The wavelet functions $\psi_{j,k}(x)$ satisfy point value vanishing property. For $j > i,\ -1 \le k \le n_j - 2$,

$$\psi_{j,k}(x_k^j) = 1,\ \ \psi_{j,k}(x_l^i) = 0, \tag{35}$$

for $-1 \le l \le n_j - 2$, if $i \ge 0$; $1 \le l \le n_j - 1$, if $i = -1$.

So $\{\phi_{0,k}(x)\}, \{\psi_{j,k}(x)\}_{j=0}^{\infty}$ form a hierarchical basis for the sobolev space $\mathcal{H}_0^2(\mathcal{I})$. Moreover, the point value vanishing property will be useful in obtaining the fast DWT.

The interpolation $\Im_{W_j} u(x)$ of the function $u(x)$ in $W_j,\ j \ge 0$, can be expressed as a linear combination of $\psi_{j,k}(x),\ k = -1, 0, ...n_j - 2$, namely,

$$\Im_{W_j} u(x) = \sum_{k=-1}^{n_j-2} \hat{u}_{j,k} \psi_{j,k}(x), \tag{36}$$

and

$$\Im_{W_j} u(x_k^j) = u(x_k^j),\ \ -1 \le k \le n_j - 2. \tag{37}$$

If we denote M_j is the $n_j th$ order matrix that relates $\hat{\mathbf{u}} = (\hat{u}_{j,-1}, ..., \hat{u}_{j,n_j-2})^T$ and $\mathbf{u} = (u^j_{-1}, ..., u^j_{n_j-2})^T$, as $u^j_k = u(x^j_k)$, then

$$\mathbf{u} = M_j \hat{\mathbf{u}}, \tag{38}$$

where

$$M_j = \begin{bmatrix} 1 & -\frac{1}{14} & & & & & \\ -\frac{1}{13} & 1 & -\frac{1}{14} & & & & \\ & -\frac{1}{14} & 1 & -\frac{1}{14} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & -\frac{1}{14} & 1 & -\frac{1}{14} & & \\ & & & -\frac{1}{14} & 1 & -\frac{1}{13} \\ & & & & -\frac{1}{14} & 1 \end{bmatrix}. \tag{39}$$

In order to calculate all the expansion coefficients, we need to solve tridiagonal system, which requires $8n_j$ operations, for each j. For any function $u(x) \in \mathcal{H}^2_0(\mathcal{I})$, we can find the wavelet interpolation at all interpolating points (30) and (34) as given below

$$\begin{aligned} u_J(x) = \mathcal{P}_J u(x) &= \hat{u}_{-1,-1}\phi_b(x) + \sum_{k=0}^{n_j-4} \hat{u}_{-1,k}\phi_{j,k}(x) + \hat{u}_{-1,n_j-3}\phi_b(1-x) \\ &+ \sum_{j=0}^{J-1}[\sum_{k=-1}^{n_j-2} \hat{u}_{j,k}\psi_{j,k}(x)] \\ &= u_{-1}(x) + \sum_{j=0}^{J-1} u_j(x), \end{aligned} \tag{40}$$

where

$$u_{-1}(x) = \Im_{V_0} u(x) \in V_0, \;\; u_j(x) = \sum_{k=-1}^{n_j-2} \hat{u}_{j,k}\psi_{j,k}(x) \in W_j, \; j \geq 0,$$

Let us denote $\mathbf{u} = (\mathbf{u}^{(-1)}, \mathbf{u}^{(0)}, ..., \mathbf{u}^{(J-1)})^T$, the value of $u(x)$ on all interpolation points, i.e

$$\mathbf{u}^{(-1)} = u(x^{-1}_k)^{n_j-1}_{k=1}; \;\; \mathbf{u}^{(j)} = u(x^j_k)^{n_j-2}_{k=-1}, \; j \geq 0,$$

and by $\hat{\mathbf{u}} = (\hat{\mathbf{u}}^{(-1)}, \hat{\mathbf{u}}^{(0)}, ..., \hat{\mathbf{u}}^{(J-1)})^T$, the wavelet coefficients, where

$$\hat{\mathbf{u}}^{(-1)} = (\hat{u}_{-1,k})^{n_j-1}_{k=1}; \;\; \hat{\mathbf{u}}^{(j)} = (\hat{u}_{j,k})^{n_j-2}_{k=-1}, \; j \geq 0.$$

After solving a tridiagonal system for $j-1$, the value of the interpolated function at level $j-1$ must be subtracted from the original value of the function at the grid points for level j, i.e $u(x^j_k) - \mathcal{P}_{j-1}u(x^j_k)$. This requires $7n_j + 5jn_j + n_j = (5j+8)n_j$ operations for each j. Therefore, total number of operation for finding all the coefficients is given by $8(n_j - 1) + \sum_{j=0}^{J-1}(5j+16)n_j$.

Lemma:- 3. *Let $0 < \alpha < 1$ and $u \in \mathcal{H}^2_0(\mathcal{I})$. If the second derivative of the function u is Hölder continuous with exponent α, at $x_0 \in \mathcal{I}$, i.e.*

$$|u''(x) - u''(x_0)| \leq C|x - x_0|^\alpha, \; x \in (x_0 - \delta, x_0 + \delta) \subset \mathcal{I},$$

for some $\delta > 0$, then for any $k \in Z, j \in Z^+$ such that $2^{-j}k \in (x_0 - \delta/2, x_0 + \delta/2)$,

$$|\hat{u}_{j,k}| = O(2^{-(\alpha+2)j}) \;\; \textit{as } j \to \infty$$

Proof can be found in [1]. Lemma implies that the wavelet coefficients $\hat{u}_{j,k}, j \geq 0$, reflect the singularity of the function to be approximated.

4. Adaptive collocation method using cubic spline and WOFD

In this section we outline adaptive collocation approach for solving singularly perturbed problems using cubic spline and WOFD. As we know that the multiresolution representation is not very convenient for the calculation of derivatives. It is the smallest scale which is important for differentiation because the basis function at small scale changes rapidly. On the other hand collocation method may be a good idea if the calculation of derivatives is performed on different scales and then added together. The collocation method proposed in [1] has the advantage that the derivatives can be calculated using the smallest scale by approximating the solution with a cubic spline. Here two different kinds of derivatives matrices have been proposed.

4.1. Adaptive choice of collocation points. We get the adaptive grid as follows:

- First we solve the given problem on a initial mesh as defined by equations (30) and (34) for the initial solution profile.
- We apply the discrete wavelet transform to the solution profile and calculate the wavelets coefficients. We locate the index (j, k) such that

  $$|\hat{u}_{j,k}| < \tau,$$

  where $\tau \geq 0$ be a prescribed tolerance. Using lemma 3 it is clear that wavelets coefficients will be smaller where the solution is smooth and large at the place of singularity like boundary layers in our case. A large value of wavelet expansion coefficient $\hat{u}_{j,k}$ is an indication that the grid spacing $1/2^j$ is too coarse to resolve u properly in the interval. Hence when a large value of $\hat{u}_{j,k}$ arises, we add points with spacing $1/2^{j+1}$ about position $k/2^j$ locally and remove the mesh points where wavelet coefficient are smaller than τ.

After getting the irregular adaptive grid, we apply the cubic spline and Lagrangian finite difference (discussed in the next section) to get the final solution.

4.2. Cubic spline. For the calculation of the derivative, we can use the fact that we have a representation of the solution on the basis consisting of cubic spline functions. Instead of differentiating the basis functions themselves, we use the continuity conditions for the derivatives. The cubic spline satisfy the following continuity conditions for the first and second derivatives:

$$u_i' = \frac{h_i}{6}u_{i-1}'' + \frac{h_i}{3}u_i'' + \frac{u_i - u_{i-1}}{h_i},\ 1 \leq i \leq L, \tag{41}$$

$$u_{i-1}' = -\frac{h_i}{3}u_{i-1}'' - \frac{h_i}{6}u_i'' + \frac{u_i - u_{i-1}}{h_i},\ 1 \leq i \leq L, \tag{42}$$

and

$$\frac{h_i}{h_i + h_{i+1}}u_{i-1}'' + 2u_i'' + \frac{h_{i+1}}{h_i + h_{i+1}}u_{i+1}'' = \frac{6}{h_i + h_{i+1}}[\frac{u_{i+1} - u_i}{h_{i+1}} - \frac{u_i - u_{i-1}}{h_i}], \tag{43}$$

where $1 \leq i \leq L,\ u_i' = u'(x_i),\ u_i'' = u''(x_i)$ and $h_i = x_i - x_{i-1}$. At boundary points $x = 0$, and $x = L$, we have

$$\frac{h_1}{3}u_0'' + \frac{h_1}{6}u_1'' = \frac{u_1 - u_0}{h_1} - \dot{u}_0, \tag{44}$$

$$\frac{h_L}{6}u''_{L-1} + \frac{h_L}{3}u''_L = \dot{u_L} - \frac{u_L - u_{L-1}}{h_L}, \tag{45}$$

where $u_0 = u_L = 0$ and $\dot{u_0}, \dot{u_L}$ are the approximations of the first derivative of $u(x)$ at x_0, x_L as

$$\begin{aligned} u'(0) &= \frac{1}{h}\sum_{k=0}^{p} c_k u(kh) + O(h^s), \\ u'(L) &= -\frac{1}{h}\sum_{k=0}^{p} c_k u(L-kh) + O(h^s), \end{aligned} \tag{46}$$

where $h > 0$ and $p \geq 3$. For $p = 3$, if we take

$$c_0 = -\frac{11}{6},\ c_1 = 3,\ c_2 = -\frac{3}{2},\ c_3 = \frac{1}{3},$$

then we have an error of $O(h^3)$.

In matrix form these relations become

$$\mathcal{T}_1\mathbf{u}'' = \mathcal{T}_2\mathbf{u} + \dot{\mathbf{u}}_b, \tag{47}$$

where $\mathbf{u} = (u(x_0), u(x_1), ..., u(x_L))^T$, $\mathbf{u}'' = (u''(x_0), u''(x_1), ..., u''(x_L))^T$ and $\dot{\mathbf{u}}_\mathbf{b} = (\dot{u}_0, 0, 0, ..., 0, \dot{u}_L)^T$, where matrices are given as

$$\mathcal{T}_1 = \begin{bmatrix} \frac{h_1}{3} & \frac{h_1}{6} & & & & & \\ \frac{h_1}{h_1+h_2} & 2 & \frac{h_2}{h_1+h_2} & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \frac{h_i}{h_i+h_{i+1}} & 2 & \frac{h_{i+1}}{h_i+h_{i+1}} & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \frac{h_{L-1}}{h_{L-1}+h_L} & 2 & \frac{h_L}{h_{L-1}+h_L} \\ & & & & & \frac{h_L}{6} & \frac{h_L}{3} \end{bmatrix}, \tag{48}$$

and

$$\mathcal{T}_2 = \begin{bmatrix} 0 & \frac{1}{h_1} & & & \\ \frac{6}{h_1(h_1+h_2)} & -\frac{6}{h_1h_2} & \frac{6}{h_2(h_1+h_2)} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{6}{h_{L-1}(h_{L-1}+h_L)} & -\frac{6}{h_{L-1}h_L} & \frac{6}{h_L(h_{L-1}+h_L)} \\ & & & \frac{1}{h_L} & 0 \end{bmatrix}, \tag{49}$$

by solving this equation for $\mathbf{u}''$, we can get the second derivative of $u(x)$. This calculation requires $(8+5)N$ operations, where N be the total number of grid points. As a result of the uniqueness of the spline $\mathcal{T}_1$ is invertible, so we have

$$u'' = \mathcal{T}_1^{-1}(\mathcal{T}_2\mathbf{u} + \dot{\mathbf{u}}_b) = \mathcal{T}u. \tag{50}$$

4.3. Wavelet optimized finite difference. WOFD works by using adaptive wavelet to generate an irregular adaptive grid, as discussed in section 4.1, which is then exploited using Lagrangian finite difference methods as discussed below.

Using mesh points (30) and (34), derivatives on non-uniform grid can also be approximated using Lagrangian interpolating polynomial through p points [2]. We consider only odd $p \geq 3$ because it makes the algorithm simpler. Let $w = \frac{p-1}{2}$ and define

$$u_I(x) = \sum_{k=i-w}^{i+w} u(x_k) \frac{P_{w,i,k}(x)}{P_{w,i,k}(x_k)}, \tag{51}$$

where

$$P_{w,i,k}(x) = \prod_{l=i-w, l \neq k}^{i+w} (x - x_l).$$

It follows that $u_I(x_i) = u(x_i)$ for $i = 0, 1, 2....N-1, N$ i.e u_I interpolates u at the grid points. Differentiation of $u_I(x)$ d times yields

$$u_I^d(x) = \sum_{k=i-w}^{i+w} u(x_k) \frac{P_{w,i,k}^d(x)}{P_{w,i,k}(x_k)}. \tag{52}$$

The derivatives $u_I^d(x)$ can then be approximated at all the grid points by

$$u_I^d = D_p^d u,$$

where the differentiation matrix D_p^d is defined by

$$[D_p^d]_{i,k} = \frac{P_{w,i,k}^d(x_i)}{P_{w,i,k}(x_k)};\ d = 1, 2.$$

First and second derivatives are

$$P_{w,i,k}^{(1)}(x) = \sum_{l=i-w, l \neq k}^{i+w} \prod_{m=i-w, m \neq k,l}^{i+w} (x - x_m), \tag{53}$$

and

$$P_{w,i,k}^{(2)}(x) = \sum_{l=i-w, l \neq k}^{i+w} \sum_{m=i-w, m \neq k,l}^{i+w} \prod_{n=i-w, n \neq k,l,m}^{i+w} (x - x_n). \tag{54}$$

5. Numerical results and discussion

Now we discuss adaptive collocation method for elliptic and parabolic problems.

5.1. Elliptic problem. We consider linear elliptic problem as

$$-\epsilon u_{xx} + u = 1 + 2\sqrt{\epsilon}(\exp(-x/\sqrt{\epsilon}) + \exp((x-1)/\sqrt{\epsilon})), \tag{55}$$

with boundary conditions

$$u(0) = 0 \text{ and } \quad u(1) = 0. \tag{56}$$

This problem has earlier been discussed in [10]. The exact solution is

$$u(x) = 1 - (1-x)\exp(-x/\sqrt{\epsilon}) - x\exp((x-1)/\sqrt{\epsilon}).$$

This problem has boundary layers at $x = 0$ and $x = 1$.

Definition:- A discretization method is said to be uniformly convergent (with respect to ϵ) of order $\varrho > 0$ in the norm $\|.\|$, if there exists a positive integer N_0, and positive numbers C, where N_0, C and ϱ are all independent of N and ϵ, such that for all $N \geq N_0$

$$\sup_{0<\epsilon\leq 1} \|u_\epsilon - U_\epsilon\| \leq CN^{-\varrho}$$

where U_ϵ is the numerical approximation for the analytical solution u_ϵ of the singularly perturbed problems in consideration and N is the number of mesh points in the interval.

Table 1 shows the maximum error obtained by the proposed methods. We have estimated all the error in the maximum norm because it is an appropriate norm for the study of boundary layer phenomena as discussed in [11]. This problem has also been studied in [5] using cubic B-spline where it was shown that the method is second order ($\varrho = 2$) uniform convergent in ϵ. If we interpolate a given function, say

TABLE 1. Maximum error for elliptic problem using adaptivity

$\epsilon = 2^{-k}$	$n_J = 2^J$	cubic spline	WOFD	$N, N_a^\star$
$k = 5$	$J = 7$	.0030	.0021	128, 56
	$J = 8$	.0030	.0024	256, 69
	$J = 10$	.0023	.0022	1024, 95
	$J = 11$	.0023	.0022	2048, 106
$k = 10$	$J = 7$	.0037	.0033	128, 61
	$J = 8$	.0025	.0012	256, 75
	$J = 10$	.0019	.0024	1024, 106
	$J = 11$	.0019	.0024	2048, 119
$k = 15$	$J = 7$	.0518	.0410	128, 61
	$J = 8$	.0455	.0171	256, 76
	$J = 10$	.0027	.0012	1024, 105
	$J = 11$	.0021	.0017	2048, 122

($\star$) N and N_a are the number of mesh points before and after adaptivity.

$g(x) \in C^4[a, b]$ on a uniform mesh $x_i = ih,\ 0 \le i \le n,\ h = \frac{(b-a)}{h}$, we get a general (not in terms of ϵ uniform convergence) fourth order convergence, which reduces to second order for non-uniform meshes as discussed in [1, 6] and [8]. Cubic B-spline gives second order uniform convergence in ϵ for piecewise uniform mesh as given by Shishkin [5]. As wavelets have a very strong advantage of data compression, therefore, at this point of time we are more concerned about this issue as compared to ϵ uniform convergence. Nevertheless, in view of the numerical results in Table 1, the proposed methods seems to have atleast first order ϵ uniform convergence on the adaptive grid and hence are able to resolve the boundary layer for all ϵ.

We define compression error as

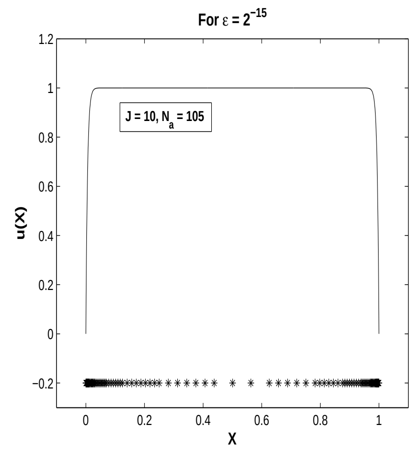
$$\mathcal{E}^\tau = \|u^\tau - u^0\|_\infty \tag{57}$$

where u^τ is the solution obtained with tolerance τ and u^0 corresponds to the solution obtained on the finest grid. In Table 2 we have given the compression error for different value of singular perturbation parameter ϵ and for different J ($n_J = 2^J$) using cubic spline and WOFD. Clearly, both the methods give almost same result but CPU time can be reduced using WOFD as compared to cubic spline because of very spares nature of the Lagrangian matrices in comparison of very dense matrices which arise in cubic spline.

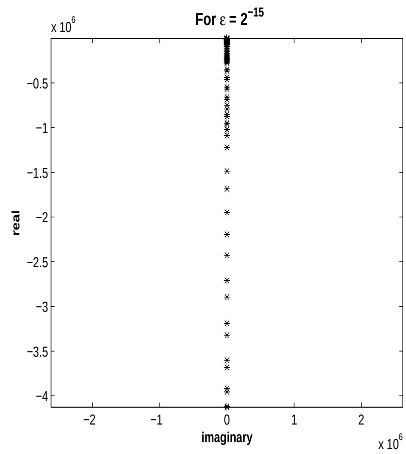
Figures 1, 2 show the solution for the elliptic problem for adaptive grid using cubic spline and WOFD. It is clear from these figures that mesh points are more concentrated at the boundaries (where the boundary layers occur). Figure 3 gives the compression for different values of tolerance τ and perturbation parameter ϵ. As we decrease the tolerance τ, significant coefficients increase almost linearly. Eigenvalues for the diffusion operator have also been plotted for both the methods in figures 1, 2 which show that all the eigenvalues are real.

TABLE 2. Compression error for elliptic problem with tolerance $\tau = 10^{-4}$

Methods	ϵ	$n_J = 2^J$	$\mathcal{E}^\tau(u)$	CPU(s) for u^0	CPU(s) for u^τ	$N,\ N_a$
Cubic Spline	2^{-10}	$J = 7$	.0004	.4360	.6420	$128, 61$
		$J = 9$	.0016	8.812	.7970	$512, 91$
		$J = 10$	.0018	80.84	.5000	$1024, 106$
		$J = 11$	.0019	526.98	1.2350	$2048, 119$
	2^{-15}	$J = 7$	.0001	.3130	.4770	$128, 61$
		$J = 9$	.0001	8.469	.4120	$512, 91$
		$J = 10$	.0008	81.60	.5000	$1024, 105$
		$J = 11$	.0014	512.9	.6489	$2048, 122$
WOFD	2^{-10}	$J = 7$	.0007	.0760	.3550	$128, 61$
		$J = 9$	.0016	.1560	.2500	$512, 91$
		$J = 10$	.0023	.3440	.4230	$1024, 106$
		$J = 11$	.0024	1.112	.3430	$2048, 119$
	2^{-15}	$J = 7$	.0001	.0971	.2370	$128, 61$
		$J = 9$	.0001	.1410	.2620	$512, 91$
		$J = 10$	.0010	.3600	.2940	$1024, 105$
		$J = 11$	.0010	1.019	.3689	$2048, 122$



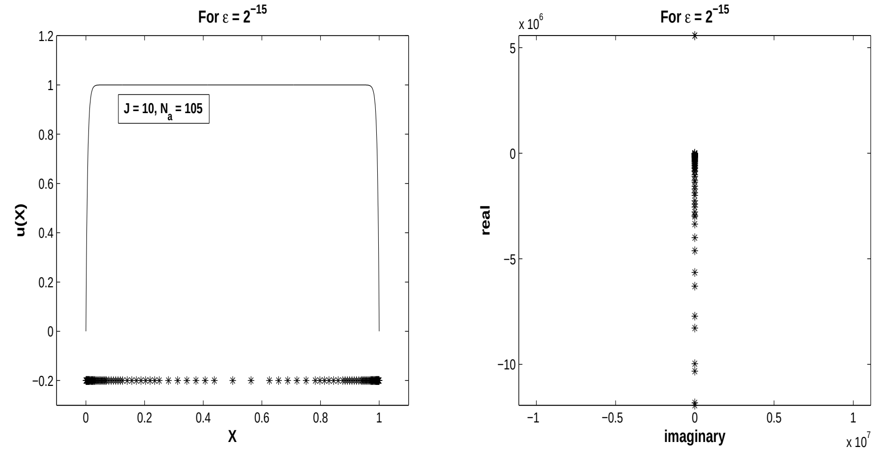
(a) Adaptive wavelet collocation solution with tolerance $\tau = 10^{-4}$ using WOFD.



(b) Eigenvalues for Lagrangian differential matrix.

FIGURE 1. Approximation for elliptic problem for $p = 3$.

5.2. Parabolic problem. The second problem is the semilinear parabolic problem with $a(x,t) = q_0(t) = q_1(t) = 0,\ b(x,t) = 1,\ f(x,t,u) = e^{-u} - 1$ and initial condition $s(x) = 1,\ 0 \le x \le 1$. Nonlinearity is dealt with quasilinearization technique of Bellman and Kalaba [12]. In the quasilinearization technique, the nonlinear differential equation is solved recursively by a sequence of linear differential equations. The main advantage of this method is that if the procedure converges, it converges quadratically to the solution of the original problem. The linear equation



(a) Adaptive wavelet collocation solution with tolerance $\tau = 10^{-4}$ using cubic spline.

(b) Eigenvalues for cubic spline differential matrix.

FIGURE 2. Approximation for elliptic problem.

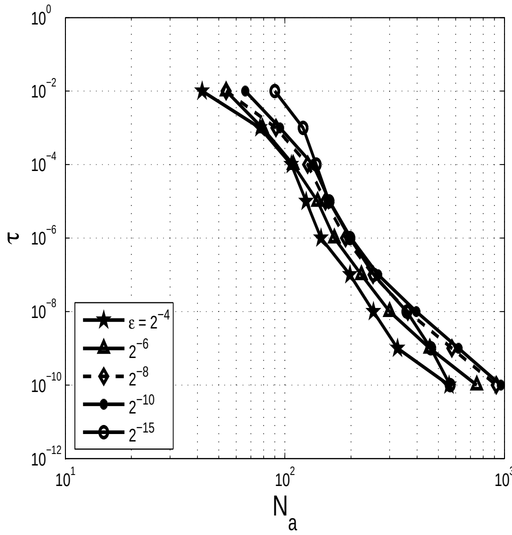


FIGURE 3. Compression for elliptic problem for various ϵ and for $N = 2^{12}$.

is obtained by using the first and second term of the Taylor's series expansion of the original non-linear differential equation. The non-linear term $f(x, u^{n+1}(x))$ can be expanded as

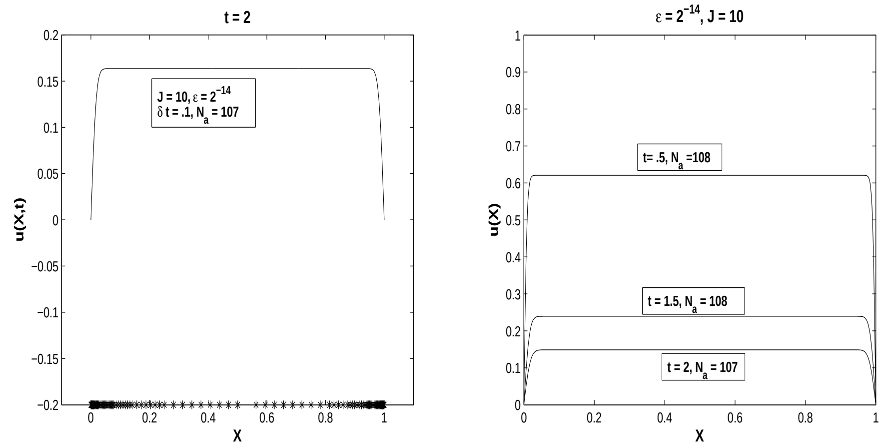
(58)

$$f(x, u^{n+1}(x)) = f(x, u^n(x)) + (u^{n+1} - u^n)f'(x, u^n(x)) + ... \text{ where } n = 0, 1, 2......;.$$

Here we consider an implicit two-level time difference scheme on the non-uniform mesh as given in equations (30) and (34). Table 3 gives the compression error for

TABLE 3. Compression error for parabolic problem with tolerance $\tau = 10^{-4}$ at time $t = 2$

Methods	ϵ	$n_J = 2^J$	$\mathcal{E}^\tau(u)$	CPU(s) for u^0	CPU(s) for u^τ	$N,\ N_a$
Cubic Spline	2^{-8}	$J = 8$	$9.053E(-5)$	11.391	4.609	$256, 71$
		$J = 9$	$8.926E(-5)$	171.64	19.032	$512, 83$
		$J = 10$	$7.720E(-5)$	1666.0	168.98	$1024, 95$
	2^{-14}	$J = 8$	$6.253E(-6)$	11.45	2.797	$256, 80$
		$J = 9$	$6.690E(-5)$	172.157	19.047	$512, 89$
		$J = 10$	$8.779E(-5)$	1674.65	169.26	$1024, 107$
WOFD	2^{-8}	$J = 8$	$1.700E(-4)$	1.266	1.719	$256, 71$
		$J = 9$	$1.200E(-4)$	2.344	2.032	$512, 83$
		$J = 10$	$9.000E(-5)$	6.188	2.969	$1024, 95$
	2^{-14}	$J = 8$	$1.344E(-5)$	1.266	1.781	$256, 77$
		$J = 9$	$1.558E(-4)$	2.438	1.953	$512, 91$
		$J = 10$	$1.288E(-4)$	6.250	3.00	$1024, 108$

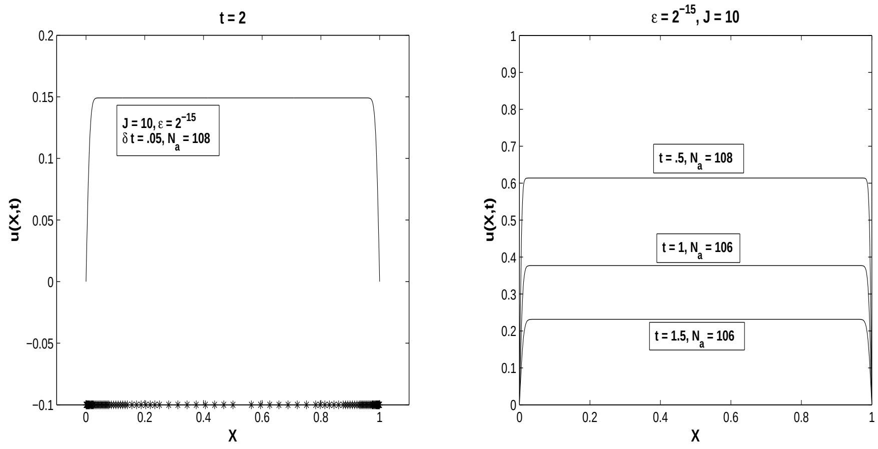


(a) Solution at $t = 2$. (b) Solution at various times.

FIGURE 4. Approximate solution with adaptive mesh for parabolic problem using cubic spline.

the parabolic equation at time $t = 2$ for both the methods. It can be seen that the proposed methods work well in term of compression error and CPU time.

Figures 4, 5 show the approximated solution for the parabolic problem at time $t = 2$ and at various times with adaptive mesh points N_a. Again we can see that more mesh points are concentrated in the area with large gradient. In order to represent the better solution in the area of boundary layers, one needs to add more points. One way to do this is to increase the resolution. With smaller and smaller scales, one can progressively resolves the boundary layers, as can be seen in all the figures. Once again WOFD gives better results in terms of CPU time as compared to cubic spline.



(a) Solution at $t = 2$.   (b) Solution at various times.

FIGURE 5. Approximate solution with adaptive mesh for parabolic problem using WOFD.

6. Conclusion

We have proposed cubic spline and WOFD using adaptivity for solving singularly perturbed reaction diffusion elliptic and parabolic problems. The adaptivity is achieved by using an interpolating wavelet basis. The novelty of this work lies in the threshholding scheme used to adaptively reducing the CPU time with wavelet compression of data. This adaption works both for features that are moving and for features that develop over time, such as boundary layers. WOFD takes the advantage over the cubic spline in term of CPU time and also can be extended easily to higher degree polynomials for different p.

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